Chapter 2
First-Order Logic: Proofs with Quantifiers

2.1 Introduction

This chapter introduces quantifiers and first-order logic. The first few sections demonstrate methods for designing proofs through preliminary versions of the Deduction Theorem for first-order logic, Substitutivity of Equivalences, and transformations into prenex forms. A final section derives features of predicates for equality and inequality, either as primitive predicate constants, or predicates defined from other primitive binary predicate constants. The prerequisite for this chapter is a working knowledge of the Classical Propositional Logic for instance, as in chapter 1.

Pure first-order logic includes *quantifiers* corresponding to phrases such as "for each object" or "there exists an (at least one) object" with templates for functions of objects and relations between objects. Applied first-order logic replaces such templates with functions and relations specific to areas such as algebra, arithmetic, geometry, or set theory.

2.2 The Pure Predicate Calculus of First Order

In grammar, the noun "predicate" designates the verb or verbal phrase that makes a statement about the subject of a clause. In logic, similarly, a predicate is a part of an atomic formula that makes a statement about individual objects in applications.

2.2.1 Logical Predicates

The logical concept of **predicate** depends upon the theory under consideration.

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2.1 Example (predicates in arithmetic). Some versions of arithmetic have only two predicates, which state that a number is the sum or product of two numbers:

$M = K + L$ (read "M equals the sum of K and L"),

$N = K * L$ (read "N equals the product of K and L"),

or equivalent formulae with a different notation [18, p. 318], [72, p. 202–203].

These predicates are called "ternary" because each involves three variables.

2.2 Example (predicates in geometry). In geometry, a predicate may state that a point is on a line, or that a point lies between two other points on the same line, or that two segments, or two angles, are congruent, or that a line lies in a plane:

$P \in L$ (read "the point P is on the line L"),

$X < Y < Z$ (read "the point Y is between the points X and Z"),

$PQ \equiv RS$ (read "the segment PQ is congruent to the segment RS"),

$\angle ABC \equiv \angle PQR$ (read "the angle ABC is congruent to the angle PQR"),

$L \subset E$ (read "the line L lies in the plane E"),

or equivalent formulations with a different notation [61, Ch. I].

2.3 Example (predicates in set theory). Some versions of set theory have only one predicate, which states that a set is an element of a set:

$X \in Y$ (read "X is an element of Y"),

$\varnothing \in Y$ (read "the empty set is an element of Y"),

$X \in \varnothing$ (read "X is an element of the empty set"),

$\varnothing \in \varnothing$ (read 'the empty set is an element of the empty set").

This predicate is called "binary" because it involves two variables, X and Y.

The formulae in the foregoing examples are called **terms** or **atomic formulae** because they are the simplest formulae in arithmetic, geometry, and set theory. Thus, $X \in Y$ is a term, or, in other words, an atomic formula.

In arithmetic, the symbols 0 and 1 are called **individual constants**, because they always denote the numbers zero and one, respectively. Similarly, in set theory, the symbol $\varnothing$ is an **individual constant**, because it always denotes the empty set. In contrast, the symbols $=$ and $\in$ are called **predicate constants**, because they always denote the relations of equality and set membership, respectively. Logics that include such constants are called **applied** predicate calculi; they may also include other **functional constants** or **relational constants** corresponding to other relations between objects. In contrast, logics that do not include any constants but allow for variables representing arbitrary individuals, predicates, functions, and relations are called **pure** predicate calculi. Thus a pure predicate calculus is a general logic that may later apply to algebra, arithmetic, geometry, and set theory as well.

In applied logics, if an atomic formula contains a variable, then it may, but need not, have a Truth value. For example, the formula $X \in Y$ has no Truth value, because different substitutions for X and Y can yield different Truth values. However some formulae may contain variables and yet have a Truth value.

2.4 Example. In logics with an "equality" relation, the formula $X = X$ is True for every X.

2.5 Example. In binary arithmetic the formula $0 * X = 0$ is True for every X.

2.6 Example. In binary arithmetic $0 * X = 1$ is False for every X.

2.7 Example. In set theory the formula $X \in \varnothing$ is False for every X.

2.8 Example. In the theory of well-formed sets the formula $X \in X$ is False for every X.

2.2.2 Variables, Quantifiers, and Formulae

The formulae studied here are those specified in definition 2.9.

2.9 Definition (well-formed formulae). Select *three* disjoint lists of symbols.

Every symbol from the *first* list of symbols, which may consist of one or more letter(s) from a specified alphabet, P, Q, ..., optionally with subscript(s) $P_\flat$, $P_{\flat\flat}$, ..., superscript(s) $P^\sharp$, $P^{\sharp\sharp}$, ..., or "middlescript(s)" $P|$, $P||$, ..., is called a **formulaic letter**. Such formulaic letters are not parts of the predicate calculus, but they help in describing the following rules to define well-formed formulae.

Also, every symbol from the *second* list of symbols, which may consist of one or more letter(s) from a specified alphabet, A, B, ..., optionally with subscript(s) $A_\flat$, $A_{\flat\flat}$, ..., superscript(s) $A^\sharp$, $A^{\sharp\sharp}$, ..., or "middlescript(s)" $A|$, $A||$, ..., is called a **propositional variable** or a **sentence symbol** [31, p. 17]. (Propositional variables may later be replaced in pure calculi by functional or relational variables, or in applied calculi by atomic formulae, which may include individual variables or constants specific to applications.)

Moreover, every symbol from the *third* list of symbols, which may consist of one or more letter(s) from a specified alphabet, X, Y, ..., optionally with subscript(s) $X_\flat$, $X_{\flat\flat}$, ..., superscript(s) $X^\sharp$, $X^{\sharp\sharp}$, ..., or "middlescript(s)" $X|$, $X||$, ..., is called an **individual variable**. (Individual variables may later be replaced by items specific to applications, for instance, numbers in arithmetic, or points in geometry.)

Every propositional variable or atomic formula is a **well-formed formula**. For all well-formed formulae P and Q, and for every individual variable X, the following four strings of symbols are also well-formed formulae:

$(W1)$	$\neg(P)$	(read "not P"),
$(W2)$	$(P) \Rightarrow (Q)$	(read "P implies Q" or "if P, then Q"),
$(W3)$	$\forall X(P)$	(read "for each X, P"),
$(W4)$	$\exists X(P)$	(read "there exists X such that P").

Furthermore, only strings of symbols built from letters or variables through applications of the rules W1–W4 can be well-formed formulae. Equivalent definitions apply to other connectives and to prefix and postfix notations.

2.2.3 Proper Substitutions of Free or Bound Variables

In the logic presented here, only individual variables may appear immediately after either quantifier, $\forall$ (read "for each") or $\exists$ (read "there exists"). Because of this restriction, this logic is a **first order logic**. Logical systems allowing for propositional variables to appear immediately after a quantifier are of second or higher order.

In Boolean logic, if a formula P is True regardless of X, but if P also contains another variable Z, then substituting Z for X can change the Truth value of P.

2.10 Counterexample. Consider any context with at least two different objects, for instance, two binary numbers in arithmetic, two points in geometry, or two sets in set theory. Thus for each object X there exists a *different* object Z, whence $\forall X\{\exists Z[\neg(X = Z)]\}$ is True. Replacing Z by X in $\exists Z[\neg(X = Z)]$ gives $\exists X[\neg(X = X)]$, which is False, because each object equals itself, by example 2.4. Thus, replacing Z by X in the True formula $\forall X\{\exists Z[\neg(X = Z)]\}$ yields the False formula $\forall X\{\exists X[\neg(X = X)]\}$. Similarly, replacing X by Z in the True formula $\forall X\{\exists Z[\neg(X = Z)]\}$ gives the False formula $\forall Z\{\exists Z[\neg(Z = Z)]\}$.

One way (not pursued here) to avoid the phenomenon exhibited in counterexample 2.10 consists of substituting parameters other than variables [117]. Alternatively, counterexample 2.10 shows that substitutions of a variable by another must obey certain rules, for instance, with the concepts introduced in definition 2.11.

2.11 Definition (free or bound variables). For each individual variable X and for each logical formula P, an occurrence of the variable X is **bound** in the formula P if and only if in P that occurrence of the variable X immediately follows $\forall$ or $\exists$, or if it appears in the **scope** of the quantifier, which is defined to be between either $\forall X($ or $\exists X($ and the corresponding right parenthesis $)$). An occurrence of the variable X is **free** in P if and only if that occurrence of X is not bound in P. A logical formula is **closed** if and only if it does not contain any free occurrence of any variable. A logical **sentence** is a closed logical formula.

2.12 Example. This example focuses on the formula from counterexample 2.10.

In the formula $\exists Z[\neg(X = Z)]$, both occurrences of the variable Z are *bound*.
In the formula $\exists Z[\neg(X = Z)]$, the only occurrence of the variable X is *free*.
The formula $\exists Z[\neg(X = Z)]$ is *not* closed, because it contains a free occurrence of X.

In contrast, the formula $\forall X\{\exists Z[\neg(X = Z)]\}$ is closed, because all occurrences of X and Z are bound.

The following definitions avoid the phenomenon in counterexample 2.10.

2.13 Definition (change of bound variables). The substitution of the variable Z for each *bound* occurrence of the variable X in P is denoted by $\mathrm{Subb}_Z^X(P)$. Such a substitution is called a **change of bound variables** if and only if Z does not occur in P. Such a change of bound variables is **proper** if and only if X does not occur freely in P and Z does not occur in P.

2.14 Example. In the formula $\exists Z[\neg(X = Z)]$ from counterexample 2.10, the substitution of X for the *bound* occurrences of Z is *not* a change of bound variables, because X already occurs in the formula.

Definition 2.13 explicitly applies only to variables. In particular, it does *not* allow for substitutions of constants for bound variables.

2.15 Remark. If $\mathrm{Subb}_Z^X(P)$ is a change of bound variables in a formula P, then $\mathrm{Subb}_X^Z[\mathrm{Subb}_Z^X(P)]$ reproduces P.

Indeed, if $\mathrm{Subb}_Z^X(P)$ is a change of bound variables, then Z does not occur in P. Consequently, the only occurrences of Z in $\mathrm{Subb}_Z^X(P)$ are those replacing bound occurrences of X. Therefore, Z does not occur freely in $\mathrm{Subb}_Z^X(P)$. Thus, $\mathrm{Subb}_X^Z[\mathrm{Subb}_Z^X(P)]$ replaces *all* the occurrences of Z in $\mathrm{Subb}_Z^X(P)$, all of which are bound, by the initially bound occurrences of X, and reverts to the initial formula P.

Depending on the axiomatic system, an axiom, theorem, or inference rule may declare that two formulae that differ from each other only by changes of bound variables are mutually equivalent, so that $(P) \Leftrightarrow [\mathrm{Subb}_Z^X(P)]$ [89, p. 181].

2.16 Definition (change of free variables). A formula P **admits** (the substitution of) Z for an individual variable X, or, in other words, Z is **free** (to be substituted) **for** X in P, if and only if in the substitution of Z for every free occurrence of X

- every free occurrence of X becomes a free occurrence of Z,

or, equivalently, if X, Z, and P satisfy the following condition:

- either Z is an individual constant, or
- Z is an individual variable, and substituting Z for every *free* occurrence of X in P does not convert any free occurrence of X into a bound occurrence of Z.

In the present exposition, the notation $\mathrm{Subf}_Z^X(P)$ states that P admits Z for X, and substitutes Z for each free occurrence of X in P.

2.17 Example. In counterexample 2.10, the formula $\exists Z[\neg(X = Z)]$ does *not* admit Z for X, or, in other words, Z is *not* free for X, because substituting Z for every free occurrence of X converts the free occurrence of X into a bound occurrence of Z in $\exists Z[\neg(Z = Z)]$.

Thus another way to avoid the phenomenon exhibited in counterexample 2.10 consists in allowing only substitutions *admitted* in the sense of definition 2.16 [72, p. 94], [84, p. 48] [108, p. 37], [110, p. 101].

2.18 Remark. If an individual variable Z does not occur in a formula P, then P admits Z for each individual variable X. Indeed, Z does not appear in P after any quantifier ($\forall$ or $\exists$); thus Z is not bound in P, so that every free occurrence of X is replaced by a free occurrence of Z in $\mathrm{Subf}_Z^X(P)$.

2.19 Remark. If an individual variable Z does not occur in a formula P, and if P does not contain any bound occurrences of an individual variable X, then $\mathrm{Subf}_X^Z[\mathrm{Subf}_Z^X(P)]$ is P. Indeed, by hypothesis all occurrences of X are free in P, and by remark 2.18 they all become free occurrences of Z in $\mathrm{Subf}_Z^X(P)$. Also, X does not occur in $\mathrm{Subf}_Z^X(P)$. Again by remark 2.18 but with X and Z swapped, all occurrences of Z in $\mathrm{Subf}_Z^X(P)$ become free occurrences of X in $\mathrm{Subf}_X^Z[\mathrm{Subf}_Z^X(P)]$, which no longer contain any occurrences of Z. Thus $\mathrm{Subf}_X^Z[\mathrm{Subf}_Z^X(P)]$ is P.

