

Chapter 2

Stable Toric Varieties

For a detailed introduction to the theory of toric varieties, one should consult the usual sources [21, 48]. Our introduction is very brief and serves mainly to set up the notation and clarify the definitions (for example, our toric varieties are normal and do not have the “origin” fixed). The theory of stable toric varieties reviewed below is contained in [4, 9].

By k we denote the base field, which is assumed to be algebraically closed.

2.1 Projective toric varieties and polytopes

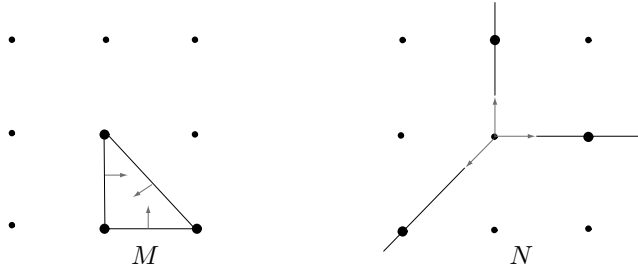
2.1.1 Toric varieties and torus embeddings

The *multiplicative group variety* $\mathbb{G}_m$ is the group variety $\mathrm{Spec} k[t, 1/t] = \mathbb{A}^1 \setminus \{0\}$. It comes with the structure morphisms $\mathrm{mult}: \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m$, inverse: $\mathbb{G}_m \rightarrow \mathbb{G}_m$, and unit: $\mathrm{Spec} k \rightarrow \mathbb{G}_m$, satisfying the group axioms. The reason not to write simply k^* is that k^* is a set but $\mathbb{G}_m$ is an algebraic variety. The set of k -points is $\mathbb{G}_m(k) = k^*$.

The *multiplicative torus* T of dimension r is $\mathbb{G}_m^r = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$. It comes with two standard lattices commonly denoted M and N , $N = \mathrm{Hom}(M, \mathbb{Z})$:

1. The lattice $N = \mathrm{Hom}(\mathbb{G}_m, T) \simeq \mathbb{Z}^r$ of *one-parameter subgroups*. An arbitrary toric variety is described by a fan (a collection of finitely generated strictly convex cones) in $N_{\mathbb{R}} = N \otimes \mathbb{R}$. The pictures recorded in this space are “inverted”. A cone τ of dimension d corresponds to a T -orbit O_τ of dimension $r-d$, and the order is reversed: τ_1 is a face of τ_2 , denoted $\tau_1 < \tau_2$, if $\overline{O_{\tau_1}} \supset O_{\tau_2}$.
2. The lattice $M = \mathrm{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$ of *characters*, or *monomials*. This space is responsible for a “direct picture”. A d -dimensional polytope in $M_{\mathbb{R}} = M \otimes \mathbb{R}$ corresponds to a d -dimensional projective toric variety, and the inclusions go the same way.

We will work exclusively with the M -lattice, which in fact is much easier.

Figure 2.1: A polytope in M and its normal fan in N .

Definition 2.1.1. A *toric variety* is a normal variety with a T -action which has a dense T -orbit O . A *torus embedding* is a normal variety with a T -action and a fixed embedding $T \subset X$ which is a dense T -orbit.

Thus, the main difference between a toric variety and a torus embedding is that the latter comes with a special point $1 \in T \subset X$, while in the former the “origin” is not chosen.

If p is any point in the dense orbit O and T_p is its stabilizer, then the orbit O is a torsor (principal homogeneous space) over a torus $T'' = T/T_p$ of dimension at most r . We allow the stabilizer T_p to be nontrivial but we assume that it is connected. In characteristic $p > 0$ we assume additionally that it is reduced. As any algebraic subgroup of a torus, T_p is the product of a torus T' and several copies of groups of roots of unity μ_{n_i} (which in characteristic p may be connected and non-reduced if n_i is a power of p). So we are saying that $T_p = T'$ and there is no finite part.

A torus embedding together with a T -action has no isomorphisms. A toric variety together with a T -action still has an automorphism group equal to T'' .

2.1.2 Polarized toric varieties vs polytopes

Definition 2.1.2. A *polarized toric variety* is a pair (X, L) of a projective toric variety X and an ample line bundle L on it.

Every line bundle on a toric variety is linearizable, and two linearizations differ by an element in $\text{Hom}(T, \mathbb{G}_m) = M$. A *linearization* of L is a lift of the action $T \curvearrowright X$ to the $\mathbb{A}^1$ -bundle $\mathbb{L} \rightarrow X$ corresponding to L . When L is ample, it is also the same as an action of T on the ring $R(X, L) = \bigoplus_{d=0}^{\infty} H^0(X, L^d)$ which induces the original T -action on $X = \text{Proj } R(X, L)$.

The main connection that we need between algebraic geometry and combinatorics is expressed in the following result.

Theorem 2.1.3. *There is a 1-to-1 correspondence between (isomorphism classes of) polarized toric varieties (X, L) with linearized L , and polytopes P with vertices in the lattice M , under which $\dim X = \dim P$.*

Furthermore, a polytope P_1 is a face of a polytope P_2 if and only if X_1 is a T -invariant subvariety of X_2 and $L_1 \simeq L_2|_{X_1}$ as T -linearized line bundles.

Translating a polytope by an element of M corresponds to another choice of the linearization.

Figure 2.2 illustrates this correspondence.

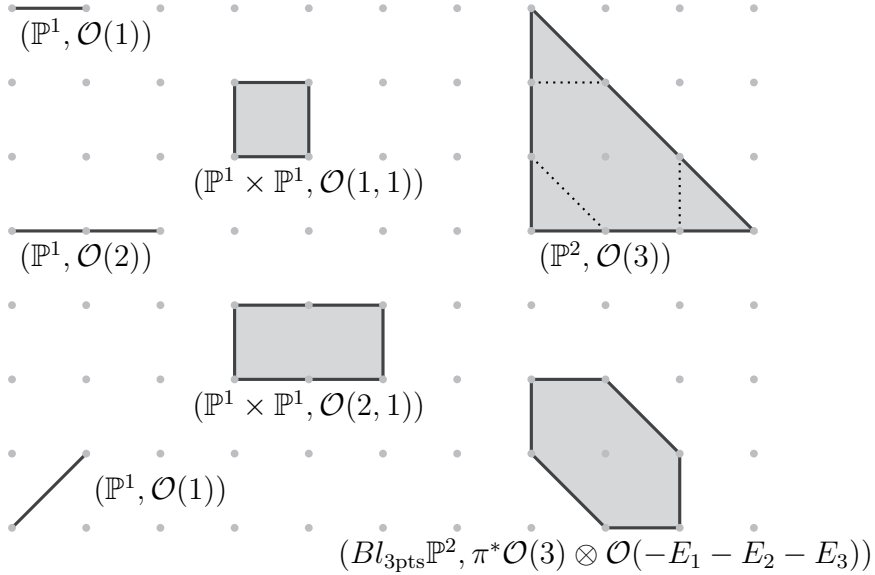


Figure 2.2: Polarized toric varieties $\leftrightarrow$ lattice polytopes.

