Chapter II

Geodesics on S^3

In this chapter we study the geodesic vector field on the tangent bundle of the 3-sphere. We examine its relation to the Kepler vector field, which governs the motion of two bodies in $\mathbf{R}^3$ under gravitational attraction. We give two methods to regularize the flow of the Kepler vector field: one energy surface by energy surface and the other for all negative energies at once.

1 The geodesic vector field

Here we find the geodesic vector field on the 3-sphere and give a formula for its flow.

We begin by discussing the geodesic vector field. Suppose that $\langle\, , \rangle$ is the Euclidean inner product on $\mathbf{R}^4$. This induces a *Riemannian metric* g on $\mathbf{R}^4$ defined by $g(x)^\sharp(y)z = \langle y, z\rangle$, where $x \in \mathbf{R}^4$ and $y, z \in T_x\mathbf{R}^4 = \mathbf{R}^4$. Pulling back the canonical symplectic 2-form on $T^*\mathbf{R}^4$ by the map $g^\sharp$, see chapter VI §2, we obtain the symplectic form $\omega_4 = -\,\mathrm{d}\langle y, \mathrm{d}x\rangle$ on $T\mathbf{R}^4$. On $(T\mathbf{R}^4, \omega_4)$ consider the Hamiltonian function

$$\mathscr{H} : T\mathbf{R}^4 \to \mathbf{R} : (x, y) \mapsto \tfrac{1}{2}\langle y, y\rangle. \tag{1}$$

Since an integral curve of the Hamiltonian vector field $X_{\mathscr{H}}$ satisfies $\dot{x} = y$ and $\dot{y} = 0$, it is a straight line on $T\mathbf{R}^4$, except when $y = 0$; then it is a point. Hence $X_{\mathscr{H}}$ describes the *motion* of a particle in $T\mathbf{R}^4$ which is not subject to any *force*. To constrain this free particle so that it moves on the 3-sphere $S^3 = \{x \in \mathbf{R}^4 \,\big|\, \langle x, x\rangle = 1\}$, we add a force $\lambda(x, \dot{x})x$ which is normal to S^3 at the point x. The motion of the particle subject to this constraining force is governed by Newton's equations

$$\ddot{x} = \lambda(x, \dot{x})\, x. \tag{2}$$

Differentiating the defining equation of S^3 twice gives

$$\langle x, \ddot{x}\rangle + \langle \dot{x}, \dot{x}\rangle = 0. \tag{3}$$

Substituting (2) into (3) and using the constraint $\langle x, x\rangle = 1$ gives $\lambda(x, \dot{x}) = -\langle \dot{x}, \dot{x}\rangle$. Hence the *motion* of the free particle constrained to S^3 is governed by the second order equation

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$$\ddot{x} = -\langle \dot{x}, \dot{x} \rangle x \tag{4}$$

subject to the constraints $\langle x,x\rangle = 1$ and $\langle \dot{x},x\rangle = 0$. Written as a first order equation on the tangent bundle $TS^3 = \{(x,y) \in T\mathbf{R}^4 \,\big|\, \langle x,x\rangle = 1 \;\&\; \langle x,y\rangle = 0\}$ of S^3, the constrained system (4) becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\langle y,y\rangle x. \end{aligned} \tag{5}$$

This defines the integral curves of the vector field $Y = \langle y, \frac{\partial}{\partial x}\rangle - \langle y,y\rangle\langle x, \frac{\partial}{\partial y}\rangle$ on TS^3. Note that TS^3 is an invariant manifold of (5), thought of as a vector field on $T\mathbf{R}^4$, since the initial conditions $\langle x,x\rangle = 1$ and $\langle x,y\rangle = 0$ are preserved under its flow. The above ▷ discussion is not at all Hamiltonian. What we want to do is to show that Y is a Hamiltonian vector field on the phase space (TS^3, Ω_4). Here Ω_4 is a suitable symplectic form.