2.20 Remark. All well-formed formulae P and Q result from the construction specified in definition 2.9, so that individual variables occur only immediately after quantifiers, or in terms (atomic formulae) that are then combined with connectives. Consequently, the following pairs of formulae are not only mutually equivalent but also mutually identical [117, p. 44]:

- $\mathrm{Subf}_Z^X[\neg(P)]$ and $\neg[\mathrm{Subf}_Z^X(P)]$.
- $\mathrm{Subf}_Z^X[(P) \Rightarrow (Q)]$ and $[\mathrm{Subf}_Z^X(P)] \Rightarrow [\mathrm{Subf}_Z^X(Q)]$.
- $\mathrm{Subb}_Z^X[\neg(P)]$ and $\neg[\mathrm{Subb}_Z^X(P)]$.
- $\mathrm{Subb}_Z^X[(P) \Rightarrow (Q)]$ and $[\mathrm{Subb}_Z^X(P)] \Rightarrow [\mathrm{Subb}_Z^X(Q)]$.

The following abbreviation is convenient.

2.21 Definition (abbreviation). The notation $\exists!X(P)$ (read "there exists a unique X such that P" or "there exists exactly one X such that P") abbreviates the formula

$$\exists X\big[(P) \wedge \big(\forall Z\{[\mathrm{Subf}_Z^X(P)] \Rightarrow (Z = X)\}\big)\big].$$

2.2.4 Axioms and Rules for the Pure Predicate Calculus

As the axioms of the propositional calculus reflect patterns of deductive reasoning with implications and negations, the axioms of the predicate calculus reflect patterns of deductive reasoning with quantifiers. There also exist several choices of initial axioms for use with quantifiers, for instance, the following axioms [18, §30, p. 171–172], [122, p. 170].

2.22 Definition. The following **axioms of the Pure Predicate Calculus** govern the use of the **universal quantifier** $\forall$ and the **existential quantifier** $\exists$.

Axiom Q0 *Axioms of the Propositional Calculus, but here with well-formed formulae as in definition 2.9, are axioms of the predicate calculus.*

Axiom Q1 (specialization) $[\forall X(P)] \Rightarrow [\mathit{Subf}_Z^X(P)]$, *if P admits Z for X.*

Axiom Q2 $\{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{(P) \Rightarrow [\forall X(Q)]\}$, *if P contains no free X.*

Axiom Q3 $\{\forall X[\neg(P)]\} \Leftrightarrow \{\neg[\exists X(P)]\}$.

Axiom Q4 $\{\exists X[\neg(P)]\} \Leftrightarrow \{\neg[\forall X(P)]\}$.

The first axiom (schema) of the predicate calculus (Q0) and the rules of inference carry all the theorems from the propositional calculus over to theorems of the predicate calculus. In particular, different propositional calculi, which may result from different axiom systems, may lead to different predicate calculi.

2.23 Example. From definition 1.4, the following formulae (P1) and (P2) form a system of two axioms for the Pure Positive Implicational Propositional Calculus:

$$\text{(P1)} \qquad (P) \Rightarrow [(Q) \Rightarrow (P)],$$

$$\text{(P2)} \qquad \{(P) \Rightarrow [(Q) \Rightarrow (R)]\} \Rightarrow \{[(P) \Rightarrow (Q)] \Rightarrow [(P) \Rightarrow (R)]\}.$$

Formulae (P1), (P2), and

$$\text{(P3)} \qquad \{[\neg(Q)] \Rightarrow [\neg(P)]\} \Rightarrow [(P) \Rightarrow (Q)],$$

form a system of three axioms for the Pure Classical Propositional Calculus.

The second axiom (schema) (Q1) corresponds to the notion that if an individual variable X may occur in a formula P, and if P is True regardless of X, in other words, if $\forall X(P)$ is True, then P remains True with X replaced by any individual variable or constant Z. If X and Z are the same variable, then axiom Q1 gives $[\forall X(P)] \Rightarrow (P)$.

The third axiom (schema) (Q2) describes the relation between the universal quantifier (“for each”) and the logical connective of the Pure Positive Implicational Propositional Calculus (“if ... then”).

The fourth axiom, for the existential quantifier (Q3), states that a formula P is False for every X if and only if there does *not* exist any X for which P is True.

Similarly, the fifth axiom (schema), for the existential quantifier (Q4), states that there exists some X for which P is False if and only if it is False that P is True for every X. Axiom Q4 asserts the existence of an object. Consequently, axiom Q4 applies neither to “empty” theories where nothing exists, nor to logics that require not only existence but also the determination of which objects satisfy a formula.

Besides the axioms, the predicate calculus allows for proofs of theorems through the following **rules of inference**.

2.24 Definition (rules of inference). The following **rules of inference** hold.

2.25 Rule (“Modus Ponens” (abbreviated by M. P.), or “Detachment”).

If P is a theorem, and
if $(P) \Rightarrow (Q)$ is a theorem,
then Q is a theorem.

2.26 Rule (Generalization).

If P is a theorem,
then $\forall X(P)$ is also a theorem.

2.27 Definition (theorems and proofs). A **proof** is a sequence of well-formed formulae H, K, L, ... P, Q, R, where each formula is either (a substitution in) an axiom (schema), or results from a previous formula in the sequence by any rule of inference (*Detachment*, *Generalization*, or Substitution).

A formula is a **theorem** if and only if it is a (usually the last) formula in a proof. The notation $\vdash R$ means that R is a theorem.

2.28 Example. Every axiom of the predicate calculus is a theorem.

2.2.5 Exercises on Quantifiers

Each of the following ten exercises lists one formula P. Identify a formula that is logically equivalent to $\neg(P)$ among the same ten exercises.

2.1 . $\forall X[\exists Y(X \in Y)]$

2.2 . $\forall X[\exists Y(Y \in X)]$

2.3 . $\forall X[(X \in A) \Rightarrow (X \in B)]$

2.4 . $\forall X\{(X \in C) \Leftrightarrow [(X \in A) \wedge (X \in B)]\}$

2.5 . $\forall X\{(X \in C) \Leftrightarrow [(X \in A) \vee (X \in B)]\}$

2.6 . $\exists X\big(\{(X \in C) \wedge [\neg(X \in A)] \wedge [\neg(X \in B)]\} \vee \{[\neg(X \in C)] \wedge [(X \in A) \vee (X \in B)]\}\big)$

2.7 . $\exists X\{(X \in A) \wedge [\neg(X \in B)]\}$

2.8 . $\exists X\{\forall Y[\neg(Y \in X)]\}$

2.9 . $\exists X\big(\big[(X \in C) \wedge \{[\neg(X \in A)] \vee [\neg(X \in B)]\}\big] \vee \{[\neg(X \in C)] \wedge [(X \in A) \wedge (X \in B)]\}\big)$

2.10 . $\exists X\{\forall Y[\neg(X \in Y)]\}$

2.2.6 Examples with Implicational and Predicate Calculi

The examples of theorems and proofs selected for this and the subsequent subsections gradually build up a tool to design proofs by substituting mutually equivalent formulae for one another. As a first step, the following derived rules of inference will simplify proofs by avoiding potentially lengthy instances of axiom Q2.

2.29 Theorem (derived rule). *If X is not free in R, and if $\forall X[(R) \Rightarrow (S)]$ is a theorem, then $(R) \Rightarrow [\forall X(S)]$ is also a theorem.*

Proof. Apply axiom Q2 and *Detachment*:

$\vdash \forall X[(R) \Rightarrow (S)]$	hypothesis,
$\vdash \{\forall X[(R) \Rightarrow (S)]\} \Rightarrow \{(R) \Rightarrow [\forall X(S)]\}$	axiom Q2, no free X in R,
$\vdash (R) \Rightarrow [\forall X(S)]$	*Detachment*.

□

2.30 Theorem (derived rule). *If X does not occur freely in R, and if $(R) \Rightarrow (S)$ is a theorem, then $(R) \Rightarrow [\forall X(S)]$ is also a theorem.*

Proof. Apply *Generalization* and theorem 2.29:

$\vdash (R) \Rightarrow (S)$	hypothesis,
$\vdash \forall X[(R) \Rightarrow (S)]$	*Generalization*,
$\vdash (R) \Rightarrow [\forall X(S)]$	theorem 2.29, no free X in R.

□

Theorem 2.31 reveals a situation where $\forall Y[\text{Subf}_Y^X(U)]$ may be replaced by $\forall X[\text{Subf}_X^X(U)]$, which is $\forall X(U)$.

2.31 Theorem (change of bound variables). *If Y does not occur in U, then $\vdash [\forall X(U)] \Rightarrow \{\forall Y[\mathit{Subf}_Y^X(U)]\}$.*

If Y does not occur in U, and if U does not contain any bound occurrence of X, then conversely $\vdash [\forall X(U)] \Leftarrow \{\forall Y[\mathit{Subf}_Y^X(U)]\}$.

Proof. This proof follows Monk's [89, p. 180, thm. 10.55].

$$\vdash \underbrace{[\forall X(U)]}_{R} \Rightarrow \underbrace{[\text{Subf}_Y^X(U)]}_{S} \qquad \text{specialization (axiom Q1), no } Y \text{ in } U,$$

$$\vdash \overbrace{[\forall X(U)]}^{R} \Rightarrow \overbrace{\{\forall Y[\text{Subf}_Y^X(U)]\}}^{\forall Y[S]} \qquad \text{theorem 2.29, no } Y \text{ in } U, \text{ so no } Y \text{ in } R.$$

For the converse, if Y does not occur in U, and if U does not contain any bound occurrence of X, then each occurrence of X in U is replaced by a free occurrence of Y in $\text{Subf}_Y^X(U)$. Consequently, if V denotes the formula $\text{Subf}_Y^X(U)$, then V contains no occurrences of X and no bound occurrences of Y. Moreover, $\text{Subf}_X^Y[\text{Subf}_Y^X(U)]$ reproduces U, by remark 2.19. Thus, applying the previous result to the formula V and swapping the roles of X and Y give $\vdash [\forall Y(V)] \Rightarrow \{\forall X[\text{Subf}_X^Y(V)]\}$. □

2.32 Remark (change of bound variables). Theorem 2.31 shows that two formulae P and Q that differ from each other only by the names of their bound variables are mutually equivalent. Indeed, if the variable Y does not occur in a formula P, and if U is any atomic formula (for instance, $X \in Z$) that occurs in P, then U does not contain any bound variables; in particular, U does not contain any bound occurrence of X. Consequently $\vdash [\forall X(U)] \Leftrightarrow \{\forall Y[\text{Subf}_Y^X(U)]\}$ by theorem 2.31. This substitution may use different new variables $Y, Y_\flat, Y_{\flat\flat}, \ldots$, for different atomic formulae $U, U_\flat, U_{\flat\flat}, \ldots$ and then proceed to more complicated components of P. Using the same new variables in P and Q results in two identical formulae.

2.33 Example. Let P denote the formula $\forall X\{\exists Z[\neg(X = Z)]\}$, and let Q denote the formula $\forall W\{\exists Y[\neg(W = Y)]\}$. The variables $Y_\flat$ and $Y_{\flat\flat}$ occur in neither P nor Q.

In P, let $U_\flat$ denote the atomic formula $X = Z$. Then $\vdash [\exists Z(U_\flat)] \Leftrightarrow \{\exists Y_\flat[\mathrm{Subf}^{Z}_{Y_\flat}(U_\flat)]\}$ by theorem 2.31. Let $U_{\flat\flat}$ denote the resulting formula $\exists Y_\flat[\neg(X = Y_\flat)]$. Then theorem 2.31 shows that $\vdash [\forall X(U_{\flat\flat})] \Leftrightarrow \{\forall Y_{\flat\flat}[\mathrm{Subf}^{X}_{Y_{\flat\flat}}(U_{\flat\flat})]\}$, which is $\forall Y_{\flat\flat}\{\exists Y_\flat[\neg(Y_{\flat\flat} = Y_\flat)]\}$. The same formula results from the same procedure applied to Q.

The following selection of theorems also relates the present axioms to other axiom systems in subsection 2.2.8. Their proofs follow Church's [18, p. 186–188].

2.34 Theorem. *For all P, Q, and X,* $\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow (Q)\}$.

Proof. Apply the Implicational Calculus with axiom Q1:

$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow [(P) \Rightarrow (Q)]$ axiom Q1,
$\vdash [\forall X(P)] \Rightarrow (P)$ axiom Q1,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow (Q)\}$ derived rule (theorem 1.31).

□

2.35 Theorem. *For all P, Q, and X,* $\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow [\forall X(Q)]\}$.

Proof. Apply the Implicational Calculus with *Generalization*, axiom Q2, and theorems 2.30 and 2.34:

$\vdash \underbrace{\{\forall X[(P) \Rightarrow (Q)]\}}_{R} \Rightarrow \underbrace{\{[\forall X(P)] \Rightarrow (Q)\}}_{S}$ theorem 2.34,

$\vdash \overbrace{\{\forall X[(P) \Rightarrow (Q)]\}}^{R} \Rightarrow \overbrace{\big(\forall X\{[\forall X(P)] \Rightarrow (Q)\}\big)}^{\forall X\{S\}}$ theorem 2.30,
$\vdash \big(\forall X\{[\forall X(P)] \Rightarrow (Q)\}\big) \Rightarrow \{[\forall X(P)] \Rightarrow [\forall X(Q)]\}$ axiom Q2,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow [\forall X(Q)]\}$ transitivity (1.16).