The correspondence proceeds as follows. If (X, L) is a polarized toric variety and L is T -linearized, then T acts on $H^0(X, L)$. An algebraic action of a torus on a vector space V is diagonalizable and decomposes V into a direct sum $\bigoplus_{m \in M} V_m$ over the character group, so that for $v \in V_m$ the action is

$$(\lambda_1, \dots, \lambda_r) \cdot v = \prod_{i=1}^r \lambda_i^{a_i} \cdot v,$$

where $m = (a_1, \dots, a_r) \in M$. The characters m are also called *weights*. It is known that, for the action $T \curvearrowright H^0(X, L)$, the weights m with $V_m \neq 0$ are in bijection with the integral points of a lattice polytope P , and for each of these weights $\dim V_m = 1$. Thus, the polytope P is the convex hull of the weights m such that $H^0(X, L)_m \neq 0$.

In the opposite direction, start with a polytope $P \subset M_{\mathbb{R}}$. Let $\text{Cone}(1, P)$ be the cone in $\mathbb{R}^{1+r}$ over the polytope $(1, P)$. We call the extra dimension the degree, so we put P in degree 1. Let S be the semigroup of integral points $\mathbb{Z}^{1+r} \cap$

$\text{Cone}(1, P)$. It is graded by the degree. The semigroup algebra $k[S]$ is a graded algebra. Then, $X = \text{Proj } k[S]$ and $L = \mathcal{O}_{\text{Proj } k[S]}(1)$.

The elements $s = (d, m) \in S$ are the monomials x^s in this algebra, $\deg s = d$. Relations between the vectors in S give relations in $k[S]$. Choosing generators of S and figuring out relations between them gives concrete coordinates and homogeneous equations for X .

Example 2.1.4. Let P be the triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$. Denote $u = x^{(0,0)}$, $v = x^{(1,0)}$, $w = x^{(0,1)}$. Then u, v, w generate $k[S]$ and there are no relations, so X is $\mathbb{P}^2$ with homogeneous coordinates u, v, w .

Let P be the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. Denote $u = x^{(0,0)}$, $v = x^{(1,0)}$, $w = x^{(0,1)}$, $t = x^{(1,1)}$. Then u, v, w, t generate $k[S]$ and there is a single relation $(0, 0) + (1, 1) = (1, 0) + (0, 1)$. So X is a subvariety of $\mathbb{P}^3$ defined by the homogeneous equation $ut = vw$. Of course, $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

If F is a polytope, then F^0 denotes its relative interior, i.e., F minus the proper faces. F^0 is a locally closed subset of $\mathbb{R}^r$.

Lemma 2.1.5. *There is a bijection between the T -orbits of X and locally closed faces F^0 of the polytope P , $O_F \hookrightarrow F$, $F < P$. Moreover, this bijection is dimension- and order-preserving:*

1. $\dim F = \dim O_F$, and
2. $F_1 < F_2$ (i.e., $F_1^0 \subset \overline{F_2^0}$) if and only if $O_{F_1} \subset \overline{O_{F_2}}$.

Note that $P = \bigsqcup F_i^0$ and $X = \bigsqcup O_{F_i^0}$.

2.2 Stable toric varieties and tilings

A stable toric variety is a seminormal union of toric varieties, glued along T -invariant subvarieties. Combinatorially, it corresponds to a union of polytopes glued along faces.

Definition 2.2.1. A variety X is said to be *seminormal* if any finite morphism $f: X' \rightarrow X$ which is a bijection is in fact an isomorphism.

An example of a *non*-seminormal variety is the cuspidal curve $y^2 = x^3$: the normalization is a bijection, but not an isomorphism. This is a good way to think about seminormal varieties: they are varieties without “cusps”. The main statement about seminormal singularities is the following one.

Lemma 2.2.2. *Any variety has a unique seminormalization $\pi^{\text{sn}}: X^{\text{sn}} \rightarrow X$, a proper bijective morphism with seminormal X^{sn} which has a universality property: any morphism $Y \rightarrow X$ from a seminormal variety factors uniquely through π^{sn} .*

A curve is seminormal if and only if it is locally analytically isomorphic to a union of n coordinate axes in $\mathbb{A}^n$ for some n . For such a curve, $n = \dim T_{X,x}$ at

the singular point. In particular, a planar curve is seminormal if it has at worst nodes as singularities.

Definition 2.2.3. A polarized stable toric variety is a pair (X, L) of a projective variety with a linearized ample line bundle such that

1. X is seminormal, and
2. the irreducible components $(X_i, L_i = L|_{X_i})$ are polarized toric varieties.

(A “variety” for us need not be irreducible, but it has to be reduced; also, recall that our toric varieties are normal by definition.)

Thus, a stable toric variety is glued from ordinary toric varieties in a generic way, without introducing “cusps”.

For every irreducible component (X_i, L_i) we have a lattice polytope. An intersection $X_i \cap X_j$ has to be T -invariant, so it is a closed union of orbits of both X_i and X_j . On the combinatorial side, this gives a closed union of faces of both P_i and P_j .

Definition 2.2.4. The *topological type* of a stable toric variety is the topological space $|\Delta| = \bigcup P_i$, a union of polytopes glued in the same way as $X = \bigcup X_i$, together with the finite map $\rho: |\Delta| \rightarrow M_{\mathbb{R}}$ such that $\rho|_{P_i}: P_i \rightarrow M_{\mathbb{R}}$ are the embeddings of lattice polytopes corresponding to (X_i, L_i) .

The easiest complex Δ is a tiling of a bigger polytope P by smaller polytopes $P = \bigcup P_i$. An example is given in Figure 2.3. In these lectures, we will only work with stable toric varieties of this form. But in principle, the images of the polytopes in $M_{\mathbb{R}}$ are allowed to intersect or cover each other, so ρ is not necessarily an inclusion. For example, we can take two copies of the same square and glue them along the boundary; in this case ρ would be generically 2-to-1.

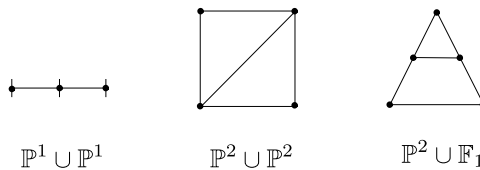


Figure 2.3: Some simple stable toric varieties.

For each tiling Δ (or a more general complex of lattice polytopes), there is generally not a single variety, but a *family* of polarized stable toric varieties. That is because there may be many ways to glue the individual toric varieties.