(1.1) **Proof**: To do this, we use modified Dirac brackets, see chapter VI §4. On the open subset $M = T(\mathbf{R}^4 \setminus \{0\})$ of $T\mathbf{R}^4$ consider the constraint functions

$$c_1 : M \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}(\langle x,x\rangle - 1) \quad \text{and} \quad c_2 : M \to \mathbf{R} : (x,y) \mapsto \langle x,y\rangle.$$

Let $\{\,,\,\}$ be the *standard Poisson bracket* on $C^\infty(T\mathbf{R}^4)$, the space of smooth functions on the symplectic manifold $(T\mathbf{R}^4, \omega_4)$, see chapter VI §4. Since the matrix $\big(\{c_i, c_j\}\big)$, which is equal to $\begin{pmatrix} 0 & \langle x,x\rangle \\ -\langle x,x\rangle & 0 \end{pmatrix}$, is invertible on M with inverse $(C_{ij}) = \frac{1}{\langle x,x\rangle}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and 0 is a regular value of the constraint map $\mathscr{C} : M \to \mathbf{R}^2 : m \mapsto \big(c_1(m), c_2(m)\big)$, the constraint manifold $TS^3 = \mathscr{C}^{-1}(0)$ is a *cosymplectic* submanifold of $(M, \omega_4|M)$. In other words, $\Omega_4 = \omega_4|TS^3$ is a symplectic form on TS^3. For $F \in C^\infty(M)$ let

$$F^* = F - \sum_{i,j=1}^{2} (\{F, c_i\} + F_i) C_{ij} c_j,$$

where the F_i lies in the ideal of $(C^\infty(M), \cdot)$ generated by c_1 and c_2. Define a Poisson bracket $\{\,,\,\}_{TS^3}$ on $C^\infty(TS^3)$ by

$$\{F|TS^3, G|TS^3\}_{TS^3} = \{F^*, G^*\}|TS^3.$$

Note that the Hamiltonian vector field $X_{F|TS^3}$ of the Hamiltonian F constrained to TS^3 is the Hamiltonian vector field X_{F^*} restricted to TS^3. Applying these remarks to the unconstrained Hamiltonian $\mathscr{H}$ (1) on M gives

$$\begin{aligned} \mathscr{H}^* &= \mathscr{H} - \sum_{i,j} (\{\mathscr{H}, c_i\} + \mathscr{H}_i) C_{ij} c_j \\ &= \tfrac{1}{2}\langle y,y\rangle + \langle x,x\rangle^{-1}\big\langle \big(\langle x,y\rangle - \mathscr{H}_1, \langle y,y\rangle - \mathscr{H}_2\big), \big(-\langle x,y\rangle, \tfrac{1}{2}(\langle x,x\rangle - 1)\big)\big\rangle \\ &= \tfrac{1}{2}\big(\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2\big), \end{aligned}$$

where we have chosen $\mathscr{H}_1 = \langle x,y\rangle(1 - \frac{1}{2}\langle x,x\rangle)$ and $\mathscr{H}_2 = -\langle y,y\rangle(\langle x,x\rangle - 1)$.

From Hamilton's equations on $(T\mathbf{R}^4, \omega_4)$ it follows that the integral curves of $X_{\mathscr{H}^*}$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = A(x,y)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\langle x,y\rangle & \langle x,x\rangle \\ -\langle y,y\rangle & \langle x,y\rangle \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{6}$$

Using (6) and the definition of TS^3, it is easy to see that the integral curves of $X_{\mathscr{H}^*}|TS^3$ satisfy (5). Because $X_{\mathscr{H}|TS^3} = X_{\mathscr{H}^*}|TS^3$, the *geodesic vector field* on TS^3 is the Hamiltonian vector field X_H on (TS^3, Ω_4) corresponding to the Hamiltonian function

$$H = \mathscr{H}^*|TS^3 : TS^3 \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}\langle y,y\rangle. \tag{7}$$

Note that H is the free particle Hamiltonian on $T\mathbf{R}^4$ restricted to TS^3. Thus the integral
▷ curves of the geodesic vector field X_H on TS^3 satisfy (5). To find the flow of the geodesic vector field X_H, we first look for *integrals* (= conserved quantities) of the vector field $X_{\mathscr{H}^*}$. From the construction of the Hamiltonian $\mathscr{H}^*$ on $T\mathbf{R}^4$, we know that TS^3 is an invariant manifold of $X_{\mathscr{H}^*}$. Therefore the functions $f_1(x,y) = \frac{1}{2}\langle x,x\rangle$ and $f_2(x,y) = \langle x,y\rangle$ are integrals of $X_{\mathscr{H}^*}$. A calculation shows that $f_3(x,y) = \frac{1}{2}\langle y,y\rangle$ is also an integral of $X_{\mathscr{H}^*}$. The integrals $\{f_1, f_2, f_3\}$ span a Lie subalgebra of $(C^\infty(T\mathbf{R}^4), \{\, , \})$, which is isomorphic to $\mathrm{sl}(2,\mathbf{R})$ since $\{f_1, f_2\} = 2f_1$, $\{f_1, f_3\} = f_2$, and $\{f_3, f_2\} = -2f_3$. Because the functions f_i are constant along the integral curves of $X_{\mathscr{H}^*}$, so is the matrix $A(x,y)$ (6). Since $A^2(x,y) = -2\mathscr{H}^*(x,y)I_2$ and $\mathscr{H}^*(x,y) \ge 0$, the flow of $X_{\mathscr{H}^*}$ is