□

2.36 Counterexample. The converse of theorem 2.35, which would be

$$[\forall X(P) \Rightarrow \forall X(Q)] \Rightarrow \{\forall X[(P) \Rightarrow (Q)]\},$$

is *False* in contexts with two *different* objects Y and Z, so that $\neg(Y = Z)$ is True:

- $\forall X[(X = Y) \Rightarrow (X = Z)]$ is False, because substituting Y for X gives $[(Y = Y) \Rightarrow (Y = Z)]$, which is False, because of the True hypothesis $Y = Y$ and the False conclusion $Y = Z$.
- $\forall X(X = Y)$ is False, because substituting Z for X gives $(Z = Y)$, which is False by the assumption that $\neg(Y = Z)$.
- $[\forall X(X = Y)] \Rightarrow [\forall X(X = Z)]$ is True, because of its False hypothesis.
- $\{[\forall X(X = Y)] \Rightarrow [\forall X(X = Z)]\} \Rightarrow \{\forall X[(X = Y) \Rightarrow (X = Z)]\}$ is False, because of the True hypothesis and the False conclusion.

2.37 Theorem (derived rule). *For all P, Q, and X, if $\vdash (P) \Rightarrow (Q)$, then $\vdash [\forall X(P)] \Rightarrow (Q)$ and $\vdash [\forall X(P)] \Rightarrow [\forall X(Q)]$.*

Proof. Apply theorems 2.34 and 2.35:

$\vdash (P) \Rightarrow (Q)$	hypothesis,
$\vdash \forall X[(P) \Rightarrow (Q)]$	*Generalization*,
$\vdash \{[\forall X(P)] \Rightarrow (Q)\}$	theorem 2.34 and *Detachment*,
$\vdash [\forall X(P)] \Rightarrow [\forall X(Q)]$	theorem 2.35 and *Detachment*.

□

2.38 Theorem. *For all P, Q, and X, if P does not contain any free occurrence of X, then $\vdash \{(P) \Rightarrow [\forall X(Q)]\} \Leftrightarrow \{\forall X[(P) \Rightarrow (Q)]\}$.*

Proof. Axiom Q2 gives $\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{(P) \Rightarrow [\forall X(Q)]\}$. For the converse, use the Pure Positive Implicational Propositional Calculus with axioms Q1 and *Generalization*:

$\vdash [\forall X(Q)]\Rightarrow(Q)$	axiom Q1,
$\vdash \{(P)\Rightarrow[\forall X(Q)]\}\Rightarrow\{(P)\Rightarrow[\forall X(Q)]\}$	theorem 1.14,
$\vdash \{(P)\Rightarrow[\forall X(Q)]\}\Rightarrow[(P)\Rightarrow(Q)]$	theorem 1.32,
$\vdash \forall X\big(\underbrace{\{(P)\Rightarrow[\forall X(Q)]\}}_{R}\Rightarrow\underbrace{[(P)\Rightarrow(Q)]}_{S}\big)$	*Generalization*,
$\vdash \overbrace{\{(P)\Rightarrow[\forall X(Q)]\}}^{R}\Rightarrow\big(\overbrace{\forall X\{[(P)\Rightarrow(Q)]\}}^{\forall X\{S\}}\big)$	theorem 2.29, no free X in $(P)\Rightarrow[\forall X(Q)]$.

□

2.39 Theorem. *For all P and X, if P does not contain any free occurrence of X, then $\vdash [\forall X(P)] \Rightarrow (P)$ and $\vdash [\forall X(P)] \Leftarrow (P)$.*

Proof. Axiom Q1 gives $\vdash [\forall X(P)] \Rightarrow (P)$. For the converse, apply theorems 1.14 and 2.30:

$\vdash (P) \Rightarrow (P)$	theorem 1.14,
$\vdash (P) \Rightarrow [\forall X(P)]$	theorem 2.30.

□

2.40 Remark. The statement of theorem 2.39 suggests that if P contains a free occurrence of X, then the implication $(P) \Rightarrow [\forall X(P)]$ may differ from the *Generalization* rule, from $\vdash P$ to infer $\vdash \forall X(P)$, which applied only if P is a theorem.

2.41 Example. If P denotes the formula $X = \varnothing$, then $(P) \Rightarrow [\forall X(P)]$ becomes $(X = \varnothing) \Rightarrow [\forall X(X = \varnothing)]$, which is not a theorem. Indeed, if $(X = \varnothing) \Rightarrow [\forall X(X = \varnothing)]$ were a theorem, then substituting $\varnothing$ for the free occurrences of X by specialization and *Detachment* would yield $(\varnothing = \varnothing) \Rightarrow [\forall X(X = \varnothing)]$, which is not a theorem, because $\varnothing = \varnothing$ is True while $\forall X(X = \varnothing)$ is False in set theory.

Theorem 2.42 provides a converse for theorem 2.35 if X is not free in P.

2.42 Theorem. *For all P, Q, and X, if X does not occur freely in P, then $\vdash \{[\forall X(P)] \Rightarrow [\forall X(Q)]\} \Rightarrow \{\forall X[(P) \Rightarrow (Q)]\}$.*

Proof. Use theorems 2.38 and 2.39, with H, K, L, M as in theorem 1.36:

$\vdash \{(\underbrace{P}_{H}) \Rightarrow \underbrace{[\forall X(Q)]}_{L}\} \Rightarrow \{\underbrace{\forall X[(P) \Rightarrow (Q)]}_{M}\}$ theorem 2.38, no free X in P,

$\vdash (\underbrace{P}_{H}) \Rightarrow \underbrace{[\forall X(P)]}_{K}$ theorem 2.39, no free X in P,

$\vdash \{\underbrace{[\forall X(P)]}_{K} \Rightarrow \underbrace{[\forall X(Q)]}_{L}\} \Rightarrow \{\underbrace{\forall X[(P) \Rightarrow (Q)]}_{M}\}$ theorem 1.36.

□

2.2.7 Examples with Pure Propositional and Predicate Calculi

The following theorems invoke the full Classical Propositional Calculus, including contraposition and its converse for negations, or Tarski's axioms for equivalences.

2.43 Theorem. *For all P, Q, and X,*
$\vdash \{\forall X[(P) \Leftrightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Leftrightarrow [\forall X(Q)]\}$.

Proof. Apply theorems 2.37 and 2.35 with the transitivity of implication:

$\vdash [(P) \Leftrightarrow (Q)] \Rightarrow [(P) \Rightarrow (Q)]$ definition of $\Leftrightarrow$,

$\vdash \{\forall X[(P) \Leftrightarrow (Q)]\} \Rightarrow \{\forall X[(P) \Rightarrow (Q)]\}$ theorem 2.37,

$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow [\forall X(Q)]\}$ theorem 2.35,

$\vdash \{\forall X[(P) \Leftrightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow [\forall X(Q)]\}$ transitivity.

The converse conclusion results from the symmetry of $\Leftrightarrow$ and swapping P and Q. The final result then follows from theorem 1.55. □

For the records, theorem 2.44 combines theorems 2.35, 2.42 and 2.43.

2.44 Theorem. *For all P, Q, and X, if X does not occur freely in P, then* $\vdash \{[\forall X(P)] \Rightarrow [\forall X(Q)]\} \Leftrightarrow \{\forall X[(P) \Rightarrow (Q)]\}$.

Proof. Apply theorems 2.35, 2.42 and 2.43. □

2.45 Theorem (derived rule). *For all P, Q, and X, if* $\vdash (P) \Leftrightarrow (Q)$*, then* $\vdash [\forall X(P)] \Leftrightarrow [\forall X(Q)]$.

Proof. Apply *Generalization*, theorem 2.43, and *Detachment*:

$\vdash (P) \Leftrightarrow (Q)$ hypothesis,

$\vdash \forall X[(P) \Leftrightarrow (Q)]$ *Generalization*,

$\vdash \{\forall X[(P) \Leftrightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Leftrightarrow [\forall X(Q)]\}$ theorem 2.43,

$\vdash [\forall X(P)] \Leftrightarrow [\forall X(Q)]$ *Detachment*.

□

Theorems 2.46 and 2.47 show that $\exists$ could be defined in terms of $\forall$ and double negation, or vice versa, provided that axiom Q0 includes the full Classical Propositional Calculus.

2.46 Theorem. *For all P and X,* $\vdash [\exists X(P)] \Leftrightarrow \big(\neg\{[\forall X[\neg(P)]\}\big)$.

Proof. Apply the full propositional calculus and axiom Q3:

$\vdash \{\neg[\exists X(P)]\} \Leftrightarrow \{\forall X[\neg(P)]\}$ axiom Q3,

$\vdash \big(\neg\{\neg[\exists X(P)]\}\big) \Leftrightarrow \big(\neg\{\forall X[\neg(P)]\}\big)$ contraposition and its converse,

$\vdash [\exists X(P)] \Leftrightarrow \big(\neg\{\forall X[\neg(P)]\}\big)$ double negation and transitivity. □

2.47 Theorem. *For all P and X,* $\vdash [\forall X(P)] \Leftrightarrow \big(\neg\{[\exists X[\neg(P)]\}\big)$.

Proof. Apply the full propositional calculus and axiom Q4:

$\vdash \{\neg[\forall X(P)]\} \Leftrightarrow \{\exists X[\neg(P)]\}$ axiom Q4,

$\vdash \big(\neg\{\neg[\forall X(P)]\}\big) \Leftrightarrow \big(\neg\{\exists X[\neg(P)]\}\big)$ contraposition and its converse,

$\vdash [\forall X(P)] \Leftrightarrow \big(\neg\{\exists X[\neg(P)]\}\big)$ double negation and transitivity. □

2.48 Theorem (existential generalization). *For all X, Y, and P,* $\vdash [Subf^X_Y(P)] \Rightarrow [\exists X(P)]$. *In particular,* $\vdash (P) \Rightarrow [\exists X(P)]$.

Proof. Apply the propositional calculus with axioms Q1 and Q3:

$\vdash \{\forall X[\neg(P)]\} \Rightarrow \{\mathrm{Subf}^X_Y[\neg(P)]\}$ axiom Q1,

$\vdash \{\neg[\exists X(P)]\} \Rightarrow \{\forall X[\neg(P)]\}$ axiom Q3,

$\vdash \{\mathrm{Subf}^X_Y[\neg(P)]\} \Rightarrow \{\neg[\mathrm{Subf}^X_Y(P)]\}$ remark 2.20,

$\vdash \{\neg[\exists X(P)]\} \Rightarrow \{\neg[\mathrm{Subf}^X_Y(P)]\}$ transitivity,

$\vdash [\mathrm{Subf}^X_Y(P)] \Rightarrow [\exists X(P)]$ converse contraposition & *Detachment*.

□

Theorem 2.49 provides a converse to theorem 2.48 if X is not free in P.

2.49 Theorem. *For all P and X, if X is not free in P, then* $\vdash [\exists X(P)] \Leftrightarrow (P)$.

Proof. Apply the propositional calculus with theorems 2.46 and 2.39:

$\vdash [\neg(P)] \Rightarrow \{\forall X[\neg(P)]\}$ theorem 2.39, no free X in P,

$\vdash \big(\neg\{\forall X[\neg(P)]\}\big) \Rightarrow \{\neg[\neg(P)]\}$ contraposition,

$\vdash [\exists X(P)] \Rightarrow \big(\neg\{\forall X[\neg(P)]\}\big)$ theorem 2.46,

$\vdash [\exists X(P)] \Rightarrow \{\neg[\neg(P)]\}$ transitivity,

$\vdash \{\neg[\neg(P)]\} \Rightarrow (P)$ double negation,

$\vdash [\exists X(P)] \Rightarrow (P)$ transitivity.

The converse is theorem 2.48. □

2.2.8 Other Axiomatic Systems for the Pure Predicate Calculus

With the rules of *Detachment* and *Generalization*, and equivalent propositional calculi, Margaris [84, p. 49] and Rosser [110, p. 101] use the following axiom schemata for the predicate calculus:

Axiom A4 $\{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow [\forall X(Q)]\}$.

Axiom A5 $[\forall X(P)] \Rightarrow [\mathit{Subf}_Y^X(P)]$.

Axiom A6 $(P) \Rightarrow [\forall X(P)]$ *if* X *does not occur freely in* P.

Margaris allows *Generalization*s only of axioms but proves a deduction theorem that then leads to the same rule of *Generalization* [84, p. 49].

In contrast, Kleene [72, p. 107] uses two axiom schemata

∀-schema $[\forall X(P)] \Rightarrow [\mathrm{Subf}_Y^X(P)]$,
∃-schema $[\mathrm{Subf}_Y^X(P)] \Rightarrow [\exists X(P)]$,

paired with two inference rules, where X does not occur freely in P:

∀-rule from $(P) \Rightarrow (Q)$ infer $(P) \Rightarrow [\forall X(Q)]$, where X is not free in P,
∃-rule from $(P) \Rightarrow (Q)$ infer $[\exists X(P)] \Rightarrow (Q)$, where X is not free in Q.

In both systems, $\exists X(P)$ is merely an abbreviation for $\neg\{\forall X[\neg(P)]\}$, as also in the systems of Church [18, p. 171] and Stoll [122, p. 115]. Reversely, in other systems $\forall X(P)$ is merely an abbreviation for $\neg\{\exists X[\neg(P)]\}$, for instance, in Kunen's [74, p. 3]. Axioms such as Q3 and Q4 partially dissociate the quantifiers from the axiom(s) for the negation in the selected propositional calculus.

Also, yet another way to define substitutions of free variables consists in performing substitutions with a different procedure, as outlined in definition 2.50.

2.50 Definition (proper substitution of free variables). A **proper substitution** of a variable Z for each *free* occurrence of a *different* variable X in a logical formula P consists of the following three steps:

(1) Identify a variable that does *not* occur in P, for example, Y.
(2) In P, replace each *bound* occurrence of Z by Y.
(3) Then replace each *free* occurrence of X by Z.

