On Ramsey $(2K_2, K_4)-$Minimal Graphs

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1 Introduction

All graphs in this paper are simple. Let $G(V, E)$ be a graph and $v \in V(G)$. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges incident to the vertex. The *degree sequence* of a graph is the nonincreasing sequence of the degrees of its vertices. If G has n vertices, the degree sequence of G is $(d_1, d_2, \ldots, d_n)$ where $d_i \geq d_{i+1}$ for every $i = 1, 2, \ldots, n-1$.

Let G and H be graphs with order m and n, respectively. The *disjoint union* G and H, denoted by $G \cup H$, is a graph with the vertex set $V(G \cup H) = V(G) \cup V(H)$ and the edge set $E(G \cup H) = E(G) \cup E(H)$. The *join graph* of G and H, denoted by $G + H$, is the graph $V(G) \cup V(H)$ and all edges joining every vertex of G to every vertex of H. Following Borowiecka-Olszewska and Haluszczak [2], we use notation $G \odot H$, for a graph obtained from disjoint graphs G and H by identifying vertices $u \in V(G)$ and $v \in V(H)$. So, the graph $G \odot H$ has $m + n - 1$ vertices. Similarly, we introduce notation $G \ominus H$ for a graph obtained from disjoint graphs G and H by identifying edges $a \in E(G)$ and $e \in E(H)$. The graph $G \ominus H$ has $m + n - 2$ vertices.

For any pair of graphs G and H, notation $F \to (G, H)$ means that in any red-blue coloring on the edges of F, there exists a red copy of G or a blue copy of H in F. A red-blue coloring in F such that neither a red G nor a blue H occurs is called a $(G, H)-$*coloring*. A graph F is called a *Ramsey* $(G, H)-$*minimal* if $F \to (G, H)$ but $(F - e) \nrightarrow (G, H)$ for all $e \in E(F)$. The set of all Ramsey $(G, H)-$minimal graphs is denoted by $\mathfrak{R}(G, H)$.

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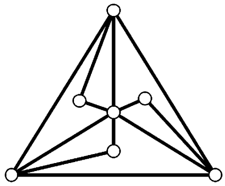

Fig. 1 Graph G in $\mathfrak{R}(2K_2, K_3)$

For a fixed pair of graphs G and H, characterizing all graphs F in $\mathfrak{R}(G, H)$ is a very interesting problem, but it is also a difficult problem, even for small graphs G and H. Burr et al. [3] proved that the set $\mathfrak{R}(mK_2, H)$ is finite for any graph H. In particular, they proved that $\mathfrak{R}(K_2, H) = \{H\}$, for every graph H, $\mathfrak{R}(2K_2, 2K_2) = \{3K_2, C_5\}$, $\mathfrak{R}(2K_2, K_3) = \{2K_3, K_5, G\}$, where G is the graph in Fig. 1. Later, Burr et al. [4] gave a characterization of all graphs in $\mathfrak{R}(tK_2, 2K_2)$ for any $t \geq 2$. Mengersen and Oeckermann [5] gave the proof of $\mathfrak{R}(2K_2, K_{1,2}) = \{2K_{1,2}, C_4, C_5\}$, which was previously mentioned in [4] without proof. In the same paper, they determined all graphs in $\mathfrak{R}(2K_2, K_{1,3})$. Baskoro and Yulianti [1] gave some necessary conditions for graphs in $\mathfrak{R}(2K_2, H)$. They proved the following theorem.

Theorem 1 ([1]). *Let H be a connected graph. Then $2H$ is the only disconnected Ramsey $(2K_2, H)$–minimal graph.* □

Next, the characterization of all graphs which belong to $\mathfrak{R}(2K_2, 2P_n)$ for $n = 4, 5$ was given by Tatanto and Baskoro [6].

In this paper, we give the necessary and sufficient conditions of graphs in $\mathfrak{R}(2K_2, mH)$ for a connected graph H. In particular, we determine all graphs in $\mathfrak{R}(2K_2, K_4)$ with at most 8 vertices. We also give a graph with 9 vertices in $\mathfrak{R}(2K_2, K_4)$. Moreover, we show that a graph obtained from any two disjoint graphs in $\mathfrak{R}(2K_2, K_4)$ by identifying vertices and edges is a member of $\mathfrak{R}(2K_2, 2K_4)$.