$$\varphi_t^{\mathscr{H}^*}(x,y) = \exp tA(x,y)\begin{pmatrix} x \\ y \end{pmatrix} = \big(\cos(t\sqrt{2\mathscr{H}^*})I_2 + \big(\sin(t\sqrt{2\mathscr{H}^*})/\sqrt{2\mathscr{H}^*}\big)A(x,y)\big)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Restricting $\varphi_t^{\mathscr{H}^*}$ to the invariant manifold TS^3 gives

$$\varphi_t^H(x,y) = \begin{pmatrix} \cos(t\sqrt{2H}) & \sin(t\sqrt{2H})/\sqrt{2H} \\ -\sqrt{2H}\sin(t\sqrt{2H}) & \cos(t\sqrt{2H}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{8}$$

which is the *flow* of the geodesic vector field X_H on TS^3. □

Clearly, all of the integral curves of X_H on the level set $H^{-1}(h)$ with $h > 0$ are periodic
▷ of *period* $2\pi/\sqrt{2h}$. In fact, when $y \ne 0$, the image of the integral curve $t \mapsto \varphi_t^H(x,y)$ under the bundle projection map $TS^3 \to S^3 : (x,y) \mapsto x$ is the *geodesic*

$$\gamma_{(x,y)} : \mathbf{R} \to S^3 : t \mapsto x\big(\cos(t\sqrt{2H})\big) + y\big((\sin(t\sqrt{2H})/\sqrt{2H}\big). \tag{9}$$

(1.2) **Proof**: To see that $\gamma_{(x,y)}$ is a geodesic on S^3 it suffices to show that

1. $\gamma_{(x,y)}$ is parametrized up to an affine transformation by arc length.
2. The *acceleration* $\ddot{\gamma}_{(x,y)}$ has no tangential component.

From the equations of motion for geodesics it follows that item 2 holds. Item 1 holds because γ is parametrized. Another argument to prove item 1 goes as follows. Differentiating (9) gives

$$\langle \dot{\gamma}_{(x,y)}, \dot{\gamma}_{(x,y)}\rangle = 2H\sin^2(t\sqrt{2H})\,\langle x,x\rangle + \cos^2(t\sqrt{2H})\,\langle y,y\rangle = \langle y,y\rangle = 2H(x,y),$$

which is a constant of motion. This constant is nonzero, since $y \neq 0$. □

The explicit formula (8) for the flow of the geodesic vector field gives *no* qualitative information about how the integral curves are organized into invariant manifolds. To understand the invariant manifolds, it is useful to explain the role of the obvious symmetry of the problem, namely, the group SO(4) of rigid motions of the 3-sphere. This will be done in the next section.

2 The SO(4)-momentum mapping

In this section we construct the momentum mapping associated to the SO(4) symmetry of the geodesic vector field on (TS^3, Ω_4) and study its geometric properties.

Recall that SO(4) is the Lie group of *orthogonal* linear mappings of $(\mathbf{R}^4, \langle\, ,\, \rangle)$ into itself with determinant 1. Consider the action of SO(4) on $\mathbf{R}^4$ given by $\varphi : \mathrm{SO}(4) \times \mathbf{R}^4 \to \mathbf{R}^4 : (A, x) \mapsto Ax$. This action lifts to an action of SO(4) on $(T\mathbf{R}^4, \omega_4)$ defined by

$$\Phi : \mathrm{SO}(4) \times T\mathbf{R}^4 \to T\mathbf{R}^4 : \big(A, (x, y)\big) \mapsto (Ax, Ay).$$

▷ Φ preserves the 1-form $\theta = \langle y, \mathrm{d}x\rangle$ on $T\mathbf{R}^4$.