Steps (1) and (2) produce a change of bound variables according to definition 2.13. After step (2), P no longer contains any bound occurrence of Z, and hence no strings of the form $\forall Z$ or $\exists Z$. Consequently, P now admits Z for X in step (3).

Definitions 2.16 and 2.50 thus provide two ways to avoid the phenomenon in counterexample 2.10. Both ways lend themselves to the same notation.

2.51 Definition. The notation $\mathrm{Subf}_Z^X(P)$ states that P admits Z for X, and substitutes Z for each free occurrence of X in P. Alternatively, the same notation $\mathrm{Subf}_Z^X(P)$ denotes the proper substitution of Z for each free occurrence of X in P. The two alternatives are compatible, by remark 2.15 and definition 2.50. For convenience, $\mathrm{Subf}_X^X(P)$ is defined to be P, and if X does *not* occur freely in P, then $\mathrm{Subf}_Z^X(P)$ is also defined to be P. The concept and notation for proper substitutions also apply to the substitution of a constant for a free variable. Because constants cannot appear immediately after a quantifier, they are not bound. Consequently, only the last step applies to the proper substitution of constants for free variables. Thus, $\mathrm{Subf}_\varnothing^X(P)$ merely substitutes $\varnothing$ for every free occurrence of X in P.

2.52 Example. For P consider the formula $\big(\forall X\{\exists Z[\neg(X = Z)]\}\big) \vee (X = \varnothing)$.

(1) Verify that the variable Y does not occur in P.
(2) Replace the bound occurrences of Z by Y, which gives the formula $(\forall X\{\exists Y[\neg(X = Y)]\}) \vee (X = \varnothing)$.
(3) Replace the free occurrence of X by Z, which gives the formula $(\forall X\{\exists Y[\neg(X = Y)]\}) \vee (Z = \varnothing)$ for $\mathrm{Subf}_Z^X(P)$.

In contrast, substituting the constant $\varnothing$ for every free occurrence of X in P yields $\big(\forall X\{\exists Z[\neg(X = Z)]\}\big) \vee (\varnothing = \varnothing)$ for $\mathrm{Subf}_\varnothing^X(P)$.

2.53 Example. A situation like that in definition 2.50 occurs with computer algorithms to swap two variables X and Z, which typically use a third variable Y distinct from X and Z as a temporary storage. First, the algorithm assigns X to Y, an operation denoted by $Y := X$. Second, the algorithm assigns Z to X, an operation denoted by $X := Z$. Finally, the algorithm assigns Y to Z, an operation denoted by $Z := Y$.

2.2.9 Exercises on Kleene's, Margaris's, and Rosser's Axioms

The following exercises show that Margaris's and Rosser's axioms A4–A6 are derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

2.11 . Prove that the abbreviation $\exists X(P)$ for $\neg\{\forall X[\neg(P)]\}$ is derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

2.12 . Prove that Margaris's and Rosser's axioms A4, A5, and A6 are theorems derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

The following exercises show that axioms Q1–Q4 are derivable from Margaris's and Rosser's axioms A4–A6 and the Classical Propositional Calculus.

2.13 . Prove that axiom Q2 is a theorem derivable from the rules of inference with Margaris's and Rosser's axioms A4–A6 and the Classical Propositional Calculus.

2.14 . Prove that axiom Q1 is a theorem derivable from the rules of inference with Margaris's and Rosser's axioms A4–A6 and the Classical Propositional Calculus.

2.15 . Prove that axiom Q4 is a theorem derivable from the rules of inference with Margaris's and Rosser's axioms A4–A6 and the Classical Propositional Calculus.

2.16 . Prove that axiom Q3 is a theorem derivable from the rules of inference with Margaris's and Rosser's axioms A4–A6 and the Classical Propositional Calculus.

The following exercises show that Kleene's schema and rules are derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

2.17 . Prove that Kleene's ∃-rule is derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

2.18 . Prove that Kleene's ∀-rule is derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

2.19 . Prove that Kleene's ∃-schema is derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

2.20 . Prove that Kleene's ∀-schema is derivable from the rules of inference with axioms Q1–Q4 and the Classical Propositional Calculus.

2.3 Methods of Proof for the Pure Predicate Calculus

If other considerations guarantee that a well-formed formula P has a proof but do not produce any proof of it, *then* writing down all the proofs of the predicate calculus, for instance, in increasing order of complexity, eventually yields among all such proofs a proof of P [18, p. 99–100, footnote 183]. However, if the shortest proof of P is very long, then this method may take longer than the time available to the user to arrive at any proof of P. Thus for all practical purposes this method may also fail to determine whether a formula is a theorem.

The problem of deciding whether a well-formed formula is a theorem, derivable from specified axioms and inference rules, is called the **decision problem**. For the pure predicate calculus, no algorithms can provide a step-by-step recipe applicable to all well-formed formulae to determine whether any such formula is a theorem, as proved by Church [16, 17]. Nevertheless, methods exist to help in deciding whether a well-formed formula is a theorem.

Trial and error is an option [114, p. 31], sometimes working backward from the particular well-formed formula as a final goal, or forward from the axioms, inference rules, and previous theorems as starting points or intermediate steps [72, p. 54–55]. The methods presented in this section guide this method of designing proofs.

2.3.1 Substituting Equivalent Formulae

One method of proof consists of replacing any occurrence of a formula by an equivalent formula, thanks to theorem 2.54 [18, p. 101, 124, 189], [108, p. 48].

2.54 Theorem (Substitutivity of Equivalence in the Pure Predicate Calculus, preliminary version). *For all well-formed logical formulae U and V, if* $\vdash (U) \Leftrightarrow (V)$*, and if a formula Q results from substituting any (zero, one, several, or all) occurrence(s) of U by V in a well-formed formula P, then* $\vdash (P) \Leftrightarrow (Q)$.

Proof (Outline). Theorems 1.29 and 1.46 have already established the conclusions for logical implications and negations.

In all cases, if Q is P, which results by substituting none of the occurrences of U by V, then $(P) \Leftrightarrow (Q)$ is $(P) \Leftrightarrow (P)$, which is theorem 1.63.

For the universal quantifier, if P is $\forall X(U)$, then Q is either $\forall X(U)$ or $\forall X(V)$. If Q is $\forall X(V)$, then P with $(U) \Rightarrow (V)$ yield Q, by theorem 2.37, and conversely, Q with $(V) \Rightarrow (U)$ yield P, by theorems 2.37, or also by theorem 2.45.

For the existential quantifier, theorem 2.46 reduces to the previous cases a formula P of the form $\exists X(U)$.

The general case follows by several applications of the previous case and the cases in theorems 1.29 and 1.46, in a way that may be specified more explicitly after the availability of the Principle of Mathematical Induction in chapter 4. □

2.55 Example. Let P denote the formula $\forall X\big([\exists Y(Y \in X)] \vee \{\forall Z[\neg(Z \in X)]\}\big)$, U the formula $\exists Y(Y \in X)$, and W the formula $\neg\{\forall Y[\neg(Y \in X)]\}$. Then $(U) \Leftrightarrow (W)$ by theorem 2.46. Moreover, let V denote the formula $\neg\{\forall Z[\neg(Z \in X)]\}$. Because Z does not occur in $\neg(Y \in X)$, theorem 2.31 shows that $(W) \Leftrightarrow (V)$. Hence $(U) \Leftrightarrow (V)$ by transitivity. Consequently, $(P) \Leftrightarrow (Q)$ where Q denotes the formula $\forall X\big(\big[\neg\{\forall Z[\neg(Z \in X)]\}\big] \vee \{\forall Z[\neg(Z \in X)]\}\big)$, which is $\forall X\big(\big[\neg\{V\}\big] \vee \{V\}\big)$. Since $(V) \vee [\neg(V)]$ is a theorem, by *Generalization* Q and hence P is also a theorem.

2.3.2 Discharging Hypotheses

A method to design a proof of an implication $(H) \Rightarrow (C)$ consists of first designing a derivation $H \vdash C$ of C from H, and then transforming the derivation $H \vdash C$ into a proof of $(H) \Rightarrow (C)$ by theorem 2.56 [108, p. 46–47].

2.56 Theorem (Deduction Theorem, preliminary version). *With any axiom system for which axioms P1, P2, and $(P) \Rightarrow (P)$ are axioms or theorems (or schema thereof), and for every derivation $H \vdash C$ of a formula C from a formula H with the propositional calculus, the rules of inference, and axioms Q1–Q4, but without* Generalization *on any free variable in H, there exists a proof of* $(H) \Rightarrow (C)$.

Proof (Outline). Every step of the derivation $H \vdash C$ is a formula, denoted here by S. If S is H, or an axiom, or results from two previous steps and *Detachment*, then the proof of the deduction theorem 1.22 for the Pure Positive Implicational Propositional Calculus shows how to replace S by $(H) \Rightarrow (S)$.

If S is a *Generalization* $\forall X(R)$ of a previous step R with a variable X that does not occur freely in H, then R has already been replaced by $(H) \Rightarrow (R)$. Hence *Generalization* gives $\forall X[(H) \Rightarrow (R)]$, whence, because X does not occur freely in H, theorem 2.29 yields a proof of $(H) \Rightarrow [\forall X(R)]$, which is $(H) \Rightarrow (S)$.

The general case follows by several applications of the previous cases in a way that may be specified more explicitly after the availability of the Principle of Mathematical Induction in chapter 4. □

Thus, the selection of axioms P1–P3 and Q1–Q4 leads to the Deduction Theorem (2.56) more directly than would other selections of otherwise equivalent axioms [108, p. 47]. In practice, however, a derivation of $H \vdash C$ of C from H may already suggest other logical steps that shortcut or bypass the entire procedure outlined in the proof of the Deduction Theorem 2.56.

To demonstrate such shortcuts, the following theorems provide means for bringing quantifiers to the "front" of a formula. For example, axioms Q3 and Q4 with theorem 2.54 already allow the replacement of $\neg[\exists X(P)]$ by $\forall X[\neg(P)]$, and of $\neg[\forall X(P)]$ by $\exists X[\neg(P)]$. In an implication $(R) \Rightarrow (S)$, each of R and S can be of the form (P), or $\forall X(P)$, or $\exists X(P)$, starting with $\forall$, or $\exists$, or no quantifiers, which leads to nine different cases. In the case where neither R nor S begins with a quantifier, then no quantifiers need to be brought to the front of $(R) \Rightarrow (S)$. The other eight cases form the object of the following theorems.

Theorem 2.57 handles a case where R is $\exists X(P)$ while S is Q.

2.57 Theorem. *If X does not occur freely in Q, then*
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{[\exists X(P)] \Rightarrow (Q)\}.$

Proof. Let H denote $\forall X[(P) \Rightarrow (Q)]$, and let C denote $[\exists X(P)] \Rightarrow (Q)$.

$\vdash \forall X[(P) \Rightarrow (Q)]$	hypothesis,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow [(P) \Rightarrow (Q)]$	specialization (Q1),
$\vdash (P) \Rightarrow (Q)$	*Detachment*,
$\vdash [\neg(Q)] \Rightarrow [\neg(P)]$	contraposition,
$\vdash \forall X\{[\neg(Q)] \Rightarrow [\neg(P)]\}$	*Generalization*,
$\vdash [\neg(Q)] \Rightarrow \{\forall X[\neg(P)]\}$	theorem 2.29, no free X in Q,
$\vdash \big(\neg\{\forall X[\neg(P)]\}\big) \Rightarrow \{\neg[\neg(Q)]\}$	contraposition,
$\vdash [\exists X(P)] \Rightarrow (Q)$	double negation (1.42) and 2.46.

Hence the Deduction Theorem (2.56) leads to a proof of $\{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\exists X(P)] \Rightarrow (Q)\}$. Yet the foregoing derivation suggests shortcuts:

$\vdash \{\forall X[(P)\Rightarrow(Q)]\}\Rightarrow[(P)\Rightarrow(Q)]$	axiom Q1,
$\vdash [(P)\Rightarrow(Q)]\Rightarrow\{[\neg(Q)]\Rightarrow[\neg(P)]\}$	contraposition,
$\vdash \{\forall X[(P)\Rightarrow(Q)]\}\Rightarrow\{[\neg(Q)]\Rightarrow[\neg(P)]\}$	transitivity,
$\vdash \{\forall X[(P)\Rightarrow(Q)]\}\Rightarrow\big(\forall X\{[\neg(Q)]\Rightarrow[\neg(P)]\}\big)$	theorem 2.30,
$\vdash \big(\forall X\{[\neg(Q)]\Rightarrow[\neg(P)]\}\big)\Rightarrow\big([\neg(Q)]\Rightarrow\{\forall X[\neg(P)]\}\big)$	theorem 2.29,
$\vdash \big([\neg(Q)]\Rightarrow\{\forall X[\neg(P)]\}\big)\Rightarrow\big(\neg\{\forall X[\neg(P)]\}\big)\Rightarrow\{\neg[\neg(Q)]\}$	contraposition,
$\vdash \big(\neg\{\forall X[\neg(P)]\}\big)\Rightarrow\{\neg[\neg(Q)]\}\Rightarrow\{[\exists X(P)]\Rightarrow(Q)\}$	1.42, 2.46,
$\vdash \{\forall X[(P)\Rightarrow(Q)]\}\Rightarrow\{[\exists X(P)]\Rightarrow(Q)\}$	transitivity.

For the converse, let H denote $[\exists X(P)] \Rightarrow (Q)$, and let C denote $\forall X[(P) \Rightarrow (Q)]$.