2 Properties of Graphs in $\mathfrak{R}(2K_2, H)$

In [1], Baskoro and Yulianti gave necessary conditions for the graphs which belong to $\mathfrak{R}(2K_2, H)$ for any graph H. In this section, we will give the necessary and sufficient conditions for those graphs. Furthermore, if $H = K_4$, then the properties of graphs in $\mathfrak{R}(2K_2, K_4)$ will be discussed. We also give a disconnected graph in $\mathfrak{R}(sK_2, mH)$ for any connected graph H.

Theorem 2. *Let H be any graph. $F \in \mathfrak{R}(2K_2, H)$ if and only if the following conditions are satisfied:*

(i) For every $v \in V(F)$, $F - v \supseteq H$.
(ii) For every K_3 in F, $F - E(K_3) \supseteq H$.
(iii) For every $e \in E(F)$, there exists $v \in V(F)$ or K_3 in F such that $(F - e) - v \nsupseteq H$ or $(F - e) - E(K_3) \nsupseteq H$.

Proof. First, suppose that either (i) is violated by some $v \in V(F)$. Then, color all edges incident to v by red and all the remaining edges by blue. Then, we have a $(2K_2, H)$-coloring of F, a contradiction. Similarly, if (ii) is violated by some K_3, then color the edges of K_3 by red and the remaining edges by blue. By this coloring, we have a $(2K_2, H)$-coloring of F, a contradiction. Furthermore, by the minimality of F, the case (iii) is satisfied.

Conversely, suppose that (i)–(iii) are satisfied. Let us consider any red-blue coloring of F not containing a red $2K_2$. Then either all edges are blue or the red edges form a star or a K_3. In both cases, the existence of a blue H is implied by (i)–(ii). So $F \rightarrow (2K_2, H)$. Next, for every $e \in E(F)$, by (iii) there exists a vertex v or a K_3 in F such that $(F - e) - v \not\supseteq H$ or $(F - e) - E(K_3) \not\supseteq H$. Now, define a coloring ϕ of $F - e$ such that $\phi(x)$ is red for all edges x incident to v or all edges $x \in E(K_3)$ and blue for the remaining edges. Then, we obtain that ϕ is a $(2K_2, H)-$coloring of $F - e$. Hence, $F \in \mathfrak{R}(2K_2, H)$. □

Lemma 1. *Let $F \in \mathfrak{R}(2K_2, K_4)$. Then the following conditions are satisfied:*

(i) $\delta(F) \geq 3$ *where δ is the minimum degree in F.*
(ii) *F is not a tree.*
(iii) *Every vertex $v \in V(F)$ is contained in some K_4 in F.*
(iv) *Every edge $e \in E(F)$ is contained in some K_4 in F.*

Proof. Theorem 2 implies $\delta(F) \geq 3$ and F is not a tree. Suppose now that there exists a vertex $v \in V(F)$ not contained in a K_4 in F. Since $F \in \mathfrak{R}(2K_2, K_4)$, then we have a $(2K_2, K_4)-$coloring of $F - v$. Use this coloring in $F - v$, and color all edges incident to v in F by blue, and we obtain a $(2K_2, K_4)-$coloring of F, a contradiction. Next, suppose that there exists an edge $e \in E(F)$ not contained in a K_4 in F. Since $F \in \mathfrak{R}(2K_2, K_4)$, then we have a $(2K_2, K_4)-$coloring of $F - e$. By using this coloring and color the edge e by blue, we obtain a $(2K_2, K_4)-$coloring of F, a contradiction. □

Theorem 3. *For any integers $s \geq 2$, $m \geq 1$, and any connected graph H, the disconnected graph $(s + m - 1)H$ is in $\mathfrak{R}(sK_2, mH)$.*

Proof. First, we prove that $(s + m - 1)H \rightarrow (sK_2, mH)$. Let $F = (s + m - 1)H$. Consider any red-blue coloring of F containing no blue mH. Therefore, there are at most $m - 1$ components of F having blue H. So, we have at least s components of F having no blue H. This means that each of these components will contain a red edge. These red edges together will form sK_2 in F.