The gluing can be understood as follows. Choose an “origin” in every irreducible component, thus fixing on each X_i the structure of a torus embedding. Then on every component of the intersections $X_i \cap X_j$ one gets two “origins”,

coming from X_i and from X_j . They differ by an element $t_{ij} \in T_{ij}$ in a corresponding torus. The collection (t_{ij}) has to satisfy the 1-cocycle condition $t_{ij}t_{jk}t_{ki} = 1$ on $X_i \cap X_j \cap X_k$. On the other hand, the “origins” in the varieties X_i can be chosen arbitrarily, up to an action of the tori T_i . Thus, the collection (t_{ij}) is defined only up to a 1-coboundary $(t_i t_j^{-1})$. Putting this together shows that the possible glued stable toric varieties X are in bijection with a certain 1-cohomology group $H^1(\Delta, \underline{T})$. Figure 2.4 gives an example where this group is one-dimensional.

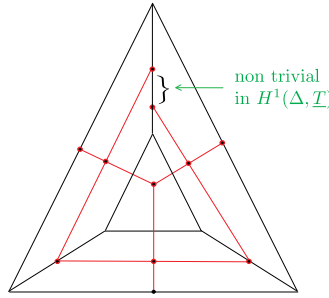


Figure 2.4: Complex Δ with a one-dimensional family of STVs.

Repeating the same argument for the *polarized* stable toric varieties we get that (X, L) are in bijection with the 1-cohomology group $H^1(\Delta, \underline{\mathbb{T}})$, where $\underline{T}$ and $\underline{\mathbb{T}}$ are constructible sheaves on $|\Delta|$ related by an exact sequence

$$1 \longrightarrow \underline{\mathbb{G}}_m \longrightarrow \underline{\mathbb{T}} \longrightarrow \underline{T} \longrightarrow 1,$$

and $\underline{\mathbb{G}}_m$ is a constant sheaf on $|\Delta|$. In particular, if $|\Delta|$ is simply connected then $H^p(\Delta, \underline{\mathbb{G}}_m) = 1$ for $p > 0$ and $H^1(\Delta, \underline{\mathbb{T}}) = H^1(\Delta, \underline{T})$.

2.3 Linear systems on toric and stable toric varieties

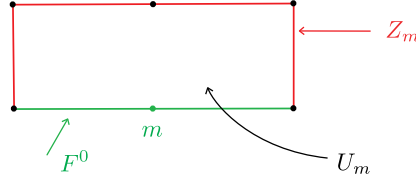
2.3.1 Linear systems on toric varieties

Let (X, L) be a polarized toric variety. Another basic fact is that the linear system $|L|$ is base-point free and defines a finite morphism $\varphi_L: X \rightarrow \mathbb{P}^N$, $N = h^0(X, L) - 1$, which however need not be a closed embedding or even generically 1-to-1.

Lemma 2.3.1. *Let m be an integral point in the lattice polytope associated to (X, L) , and $e_m \in H^0(X, L)$ a corresponding section. Let F be the minimal face of P containing m , so that $m \in F^0$. Then*

1. *the open subset $U_m = \{e_m \neq 0\}$ is $U_m = \bigcup_{m \in F_i} O_{F_i}^0$, and*
2. *the closed subset $Z_m = (e_m)$ is $Z_m = \bigcup_{m \notin F_i} O_{F_i}^0$.*

This lemma is illustrated in Figure 2.5.

Figure 2.5: Open subset U_m and closed subset Z_m .

Definition 2.3.2. Let $A \subset P \cap M$ be an arbitrary subset of integral vectors in P . Let $V_A = \bigoplus_{m \in A} ke_m \subset H^0(X, L)$ be a linear system and $\varphi_A: X \rightarrow \mathbb{P}^{|A|-1}$ be the corresponding rational map.

We denote by $\mathbb{R}_A$ the vector subspace of $\mathbb{R}^{1+r}$ generated by the vectors $(1, m)$, $m \in A$, and by $\mathbb{Z}_A = \mathbb{R}_A \cap \mathbb{Z}^{1+r}$ the corresponding saturated sublattice.

We denote by $\mathbb{R}_P$ and $\mathbb{Z}_P$ the corresponding sets for $A = P \cap M$.

Theorem 2.3.3. *The following holds:*

1. *The rational map φ_A is regular, i.e., the base locus of the linear system V_A is empty if and only if $A \supset \text{Vertices}(P)$. In this case, φ_A is a finite map of degree $|\mathbb{Z}_P : \langle (1, m), m \in A \rangle|$.*
2. *Assuming $A \supset \text{Vertices}(P)$, the map φ_A is a closed embedding if and only if for every vertex v , the semigroup of integral vectors in the cone $\mathbb{R}_{\geq 0}(P - v) \subset M_{\mathbb{R}}$ is generated by the vectors $a - v$, $a \in A$.*
3. *In particular, if the semigroup $S_P = \text{Cone}(1, P) \cap \mathbb{Z}^{1+r}$ is generated by the vectors $(1, m)$, $m \in A$, then φ_A is a closed embedding.*

Definition 2.3.4. We call a set $A \subset P \cap M$ *generating* if the group $\mathbb{Z}_P$ is generated by the vectors $(1, m)$, $m \in A$, and *totally generating* if the semigroup S_P is generated by $(1, m)$, $m \in A$. We call a lattice polytope *generating* (resp. *totally generating*) if the set $\text{Vertices}(P)$ is generating (resp. totally generating).

Thus, for a generating polytope, the map $\varphi_A: X \rightarrow \mathbb{P}^{|A|-1}$ is generically 1-to-1 for any $A \supset \text{Vertices}(P)$ and, for a totally generating subset, φ_A is a closed embedding.

2.3.2 Linear systems on stable toric varieties

The following theorem is contained in [4]:

Theorem 2.3.5. *Let (X, L) be a stable toric variety. Then:*

1. $H^p(X, L) = 0$ for $p > 0$;
2. $H^0(X, L) = \bigoplus_{\rho^{-1}(M) \cap |\Delta|} ke_m$. In other words, $H^0(X, L)$ is a direct sum of one-dimensional eigenspaces, one for each “integral” point of the topological space $|\Delta|$. Thus, $H^0(X, L)$ is the union of $H^0(X_i, L_i)$ for the irreducible components X_i , with subspaces corresponding to $X_i \cap X_j$ identified.

2.4 Stable toric varieties over a projective variety V

2.4.1 Definition and main result

Let $\mathbb{P}^n$ be a projective space together with a T -linearized sheaf $\mathcal{O}(1)$. The linearization is the same as an assignment $z_j \mapsto m_j = \text{wt}(z_j) \in M$, $j = 1, \dots, n$.

Definition 2.4.1. A (stable) toric variety *over* $\mathbb{P}^{n-1}$ is a (stable) toric variety X together with a finite morphism $f: X \rightarrow \mathbb{P}^{n-1}$ and an isomorphism $L \simeq f^*\mathcal{O}(1)$ of T -linearized ample sheaves.

The homomorphism f is the same as a homomorphism of graded vector spaces $H^0(\mathbb{P}^{n-1}, \mathcal{O}(1)) = \bigoplus_{j=1}^n kz_j \rightarrow H^0(X, L)$. It gives a homomorphism

$$f^*: \bigoplus_{d \geq 0} H^0(\mathbb{P}^{n-1}, \mathcal{O}(d)) = k[z_1, \dots, z_n] \longrightarrow R(X, L) = \bigoplus_{d \geq 0} H^0(X, L^d)$$

and the map $f: (X, L) \rightarrow (\mathbb{P}^{n-1}, \mathcal{O}(1))$ in the opposite direction. Thus, the morphism f is equivalent to picking n homogeneous eigenvectors $f^*(z_{m_j}) = e_{m_j} \in H^0(X, L)$ with $\text{wt}(e_{m_j}) = m_j$.