$\vdash [\exists X(P)] \Rightarrow (Q)$	hypothesis,
$\vdash (P) \Rightarrow [\exists X(P)]$	theorem 2.48,
$\vdash (P) \Rightarrow (Q)$	transitivity,
$\vdash \forall X[(P) \Rightarrow (Q)]$	*Generalization*.

Again the Deduction Theorem (2.56) leads to a proof of $\{[\exists X(P)] \Rightarrow (Q)\} \Rightarrow \{\forall X[(P) \Rightarrow (Q)]\}$ but the foregoing derivation suggests shortcuts:

$\vdash (P) \Rightarrow [\exists X(P)]$	theorem 2.48,
$\vdash \big(\{[\exists X(P)] \Rightarrow (Q)\}\big) \Rightarrow [(P) \Rightarrow (Q)]$	theorem 1.34,
$\vdash \big(\{[\exists X(P)] \Rightarrow (Q)\}\big) \Rightarrow \{\forall X[(P) \Rightarrow (Q)]\}$	theorem 2.30.

□

The proofs of the following theorems emerge from similar outlines, starting with a derivation $H \vdash C$ of C from H, and transforming it into a proof of $\vdash (H) \Rightarrow (C)$ by shortcuts suggested by the derivation or by the Deduction Theorem (2.56). To this end, the following derived rule proves useful.

2.58 Theorem (derived rule). *If X is not free in Q, and if $\vdash (P) \Rightarrow (Q)$, then $\vdash [\exists X(P)] \Rightarrow (Q)$.*

Proof. Apply theorem 2.57:

$\vdash (P) \Rightarrow (Q)$	hypothesis,
$\vdash \forall X[(P) \Rightarrow (Q)]$	*Generalization*,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\exists X(P)] \Rightarrow (Q)\}$	theorem 2.57, no free X in Q,
$\vdash [\exists X(P)] \Rightarrow (Q)$	*Detachment*.

□

2.59 Theorem. *If X does not occur freely in P, then $\vdash (P) \Leftrightarrow [\exists X(P)]$.*

Proof. Apply theorems 2.48, 1.12, and 2.58.

$\vdash (P) \Rightarrow [\exists X(P)]$	theorem 2.48,
$\vdash (P) \Rightarrow (P)$	theorem 1.12,
$\vdash [\exists X(P)] \Rightarrow (P)$	theorem 2.58, no free X in P.

□

2.60 Theorem. *For all P, Q, and X, $\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\exists X(P)] \Rightarrow [\exists X(Q)]\}$.*

Proof. Let H denote $\forall X[(P) \Rightarrow (Q)]$, and let C denote $[\exists X(P)] \Rightarrow [\exists X(Q)]$.

$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow [(P) \Rightarrow (Q)]$	axiom Q1,
$\vdash [(P) \Rightarrow (Q)] \Rightarrow \{[\neg(Q)] \Rightarrow [\neg(P)]\}$	contraposition,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\neg(Q)] \Rightarrow [\neg(P)]\}$	transitivity,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \big(\forall X\{[\neg(Q)] \Rightarrow [\neg(P)]\}\big)$	theorem 2.30,
$\vdash \big(\forall X\{[\neg(Q)] \Rightarrow [\neg(P)]\}\big)$ $\Rightarrow \big(\{\forall X[\neg(Q)]\} \Rightarrow \{\forall X[\neg(P)]\}\big)$	theorem 2.35
$\vdash \big(\{\forall X[\neg(Q)]\} \Rightarrow \{\forall X[\neg(P)]\}\big)$ $\Rightarrow \big[\big(\neg\{\forall X[\neg(P)]\}\big) \Rightarrow \big(\neg\{\forall X[\neg(Q)]\}\big)\big]$	contraposition,
$\vdash \big[\big(\neg\{\forall X[\neg(P)]\}\big) \Rightarrow \big(\neg\{\forall X[\neg(Q)]\}\big)\big]$ $\Rightarrow \{[\exists X(P)] \Rightarrow [\exists X(Q)]\}$	1.42, 2.46,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\exists X(P)] \Rightarrow [\exists X(Q)]\}$	transitivity.

□

2.61 Theorem (derived rule). *If $(P) \Leftrightarrow (Q)$ is a theorem, then $[\exists X(P)] \Leftrightarrow [\exists X(Q)]$ is a theorem.*

Proof. Apply *Generalization*, theorem 2.60, and *Detachment*:

$\vdash (P) \Leftrightarrow (Q)$	hypothesis,
$\vdash \forall X[(P) \Leftrightarrow (Q)]$	*Generalization*,
$\vdash \{\forall X[(P) \Leftrightarrow (Q)]\} \Rightarrow \{[\exists X(P)] \Rightarrow [\exists X(Q)]\}$	theorem 2.60,
$\vdash [\exists X(P)] \Rightarrow [\exists X(Q)]$	*Detachment*.

□

2.62 Theorem. $\vdash \{\exists X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{[\forall X(P)] \Rightarrow [\exists X(Q)]\}$.

Proof. This proof follows Church's [18, p. 205]. Apply the law of denial of the antecedent (theorem 1.40) and the law of proof by cases subject to hypotheses (theorem 1.50):

$\vdash [\neg(P)] \Rightarrow [(P) \Rightarrow (Q)]$	theorem 1.40,
$\vdash \forall X\{[\neg(P)] \Rightarrow [(P) \Rightarrow (Q)]\}$	*Generalization*,
$\vdash \{\exists X[\neg(P)]\} \Rightarrow \{\exists X[(P) \Rightarrow (Q)]\}$	theorem 2.60 and *Detachment*,
$\vdash \{\neg[\forall X(P)\} \Rightarrow \{\exists X[(P) \Rightarrow (Q)]\}$	axiom Q4 and *Detachment*,
$\vdash (Q) \Rightarrow [(P) \Rightarrow (Q)]$	axiom P1,
$\vdash \forall X\{(Q) \Rightarrow [(P) \Rightarrow (Q)]\}$	Generalization,
$\vdash [\exists X(Q)] \Rightarrow \{\exists X[(P) \Rightarrow (Q)]\}$	theorem 2.60 and *Detachment*,
$\vdash \{[\forall X(P)] \Rightarrow [\exists X(Q)]\} \Rightarrow \{\exists X[(P) \Rightarrow (Q)]\}$	theorem 1.50 and *Detachment*.

For the converse, use theorems 2.48 and 2.57:

$\vdash (P) \Rightarrow \{[(P) \Rightarrow (Q)] \Rightarrow (Q)\}$	law of assertion (1.38),
$\vdash [\forall X(P)] \Rightarrow (P)$	axiom Q1,
$\vdash [\forall X(P)] \Rightarrow \{[(P) \Rightarrow (Q)] \Rightarrow (Q)\}$	transitivity,
$\vdash [(P) \Rightarrow (Q)] \Rightarrow \{[\forall X(P)] \Rightarrow (Q)\}$	commutation (1.37),
$\vdash (Q) \Rightarrow [\exists X(Q)]$	theorem 2.48,
$\vdash [(P) \Rightarrow (Q)] \Rightarrow \{[\forall X(P)] \Rightarrow [\exists X(Q)]\}$	derived rule,
$\vdash \forall X\big([(P) \Rightarrow (Q)] \Rightarrow \{[\forall X(P)] \Rightarrow [\exists X(Q)]\}\big)$	*Generalization*,
$\vdash \{\exists X[(P) \Rightarrow (Q)]\} \Rightarrow \{[\forall X(P)] \Rightarrow [\exists X(Q)]\}$	theorem 2.57 and *Detachment*.

□

2.63 Theorem. *If X does not occur freely in P, then*
$\vdash \{\exists X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{(P) \Rightarrow [\exists X(Q)]\}$.

Proof. This proof follows Church's [18, p. 205]. Apply theorems 2.39 and 2.62:

$\vdash \{\exists X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{[\forall X(P)] \Rightarrow [\exists X(Q)]\}$	theorem 2.62,
$\vdash (P) \Leftrightarrow [\forall X(P)]$	theorem 2.39, no free X in P,
$\vdash \{\exists X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{(P) \Rightarrow [\exists X(Q)]\}$	derived rules.

□

Similar to theorem 2.57, theorem 2.64 handles a case where R is $\forall X(P)$ while S is Q.

2.64 Theorem. *If X does not occur freely in Q, then*
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{[\forall X(P)] \Rightarrow (Q)\}$.

Proof. Apply theorems 2.59, 2.62, and 2.54:

$\vdash (Q) \Leftrightarrow [\exists X(Q)]$	theorem 2.59, no free X in Q,
$\vdash \{\exists X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{[\forall X(P)] \Rightarrow [\exists X(Q)]\}$	theorem 2.62,
$\vdash \{\forall X[(P) \Rightarrow (Q)]\} \Leftrightarrow \{[\forall X(P)] \Rightarrow (Q)\}$	theorem 2.54.

□

2.3.3 Prenex Normal Form

Yet another method of proof consists in transforming a formula into an equivalent formula in which all the quantifiers, if any, are at the beginning.

2.65 Definition. A formula Q is in **prenex normal form** if and only if Q is of the form

$$Q_{\flat}X_{\flat}Q_{\flat\flat}X_{\flat\flat}\ldots Q_{\flat\ldots\flat}X_{\flat\ldots\flat}(R)$$

optionally with brackets and parentheses, where R is a well-formed formula without quantifiers and each string $Q_{\flat\ldots\flat}$ is either $\forall$ or $\exists$. The formula R is called the **matrix** of P while the string $Q_{\flat}X_{\flat}Q_{\flat\flat}X_{\flat\flat}\ldots Q_{\flat\ldots\flat}X_{\flat\ldots\flat}$ is called the **prefix** of P.

2.66 Example. The formula $\forall X \exists Z[\neg(Z = X)]$ is in prenex normal form. Its prefix is $\forall X \exists Z$ while its matrix is $\neg(Z = X)$.

Theorem 2.67 reveals that every well-formed formula is equivalent to a formula in prenex normal form [18, §39], [108, p. 49].

2.67 Theorem (prenex normal form, preliminary version). *For every well-formed formula P there exists a well-formed formula Q in prenex normal form such that* $(P) \Leftrightarrow (Q)$.

Proof (Outline). In P, replace bound variables so that different quantifiers bind different variables, which gives a formula equivalent to P, by theorems 2.31 and 2.54.

With different quantified variables, theorem 2.38 then provides a means to bring quantifiers in front of an implication.

Axioms Q3 and Q4 also provide a means to bring any quantifier in front of any negation.

The general case follows by several applications of the previous cases in a way that may be specified more explicitly after the availability of the Principle of Mathematical Induction in chapter 4. □

2.68 Example. The formula $\neg\{\exists X[\forall Z(Z = X)]\}$ is *not* in prenex normal form. Nevertheless, axiom Q4 gives the equivalent formula $\forall X\{\neg[\forall Z(Z = X)]\}$, whence axiom Q3 and theorem 2.54 yield the equivalent formula $\forall X\{\exists Z[\neg(Z = X)]\}$, which *is* in prenex normal form.

Transforming a logical formula P into an equivalent formula Q in prenex normal form, or partly so, may reveal a proof of Q and hence also of P, as demonstrated in example 2.55. Example 2.69 completes the transformation into prenex normal form, which reveals a propositional theorem in the matrix.

2.69 Example. The formula $\forall X\big([\exists Y(Y \in X)] \vee \{\forall Z[\neg(Z \in X)]\}\big)$ is *not* in prenex normal form. Nevertheless, by the definition of $\vee$ in terms of $\neg$ and $\Rightarrow$ the formula becomes $\forall X\big(\{\neg[\exists Y(Y \in X)]\} \Rightarrow \{\forall Z[\neg(Z \in X)]\}\big)$.

Axiom Q3 gives $\forall X\big(\{\forall Y[\neg(Y \in X)]\} \Rightarrow \{\forall Z[\neg(Z \in X)]\}\big)$.

Theorem 2.38 gives $\forall X\big[\forall Z\big(\{\forall Y[\neg(Y \in X)]\} \Rightarrow [\neg(Z \in X)]\big)\big]$.

Theorem 2.64 yields $\forall X\big[\forall Z\big(\exists Y\{[\neg(Y \in X)] \Rightarrow [\neg(Z \in X)]\}\big)\big]$, which is a theorem: selecting Z for Y gives $[\neg(Z \in X)] \Rightarrow [\neg(Z \in X)]$, which has the pattern $(P) \Rightarrow (P)$ of theorem 1.12.

Besides providing transformations that may facilitate proofs, as in example 2.55, bringing formulae into prenex normal form, in particular, Skolem's normal form with all the existential quantifiers preceding all the universal quantifiers,

$$\exists X_{\flat} \ldots \exists X_{\flat\ldots\flat} \forall Y^{\sharp} \ldots \forall Y^{\sharp\cdots\sharp}(R),$$

leads to Gödel's Completeness Theorem, that a formula is a theorem if and only if it is valid in all applications, even though no mechanical ways to check either may exist [18, §42–§44].

2.3.4 *Proofs with More than One Quantifier*

The following theorems are examples of theorems involving more than one quantifier. The first theorem allows for the deletion of a redundant universal quantifier.

2.70 Theorem. $\vdash [\forall X(Q)] \Leftrightarrow \{\forall X[\forall X(Q)]\}$.

Proof. Apply theorem 2.39 to $\forall X(Q)$, which has no free X. □

The second theorem allows for the swap of two consecutive universal quantifiers.