(2.1) **Proof**: We compute

$$\Phi_A^* \theta \,=\, \langle Ay, \mathrm{d}Ax\rangle \,=\, \langle Ay, A\,\mathrm{d}x\rangle \,=\, \langle A^t Ay, \mathrm{d}x\rangle \,=\, \langle y, \mathrm{d}x\rangle \,=\, \theta.$$

The second to last equality follows because $A \in \mathrm{SO}(4)$. □

Thus the action Φ is symplectic, for

$$\Phi_A^* \omega_4 = -\Phi_A^*(\mathrm{d}\theta) \,=\, -\mathrm{d}(\Phi_A^*\theta) \,=\, -\mathrm{d}\theta \,=\, \omega_4.$$

▷ To show that Φ is a Hamiltonian action, we must verify that for every $a \in \mathrm{so}(4)$, the Lie algebra of SO(4), the vector field

$$X^a(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_{\exp ta}(x,y) \,=\, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big((\exp ta)x, (\exp ta)y\big) = (ax, ay) \,=\, (X_a(x), ay),$$

which is the infinitesimal generator of Φ in the direction a, is a Hamiltonian vector field on $(T\mathbf{R}^4, \omega_4)$.

(2.2) **Proof**: From the momentum lemma, see chapter VII ((5.7)), it follows that $X^a = X_{J^a}$ where

$$J^a : T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \theta(x,y)X_a(x) = \langle ax, y\rangle. \tag{10}$$

Thus the action Φ has momentum mapping $J : T\mathbf{R}^4 \to \mathrm{so}(4)^*$ defined by $J(x,y)a = J^a(x,y)$. Choose a basis $\{e_{ij}\}_{1\le i<j\le 4}$ of so(4) where the $(k,\ell)^{th}$ entry of the 4×4 matrix e_{ij} is 1 if $(k,\ell) = (i,j)$, -1 if $(k,\ell) = (j,i)$, and 0 otherwise. Then

$$J^{e_{ij}}(x,y) = \langle e_{ij}x, y\rangle \,=\, x_i y_j - x_j y_i = S_{ij}(x,y). \tag{11}$$

▷ The mapping J is coadjoint equivariant.

(2.3) **Proof**: We compute

$$\begin{aligned} J(\Phi_A(x,y))a = J(Ax,Ay)a &= \langle aAx, Ay\rangle = \langle A^{-1}aAx, y\rangle \\ = J(x,y)(Ad_{A^{-1}}a) &= Ad^t_{A^{-1}}(J(x,y))a. \end{aligned}$$ □

Since Φ_A maps TS^3 into itself for every $A \in \mathrm{SO}(4)$, Φ restricts to an action $\widehat{\Phi}$ on TS^3 given by $\widehat{\Phi} : \mathrm{SO}(4) \times TS^3 \to TS^3 : \big(A,(x,y)\big) \mapsto (Ax,Ay)$. For every $a \in \mathrm{so}(4)$ the infinitesimal generator X^a of the SO(4)-action Φ leaves TS^3 invariant because

$$\begin{aligned} \frac{\mathrm{d}\langle x,x\rangle}{\mathrm{d}t} &= 2\langle x,\dot{x}\rangle = 2\langle x,ax\rangle = 0, \\ \frac{\mathrm{d}\langle x,y\rangle}{\mathrm{d}t} &= \langle \dot{x},y\rangle + \langle x,\dot{y}\rangle = \langle x,ay\rangle + \langle ax,y\rangle = 0, \\ \frac{\mathrm{d}\langle y,y\rangle}{\mathrm{d}t} &= 2\langle y,\dot{y}\rangle = 2\langle y,ay\rangle = 0, \end{aligned}$$

since $a^t = -a$. Therefore $X^a|TS^3$ is a vector field on TS^3. The action $\widehat{\Phi}$ preserves the symplectic form Ω_4 on TS^3 because

$$\widehat{\Phi}^*_A\Omega_4 = \widehat{\Phi}^*_A(\omega_4|TS^3) = (\Phi^*_A\omega_4)|TS^3 = \omega_4|TS^3 = \Omega_4.$$

Claim: The action $\widehat{\Phi}$ on (TS^3,Ω_4) is Hamiltonian with *momentum mapping*

$$\mathscr{J} = J|TS^3 : TS^3 \subseteq T\mathbf{R}^4 \to \mathrm{so}(4)^*. \qquad (12)$$