Let $A \subset \{1, \dots, n\}$ be the subset of those m_j such that $e_{m_j} \neq 0$. For each irreducible component X_i of X we get a set $A_i = A \cap P_i$. By the previous section, one has $A_i \supset \text{Vertices}(P_i)$, otherwise the map f is not regular on X_i .

One can easily generalize the above definition by considering a T -invariant subvariety $V \subset \mathbb{P}^{n-1}$ with the sheaf $\mathcal{O}_V(1) = \mathcal{O}_{\mathbb{P}^{n-1}}|_V$.

Definition 2.4.2. A (stable) toric variety *over* $V \subset \mathbb{P}^{n-1}$ is a (stable) toric variety X together with a finite morphism $f: X \rightarrow V$ and an isomorphism $L \simeq f^*\mathcal{O}(1)$ of T -linearized ample sheaves.

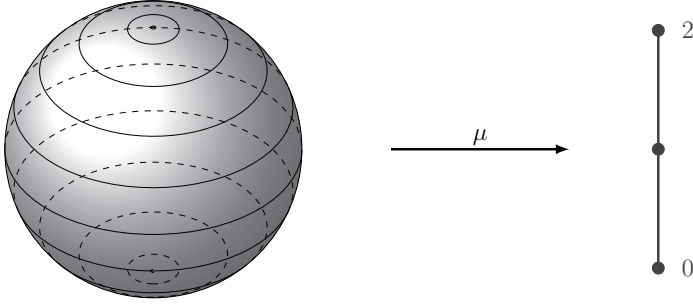
The main result about stable toric varieties over $V \subset \mathbb{P}^{n-1}$ is the following:

Theorem 2.4.3 ([9]). *For each topological type $|\Delta|$ there exists a coarse moduli space $M_{|\Delta|}^T(V)$ of stable toric varieties over V . Further, $M_{|\Delta|}^T(V)$ is a projective scheme.*

Each point $[f: X \rightarrow V] \in M_{|\Delta|}^T(V)$ defines a tiling $\bigcup P_i$ of $|\Delta|$ by lattice polytopes, and the sets $A_i \supset \text{Vertices}$. The points of $M_{|\Delta|}^T(V)$ with the same $\bigcup(P_i, A_i)$ form a locally closed stratum. This gives a stratification of $M_{|\Delta|}^T(V)$.

2.4.2 Moment map

When working over $\mathbb{C}$, there is an even nicer geometric connection between (X, L) and a lattice polytope P : there is a natural *moment map* $\mu: X(\mathbb{C}) \rightarrow M_{\mathbb{R}}$ whose image is P . So, $X(\mathbb{C})$ is fibered over the polytope P . [Figure 2.6](#) gives an illustration for $(X, L) = (\mathbb{P}^1, \mathcal{O}(2))$.

Figure 2.6: Moment map for $(\mathbb{P}^1, \mathcal{O}(2))$.

The moment map for $(\mathbb{P}^{n-1}, \mathcal{O}(1))$ with a T -linearized sheaf $\mathcal{O}(1)$ is defined by the formula

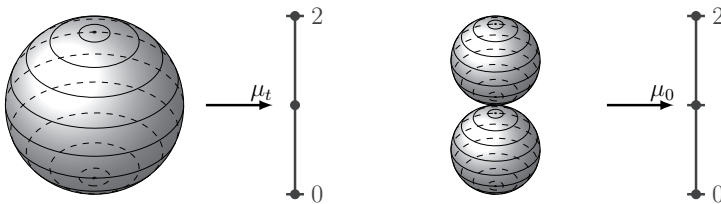
$$\mu(z_1, \dots, z_n) = \frac{\sum |z_j|^2 \cdot m_j}{\sum |z_j|^2}.$$

The moment map for a (stable) toric variety over $\mathbb{P}^{n-1}$ is the composition $X(\mathbb{C}) \rightarrow \mathbb{P}^{n-1}(\mathbb{C}) \rightarrow M_{\mathbb{R}}$. Thus, if f is given by $f^*: z_j \mapsto c_j e_j \in H^0(X, L)$, where e_j is a homogeneous basis of $H^0(X, L)$, then the moment map $\mu: X(\mathbb{C}) \rightarrow M_{\mathbb{R}}$ is defined by the formula

$$\mu(p) = \frac{\sum |c_m e_m(p)|^2 \cdot m}{\sum |c_m e_m(p)|^2}, \quad m \in |\Delta| \cap \rho^{-1}(M).$$

The preimage $\mu^{-1}(y)$ over a point in an a -dimensional face of P is isomorphic to the compact real torus $(S^1)^a$.

The moment map gives a nice representation of the T -orbits in X . The T -orbits are $\mu^{-1}(F^0)$, for all faces F in the tiling $\cup P_i$ of $|\Delta|$. Figure 2.7 gives an example of moment maps for the family of quadrics $x_0 x_2 = t x_1^2$ in $\mathbb{P}^2$ and its degeneration $x_0 x_2 = 0$.

Figure 2.7: Moment map for $(\mathbb{P}^1, \mathcal{O}(2))$ and of its degeneration $\mathbb{P}^1 \cup \mathbb{P}^1$.

2.5 Stable toric pairs vs stable toric varieties over $\mathbb{P}^{n-1}$

Consider a toric variety (X, L) with a T -linearized sheaf L . Pick a homogeneous basis $H^0(X, L) = \bigoplus_{m \in P \cap M} k e_m$. Any section $s \in H^0(X, L)$ can be uniquely written as $s = \sum c_m e_m$. Let $A \subset P \cap M$ be the set of those m for which $c_m \neq 0$.

Theorem 2.5.1. *The following conditions are equivalent:*

1. $A \supset \text{Vertices}(P)$;
2. the rational map $X \mapsto \mathbb{P}^{n-1}$, $n = h^0(X, L)$, defined by $z_m \rightarrow c_m e_m$ is regular;
3. the divisor $D = (s)$ does not contain any T -orbits.

Proof. We already saw the equivalence of 1. and 2.

To see the equivalence of 1. and 3., observe that condition (3) is equivalent to requiring that D does not contain any zero-dimensional T -orbits. The zero-dimensional orbits are in bijection with the vertices of P , $v \preceq Q_v$. For each vertex v of P , all the sections e_m with $m \neq v$ vanish at the point Q_v . So, $s(Q_v) \neq 0$ if and only if $c_v \neq 0$. $\square$

This shows that the moduli space $M_P^T(\mathbb{P}^{n-1})$ of stable toric varieties over $\mathbb{P}^{n-1}$ in this case is equivalent to the moduli space of stable toric *pairs* (X, D) of topological type P which satisfy condition (3) in the above theorem 2.5.1.

