Preface

Leonhard Euler introduced the numbers that are at the heart of this book in 1755. His motivation seems to have been to obtain a formula for the alternating sums of powers $(1^n - 2^n + 3^n - \cdots)$ in a manner analogous to what Jacques Bernoulli had done for the unsigned sums of powers. The connection with Bernoulli numbers motivated work of Julius Worpitzky in 1883 and Georg Frobenius in 1910. In the mid-twentieth century Leonard Carlitz wrote many papers surrounding Eulerian numbers and their use in number theory. We will discuss almost none of these topics in this book.

Rather, our starting point comes from later work of Carlitz and his collaborators, who began to study the Eulerian numbers as *combinatorial* quantities, following in the vein of late 19th and early 20th century combinatorialists like Simon Newcomb and Percy MacMahon. As explained by John Riordan in his 1958 textbook, a wonderful way to encounter the Eulerian numbers is as the answer to *Simon Newcomb's problem*:

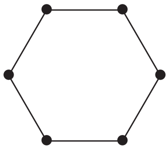
> ... a deck of cards of arbitrary specification is dealt out into a single pile so long as cards are in rising order, with like cards counted as rising, and a new pile is started whenever a non-rising card appears; with all possible arrangements of the deck, in how many ways do k piles appear?

If there are no ties among the cards (if we order the suits as well as the face values of the cards, say), then we can consider the deck of cards as a permutation, and the stacks correspond to maximal increasing runs in the permutation. The Eulerian numbers count the number of permutations of fixed size with a given number of increasing runs.

This book is not the first book written on the topic of Eulerian numbers. Dominique Foata and Marcel-Paul Schützenberger wrote "Théorie géométrique des polynômes eulériens" in 1970. This wonderful book collected and expanded upon many of the ideas surrounding the combinatorics of Eulerian numbers. Despite the title, there is little geometry (in the usual sense) in the book of Foata and Schützenberger. As they themselves explain, the title of their book comes from the fact that they use "propriétés géométriques (combinatoires) des permutations" to obtain their results. For them, "geometric" was synonymous with "combinatorial," which in this case meant a visual, almost tactile understanding of permutations and transformations of permutations.

In the decades since that book, geometry, in the usual sense, has most definitely entered the story of Eulerian numbers. For example, we now know how the Eulerian numbers arise in problems of counting integer points in polytopes, computing volumes of slices of a cube, and counting faces of simplicial complexes. Moreover, the Eulerian numbers can be understood in a larger context of finite reflection groups, known as *Coxeter groups*, where the geometry of hyperplane arrangements plays a major role.

To get a taste of the form some of these connections take, consider the following 1-dimensional simplicial complex:



It has one empty face, six vertices, and six edges. We can record this information in its *f-vector*, $(1, 6, 6)$, or *f-polynomial*, $1 + 6t + 6t^2$. Next we'll rewrite the f-vector in another basis, as a linear combination of rows of Pascal's triangle (right justified):

$$\begin{array}{lr} & (1,6,6) \\ \hline 1\times & (1,2,1) \\ 4\times & (1,1) \\ 1\times & (1) \end{array}$$

The coefficients of this expansion we will put into the *h-vector*, $(1, 4, 1)$, or *h-polynomial*, $1 + 4t + t^2$.

Now let's do something completely different. List out all permutations of $\{1, 2, 3\}$ and count their *descents*, i.e., the number of positions i such that $w(i) > w(i + 1)$:

w	des(w)
123	0
132	1
213	1
231	1
312	1
321	2

If we record the number of permutations with zero, one, and two descents in a vector, we get $(1, 4, 1)$. The polynomial with these coefficients is known as the Eulerian polynomial $S_3(t) = 1 + 4t + t^2$, which we observe is the same as the

h-polynomial of the hexagon above. This is not a coincidence! Moreover, that hexagon can be interpreted as the *Coxeter complex* of the symmetric group S_3. A big part of this book seeks to generalize and explain this example.

Another thing that was probably not apparent in 1970 but has since come to the forefront of this subject is that the Eulerian numbers have close cousins known as the *Narayana numbers*. These are named after Tadepalli Venkata Narayana, who described these numbers in a 1959 paper by counting certain types of lattice paths. The Narayana numbers possess many of the same properties as the Eulerian numbers and have many of the same geometric connections. Just as with Eulerian numbers, we can obtain the Narayana numbers by counting permutations according to descents. Here we only consider a certain subset of "pattern-avoiding" permutations that are in bijection with the paths studied by Narayana. The cardinality of this subset is given by the *Catalan numbers*. These numbers are ubiquitous in combinatorial mathematics. In fact, Richard Stanley has a book with a catalogue of objects counted by the Catalan numbers that includes over two hundred distinct entries!

This book has fourteen chapters split into three parts. Chapter 1 is a brief introduction to the classical Eulerian numbers from a modern, combinatorial point of view. Chapter 2 introduces the Catalan numbers and Narayana numbers, including a few different combinatorial models counted by these numbers. Chapter 3 discusses partially ordered sets, a topic that is central to modern enumerative combinatorics, and one that will be important for later chapters. Chapter 4 discusses a strong sort of symmetry property possessed by both the Eulerian numbers and the Narayana numbers. The first real connection to geometry comes in Chapter 5, where we discuss the geometric underpinnings for many of the later chapters. Chapter 6 is a brief diversion into refined enumeration and q-analogues for the Eulerian and Narayana numbers.

Part 2 consists of Chapters 8 and 9, with a supplementary Chapter 10*. In Chapter 8 we discuss some background from combinatorial topology, including simplicial complexes and the Dehn-Sommerville relations. Chapter 9 studies in detail how Eulerian numbers arise when counting faces of simplicial complexes.

Part 3 consists of Chapter 11, Chapter 12, and two supplementary chapters: 13* and 14*. Chapter 11 provides some background on Coxeter groups and discusses how there exist analogues of Eulerian numbers associated with any finite reflection group. Chapter 11 shows how the Narayana numbers can be similarly generalized to Coxeter groups. There are four supplementary Chapters sprinkled throughout the book, covering special topics. These are Chapters 7*, 10*, 13*, and 14*.

The book is primarily intended for a graduate student of combinatorics, or perhaps even an advanced undergraduate. Chapters 1–6, for example, would make for a good one-semester topics in combinatorics course. Very little in the way of background is assumed, particularly in the first four chapters.

Notes and literature references are included at the end of each chapter, along with some relevant problems to work on. Hints and solutions for the problems are given at the end of the book.

The writing style is meant to be expository. Rather than a "Definition-Theorem-Proof" format, I lean towards a more narrative style of writing. My hope is to focus on two main questions:

What is the truth? and Why is it true?

(Hyman Bass once told me that in any human endeavor these are the only two questions that matter.) In some cases answering these questions calls for a completely rigorous proof, but in others I find that a clearly explained "proof by generic example" does a better job of conveying the heart of the matter.

Finally, let me say this book represents my taste and knowledge in algebraic and enumerative combinatorics, omitting many interesting topics that can be related to Eulerian numbers (connections with number theory, topics in the statistics of permutations and words, theory of symmetric and quasisymmetric functions, juggling (!), card shuffling (!), and more). For me the Eulerian and Narayana numbers provide an interesting way to learn about various overlapping topics in modern combinatorial mathematics. I hope that this book can serve as an introduction to a circle of ideas that has been growing for the past few decades. Students can get a glimpse at recent developments while learning more general combinatorial techniques in a motivated way. For those in the research community, I hope the book can serve as a reference, by collecting many of these results in one place. I certainly look forward to having a copy on my shelf.

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