(2.4) **Proof**: Because X^a leaves TS^3 invariant and $\Omega_4 = \omega_4|TS^3$, it follows that $X^a|TS^3 = X_{J^a|TS^3}$. Thus $X^a|TS^3$ is the infinitesimal generator of $\widehat{\Phi}$ on TS^3 in the direction a. □

So far the SO(4) symmetry is not related to the geodesic flow on TS^3. But note, the Hamiltonian $\mathscr{H}^*$ is preserved by the action Φ, because for every $A \in \mathrm{SO}(4)$

$$\mathscr{H}^*(\Phi_A(x,y)) = \tfrac{1}{2}\big(\langle Ax,Ax\rangle\langle Ay,Ay\rangle - \langle Ax,Ay\rangle^2\big) = \mathscr{H}^*(x,y).$$

▷ Thus the function J^a (10) is an integral of the vector field $X_{\mathscr{H}^*}$ for every $a \in \mathrm{so}(4)$.

(2.5) **Proof**: For every $a \in \mathrm{so}(4)$ we have $\Phi^*_{\exp ta}\mathscr{H}^* = \mathscr{H}^*$. Therefore

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi^*_{\exp ta}\mathscr{H}^* = L_{X^a}\mathscr{H}^* = L_{X_{J^a}}\mathscr{H}^* = -L_{X_{\mathscr{H}^*}}J^a. \qquad (13)$$

From the fact that Φ preserves both the Hamiltonian $\mathscr{H}^*$ and the manifold TS^3, it follows that $\widehat{\Phi}$ preserves the geodesic Hamiltonian $H = \mathscr{H}^*|TS^3$. Therefore for every $a \in \mathrm{so}(4)$ the function $J^a|TS^3$ is an integral of the geodesic vector field X_H. □

In order to study the geometry of the momentum mapping $\mathscr{J}$ (12), we transform it into an easier to understand mapping, see (16) below. We begin by recalling that the 4×4 skew symmetric matrices $\{e_{ij}\}_{1 \le i < j \le 4}$ form a basis for the Lie algebra $(\mathrm{so}(4), [\,,\,])$. The covectors $\{e^*_{ij}\}_{1\le i<j\le 4}$, where $e^*_{ij} = e^t_{ij}$, form the standard dual basis for $\mathrm{so}(4)^*$. The Lie bracket $\{\, , \}_{\mathrm{so}(4)^*}$ on $\mathrm{so}(4)^*$ is defined by $\{e^*_{ij}, e^*_{\ell k}\}_{\mathrm{so}(4)^*} = \sum_{m,n} c^{mn}_{ij,\ell k}\, e^*_{mn}$, where $[e_{ij}, e_{\ell k}] = \sum_{m,n} c^{mn}_{ij,\ell k}\, e_{mn}$.

For $u, v, w \in \mathbf{R}^4$ consider the map $\vartheta : \bigwedge^2 \mathbf{R}^4 \to \mathrm{so}(4) : u \wedge w \mapsto \ell_{u,w}$, where $\ell_{u,w} : \mathbf{R}^4 \to \mathbf{R}^4 : v \mapsto \langle v, w\rangle\, u - \langle v, u\rangle\, w$ is a linear mapping, which is skew symmetric, that is, $\langle \ell_{u,w} x, y\rangle = -\langle x, \ell_{u,w} y\rangle$ for every $x, y \in \mathbf{R}^4$. Using the basis $\{e_i \wedge e_j\}_{1\le i<j\le 4}$ of $\bigwedge^2 \mathbf{R}^4$, we see that $\vartheta(e_i \wedge e_j) = e_{ij}$. Thus ϑ is an isomorphism. Consequently the mapping $\vartheta^t : \mathrm{so}(4)^* \to (\bigwedge^2 \mathbf{R}^4)^* = \bigwedge^2 (\mathbf{R}^4)^* : e^*_{ij} \mapsto e^*_i \wedge e^*_j$. Since

$$\vartheta^t(e^*_{ij})(x,y) = (e^*_i \wedge e^*_j)(x,y) = e^*_i(x)\, e^*_j(y) - e^*_i(y) e^*_j(x) = x_i y_j - x_j y_i = S_{ij}(x,y), \quad (14)$$