2.6 Singularities of stable toric varieties

2.6.1 Depth properties

It turns out that the singularities of a stable toric variety are completely determined by the topological space $|\Delta|$.

Definition 2.6.1. Let S be a topological space which has a structure of a finite simplicial complex. For a point $s \in S$, the link Link_s is the intersection of S with a small sphere centered at s .

The space S is called *Cohen–Macaulay* over the base field k if for all $s \in S$, one has $H_0(\text{Link}_s, k) = k$ and $H_p(\text{Link}_s, k) = 0$ for $0 < p < \dim \text{Link}_s$.

Theorem 2.6.2. *The support $|\Delta|$ of a stable toric variety (X, L) is Cohen–Macaulay if and only if X is Cohen–Macaulay.*

Corollary 2.6.3. *If the support $|\Delta|$ of a stable toric variety (X, L) is a polytope, then X is Cohen–Macaulay.*

Proof. Indeed, in this case Link_s is either a sphere, or a disk of dimension $d = \dim X - 1$, so $H_0 = k$ and $H_p = 0$ for $0 < p < d$. $\square$

Example 2.6.4. The complex in Figure 2.8 is *not* Cohen–Macaulay, since the link at the origin is two closed intervals, and $H_0 = k^2$. This stable toric variety is a union of two normal surfaces glued at a single point. One can also see this directly: Cohen–Macaulay implies connected in codimension 1. The surface in Figure 2.8 is not connected in codimension 1.

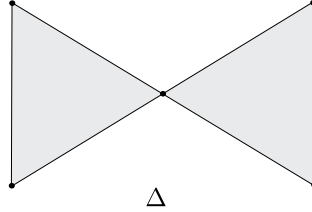


Figure 2.8: A non Cohen–Macaulay stable toric variety.

2.6.2 Log canonical and semi-log canonical

Toric varieties provide some of the easiest examples of log canonical singularities.

Lemma 2.6.5. *Let X be a toric variety and Δ be the union of the boundary divisors, the complement of the dense orbit. Then $K_X + \Delta \sim 0$ and the pair (X, Δ) is log canonical.*

Proof. The first property is well known, and the second property is a consequence of the first. Indeed, any toric variety has a toric resolution obtained by subdividing the cone. Let $f: Y \rightarrow X$ be such a resolution. Then Y is smooth and Δ^Y is a normal crossing divisor. Then,

$$f^*(K_X + \Delta) = f^*(0) = 0 = K_Y + \Delta^Y.$$

Since $\Delta^Y = \sum D_i$ is the sum of the boundary divisors with coefficients 1, (X, Δ) is lc. □

What if we want a stable pair, i.e., it should be lc and $K + B$ should be ample? Then we need to add something ample to $K_X + \Delta$.

Lemma 2.6.6. *Let B be an ample effective divisor. Then*

1. $K_X + \Delta + \epsilon B$ is ample for any $\epsilon > 0$;
2. the pair $(X, \Delta + \epsilon B)$ is lc for $0 < \epsilon \ll 1$ if and only if B does not contain any T -orbits, and if and only if B does not contain any zero-dimensional T -orbits.

The proof of (2) follows by continuity: the only places where $(X, \Delta + \epsilon B)$ is not lc for $0 < \epsilon \ll 1$ are the places where we are already at the limit, i.e., the discrepancy is $a_D = -1$. These are precisely the boundary divisors and their intersections, i.e., the closures of the T -orbits. And the only closed T -orbits are zero-dimensional.

Similarly, stable toric varieties provide some of the easiest examples of semi log canonical singularities.

Lemma 2.6.7. *Let X be a stable toric variety whose topological type is a manifold with boundary (for example, a polytope). Let Δ be the union of its outside boundary divisors (as illustrated in Figure 2.9). Then,*

1. $K_X + \Delta = 0$ and the pair (X, Δ) is slc;
2. for an effective divisor B , the pair $(X, \Delta + \epsilon B)$ is slc if and only if B does not contain any T -orbits, and if and only if B does not contain any zero-dimensional orbits.

Indeed, normalizing reduces the situation to the toric case, and $\nu^*(K_X + \Delta_{\text{outside}}) = K_{X^\nu} + \Delta_{\text{all}}$.

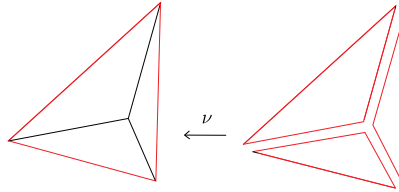


Figure 2.9: Outside boundary of a stable toric variety.

2.7 One-parameter degenerations

Many of the combinatorial constructions in this section originated in the work of Gelfand–Kapranov–Zelevinsky [23]. The conclusions from these combinatorial constructions differ in the following respect: [23] works with families of embedded cycles in $\mathbb{P}^{n-1}$, and some of these cycles may have multiplicities. However, the resulting family below is a family of stable toric varieties *over* $\mathbb{P}^{n-1}$ or, equivalently, the family of stable toric pairs (X_t, D_t) in which the varieties X_t are reduced, so there are no multiplicities.

Let us go in detail through a simple example, which hopefully illustrates everything there is to understand about one-parameter degenerations of stable toric varieties.

Consider a line $\mathbb{P}^1$ together with a divisor $\Delta + \epsilon B_t$, where $\Delta = P_0 + P_\infty$ is the boundary divisor, the complement of the dense torus orbit, and B_t is given by the

equation

$$f_t = c_0 t^2 x_0^5 + c_1 t x_0^4 x_1 + c_2 x_0^3 x_1^2 + c_3 x_0^2 x_1^3 + c_4 t x_0 x_1^4 + c_5 t x_1^5$$

for some fixed constants c_i . If all $c_i \neq 0$, then for any $t \neq 0$ the pair $(\mathbb{P}^1, \Delta + \epsilon B_t)$ is stable: it has lc singularities and ample $K_X + \Delta + \epsilon B_t$. Clearly, $\mathcal{O}_{\mathbb{P}^1}(B_t) \simeq \mathcal{O}_{\mathbb{P}^1}(5)$. It is a toric variety corresponding to the polytope $[0, 5]$ in $M_{\mathbb{R}}$, $M = \mathbb{Z}$.

We would like to understand the limit of this pair as $t \rightarrow 0$.

2.7.1 Complimentary degenerations

Let us associate to this family the following graph. For each of the points $m = 0, 1, \dots, 5$ let $h(m)$ be the *height*, the valuation at t of the coefficient of $x_0^{5-m} x_1^m$ in f_t . This graph is shown in [Figure 2.10](#).

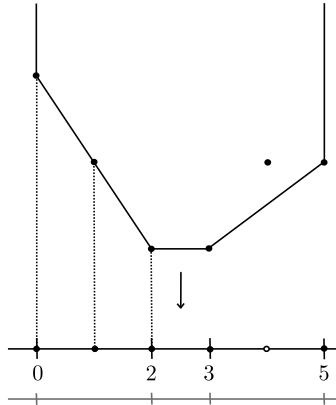


Figure 2.10: One-parameter degeneration of stable toric varieties over V .