2.71 Theorem. $\vdash \{\forall X[\forall Y(P)]\} \Leftrightarrow \{\forall Y[\forall X(P)]\}$.

Proof. Apply axiom Q1, theorem 2.29, and *Generalization*:

$\vdash \{\forall X[\forall Y(P)]\} \Rightarrow [\forall Y(P)]$	axiom Q1,
$\vdash [\forall Y(P)] \Rightarrow (P)$	axiom Q1,
$\vdash \{\forall X[\forall Y(P)]\} \Rightarrow (P)$	transitivity (theorem 1.16),
$\vdash \{\forall X[\forall Y(P)]\} \Rightarrow [\forall X(P)]$	theorem 2.37,
$\vdash \{\forall X[\forall Y(P)]\} \Rightarrow \{\forall Y[\forall X(P)]\}$	theorem 2.29.

□

The third theorem allows for the swap of two consecutive existential quantifiers.

2.72 Theorem. $\vdash \{\exists X[\exists Y(P)]\} \Leftrightarrow \{\exists Y[\exists X(P)]\}$.

Proof. Apply the complete law of double negation, axiom Q3, and theorem 2.71:

$$
\begin{array}{rl}
\exists X[\exists Y(P)] & \\
\Updownarrow & \text{double negation,} \\
\neg\big(\neg\{\exists X[\exists Y(P)]\}\big) & \\
\Updownarrow & \text{axiom Q3 and theorem 2.54,} \\
\neg\big(\forall X\{\neg[\exists Y(P)]\}\big) & \\
\Updownarrow & \text{axiom Q3 and theorem 2.54,} \\
\neg\big(\forall X\{\forall Y[\neg(P)]\}\big) & \\
\Updownarrow & \text{theorem 2.71,} \\
\neg\big(\forall Y\{\forall X[\neg(P)]\}\big) & \\
\Updownarrow & \text{axiom Q3 and theorem 2.54,} \\
\neg\big(\forall Y\{\neg[\exists X(P)]\}\big) & \\
\Updownarrow & \text{axiom Q3 and theorem 2.54,} \\
\neg\big(\neg\{\exists Y[\exists X(P)]\}\big) & \\
\Updownarrow & \text{double negation (theorems 1.41 and 1.42).} \\
\exists Y[\exists X(P)] &
\end{array}
$$

□

The fourth theorem allows for the swap of different quantifiers in an implication.

2.73 Theorem. $\vdash \{\exists X[\forall Y(P)]\} \Rightarrow \{\forall Y[\exists X(P)]\}$.

Proof. Apply theorems 2.48, 2.35, and 2.57:

$$
\begin{array}{ll}
\vdash (P) \Rightarrow [\exists X(P)] & \text{theorem 2.48,} \\
\vdash [\forall Y(P)] \Rightarrow \{\forall Y[\exists X(P)]\} & \text{theorem 2.37,} \\
\vdash \{\exists X[\forall Y(P)]\} \Rightarrow \{\forall Y[\exists X(P)]\} & \text{theorem 2.57, no free } X \text{ in } \{\forall Y[\exists X(P)]\}.
\end{array}
$$

□

2.74 Counterexample. The converse of theorem 2.73, which would be

$$\{\forall Y[\exists X(P)]\} \Rightarrow \{\exists X[\forall Y(P)]\},$$

can be *False*. For instance, in every context with at least two *different* objects V and W, consider the logical formula $X = Y$ for P.

$$
\begin{array}{ll}
\vdash \forall Y[\exists X(X = Y)] & \text{for each } Y, \text{ choose } X := Y; \\
\exists X[\forall Y(X = Y)] & \text{is False: no } X \text{ equals } V \text{ and } W; \\
\{\forall Y[\exists X(X = Y)]\} \Rightarrow \{\exists X[\forall Y(X = Y)]\} & \text{is False because } (T) \not\Rightarrow (F).
\end{array}
$$

2.3.5 Exercises on the Substitutivity of Equivalence

The following exercises focus on details of the proof of theorem 2.54, with the logical equivalence $\Leftrightarrow$ defined either by Tarski's axioms IV, V, VI in example 1.87 on page 55, or with $\Rightarrow$ and $\wedge$ in definition 1.51 on page 38.

2.21 . Prove that if P denotes $(U) \Rightarrow (W)$, if Q denotes $(V) \Rightarrow (W)$, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (P) \Rightarrow (Q)$.

2.22 . Prove that if P denotes $(W) \Rightarrow (U)$, if Q denotes $(W) \Rightarrow (V)$, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (P) \Rightarrow (Q)$.

2.23 . Prove that if P denotes $(U) \Rightarrow (W)$, if Q denotes $(V) \Rightarrow (W)$, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (Q) \Rightarrow (P)$.

2.24 . Prove that if P denotes $(W) \Rightarrow (U)$, if Q denotes $(W) \Rightarrow (V)$, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (Q) \Rightarrow (P)$.

2.25 . Prove that if P denotes $\forall X(U)$, if Q denotes $\forall X(V)$, without free occurrences of X in U and V, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (P) \Rightarrow (Q)$.

2.26 . Prove that if P denotes $\forall X(U)$, if Q denotes $\forall X(V)$, without free occurrences of X in U and V, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (Q) \Rightarrow (P)$.

2.27 . Prove that if P denotes $\neg(U)$, if Q denotes $\neg(V)$, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (P) \Rightarrow (Q)$.

2.28 . Prove that if P denotes $\neg(U)$, if Q denotes $\neg(V)$, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (Q) \Rightarrow (P)$.

2.29 . Prove that if P denotes either $(U) \Rightarrow (W)$ or $(W) \Rightarrow (U)$, if Q denotes either $(V) \Rightarrow (W)$ or $(W) \Rightarrow (V)$, respectively, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (P) \Leftrightarrow (Q)$.

2.30 . Prove that if P denotes $\neg(U)$, if Q denotes $\neg(V)$, and if $\vdash (V) \Leftrightarrow (U)$, then $\vdash (P) \Leftrightarrow (Q)$.

2.4 Predicate Calculus with Other Connectives

This section introduces theorems with quantifiers and conjunctions or disjunctions.

2.4.1 Universal Quantifiers and Conjunctions or Disjunctions

This subsection presents theorems involving the universal quantifier ($\forall$) and a conjunction ($\wedge$) or disjunction ($\vee$), beginning with an equivalence with a conjunction.

2.75 Theorem. $\vdash \{\forall X[(P) \wedge (Q)]\} \Rightarrow \{[\forall X(P)] \wedge [\forall X(Q)]\}$.

Proof. Apply theorems 1.53, 2.37, 1.52, 1.55:

$\vdash [(P) \wedge (Q)] \Rightarrow (P)$	theorem 1.53,
$\vdash \{\forall X[(P) \wedge (Q)]\} \Rightarrow \{\forall X(P)\}$	theorem 2.37,
$\vdash [(P) \wedge (Q)] \Rightarrow (Q)$	theorem 1.52,

$\vdash \{\forall X[(P) \wedge (Q)]\} \Rightarrow \{\forall X(Q)\}$ theorem 2.37,
$\vdash \{\forall X[(P) \wedge (Q)]\} \Rightarrow \{[\forall X(P)] \wedge [\forall X(Q)]$ theorem 1.55.

□

The converse implication forms the object of the following theorem.

2.76 Theorem. $\vdash \{[\forall X(P)] \wedge [\forall X(Q)]\} \Rightarrow \{\forall X[(P) \wedge (Q)]\}$.

Proof. Apply axiom Q1 with theorems 1.82 and 2.30:

$\vdash [\forall X(P)] \Rightarrow (P)$ axiom Q1,
$\vdash [\forall X(Q)] \Rightarrow (Q)$ axiom Q1,
$\vdash \{[\forall X(P)] \wedge [\forall X(Q)]\} \Rightarrow [(P) \wedge (Q)]$ theorem 1.82,
$\vdash \{[\forall X(P)] \wedge [\forall X(Q)]\} \Rightarrow \{\forall X[(P) \wedge (Q)]\}$ theorem 2.30.

□

The following theorem gives an implication with a disjunction.

2.77 Theorem. $\vdash \{[\forall X(P)] \vee [\forall X(Q)]\} \Rightarrow \{\forall X[(P) \vee (Q)]\}$.

Proof. Apply axiom Q1, exercise 1.57, theorem 2.29, and *Generalization*:

$\vdash [\forall X(P)] \Rightarrow (P)$ axiom Q1,
$\vdash [\forall X(Q)] \Rightarrow (Q)$ axiom Q1,
$\vdash \{[\forall X(P)] \vee [\forall X(Q)]\} \Rightarrow [(P) \vee (Q)]$ exercise 1.57,
$\vdash \forall X\big(\{[\forall X(P)] \vee [\forall X(Q)]\} \Rightarrow [(P) \vee (Q)]\big)$ *Generalization*,
$\vdash \{[\forall X(P)] \vee [\forall X(Q)]\} \Rightarrow \{\forall X[(P) \vee (Q)]\}$ theorem 2.29.

□

2.78 Counterexample. The converse of theorem 2.77, which would be

$$\{\forall X[(P) \vee (Q)]\} \Rightarrow \{[\forall X(P)] \vee [\forall X(Q)]\},$$

may be *False*. For instance, in every context with exactly *two different* objects V and W, consider the formulae $X = V$ for P and $X = W$ for Q:

$\vdash \forall X[(X = V) \vee (X = W)]$ because either $(X = V)$ or $(X = W)$;
$\forall X(X = V)$ is False if $X := W$;
$\forall X(X = W)$ is False if $X := V$;
$[\forall X(X = V)] \vee [\forall X(X = W)]$ is False by the preceding two lines;
$\{\forall X[(X = V) \vee (X = W)]\}$
$\quad \Rightarrow \{[\forall X(X = V)] \vee [\forall X(X = W)]\}$ is False because $(T) \not\Rightarrow (F)$.

However, theorem 2.79 shows a converse of theorem 2.77 in a particular case.

2.79 Theorem. *If P has no free X, then* $\vdash \{\forall X[(P) \vee (Q)]\} \Rightarrow \{(P) \vee [\forall X(Q)]\}$ *and* $\vdash \{\forall X[(P) \vee (Q)]\} \Rightarrow \{[\forall X(P)] \vee [\forall X(Q)]\}$

Proof. Apply the definition of $\vee$:

$$\begin{array}{rll}
\forall X[(P) \vee (Q)] & & \\
\Updownarrow & & \text{definition of } (P) \vee (Q), \\
\forall X\{[(P) \Rightarrow (Q)] \Rightarrow (Q)\} & & \\
\Downarrow & & \text{theorem 2.35}, \\
\{\forall X[(P) \Rightarrow (Q)]\} \Rightarrow [\forall X(Q)] & & \\
\Updownarrow & & \text{theorems 2.38, 2.54, no free } X \text{ in } P, \\
\{(P) \Rightarrow [\forall X(Q)]\} \Rightarrow [\forall X(Q)] & & \\
\Updownarrow & & \text{definition of } \vee, \\
(P) \vee [\forall X(Q)] & & \\
\Updownarrow & & \text{theorem 2.39, 2.54, no free } X \text{ in } P. \\
[\forall X(P)] \vee [\forall X(Q)] & &
\end{array}$$

□

2.4.2 Existential Quantifiers and Conjunctions or Disjunctions

This subsection presents theorems involving the existential quantifier ($\exists$) and a conjunction ($\wedge$) or disjunction ($\vee$), beginning with an equivalence with a disjunction.

2.80 Theorem. $\vdash \{[\exists X(P)] \vee [\exists X(Q)]\} \Leftrightarrow \{[\exists X[(P) \vee (Q)]\}$.

Proof. Apply contraposition with theorems 2.75, 2.76, and 2.45:

$$\begin{array}{rll}
\vdash \big(\forall X\{[\neg(P)] \wedge [\neg(Q)]\}\big) \Leftrightarrow \big(\{\forall X[\neg(P)]\} \wedge \{\forall X[\neg(Q)]\}\big) & & 2.75, 2.76, \\
\Updownarrow & & \text{contraposition}, \\
\big[\neg\big(\{\forall X[\neg(P)]\} \wedge \{\forall X[\neg(Q)]\}\big)\big] \Leftrightarrow \big[\neg\big(\forall X\{[\neg(P)] \wedge [\neg(Q)]\}\big)\big] & & \\
\Updownarrow & & 2.45, \\
\big[\big(\neg\{\forall X[\neg(P)]\}\big) \vee \big(\neg\{\forall X[\neg(Q)]\}\big)\big] \Leftrightarrow \big[\neg\big(\forall X\{\neg[(P) \vee (Q)]\}\big)\big] & & \\
\Updownarrow & & \text{axiom Q4}, \\
\big(\neg\{\neg[\exists X(P)]\}\big) \vee \big(\neg\{\neg[\exists X(Q)]\}\big) \Leftrightarrow \big[\neg\big(\neg\{\exists X[(P) \vee (Q)]\}\big)\big] & & \\
\Updownarrow & & \text{double negations}. \\
\{[\exists X(P)] \vee [\exists X(Q)]\} \Leftrightarrow \{\exists X[(P) \vee (Q)]\} & &
\end{array}$$

□

A similar equivalence with a conjunction requires that X be not free in P.

2.81 Theorem. *If P has no free X, then* $\vdash \{\exists X[(P) \wedge (Q)]\} \Leftrightarrow \{(P) \wedge [\exists X(Q)]\}$.