Next, we show the minimality. Let $e \in E(F)$. We will prove that $F - e \nrightarrow (sK_2, mH)$. Since $F - e = (s - 1)H \cup (m - 1)H \cup (H - e)$, then define an edge coloring ϕ on $F - e$ such that $\phi(x)$ is blue if $x \in E((m - 1)H \cup s(H - e))$ and red otherwise. Then, it is easy to verify that ϕ is a $(sK_2, mH)-$coloring. □

Corollary 1. $2K_4$ *is the only disconnected graph in $\mathfrak{R}(2K_2, K_4)$.*

Proof. By Theorem 1. □

Theorem 4. *Let H be a connected graph. Let $F_1, F_2, \ldots, F_m$ be connected graphs in $\mathfrak{R}(2K_2, H)$. Then, graph $F = F_1 \cup F_2 \cup \cdots \cup F_m$ is in $\mathfrak{R}(2K_2, mH)$.*

Proof. Suppose that $F \nrightarrow (2K_2, mH)$, then there exists a $(2K_2, mH)-$coloring of F. It means that there is a $(2K_2, H)-$coloring of F_i for some $i \in \{1, 2, \ldots, m\}$, a contradiction.

Now, we prove that $F - e \nrightarrow (2K_2, mH)$, for any edge e. Let $F - e = F_1 \cup F_2 \cup \cdots \cup (F_i - e) \cup \cdots \cup F_m$, for some $i \in \{1, 2, \ldots, m\}$. Then there exists a $(2K_2, H)-$coloring of $F_i - e$. We use such a coloring in $F_i - e$, and all edges in $(F - e) - (F_i - e)$ are colored by blue. Then, we obtain a $(2K_2, mH)-$coloring of $F - e$. □

3 The Set $\mathfrak{R}(2K_2, K_4)$

We determine all graphs in $\mathfrak{R}(2K_2, K_4)$ with at most 8 vertices. We also give a graph with 9 vertices in $\mathfrak{R}(2K_2, K_4)$.

Theorem 5. *K_6 is the only graph in $\mathfrak{R}(2K_2, K_4)$ with 6 vertices.*

Proof. First, we prove that K_6 satisfies three conditions in Theorem 2. Since $K_6 - v = K_5$, then $K_6 - v \supseteq K_4$. Since $K_6 - E(K_3) = K_3 + \overline{K}_3$, then $K_6 - E(K_3) \supseteq K_4$. Next, for every $e \in E(K_6)$, $K_6 - e = K_4 + \overline{K}_2$. So there exists a K_3 in $K_6 - e$ such that $(K_6 - e) - E(K_3) = K_1 + \overline{K}_3 + \overline{K}_2$ does not contain K_4. Since K_6 is a graph with the maximum number of edges, then K_6 is the only graph in $\mathfrak{R}(2K_2, K_4)$ with 6 vertices. □

Theorem 6. *$\mathfrak{R}(2K_2, K_4)$ contains no connected graphs with 7 vertices.*

Proof. Let F be a connected graph with 7 vertices. If $F \in \mathfrak{R}(2K_2, K_4)$, then F contains a K_4, but it does not contain a K_6. We will show that no graph $F \in \mathfrak{R}(2K_2, K_4)$ on 7 vertices. Since both graphs K_7 and $K_7 - e$ contain K_6, then $K_7, K_7 - e \notin \mathfrak{R}(2K_2, K_4)$, for any edge $e \in E(K_7)$. Hence, F must be a subgraph of $K_7 - e$. Now, we consider $F = K_7 - 2e$. There are two non-isomorphic graphs $K_7 - 2e$, namely, F_a with a degree sequence $(6, 6, 6, 6, 5, 5, 4)$ and F_b with a degree sequence $(6, 6, 6, 5, 5, 5, 5)$.

Now, let us consider the graph F_a with a degree sequence $(6, 6, 6, 6, 5, 5, 4)$. Since F_a contains a K_6, then F_a is not in $\mathfrak{R}(2K_2, K_4)$. Therefore, F must be a subgraph of F_a. So, consider now $F = F_a - e$ which contain no K_6. Then, we obtain such a graph with its degree sequence $(6, 6, 6, 6, 4, 4, 4)$, $(6, 6, 6, 5, 5, 4, 4)$, or $(6, 6, 5, 5, 5, 5, 4)$. We assume $d(v_i) \geq d(v_{i+1})$ for $i = 1, 2, \ldots, 6$. From all graphs, for $V(K_3) = \{v_1, v_2, v_3\}$, we obtain $F - E(K_3)$ does not contain a K_4. So, $F \nrightarrow (2K_2, K_4)$. Therefore, any subgraph F^* with 7 vertices of F will satisfy that $F^* \nrightarrow (2K_2, K_4)$.