for every $x, y \in \mathbf{R}^4$, it follows that $(\bigwedge^2 \mathbf{R}^4)^*$ is the space $\mathscr{S}$ of homogeneous quadratic functions on $T\mathbf{R}^4$, which is spanned by $\{S_{ij}\}_{1\le i<j\le 4}$. As a subspace of $C^\infty(T\mathbf{R}^4)$, $\mathscr{S}$ has a Poisson bracket $\{\, , \}_{\mathscr{S}}$, which is induced from the standard Poisson bracket $\{\, , \}$ on the space of smooth functions on $(T\mathbf{R}^4, \omega_4)$. In other words, for every $(x,y) \in T\mathbf{R}^4$

$$\{S_{ij}, S_{\ell k}\}_{\mathscr{S}}(x,y) = \omega_4\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big), \quad (15)$$

where $X_{S_{rs}}$ is the Hamiltonian vector field on $(T\mathbf{R}^4, \omega_4)$ corresponding to the Hamiltonian function S_{rs}. A calculation using (15) gives table 2.1.

$\{A,B\}_{\mathscr{S}}$	S_{12}	S_{13}	S_{14}	S_{23}	S_{24}	S_{34}	B
S_{12}	0	S_{23}	S_{24}	$-S_{13}$	$-S_{14}$	0	
S_{13}	$-S_{23}$	0	S_{34}	S_{12}	0	$-S_{14}$	
S_{14}	$-S_{24}$	$-S_{34}$	0	0	S_{12}	S_{13}	
S_{23}	S_{13}	$-S_{12}$	0	0	S_{34}	$-S_{24}$	
S_{24}	S_{14}	0	$-S_{12}$	$-S_{34}$	0	S_{23}	
S_{34}	0	S_{14}	$-S_{13}$	S_{24}	$-S_{23}$	0	
A							

Table 2.1. The Poisson bracket on $\mathscr{S}$.

Because the functions $f_1 = \frac{1}{2}\langle x, x\rangle$, $f_2 = \langle x, y\rangle$, and $f_3 = \frac{1}{2}\langle y, y\rangle$ are invariant under the $\mathrm{SO}(4)$ action Φ on $T\mathbf{R}^4$, the function J^a (10) is an integral of X_{f_i} for every $a \in \mathrm{so}(4)$. In other words, $\{f_i, J^a\} = 0$ for $i = 1,2,3$ and $a \in \mathrm{so}(4)$. Thus the Lie algebra $(\mathrm{sl}(2,\mathbf{R}), \{\, , \})$ spanned by $\{f_i\}_{1\le i\le 3}$ and the Lie algebra $(\mathscr{S}, \{\, , \}_{\mathscr{S}})$ are *dual pairs* in the Lie algebra of homogeneous quadratic functions on $T\mathbf{R}^4$ with Poisson bracket $\{\, , \}$. In other words, they have the following properties:

1. They centralize $\mathscr{H}^*$, that is, $\{\mathscr{H}^*, f_i\} = 0 = \{\mathscr{H}^*, S_{jk}\}$.
2. They centralize each other, that is, $\{f_i, S_{jk}\} = 0$.

$\triangleright$ We now show that the Lie algebras $(\mathscr{S}, \{\, , \}_{\mathscr{S}})$ and $(\mathrm{so}(4)^*, \{\, , \}_{\mathrm{so}(4)^*})$ are isomorphic.

(2.6) **Proof**: From the definition of S_{ij} (11) we obtain $\mathrm{d}S_{ij}(x,y) = -\langle e_{ij}(y), \mathrm{d}x\rangle + \langle e_{ij}(x), \mathrm{d}y\rangle$.

Since $\omega_4^\flat(\mathrm{d}x) = -\frac{\partial}{\partial y}$ and $\omega_4^\flat(\mathrm{d}y) = \frac{\partial}{\partial x}$, we find that

$$X_{S_{ij}}(x,y) = \omega_4^\flat(\mathrm{d}S_{ij})(x,y) = \langle e_{ij}(x), \frac{\partial}{\partial x}\rangle + \langle e_{ij}(y), \frac{\partial}{\partial y}\rangle.$$

Therefore

$$\begin{aligned}\{S_{ij}, S_{\ell k}\}_{\mathscr{S}}(x,y) = (X_{S_{\ell k}} \lrcorner\, \mathrm{d}S_{ij})(x,y) &= -\langle e_{\ell k}(x), e_{ij}(y)\rangle + \langle e_{\ell k}(y), e_{ij}(x)\rangle \\ &= \langle (e_{ij}e_{\ell k} - e_{\ell k}e_{ij})x, y\rangle = \langle [e_{ij}, e_{\ell k}]x, y\rangle = \vartheta^t([e_{ij}, e_{\ell k}]^*)(x,y) \\ &= \vartheta^t(\{e^*_{ij}, e^*_{\ell k}\}_{\mathrm{so}(4)^*})(x,y).\end{aligned}$$