Proof. Apply the full propositional calculus with theorems 2.38, 2.61, 2.46, and axiom Q4, and theorem 1.69:

$$\begin{array}{rll}
\exists X[(P) \wedge (Q)] & & \\
\Updownarrow & & \text{double negation}, \\
\exists X\big(\neg\{\neg[(P) \wedge (Q)]\}\big) & & \\
\Updownarrow & & \text{De Morgan's first law and theorem 2.61},
\end{array}$$

$$
\begin{array}{rl}
\exists X\big(\neg\{[\neg(P)] \vee [\neg(Q)]\}\big) & \\
\Updownarrow & \text{definition of } \vee, \\
\exists X\big(\neg\{(P) \Rightarrow [\neg(Q)]\}\big) & \\
\Updownarrow & \text{axiom Q4}, \\
\neg\big(\forall X\{(P) \Rightarrow [\neg(Q)]\}\big) & \\
\Updownarrow & \text{theorem 2.38}, \\
\neg\big[(P) \Rightarrow \{\forall X[\neg(Q)]\}\big] & \\
\Updownarrow & \text{definition of } \vee \text{ by theorem 1.69}, \\
\neg\big([\neg(P)] \vee \{\forall X[\neg(Q)]\}\big) & \\
\Updownarrow & \text{De Morgan's second law and double negation}, \\
(P) \wedge \big(\neg\{\forall X[\neg(Q)]\}\big) & \\
\Updownarrow & \text{theorem 2.46}. \\
(P) \wedge [\exists X(Q)] &
\end{array}
$$

□

2.4.3 Exercises on Quantifiers with Other Connectives

For the following exercises, prove that the stated formulae are theorem schema.

2.31 . $\{\exists X[(P) \vee (Q)]\} \Leftrightarrow \{\exists X[(Q) \vee (P)]\}$.

2.32 . $\{\forall X[(P) \wedge (P)]\} \Leftrightarrow \{\forall X(P)\}$.

2.33 . $\{\exists X[(P) \vee (P)]\} \Leftrightarrow \{\exists X(P)\}$.

2.34 . $(\exists X\{[(P) \vee (Q)] \vee (R)\}) \Leftrightarrow (\exists X\{(P) \vee [(Q) \vee (R)]\})$.

2.35 . $(\forall X\{[(P) \wedge (Q)] \vee (R)\}) \Leftrightarrow (\{\forall X[(P) \vee (R)]\} \wedge \{\forall X[(Q) \vee (R)]\})$.

2.36 . $(\exists X\{[(P) \vee (Q)] \wedge (R)\}) \Leftrightarrow (\{\exists X[(P) \wedge (R)]\} \vee \{\exists[(Q) \wedge (R)]\})$.

2.37 . $[\exists X(Q)] \Leftrightarrow [\exists X\,(\exists X(Q))]$.

2.38 . If P has no free X, then $\{(P) \wedge [\forall X(Q)]\} \Leftrightarrow \{\forall X[(P) \wedge (Q)]\}$.

2.39 . If P has no free X, then $\{(P) \vee [\forall X(Q)]\} \Leftrightarrow \{\forall X[(P) \vee (Q)]\}$.

2.40 . If P has no free X, then $\{(P) \vee [\exists X(Q)]\} \Leftrightarrow \{\exists X[(P) \vee (Q)]\}$.

2.5 Equality-Predicates

Applications of logic, for instance, algebra, arithmetic, and geometry, may include concepts of "equality" that allow for substitutions of mutually *equal* objects in statements and formulae, which results in mutually *equivalent* statements and formulae.

2.5.1 First-Order Predicate Calculi with an Equality-Predicate

Different applications may define equality differently [8, p. 6–7]. For instance, in some versions of integer arithmetic, the equality $a = b$ means that a and b are two symbols for *one* integer [25, p. 44], [76, p. 1]. In contrast, in some versions of set theory, the equality $A = B$ means that A and B denote sets with identical set-theoretical features: they have the same elements, and they are elements of the same sets [8, 35, p. 6–7]; the question whether A and B denote the same set does not arise in the theory. Nevertheless, such different concepts of equality happen to conform to a logical predicate, denoted by $\mathscr{I}$ to suggest identity, subject to the following axioms (which might also be called *postulates* to distinguish them from logical axioms) [18, § 48].

Axiom $\mathscr{I}$1 (reflexivity of equality) $\vdash \mathscr{I}(X,X)$.

Axiom $\mathscr{I}$2 (substitutivity of equality) $\vdash [\mathscr{I}(X,Y)] \Rightarrow [(P) \Rightarrow (Q)]$ *for all well-formed formula P and Q such that Q results from the substitution of any one free occurrence of X in P by Y, provided that the resulting occurrence of Y is also free, or, in other words, provided that in P this occurrence of X is not within the scope of a quantifier* $(\forall X, \forall Y, \exists X, \exists Y)$ *bounding X or Y.*

The condition stipulated in axiom $\mathscr{I}$2 is similar to the requirement that P admit Y for X, or that Y be free for X, but only for one particular occurrence of X in P.

Using only the Pure Positive Implicational Propositional Calculus, theorems 2.82 and 2.83 show that every predicate $\mathscr{I}$ satisfying axioms $\mathscr{I}$1 and $\mathscr{I}$2 is symmetric and transitive [84, p. 104].

2.82 Theorem (symmetry of equality). $\vdash [\mathscr{I}(X,Y)] \Rightarrow [\mathscr{I}(Y,X)]$.

Proof. In axiom $\mathscr{I}$2, substitute the terms $\mathscr{I}(X,X)$ for P and $\mathscr{I}(Y,X)$ for Q:

$\vdash [\mathscr{I}(X,Y)] \Rightarrow \{[\mathscr{I}(X,X)] \Rightarrow [\mathscr{I}(Y,X)]\}$	axiom $\mathscr{I}$2, ,
$\vdash \mathscr{I}(X,X)$	axiom $\mathscr{I}$1,
$\vdash [\mathscr{I}(X,Y)] \Rightarrow [\mathscr{I}(Y,X)]$	derived rule (theorem 1.15).

□

2.83 Theorem (transitivity of equality). $\vdash [\mathscr{I}(X,Y)] \Rightarrow \{[\mathscr{I}(Y,Z)] \Rightarrow [\mathscr{I}(X,Z)]\}$. *Hence, if* $\vdash \mathscr{I}(X,Y)$ *and* $\vdash \mathscr{I}(Y,Z)$*, then* $\vdash \mathscr{I}(X,Z)$.

Proof. Use the symmetry of equality (theorem 2.82) and axiom $\mathscr{I}$2:

$\vdash [\mathscr{I}(X,Y)] \Rightarrow [\mathscr{I}(Y,X)]$	theorem 2.82,
$\vdash [\mathscr{I}(Y,X)] \Rightarrow \{[\mathscr{I}(Y,Z)] \Rightarrow [\mathscr{I}(X,Z)]\}$	axiom $\mathscr{I}$2,
$\vdash [\mathscr{I}(X,Y)] \Rightarrow \{[\mathscr{I}(Y,Z)] \Rightarrow [\mathscr{I}(X,Z)]\}$	transitivity (theorem 1.16),
$\vdash \mathscr{I}(X,Y)$	hypothesis,
$\vdash [\mathscr{I}(Y,Z)] \Rightarrow [\mathscr{I}(X,Z)]$	*Detachment*,
$\vdash \mathscr{I}(Y,Z)$	hypothesis,
$\vdash \mathscr{I}(X,Z)$	*Detachment*.

□

Using only the Pure Positive Implicational Propositional Calculus, theorem 2.84 extends axiom $\mathscr{I}2$ to a converse implication, so that substituting mutually equal objects results in mutually equivalent formulae.

2.84 Theorem (substitutivity of equality). $\vdash [\mathscr{I}(X,Y)] \Rightarrow [(P) \Leftrightarrow (Q)]$ *for all well-formed formula P and Q such that Q results from the substitution of any one free occurrence of X in P provided that in P this occurrence of X is not within the scope of a quantifier* $(\forall X, \forall Y, \exists X, \exists Y)$ *bounding X or Y.*

Proof. The implication $\vdash [\mathscr{I}(X,Y)] \Rightarrow [(P) \Rightarrow (Q)]$ is axiom $\mathscr{I}2$.

For the converse, the hypothesis also states that in Q the resulting occurrence of Y is not within the scope of a quantifier $(\forall X, \forall Y, \exists X, \exists Y)$ bounding Y or X, which allows swapping X and Y, and swapping P and Q, in axiom $\mathscr{I}2$, so that $\vdash [\mathscr{I}(Y,X)] \Rightarrow [(Q) \Rightarrow (P)]$. Hence the conclusion follows from the symmetry $\vdash [\mathscr{I}(X,Y)] \Rightarrow [\mathscr{I}(Y,X)]$ by theorem 2.82 and the transitivity of implication. □

Repeated applications of theorem 2.84 and the proof of substitutivity of equivalence then show that substituting mutually equal objects in a formula leads to an equivalent formula.

2.5.2 *Simple Applied Predicate Calculi with an Equality-Predicate*

Some applications of logic might omit all propositional variables and instead have only atomic formulae with a few predicates, or perhaps only one predicate, which might be denoted by some constant $\mathscr{E}$. Such applications are called simple applied predicate calculi. For instance, a version of set theory has no propositional variables and only one predicate, for set membership, so that $\mathscr{E}(X,Y)$ stands for $X \in Y$. In such applications, an additional equality predicate $\mathscr{I}$ allows for substitutions of mutually equivalent objects in statements and formulae if and only if $\mathscr{I}$ is reflexive (a condition that replaces axiom $\mathscr{I}1$), symmetric, transitive, and satisfies the following two conditions, which replace axiom $\mathscr{I}2$ [18, p. 283, exercise 48.3]:

$$[\mathscr{I}(A,B)] \Rightarrow \{[\mathscr{E}(X,A)] \Rightarrow [\mathscr{E}(X,B)]\},$$
$$[\mathscr{I}(A,B)] \Rightarrow \{[\mathscr{E}(A,Y)] \Rightarrow [\mathscr{E}(B,Y)]\}.$$

In applied logics with other predicates, for instance, predicates for the sum and products of integers in arithmetic, two similar conditions must be appended for each predicate to ensure the substitutivity of mutually equal objects. By the postulated symmetry of the equality predicate $\mathscr{I}$ these two conditions are equivalent to

$$[\mathscr{I}(A,B)] \Rightarrow \{[\mathscr{E}(X,A)] \Leftrightarrow [\mathscr{E}(X,B)]\},$$
$$[\mathscr{I}(A,B)] \Rightarrow \{[\mathscr{E}(A,Y)] \Leftrightarrow [\mathscr{E}(B,Y)]\}.$$

These conditions suffice to ensure that if $\mathscr{I}(A,B)$ holds, then substituting any free occurrence of A for any free occurrence of B according to the conditions stipulated by axiom $\mathscr{I}2$ in any formula P produces an equivalent formula Q, because well-formed formulae include only atomic formulae of the form $\mathscr{E}(Z,W)$. The proof follows the pattern of the proof of the substitutivity of equivalence. The resulting theorem is called the *substitutivity of equality*.

In particular, if a simple applied predicate calculus has exactly one predicate, $\mathscr{E}$, which is binary (involving exactly two individual variables), then the same conditions may serve to *define* an equality predicate $\mathscr{I}$ so that $\mathscr{I}(A,B)$ is merely an abbreviation for

$$\big(\forall X\{[\mathscr{E}(X,A)] \Leftrightarrow [\mathscr{E}(X,B)]\}\big) \wedge \big(\forall Y\{[\mathscr{E}(A,Y)] \Leftrightarrow [\mathscr{E}(B,Y)]\}\big). \tag{2.1}$$

Formula (2.1), abbreviated by $\mathscr{I}(A,B)$, satisfies axiom $\mathscr{I}2$ and is reflexive, symmetric, and transitive, because so is the equivalence $\Leftrightarrow$ in the full propositional calculus. In particular, the theorem on substitutivity of equality holds. An equality predicate $\mathscr{I}$ defined in this manner from another binary predicate $\mathscr{E}$ thus does not add anything to the theory except convenience.

2.85 Example (equality in set theory). In a version of set theory, with the single binary predicate $\mathscr{E}(Z,W)$, the postulate (or axiom) of extensionality states that

$$\big(\forall X\{[\mathscr{E}(X,A)] \Leftrightarrow [\mathscr{E}(X,B)]\}\big) \Leftrightarrow \big(\forall Y\{[\mathscr{E}(A,Y)] \Leftrightarrow [\mathscr{E}(B,Y)]\}\big). \tag{2.2}$$

The equality $\mathscr{I}(A,B)$ of sets A and B is then an abbreviation of each of the formulae $\forall X\{[\mathscr{E}(X,A)] \Leftrightarrow [\mathscr{E}(X,B)]\}$ and $\forall Y\{[\mathscr{E}(A,Y)] \Leftrightarrow [\mathscr{E}(B,Y)]\}$.

The following theorems confirm that every equality-predicate defined by formula (2.1) is reflexive, symmetric, and transitive.