Next, we observe the graph F_b with a degree sequence $(6, 6, 6, 5, 5, 5, 5)$. Let $V(F_b) = \{v_1, v_2, \ldots, v_7\}$ where $d(v_i) = 6$ for $i = 1, 2, 3$ and $d(v_i) = 5$

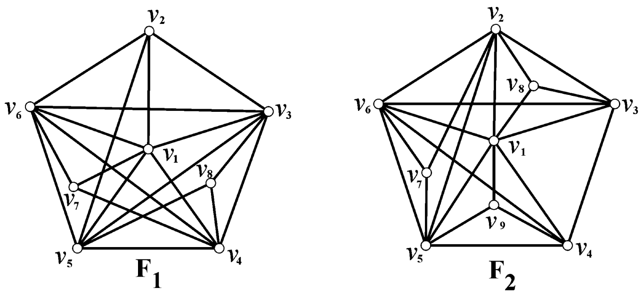

Fig. 2 Graphs in $\mathfrak{R}(2K_2, K_4)$

for $i = 5, 6, 7$. Let ϕ be a coloring of F_b such that $\phi(e)$ is red if $e \in \{v_1v_2, v_1v_3, v_2v_3\}$ and blue otherwise. We obtain a $(2K_2, K_4)$−coloring of F_b. Thus, $F_b \nrightarrow (2K_2, K_4)$. Therefore, any subgraph F^* with 7 vertices of F_b will satisfy that $F^* \nrightarrow (2K_2, K_4)$. This concludes that no graph with 7 vertices is in $\mathfrak{R}(2K_2, K_4)$. □

Now, consider graphs F_1 and F_2 in Fig. 2. Graph F_1 has the vertex set $V(F_1) = \{v_1, v_2, \ldots, v_8\}$ and the edge set $E(F_1) = \{v_iv_j \mid i, j = 1, 2, \ldots, 8,\ i \neq j\} - \{v_1v_8, v_2v_4, v_2v_7, v_2v_8, v_3v_7, v_5v_7, v_6v_8, v_7v_8\}$. Graph F_2 has the vertex set $V(F_2) = \{v_1, v_2, \ldots, v_9\}$ and the edge set $E(F_2) = \{v_iv_j \mid i, j = 1, 2, \ldots, 9,\ i \neq j\} - \{v_iv_7 \mid i = 1, 3, 4\} - \{v_iv_8 \mid i = 4, 5, 6, 7\} - \{v_iv_9 \mid i = 2, 3, 6, 7, 8\} - \{v_2v_4, v_3v_5\}$. We prove that graphs F_1 and F_2 are members of $\mathfrak{R}(2K_2, K_4)$ in the following lemma.

Theorem 7. *The graph F_1 in Fig. 2 is the only graph with 8 vertices in $\mathfrak{R}(2K_2, K_4)$.*

Proof. First, we prove that $F_1 \rightarrow (2K_2, K_4)$. We can see that for every $i \in \{1, 2, \ldots, 8\}$, $F_1 - v_i$ contains a K_4. For every K_3 in F_1, $F_1 - E(K_3)$ contains a K_4. Hence, $F_1 \rightarrow (2K_2, K_4)$. Next, since $F_1 \nsupseteq 2K_4$ and $F_1 \nsupseteq K_6$, then $F_1 \in \mathfrak{R}(2K_2, K_4)$.

Now, suppose there exists a connected graph F with 8 vertices in $\mathfrak{R}(2K_2, K_4)$ but $F \neq F_1$. Let $V(F) = \{v_1, v_2, \ldots, v_8\}$. By Theorem 2, F must contain a K_4,, and we may assume $V(K_4) = \{v_1, v_2, v_3, v_4\}$. By the minimality of F, then F does not contain both $2K_4$ and K_6. By Theorem 2(i), for $i = 1, 2, 3$, $F - v_i$ must contain a K_4. Then (up to isomorphism) the new K_4 in F is formed by the vertex set $\{v_4, v_5, v_6, v_7\}$. So, v_4 is contained in two K_4 in F. Next, there must be a K_4 in $F - v_4$ by Theorem 2(i). Since $\delta(F) \geq 3$ by Lemma 1(i), then (up to isomorphism) the K_4 in $F - v_4$ is formed by the vertex set $\{v_1, v_6, v_7, v_8\}$. Next, by Theorem 2(ii), for $V(K_3) = \{v_1, v_4, v_7\}$, $F - E(K_3)$ must contain a K_4. Then the K_4 in $F - E(K_3)$ can be formed by the vertex set $\{v_2, v_3, v_4, v_6\}$, $\{v_1, v_2, v_5, v_6\}$, or $\{v_3, v_4, v_5, v_6\}$. Otherwise F is the graph F_1 or is not minimal. For all cases, by Theorem 2(ii), $F - E(K_3)$ must contain a K_4, for $V(K_3) = \{v_1, v_4, v_6\}$. But the new K_4 causes F which is not minimal, a contradiction. □