The easiest way to degenerate the pair $(\mathbb{P}^1, B_t)$ is to simply look at the limit of the equation f_t as $t \rightarrow 0$. It is

$$f_t(0) = c_2 x_0^2 x_1^2 (c_2 x_0 + c_3 x_1).$$

The pair $(\mathbb{P}^1, \Delta + \epsilon B_0)$ is not lc, since the coefficients of P_0 and P_∞ are $1 + 2\epsilon$. The log canonical model of $(\mathbb{P}^1, \Delta + \epsilon B_0)$ (see our Definition 1.2.5) is $(X_0^{(1)}, \Delta + \epsilon B_0^{(1)})$, where $X_0^{(1)} = \mathbb{P}^1$ and $B_0^{(1)} = (c_2 x_0 + c_3 x_1)$, which is a point distinct from $0, \infty$.

To be absolutely clear, $X_0^{(1)}$ does not correspond to the polytope $[0, 5]$ as the original toric variety $\mathbb{P}^1$. Instead, it corresponds to the polytope $[2, 3]$.

However, let us rescale the coordinates as follows: $y_0 = x_0$, $y_1 = t^{-1} x_1$. In the new coordinates, our family becomes

$$f_t = t^2 (c_0 y_0^5 + c_1 y_0^4 y_1 + c_2 y_0^3 y_1^2 + c_3 t y_0^2 y_1^3 + c_4 t^3 y_0 y_1^4 + c_5 t^4 y_1^5).$$

The height function is obtained from the previous one by adding a linear function $h_2(m) = h(m) - t - 2$. The limit now is

$$t^{-2}f_t(0) = y_0^3(c_0y_0^2 + c_1y_0y_1 + c_2y_1^2).$$

Again, the pair $(\mathbb{P}^1, \Delta + \epsilon(t^{-2}f_t(0)))$ is not lc, since the coefficient of P_∞ is $1 + 2\epsilon$. Its log canonical model is $(X_0^{(2)}, \Delta + \epsilon B_0^{(2)})$, where $X_0^{(2)} = \mathbb{P}^1$ and $B_0^{(2)} = (c_0y_0^2 + c_1y_0y_1 + c_2y_1^2)$, which is two points distinct from $0, \infty$. The new variety $X_0^{(2)}$ corresponds to the polytope $[0, 2]$.

Finally, we can rescale as follows: $z_0 = x_0$, $z_1 = tx_1$. In the new coordinates, our family becomes

$$f_t = t^3(c_0t^5z_0^5 + c_1t^3z_0^4z_1 + c_2t^3z_0^3z_1^2 + c_3z_0^2z_1^3 + c_4tz_0z_1^4 + c_5z_1^5).$$

The height function is obtained from the previous one by adding a linear function $h_3(m) = h(m) + t - 3$. The limit now is

$$t^{-3}f_t(0) = z_1^3(c_3z_0^2 + c_5z_1^2).$$

Again, the pair $(\mathbb{P}^1, \Delta + \epsilon(t^{-3}f_t(0)))$ is not lc, since the coefficient of P_0 is $1 + 3\epsilon$. Its log canonical model is $(X_0^{(3)}, \Delta + \epsilon B_0^{(3)})$, where $X_0^{(3)} = \mathbb{P}^1$ and $B_0^{(3)} = (c_3z_0^2 + c_5z_1^2)$, which is two points distinct from $0, \infty$. The new variety $X_0^{(3)}$ corresponds to the polytope $[3, 5]$.

Clearly, the three degenerations above correspond to different choices of adjusting the height function by a linear homogeneous function $h(m) \mapsto h(m) + \ell(m)$ and then taking the minimum.

2.7.2 Stable degeneration

The polytope $P = [0, 5]$ lies in the space $M_{\mathbb{R}}$, where $M = \text{Hom}(T, \mathbb{G}_m)$ is the character lattice of the torus acting on the varieties X_t . The polytope P gives a cone $\text{Cone}(1, P)$ in $(\mathbb{Z} \oplus M) \otimes \mathbb{R}$, and each variety X_t , $t \neq 0$, can be written as the Proj of the semigroup algebra $k[\text{Cone}(1, P) \cap (\mathbb{Z} \oplus M)]$.

Now, to the family X_t let us associate an infinite polyhedron lying in the space $M \oplus \mathbb{Z}$. The additional $\mathbb{Z}$ corresponds to the height and not to the degree.

Definition 2.7.1. The polyhedron P^+ is the lower convex envelope of the rays $(m, h(m) + \mathbb{R}_{\geq 0})$. It is depicted in [Figure 2.10](#), and is semi-infinite in the upward direction.

Consider a cone over $(1, P^+)$ lying in the space $(\mathbb{Z} \oplus M \oplus \mathbb{Z}) \otimes \mathbb{R}$. Let R^+ be the semigroup algebra

$$R^+ = k[\text{Cone}(1, P^+) \cap (\mathbb{Z} \oplus M \oplus \mathbb{Z})].$$

A point (d, m, h) corresponds to a monomial $t^h x^{(d, m)}$ of degree $d \geq 0$. Let $X^+ = \text{Proj } R^+$. Since R^+ is a $k[t]$ -algebra, X^+ has a natural morphism $f: X^+ \rightarrow \text{Spec } k[t] = \mathbb{A}_t^1$. This is our degenerating family.

The central fiber of this family is a scheme with three irreducible components $X^{(1)}, X^{(2)}, X^{(3)}$, corresponding to the lower faces of P^+ . Projecting these faces down to $[0, 5]$ gives a polyhedral subdivision of it into the intervals $[0, 2]$, $[2, 3]$ and $[3, 5]$.

Definition 2.7.2. For any lattice polytope P , a polyhedral subdivision obtained by projecting down the lower envelope of the polyhedron P^+ for some height function $h: P \cap M \rightarrow \mathbb{R}$ is called a *convex subdivision*. Other names used for such subdivisions are *regular*, or *coherent*.

Remark 2.7.3. Not every polyhedral subdivision is convex. The standard counterexample is depicted in Figure 2.11.

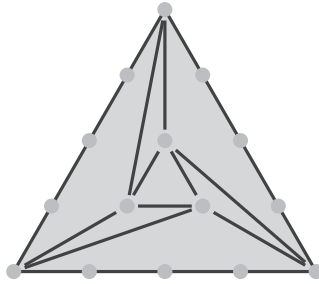


Figure 2.11: A non-convex tiling.

The corresponding stable toric variety does not appear as the limit of a one-parameter family of $(\mathbb{P}^2, D_t)$ with $\mathcal{O}_{\mathbb{P}^2}(D_t) \simeq \mathcal{O}_{\mathbb{P}^2}(4)$.