2.86 Theorem (reflexivity of defined equality-predicates). *Every equality-predicate* $\mathscr{I}(A,B)$ defined *by formula* (2.1) *is reflexive:*

$$\vdash \forall C[\mathscr{I}(C,C)].$$

Proof. One method to *design* a formal proof transforms the objective, here the *yet unproved* formula $\forall C[\mathscr{I}(C,C)]$, first into its defining formula (2.1), and then into logically equivalent formulae, for instance, in prenex form, until one such equivalent formula appears that is a theorem, thanks to an axiom or to a previously proven theorem. For instance, substituting C for A and also C for B in the defining formula (2.1) gives

$$\begin{array}{rl}
\mathscr{I}(C,C) & \textbf{yet unproved,} \\
\Updownarrow & \text{definition of } \mathscr{I} \\
\big(\forall X\{[\mathscr{E}(X,C)] \Leftrightarrow [\mathscr{E}(X,C)]\}\big) \wedge \big(\forall Y\{[\mathscr{E}(C,Y)] \Leftrightarrow [\mathscr{E}(C,Y)]\}\big) & \\
\Updownarrow & \text{theorem 2.75,} \\
\forall X \forall Y\big(\{[\mathscr{E}(X,C)] \Leftrightarrow [\mathscr{E}(X,C)]\} \wedge \{[\mathscr{E}(C,Y)] \Leftrightarrow [\mathscr{E}(C,Y)]\}\big) &
\end{array}$$

where each logical formula $[\mathscr{E}(W,Z)] \Leftrightarrow [\mathscr{E}(W,Z)]$ has the pattern of the reflexivity of the logical implication $(P) \Leftrightarrow (P)$ (theorem 1.63). Thus, a complete proof may proceed as follows:

$\vdash (P) \Leftrightarrow (P)$	theorem 1.63,
$\vdash [\mathscr{E}(X,C)] \Leftrightarrow [\mathscr{E}(X,C)]$	substitution in $(P) \Leftrightarrow (P)$,
$\vdash [\mathscr{E}(C,Y)] \Leftrightarrow [\mathscr{E}(C,Y)]$	substitution in $(P) \Leftrightarrow (P)$,
$\vdash \{[\mathscr{E}(X,C)] \Leftrightarrow [\mathscr{E}(X,C)]\} \wedge \{[\mathscr{E}(C,Y)] \Leftrightarrow [\mathscr{E}(C,Y)]\}$	theorem 1.54,
$\vdash \mathscr{I}(C,C)$	formula (2.1).

Hence $\vdash \forall C[\mathscr{I}(C,C)]$ results by *Generalization* and theorem 2.75. □

2.87 Theorem (symmetry of defined equality-predicates). *Every equality-predicate* defined *by formula* (2.1) *is symmetric: if* $\vdash \mathscr{I}(A,B)$*, then* $\vdash \mathscr{I}(B,A)$*; moreover.*

$$\vdash \forall A \forall B\{[\mathscr{I}(A,B)] \Rightarrow [\mathscr{I}(B,A)]\}.$$

Proof. One method to *design* a formal proof transforms the objective, here the *yet unproved* formula $\vdash \forall A \forall B\{[\mathscr{I}(A,B)] \Rightarrow [\mathscr{I}(B,A)]\}$, first into its defining formula (2.1), and then into logically equivalent formulae, for instance, in prenex form, until one such equivalent formula appears that is a theorem, thanks to an axiom or to a previously proven theorem. Here an equivalence will emerge:

$$[\mathscr{I}(A,B)] \Leftrightarrow [\mathscr{I}(B,A)] \quad \textbf{yet unproved,}$$
$$\Updownarrow \quad \text{definition of } \mathscr{I}$$
$$\big(\forall X\{[\mathscr{E}(X,A)] \Leftrightarrow [\mathscr{E}(X,B)]\}\big) \wedge \big(\forall Y\{[\mathscr{E}(A,Y)] \Leftrightarrow [\mathscr{E}(B,Y)]\}\big)$$
$$\Leftrightarrow \big(\forall X\{[\mathscr{E}(X,B)] \Leftrightarrow [\mathscr{E}(X,A)]\}\big) \wedge \big(\forall Y\{[\mathscr{E}(B,Y)] \Leftrightarrow [\mathscr{E}(A,Y)]\}\big),$$

which suggests invoking the symmetry of the logical equivalence $[(P) \Leftrightarrow (Q)] \Leftrightarrow [(Q) \Leftrightarrow (P)]$ (theorem 1.64). Thus, a complete proof may proceed as follows:

$\vdash [(P) \Leftrightarrow (Q)] \Leftrightarrow [(Q) \Leftrightarrow (P)]$	theorem 1.64,
$\vdash \{[\mathscr{E}(X,A)] \Leftrightarrow [\mathscr{E}(X,B)]\} \Leftrightarrow \{[\mathscr{E}(X,B)] \Leftrightarrow [\mathscr{E}(X,A)]\}$	substitution,
$\vdash [(R) \Leftrightarrow (S)] \Leftrightarrow [(S) \Leftrightarrow (R)]$	theorem 1.64,
$\vdash \{[\mathscr{E}(A,Y)] \Leftrightarrow [\mathscr{E}(B,Y)]\} \Leftrightarrow \{[\mathscr{E}(B,Y)] \Leftrightarrow [\mathscr{E}(A,Y)]\}$	substitution,
$\vdash \{[(P) \Leftrightarrow (Q)] \wedge [(R) \Leftrightarrow (S)]\} \Leftrightarrow \{[(Q) \Leftrightarrow (P)] \wedge [(S) \Leftrightarrow (R)]\}$	theorem 1.82.

Hence the conclusion results by *Generalization* and theorem 2.75. □

2.88 Theorem (transitivity of defined equality-predicates). *Every equality-predicate* defined *by formula* (2.1) *is transitive: if* $\vdash \mathscr{I}(A,B)$ *and* $\vdash \mathscr{I}(B,C)$*, then* $\vdash \mathscr{I}(A,C)$*; moreover,*

$$\vdash \forall A \forall B \forall C\big(\{[\mathscr{I}(A,B)] \wedge [\mathscr{I}(B,C)]\} \Rightarrow [\mathscr{I}(A,C)]\big).$$

Proof. One method to *design* a formal proof transforms the objective, here the *yet unproved* formula $\vdash \forall A \forall B \forall C\big(\{[\mathscr{I}(A,B)] \wedge [\mathscr{I}(B,C)]\} \Rightarrow [\mathscr{I}(A,C)]\big)$, first into its defining formula (2.1), and then into logically equivalent formulae, for instance, in prenex form, until one such equivalent formula appears that is a theorem, thanks to an axiom or to a previously proven theorem.

$$\begin{array}{rl} \{[\mathscr{I}(A,B)] \wedge [\mathscr{I}(B,C)]\} \Rightarrow [\mathscr{I}(A,C)] & \textbf{yet unproved,} \\ \Updownarrow & \text{definition of } \mathscr{I} \\ \big[\big(\forall X\{[\mathscr{E}(X,A)]\Leftrightarrow[\mathscr{E}(X,B)]\}\big) \wedge \big(\forall Y\{[\mathscr{E}(A,Y)]\Leftrightarrow[\mathscr{E}(B,Y)]\}\big) & \\ \wedge\big(\forall X\{[\mathscr{E}(X,B)]\Leftrightarrow[\mathscr{E}(X,C)]\}\big) \wedge \big(\forall Y\{[\mathscr{E}(B,Y)]\Leftrightarrow[\mathscr{E}(C,Y)]\}\big)\big] & \\ \Rightarrow \big(\forall X\{[\mathscr{E}(X,A)]\Leftrightarrow[\mathscr{E}(X,C)]\}\big) \wedge \big(\forall Y\{[\mathscr{E}(A,Y)]\Leftrightarrow[\mathscr{E}(C,Y)]\}\big), & \end{array}$$

which suggests invoking the transitivity of the logical equivalence (theorem 1.65):

$$\{[(H) \Leftrightarrow (K)] \wedge [(K) \Leftrightarrow (L)]\} \Rightarrow [(H) \Leftrightarrow (L)],$$

$$\{[(P) \Leftrightarrow (Q)] \wedge [(Q) \Leftrightarrow (R)]\} \Rightarrow [(P) \Leftrightarrow (R)],$$

with the commutativity and associativity of the logical conjunction (theorems 1.57 and 1.66) combined with theorem 1.82:

$$\big(\{[(H) \Leftrightarrow (K)] \wedge [(K) \Leftrightarrow (L)]\} \wedge \{[(P) \Leftrightarrow (Q)] \wedge [(Q) \Leftrightarrow (R)]\}\big) \\ \Rightarrow \{[(H) \Leftrightarrow (L)] \wedge [(P) \Leftrightarrow (R)]\}.$$

The conclusion results by *Generalization* and theorem 2.75. □

2.5.3 Other Axiom Systems for the Equality-Predicate

Other axioms systems exist to specify the identity predicate.

Axiom $\mathscr{I}$1 (reflexivity of equality) $\vdash \forall X\, \mathscr{I}(X,X)$.

Axiom $\mathscr{I}$2 (substitutivity of equality) *For every unary predicate variable or predicate constant $\mathscr{F}$, involving only one individual variable,*

$$\vdash \forall X \forall Y\big([\mathscr{I}(X,Y)] \Rightarrow \{[\mathscr{F}(X)] \Rightarrow [\mathscr{F}(Y)]\}\big).$$

For every binary predicate variable or predicate constant $\mathscr{F}$, involving only two individual variables,

$$\vdash \forall X \forall Y \forall W \forall Z\big\{[\mathscr{I}(X,Y)] \Rightarrow \big([\mathscr{I}(W,Z)] \Rightarrow \{[\mathscr{F}(X,W)] \Rightarrow [\mathscr{F}(Y,Z)]\}\big)\big\}.$$

For every ternary predicate variable or predicate constant $\mathscr{F}$, involving only three individual variables,

$$\vdash \forall U \forall V \forall X \forall Y \forall W \forall Z$$
$$\Big([\mathscr{I}(U,V)] \Rightarrow \big\{[\mathscr{I}(X,Y)] \Rightarrow \big([\mathscr{I}(W,Z)] \Rightarrow \{[\mathscr{F}(U,X,W)] \Rightarrow [\mathscr{F}(V,Y,Z)]\}\big)\big\}\Big).$$

Similar stipulations also hold for predicate variables or constants involving more than three individual variables.

2.5.4 Defined Ranking-Predicates

Each application with a binary predicate constant $\mathscr{E}$ allows for a corresponding predicate $\mathscr{R}$ of **ranking**, also called **ordering** or **inequality**, defined in terms of $\mathscr{E}$ so that $\mathscr{R}(A,B)$ is an abbreviation of the formula

$$\forall X\{[\mathscr{E}(X,A)] \Rightarrow [\mathscr{E}(X,B)]\}. \tag{2.3}$$

Formula (2.3) is also denoted by $A \preceq B$ (read "A precedes B") instead of $\mathscr{R}(A,B)$. The resulting predicate $\mathscr{R}$ is reflexive and transitive, but not necessarily symmetric, as verified in the exercises.

2.5.5 Exercises on Equality-Predicates

2.41 . Verify that the ranking-predicate $\mathscr{R}(A,B)$ defined by formula (2.3) is reflexive: prove $\vdash \forall A[\mathscr{R}(A,A)]$.

2.42 . Investigate whether the ranking-predicate $\mathscr{R}(A,B)$ defined by formula (2.3) is symmetric: is $\forall A \forall B\{[\mathscr{R}(A,B)] \Rightarrow [\mathscr{R}(B,A)]\}$ is a theorem?

2.43 . Verify that the ranking-predicate $\mathscr{R}(A,B)$ defined by formula (2.3) is transitive: prove $\vdash \forall A \forall B \forall C\big([\mathscr{R}(A,B)] \wedge [\mathscr{R}(B,C)]\} \Rightarrow [\mathscr{R}(A,C)]\big)$.

Exercises 2.45, 2.44, and 2.46 focus on the alternative ranking predicate $\mathscr{A}(A,B)$ defined in terms of the same binary predicate constant $\mathscr{E}$ as an abbreviation of formula (2.4):

$$\forall X\{[\mathscr{E}(B,Y)] \Rightarrow [\mathscr{E}(A,Y)]\}. \tag{2.4}$$

2.44 . Verify that the alternative ranking-predicate $\mathscr{A}(A,B)$ defined by formula (2.4) is transitive: prove $\vdash \forall A \forall B \forall C\big([\mathscr{A}(A,B)] \wedge [\mathscr{A}(B,C)]\} \Rightarrow [\mathscr{A}(A,C)]\big)$.

2.45 . Verify that the alternative ranking-predicate $\mathscr{A}(A,B)$ defined by formula (2.4) is reflexive: prove $\vdash \forall A[\mathscr{A}(A,A)]$.

2.46 . Investigate whether the alternative ranking-predicate $\mathscr{A}(A,B)$ defined by formula (2.4) is symmetric: determine whether $\forall A \forall B\{[\mathscr{A}(A,B)] \Rightarrow [\mathscr{A}(B,A)]\}$ is a theorem.

2.47 . Verify that the equality predicate defined as in example 2.85 for set theory satisfies the alternative axioms $\mathscr{J}1$ and $\mathscr{J}2$ from subsection 2.5.3.

2.48 . Verify that the equality predicate defined as in example 2.85 for set theory satisfies axioms $\mathscr{I}1$ and $\mathscr{I}2$. from subsection 2.5.1.

2.49 . Verify that the equality predicate defined by axioms $\mathscr{I}1$ and $\mathscr{I}2$ from subsection 2.5.1 also satisfies the alternative axioms $\mathscr{J}1$ and $\mathscr{J}2$ from subsection 2.5.3.

2.50 . Verify that the equality predicate defined by axioms $\mathscr{J}1$ and $\mathscr{J}2$ from subsection 2.5.3 also satisfies axioms $\mathscr{I}1$ and $\mathscr{I}2$ from subsection 2.5.1.

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