For graphs with 9 vertices, it is not difficult to verify that the graph F_2 in Fig. 2 is a Ramsey $(2K_2, K_4)$-minimal graph. However, characterizing all $(2K_2, K_4)$-minimal graphs is a open problem.

4 Constructing Graphs in $\mathfrak{R}(2K_2, 2K_4)$ by Operations over Graphs in $\mathfrak{R}(2K_2, K_4)$

In this section, we show that a graph obtained from two connected graphs in $\mathfrak{R}(2K_2, K_4)$ by identifying vertices or edges is a member of $\mathfrak{R}(2K_2, 2K_4)$.

Corollary 2. $\{3K_4, 2K_6\} \subseteq \mathfrak{R}(2K_2, 2K_4)$.

Proof. By Theorem 3, we obtain $3K_4 \in \mathfrak{R}(2K_2, 2K_4)$. By Theorem 4, we obtain $2K_6 \in \mathfrak{R}(2K_2, 2K_4)$. □

Theorem 8. *Let $G, H \in \mathfrak{R}(2K_2, K_n)$ be connected graphs and $u \in V(G)$, $v \in V(H)$. If $G \odot H$ is a graph obtained by identifying vertices u and v, then $G \odot H \in \mathfrak{R}(2K_2, 2K_n)$.*

Proof. Let $w \in V(G \odot H)$. Then $(G \odot H) - w$ is either connected or disconnected, depending of the choice of w. If $w = u$ is the identified vertex then $(G \odot H) - w$ is disconnected, that is $(G \odot H) - w = (G - u) \cup (H - v)$. Since $G - u \supseteq K_n$ and $H - v \supseteq K_n$ then $(G \odot H) - w \supseteq 2K_n$. If w is not the identified vertex then $(G \odot H) - w$ is connected, we may assume $(G \odot H) - w = (G - w) \odot H$. Since $G - w \supseteq K_n$ and $H \supseteq K_n$, then $(G \odot H) - w \supseteq 2K_n$. Next, let K_3 in $G \odot H$. Then K_3 is in G or H. Suppose that K_3 is in G, then $(G \odot H) - E(K_3) = (G - E(K_3)) \odot H$. Since $G - E(K_3) \supseteq K_n$ and $H \supseteq K_n$, then $(G \odot H) - E(K_3) \supseteq 2K_n$. So, $G \odot H \to (2K_2, 2K_n)$.

Next, let $e \in E(G \odot H)$, and then $e \in E(G)$ or $e \in E(H)$. We assume $e \in E(G)$. Then, there exists a $(2K_2, K_n)$−coloring ϕ_1 of $G - e$. Now, we define ϕ as a coloring of $(G \odot H) - e$ such that $\phi(a) = \phi_1(a)$ for $a \in E(G - e)$ and blue otherwise. We obtain a $(2K_2, 2K_n)$−coloring of $(G \odot H) - e$. So, $(G \odot H) - e \nrightarrow (2K_2, 2K_n)$. □

Theorem 9. *Let $G, H \in \mathfrak{R}(2K_2, K_n)$ be connected graphs and $a \in E(G)$, $e \in E(H)$. If $(G \ominus H)$ is a graph obtained by identifying edges a and e, then $(G \ominus H) \in \mathfrak{R}(2K_2, 2K_n)$.*

Proof. First, we prove that $(G \ominus H) \to (2K_2, 2K_n)$. Let $w \in V(G \ominus H)$. Then, $(G \ominus H) - w$ is connected. If $w \in V(G)$ is not incident to edge a, then $(G \ominus H) - w = (G - w) \ominus H$. Since $G - w \supseteq K_n$ and $H \supseteq K_n$, then $(G \ominus H) - w \supseteq 2K_n$. If w is incident to edge a, let $a = vw$ then $(G \ominus H) - w = (G - w) \odot (H - w)$ by identifying vertex v. Since $G - w \supseteq K_n$ and $H - w \supseteq K_n$, then $(G \ominus H) - w \supseteq 2K_n$. Let K_3 in $G \ominus H$. Then, this K_3 can contain the edge a or not. If K_3 does not contain a, then $(G \ominus H) - E(K_3) = (G - E(K_3)) \ominus H$. Since $G - E(K_3) \supseteq K_n$ and $H \supseteq K_n$, then $(G \ominus H) - E(K_3) \supseteq 2K_n$. If K_3 contains $a = v_1 w$, then

