

# Preface

Our work on rate-independent systems was stimulated by our search for evolutionary material models for shape-memory alloys that are flexible enough to encompass nonlinear and nonconvex phenomena. There are two difficulties, namely (i) modeling the nonlinear material behavior and (ii) modeling the dissipative processes. Under the guidance of classical thermodynamics, most material models assume that the forces induced by dissipative effects are linear in the rates of the associated variables, such as in viscoelastic friction. However, there are certain effects, such as dry friction, plasticity, and fracture, for which the force depends on the direction of the rate but not on its magnitude. Such processes are called rate-independent and include the class of hysteresis operators. The latter may relate the material response (such as stress or magnetization) to the changes of the material state (such as strain or magnetic field, respectively). In general, there may be memory effects, and rate-independence means that any monotone temporal rescaling of the input leads to the same correspondingly rescaled output. A first mathematical formulation of such an invariance of the material response under temporal rescalings is contained in [601, § 99] in the context of hypoelastic materials.

Of course, rate-independence is an idealization that is admissible only if the relevant processes of a system take place on a much slower time scale than the internal relaxation processes. In this book, we restrict our attention primarily to fully rate-independent systems, which can be justified in two ways. First, many mechanical processes can be well described by fully rate-independent models. Second, the restriction to such models allows for new mathematical tools for facilitating rigorous mathematical and numerical analysis that are not available for models with viscous dissipation. For instance, so far, the only existence result for the time-continuous evolutionary system for finite-strain elastoplasticity was obtained in the rate-independent case; see Section 4.2 and [374].

The description and discussion of rate-independent mechanical behavior has a relatively long history. Already in the eighteenth century, C.-A. Coulomb treated, besides his famous contribution to electricity, dry friction in [138], stating his *Law of Friction*: “*Kinetic friction is independent of the sliding velocity*”. In the early twentieth century, the theory of elastoplasticity became a major topic in mechanics

(e.g., [246, 265, 490, 617, 618]), where again rate-independent effects for the flow rule occur. However, the mathematical treatment of the associated evolutionary systems was not attacked until the 1970s, when J.-J. Moreau introduced methods from convex analysis to study rate-independent models, which he called sweeping processes; see [434, 435]. These works and some parallel works on monotone operator theory triggered the development of a whole mathematical field on analysis and numerics of elastoplasticity (at small strains); see [291, 578, 589] and the largely unnoticed parallel work [235, 237, 238, 240, 276] at *Zentralinstitut für Mathematik und Mechanik* of the *Akademie der Wissenschaften der DDR*, which can be seen as a predecessor of the *Weierstraß-Institut für Angewandte Analysis und Stochastik* (WIAS). See also [8, 257] for more information on the historical and more recent development of small-strain elastoplasticity.

Roughly at the same time, the modeling of materials was put on a more general level through the introduction of the notion of *generalized standard materials* by B. Halphen and Q.C. Nguyen [254], cf. also [218, 629], using convex dissipation potentials for general *internal variables*, the concept having been introduced earlier by P.W. Bridgman [97]. Thus, besides plasticity, one is also able to treat phase transformations, magnetization, piezoelectric effects, damage, and fracture; see [204, 383] and [101, 207] for relevant models for shape-memory materials.

The name *hysteresis* was probably introduced by Sir J.A. Ewing in his report “On hysteresis in the relation of strain to stress” (British Association Reports, 502, 1889). While the systematic study of hysteresis goes back to Prandtl, Preisach, and Ishlinskii, see [279, 491, 492], the mathematical theory of hysteresis operators was begun by M.A. Krasnosel’skiĭ and A.V. Pokrovskiĭ [316] with the property of rate-independence playing a major role in the definition. We refer to [101, 317, 320, 608] for recent expositions. In the present work, we do not rely on the notion of memory or hysteresis operators, since we are always including the internal variables (also called memory variables) in the state of the system to be described, thus obtaining evolutionary systems; see the discussion in Section 1.2.

Except for Chapter 5, this book focuses on quasistatic evolutionary systems that are rate-independent. Mostly, we confine ourselves to rate-independent systems  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ , where  $\mathcal{Q}$  is the state space,  $\mathcal{E}$  is an *energy-storage functional*, and  $\mathcal{D}$  is a *dissipation distance*; we will call such systems *energetic rate-independent systems* (ERIS) because both  $\mathcal{E}$  and  $\mathcal{D}$  are physically in concrete applications indeed energies. The natural (and, considering the amenability for theoretical analysis, the only) solutions generalizing other standard concepts for this general situation are the so-called *energetic solutions*  $q : [0, T] \rightarrow \mathcal{Q}$  as introduced in [426]. They are characterized by the static stability condition (S) and the total energy balance (E) for all  $t \in [0, T]$ :

$$(S) \quad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{D}(q(t), \tilde{q}) \quad \text{for all } \tilde{q} \in \mathcal{Q},$$

$$(E) \quad \mathcal{E}(t, q(t)) + \text{Diss}_{\mathcal{D}}(q, [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_s \mathcal{E}(s, q(s)) \, ds,$$