To continue with our construction, there is yet another twist: the central fiber of $\text{Proj } R^+ \rightarrow \mathbb{A}_t^1$ is not reduced, because the component X_3 appears in it with multiplicity 2. To see this, note that the monomial $tx_0x_1^4 \in R^+$ does not lie in the ideal (t) , but its square does: $t^2x_0^2x_1^8 = t \cdot x_0^2x_1^3 \cdot tx_1^5$. The reason for this is that the set $\{3, 5\}$ of the polytope $[3, 5]$ is not generating, the vertices $(1, 3), (1, 5)$ generate a sublattice of index 2 in $\mathbb{Z} \oplus M$.

This changes, however, by making a finite ramified base change $\mathbb{A}_s^1 \rightarrow \mathbb{A}_t^1$, $t = s^2$. After this base change the central fiber is a reduced seminormal union of the toric varieties $X^{(1)}, X^{(2)}, X^{(3)}$, each isomorphic to $\mathbb{P}^1$. No further finite base change changes the central fiber.

The original equation f_t defines a section of the invertible sheaf $\mathcal{O}(1)$ on $\mathcal{X} = \text{Proj } R^+$, a relative Cartier divisor B on $\mathcal{X}$ which restricts to the divisor $B_0 = B_0^{(1)} \cup B_0^{(2)} \cup B_0^{(3)}$ on the central fiber X_0 . Moreover, some thinking about this will convince the reader that R^+ is the *only* graded subalgebra of $k[M \oplus \mathbb{Z}]$ for

which f_t stays a regular section of $\mathcal{O}(1)$ and satisfies the following condition on the central fiber: “ B_0 does not contain any T -orbits”. This condition was part of our definition of stable toric varieties; so, the moduli functor of stable toric pairs is proper.

2.7.3 Maximal and higher codimension degenerations

What will happen if we rescale the height function by $h(m) \mapsto h(m) + \frac{1}{2}m$? For this to make sense, we will have to first make the base change $t = s^2$. The new height function will be $h'(m) = 2h(m)$, and then we will rescale it by $h'(m) \mapsto h'(m) + m$.

The minimum of $h'(m)$ is achieved at the unique point $\{2\}$. The limit of the divisors B_t is given by the equation $x_0^3 x_1^2$. The corresponding pair is $(\mathbb{P}^1, (1 + 3\epsilon)P_0 + (1 + 2\epsilon)P_\infty)$ and it is not lc.

If we attempt to find its log canonical model, then it is not going to work because the “round-down” pair $(\mathbb{P}^1, P_0 + P_\infty)$ is not of general type, the divisor $K_{\mathbb{P}^1} + P_0 + P_\infty$ is not of general type. The image of its Iitaka fibration, however, is a point.

Other “non-maximal” choices give the toric varieties, each a single points, corresponding to the polytopes $\{0\}$, $\{3\}$, and $\{5\}$. Clearly, the variety X is glued from the three irreducible components $X_0^{(1)}$, $X_0^{(2)}$, $X_0^{(3)}$ along these smaller-dimensional varieties.

How do we recognize if a certain choice of a height function is going to give us an irreducible component in the central fiber or a smaller-dimensional stratum of the limit variety X_0 ?

In terms of the graph, the answer is obvious: it is whether the corresponding polytope is maximal-dimensional or not.

In terms of the pair, the answer is as follows: the maximal-dimensional degenerations correspond to the pairs $(X, \Delta + \epsilon B)$ for which the automorphism group is finite. For $X_0^{(1)}$, $X_0^{(2)}$, $X_0^{(3)}$ the groups are 1, 1, and $\mu_2 \simeq \mathbb{Z}_2$. For the degenerations giving lower-dimensional strata of the central fiber X_0 , the group contains $\mathbb{G}_m$ and is infinite. The dimension of the group equals the codimension of the stratum.

2.8 Toric varieties associated to hyperplane arrangements

A hyperplane arrangement is a collection of n hyperplanes $B_1, \dots, B_n$ in a projective space $\mathbb{P}^{r-1}$; these hyperplanes are allowed to coincide. We consider them up to an isomorphism of pairs, i.e., $(\mathbb{P}^{r-1}, B_1, \dots, B_n)$ is isomorphic to $(\mathbb{P}^{r-1}, B'_1, \dots, B'_n)$ if and only if there exists an automorphism $g \in \text{PGL}(r)$ of $\mathbb{P}^{r-1}$ such that $g(B_1) = B'_1, \dots, g(B_n) = B'_n$.

We will be concerned with complete moduli of hyperplane arrangements. On the boundary of this moduli space the projective space $\mathbb{P}^{r-1}$ will split up somehow and degenerate to some non-normal variety, a higher-dimensional analogue of a stable curve.

The general idea is very simple. Since we understand so well degenerations of toric varieties, let us associate to a hyperplane arrangement $(\mathbb{P}^{r-1}, B_1, \dots, B_n)$ a toric pair $(Y, \Delta + \epsilon D)$ or a toric variety $Y \rightarrow V$ over some projective variety. Let us do this in a reversible way, so that we can go back from Y to $(\mathbb{P}^{r-1}, B_1, \dots, B_n)$.

Then, degenerations of toric varieties will give us degenerations of hyperplane arrangements, and the complete moduli spaces of stable toric varieties will give us complete moduli spaces of stable hyperplane arrangements.

2.8.1 The Gelfand–MacPherson correspondence

There are two dual ways to work with hyperplane arrangements, related by the Gelfand–MacPherson correspondence, which we explain now.

Consider an $(r \times n)$ -matrix A of rank r with nonzero columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rn} \end{pmatrix}.$$

The columns of this matrix, considered as linear functions $f_i(x) = a_{1i}x_1 + \dots + a_{ri}x_r$, define n hyperplanes $B_1, \dots, B_n$ in a projective space $\mathbb{P}^{r-1}$. The condition $\text{rank } A = r$ is equivalent to the following condition, which we want to stress:

$$\bigcap_{i=1}^n B_i = \emptyset.$$

Let $\text{Mat}^0(r, n)$ be the set of all such matrices. Let $\mathbb{P}(\text{Mat}^0(r, n))$ be the corresponding projective space of dimension $rn - 1$. The set $\text{HA}(r, n)$ of isomorphism classes of n hyperplanes in $\mathbb{P}^{r-1}$ is the quotient

$$\text{HA}(r, n) = \text{PGL}(r) \setminus \mathbb{P}(\text{Mat}^0(r, n)) / T,$$

where $\text{PGL}(r) = \text{GL}(r)/\mathbb{G}_m$ and $T = (\mathbb{G}_m^n)/\text{diag } \mathbb{G}_m$ is a torus of dimension $n - 1$. The group $\text{PGL}(r)$ acts on the rows by changing the basis in $\mathbb{A}^r$, and the torus T acts by scalar multiplication on the columns, rescaling the equations without changing the hyperplanes B_i they define.