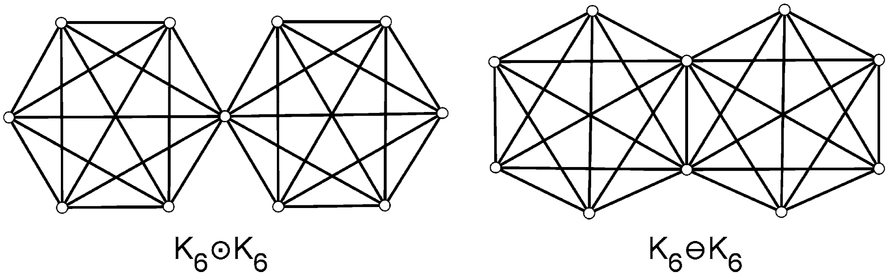

Fig. 3 $K_6 \odot K_6, K_6 \ominus K_6 \in \mathfrak{R}(2K_2, 2K_4)$

$(G \ominus H) - E(K_3) = (G - E(K_3)) \odot (H - e)$ for some $e = v_2w$ by identifying two vertices $v_1 = v_2$ and w. Since $G - E(K_3) \supseteq K_n$ and $(H - e) \supseteq K_n$, then $(G \ominus H) - E(K_3) \supseteq 2K_n$.

Next, we show the minimality. Let $b \in E(G \ominus H)$. Then the edge b can be the identified edge or not. If b is the identified edge, let $b = vw$, then $(G \ominus H) - b = (G - b) \odot (H - b)$ by identifying two vertices v and w. So, there exists a $(2K_2, K_n)$−coloring of both $G - b$ and $H - b$. Clearly, there exists a $(2K_2, 2K_n)$−coloring of $(G \ominus H) - b$. If b is not the identified edge, we may assume $b \in E(G)$, and then there exists a $(2K_2, K_n)$−coloring ϕ_1 of $G - b$. We define ϕ as a coloring of $(G \ominus H) - b$ such that $\phi(e) = \phi_1(e)$ for $e \in E(G - b)$ and blue otherwise. Then, it is easy to verify that ϕ is a $(2K_2, 2K_n)$−coloring of $(G \ominus H) - b$. □

By Theorems 8 and 9, we have the following corollary.

Corollary 3. $\{K_6 \odot K_6, K_6 \odot F_1, K_6 \odot F_2, F_1 \odot F_1, F_1 \odot F_2, F_2 \odot F_2\} \subseteq \mathfrak{R}(2K_2, 2K_4)$ *and* $\{K_6 \ominus K_6, K_6 \ominus F_1, K_6 \ominus F_2, F_1 \ominus F_1, F_1 \ominus F_2, F_2 \ominus F_2\} \subseteq \mathfrak{R}(2K_2, 2K_4)$. □

References

1. Baskoro, E.T., Yulianti, L.: On Ramsey minimal graphs for $2K_2$ versus P_n. Adv. Appl. Discret. Math. **8**(2), 83–90 (2011)
2. Borowiecka-Olszewska, M., Haluszczak M.: On Ramsey $(K_{1,m}, \mathcal{G})$−minimal graphs. Discret. Math. **313**(19), 1843–1855 (2012)
3. Burr, S.A., Erdös, P., Faudree, R.J., Schelp, R.H.: A class of Ramsey-finite graphs. In: Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory and Computing, Boca Raton, pp. 171–180 (1978)
4. Burr, S.A., Erdös, P., Faudree, R.J., Schelp, R.H.: Ramsey minimal graphs for matchings. In: The Theory and Applications of Graphs (Kalamazoo, 1980), pp. 159–168. Wiley, New York (1981)
5. Mengersen, I., Oeckermann, J.: Matching-star Ramsey sets. Discret. Appl. Math. **95**, 417–424 (1999)
6. Tatanto, D., Baskoro, E.T.: On Ramsey $(2K_2, 2P_n)$−minimal graphs. AIP Conf. Proc. **1450**, 90–95 (2012)

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