with  $\text{Diss}_{\mathcal{D}}(q, [0, t])$  the total dissipation ( $\sim$  variation) induced by  $\mathcal{D}$  of the process  $q$  on the time interval  $[0, t]$ . These solutions are also called *irreversible quasistatic evolutions* in the fracture models studied in [149, 152]. In fact, the general existence result presented in this work relies in part on the methods developed in these works. The advantage of this definition is that it does not involve derivatives of  $\mathcal{E}$ ,  $\mathcal{D}$ , and  $q$ , and hence is very flexible for treating nonsmooth and nonconvex situations, even applications without an underlying linear structure. Moreover, approximation results in the sense of  $\Gamma$ -convergence or numerical finite- or boundary-element methods can be developed quite easily. On the other hand, energetic solutions are intimately based on global energy minimization, which might be not an entirely relevant concept in some applications, especially if there is no underlying convexity structure. Therefore, another focus is on “more local” concepts of solutions. A large portion of the book deals with a more special situation whereby some natural underlying linear structure is at one’s disposal, and we can (at least formally) specify the time derivative  $\dot{q}$  and the driving force  $-\partial_q \mathcal{E}$ , and the dissipation distance  $\mathcal{D}$  can be induced by a dissipation metric  $\mathcal{R}(q, \dot{q})$ . Instead of ERIS  $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ , we will then speak just about a *rate-independent system* (RIS) determined by the triple  $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$  and governed (again at least formally) by an abstract doubly nonlinear evolution inclusion

$$\partial_q \mathcal{R}(q, \dot{q}) + \partial_q \mathcal{E}(t, q) \ni 0.$$

Physically, in concrete applications,  $\mathcal{R}$  is a power, whereas as already stated,  $\mathcal{E}$  and  $\mathcal{D}$  represent energies. The rate-independence is reflected by the property of the dissipation metric  $\mathcal{R}$  being 1-homogeneous in terms of the rate  $\dot{q}$ , or in other words, the dissipation potential  $\mathcal{R}(q; \cdot)$  is positively homogeneous of degree 1, so that particular situations may be very irregular, exhibiting, for example, jumps even if the external loading evolves smoothly. In addition to the energetic solutions, there is a menagerie of other solution concepts all of which fall essentially into a conventional weak-solution concept if  $q$  is absolutely continuous in time, but which may exhibit very different features, in particular in the aforementioned irregular situations.

The book is organized as follows. In Chapter 1, we introduce the subject on a rather intuitive and elementary level. The general existence theory for ERIS, which is based on incremental minimization problems, is presented in Chapter 2. Moreover, a general theory of  $\Gamma$ -convergence is developed there. While that chapter does not rely on any underlying linear structure, Chapter 3 addresses RIS  $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$  and the situations in which  $\mathcal{Q}$  is a weakly closed subset of a Banach space  $\mathbf{Q}$ , which allows for more specific considerations exploiting the concepts of differentials, convexity, and duality. To some extent, the origin and the prominent application of the above theory is the continuum mechanics of deformable solids or of undeformable ferroic materials. Relying on the concept of internal variables, in Chapter 4 we discuss specific inelastic processes in the bulk as plasticity, damage, and phase transformations, as well as some inelastic processes on the boundary, both at large and at small strains. Selected applications are discussed in all detail, whereas a greater number of applications are treated in only a summary fashion.

Finally, rate-independent systems may occur only as a subsystem of a larger system whose other parts exhibit responses on different time scales, and thus are to be considered rate-dependent. Even in such overall rate-dependent situations, one can benefit from the theory of rate-independent processes in an essential way. In continuum-mechanical applications, this typically includes inertial, viscous, or thermal effects. This combination widens the applicability of the presented material, which is demonstrated in Chapter 5 by the presentation of a few selected examples.

Because of the wide range of topics covered in this book, the necessary prerequisites are in the areas of functional analysis, partial differential equations, calculus of variations, and convex analysis. Without specifying the corresponding places, we expect the reader to be familiar with Hilbert and Banach spaces together with their weak and strong topologies and to have some basic knowledge of topological spaces; see, e.g., [553]. In the field of partial differential equations, we begin at the level of intermediate textbooks, including existence theory in Sobolev spaces, e.g., [179, 361]. For Chapter 5, we suggest advanced knowledge from monographs such as [532]. From the theory of calculus of variations, we mainly rely on the direct methods, which are well explained in [140, 534], as well as variational convergence [94, 95, 141]. For applications in continuum mechanics, [125] is a good starting point. The basic theory of convex analysis is nicely introduced in [84], while the infinite-dimensional theory is developed in [173]. Some basic material is also briefly surveyed in Appendices A and B, which include some further references.

This monograph reflects and summarizes the authors' research on rate-independent systems that has been carried out over the last 15 years, mostly in close collaboration. However, the work would not have been possible without the help of many colleagues, some acting as coauthors and others simply sharing their knowledge and enthusiasm. We have benefited greatly from the international mathematical community, in particular from collaborations in Germany, Czech Republic, Italy, France, and Spain. In particular, we especially acknowledge the collaborators who provided us with 2- or 3-dimensional numerical simulations and visualizations for particular applications, namely Marcel Arndt, Soeren Bartels, Barbora Benešová, Dušan Gabriel, Michal Kočvara, Martin Kružík, Christos G. Panagiotopoulos, Jan Valdman, Roman Vodička, and Jan Zeman. Moreover, we acknowledge the experimental physicists Oleg Heczko, Silvia Ignacová, Václav Novák, and Petr Šittner, of the Institute of Physics of the Czech Academy of Sciences, who kindly provided us snapshots of their experiments on shape-memory alloys.

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