Now, if we take the quotient $\mathbb{P}(\text{Mat}^0(r, n))/T$ first, we obtain $(\mathbb{P}^{r-1 \vee})^n$, the set of hyperplanes in a fixed projective space $\mathbb{P}^{r-1}$. Then

$$\text{HA}(r, n) = \text{PGL}(r) \backslash (\mathbb{P}^{r-1 \vee})^n.$$

If we take the quotient $\mathrm{PGL}(r) \backslash \mathbb{P}(\mathrm{Mat}^0(r, n))$ first, we obtain $G^0(r, n)$, the open subset of the Grassmannian $G(r, n)$ of r -dimensional quotient spaces $\mathbb{A}^n \rightarrow V^*$ of a fixed n -dimensional space with the additional condition that V are not contained in any of the n coordinate hyperplanes. (Alternatively, $G(r, n)$ parameterizes the subspaces $V \subset \mathbb{A}^n$, but we treat the columns as linear equations, so the interpretation with the quotients suits us better.) Then $\mathrm{HA}(r, n) = G^0(r, n)/T$.

So a single hyperplane arrangement, up to an isomorphism, is the same as a T -orbit inside $G^0(r, n)$. A point in the Grassmannian is $\mathbb{A}^n \rightarrow V^*$, i.e., $\mathbb{P}V \subset \mathbb{P}^{n-1}$. The hyperplanes B_i are the intersections of the n coordinate hyperplanes $H_i = \{z_i = 0\} \subset \mathbb{P}^{n-1}$ with $\mathbb{P}V$. The torus $T = \mathbb{G}_m^n / \mathrm{diag} \mathbb{G}_m$ acts by rescaling the n homogeneous coordinates z_i .

Let $T \cdot [V]$ be a single orbit. Its closure $Y = \overline{T \cdot [V]}$ is then a projective toric variety. A priori, it may be non-normal. It turns out, however, that the Grassmannians are very special and nice, and Y is indeed an ordinary normal toric variety (see Theorem 4.1.6).

Note that the orbit does not have a special “origin”, so Y is a toric variety and not a torus embedding. It is a toric variety $Y \rightarrow G(r, n)$ over the Grassmannian.

In order to recover the hyperplane arrangement from Y , let $P \rightarrow G(r, n)$ be the universal family, $P \subset \mathbb{P}^{n-1} \times G(r, n)$, whose fiber over $[\mathbb{P}V \subset \mathbb{P}^{n-1}]$ is $\mathbb{P}V \subset \mathbb{P}^{n-1}$. If Y^0 is the dense T -orbit of Y and

$$P_{Y^0} := p_2^{-1}(Y^0) = P \times_{G(r, n)} Y^0 \subset P,$$

then the hyperplane arrangement is P_{Y^0}/T .

So, the set $\mathrm{HA}(r, n)$ is the quotient set $G(r, n)/T$, and the hyperplane arrangements are the T -quotients of the preimages of these orbits in the universal family $P \rightarrow G(r, n)$.

Of course, the quotient set $G(r, n)/T$ is extremely nasty and has no algebraic variety structure. To get a nice space, we will have to take the GIT quotient. To recover the hyperplane arrangements themselves, we will have to take the GIT quotients of projective varieties $P_Y = P \times_{G(r, n)} Y$. So we will need to understand the issues involved with taking such quotients.

The degenerations of hyperplane arrangements will correspond to degenerations of toric varieties, which will be some stable toric varieties over $G(r, n)$. Thus, irreducible components X_i of a degeneration $X = \bigcup X_i$ will correspond to some toric varieties $Y_i \subset G(r, n)$, which in turn means that they will correspond to some new hyperplane arrangements $[\mathbb{P}V_i \subset \mathbb{P}^{n-1}]$. So, somehow, a limit of a family of hyperplane arrangements will be glued from several other hyperplane arrangements.

2.8.2 Torus action on the Grassmannian

Let us explain this torus action in more detail. There is a natural action of the torus $\tilde{T} = (\mathbb{G}_m)^n$ on $\mathbb{A}^n$:

$$(\lambda_1, \dots, \lambda_n) \cdot (z_1, \dots, z_n) = (\lambda_1 z_1, \dots, \lambda_n z_n).$$

This defines an action of $\tilde{T}$ on the Grassmannian $G(r, n)$, as follows. If $G(r, n) \subset \mathbb{P}^N$, $N = \binom{n}{r} - 1$, is the Plücker embedding with Plücker coordinates p_I for all $I \subset \bar{n}$, $|I| = r$, then the induced action is

$$(\lambda_1, \dots, \lambda_n) \cdot p_I = \left(\prod_{i \in I} \lambda_i \right) p_I.$$

An algebraic action of a torus on any vector space A is diagonalizable, and one gets a decomposition $A = \bigoplus_{\chi \in \Lambda_{\tilde{T}}} A_{\chi}$ into eigenspaces. Here, $\Lambda_{\tilde{T}} = \text{Hom}(\tilde{T}, \mathbb{G}_m) = \mathbb{Z}^n$ is the character group of $\tilde{T}$. Thus, to every eigenvector v one assigns a character, also called its *weight* $\text{wt}(v) \in \mathbb{Z}^n$.

In these terms, one has $\text{wt}(z_i) = \mathbf{e}_i$ and $\text{wt}(p_I) = \mathbf{e}_I = \sum_{i \in I} \mathbf{e}_i$. Since $\text{diag } \mathbb{G}_m \subset \tilde{T}$ acts trivially on $\mathbb{P}^{n-1}$, the torus $T = \tilde{T} / \text{diag } \mathbb{G}_m \simeq \mathbb{G}_m^{n-1}$ also acts on $\mathbb{P}^{n-1}$ and $G(r, n)$. The character group of T is

$$\Lambda_T = \left\{ \sum n_i \mathbf{e}_i \mid \sum n_i = 0 \right\}.$$

We do not have a natural T -action on $\mathbb{A}^n$, so there are no weights in Λ_T assigned to the homogeneous coordinates z_i, p_I . However, for the coordinates on the standard affine covers one has $\text{wt}(z_i/z_j) = \mathbf{e}_i - \mathbf{e}_j$ and $\text{wt}(p_I/p_J) = \mathbf{e}_I - \mathbf{e}_J$.

2.8.3 Moment polytope of a hyperplane arrangement

Under the Plücker embedding $Y \subset G(r, n) \subset \mathbb{P}^{N-1}$, the moment polytope of the toric variety $Y = \overline{T \cdot [V]}$ is the convex hull of the vectors $\mathbf{e}_I$ for all $I \subset \bar{n}$, $|I| = r$, such that the corresponding Plücker coordinate $p_I(V)$ of the space $V \subset \mathbb{A}^n$ is nonzero.

This condition is equivalent to any of the following two conditions:

1. $\bigcap_{i \in I} B_i = \emptyset$ in $\mathbb{P}V$;
2. the linear equations f_i of the hyperplanes B_i form a basis in the dual space V^* .

This moment polytope is called a *matroid polytope*. The collection of vectors $f_i \in V^*$ is called a *vector matroid*.

Thus, to understand the compactified moduli spaces of hyperplane arrangements, we will have to understand matroid polytopes, and matroids in general.

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