The last equality above follows by definition of the Poisson bracket $\{\,,\,\}_{\mathrm{so}(4)^*}$. Hence ϑ^t is a Lie algebra isomorphism. □

On $\bigwedge^2\mathbf{R}^4$ define an inner product $B : \bigwedge^2\mathbf{R}^4 \times \bigwedge^2\mathbf{R}^4 \to \mathbf{R} : (u\wedge v, x\wedge y) \mapsto \det\begin{pmatrix}\langle u,x\rangle & \langle u,y\rangle \\ \langle v,x\rangle & \langle v,y\rangle\end{pmatrix}$. Since $\{e_i\wedge e_j\}_{1\le i<j\le 4}$ is an orthonormal basis of $(\bigwedge^2\mathbf{R}^4, B)$, we may identify $\bigwedge^2\mathbf{R}^4$ with

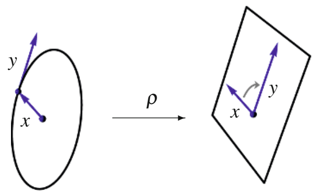

Figure 2.1. The mapping ρ.

$(\bigwedge^2\mathbf{R}^4)^*$. Instead of studying the momentum mapping $\mathscr{J}$ (12) we study the mapping

$$\rho : TS^3 \subseteq T\mathbf{R}^4 \to \bigwedge^2\mathbf{R}^4 : (x,y) \mapsto x\wedge y = \sum_{1\le i<j\le 4} S_{ij}(x,y)\, e_i\wedge e_j, \tag{16}$$

which is nothing but $B^\flat\circ\vartheta^t\circ\mathscr{J}$. The S_{ij} are the *Plücker coordinates* of the oriented 2-plane spanned by $\{x,y\}$ corresponding to the 2-vector $x\wedge y$. In other words, S_{ij} is the 2×2 minor formed from the i^{th} and j^{th} columns of the 2×4 matrix with rows x and y, that is, $S_{ij} = \det\begin{pmatrix} x_i & x_j \\ y_i & y_j\end{pmatrix}$. Because

$$0 = (x\wedge y)\wedge(x\wedge y) = (S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23})\, e_1\wedge e_2\wedge e_3\wedge e_4,$$

the Plücker coordinates of $x\wedge y$ satisfy *Plücker's equation*

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0. \tag{17}$$

Let C be the set of all nonzero 2-vectors on $\mathbf{R}^4$ whose Plücker coordinates satisfy (17). By
▷ definition $\rho(TS^3) \subseteq C$. Actually, C is the image of ρ.

(2.7) **Proof**: Suppose that $\theta \in C$. Then θ is decomposable, that is, there are vectors $u, v \in \mathbf{R}^4$ such that $\theta = u\wedge v$. To see this, let (S_{ij}) be the Plücker coordinates of θ. Since $\theta \neq 0$

not every S_{ij} is zero. Suppose that S_{12} is nonzero. Let $u = (1, 0, -S_{23}/S_{12}, -S_{24}/S_{12})$ and $v = (0, S_{12}, S_{13}, S_{14})$. Using Plücker's equation (17) it is easy to check that the Plücker coordinates of the 2-vector $u \wedge v$ are (S_{ij}). Therefore $\theta = u \wedge v$. A similar argument, which we omit, works in the other cases. Let $\{x, y\}$ be an orthonormal basis of the 2-plane spanned by $\{u, v\}$. Then $u \wedge v = \lambda\, x \wedge y$ for some nonzero λ. Therefore $\rho(x, \lambda y) = \theta$. $\square$

For $h > 0$ let $H^{-1}(h) = \{(x,y) \in TS^3 \subseteq T\mathbf{R}^4 \,\big|\, \frac{1}{2}\langle y, y\rangle = h\}$ be the h-level set of the geodesic Hamiltonian H (7). Consider the mapping

$$\rho_h : H^{-1}(h) \subseteq TS^3 \to C \subseteq \bigwedge\nolimits^2 \mathbf{R}^4 : (x,y) \mapsto x \wedge y, \tag{18}$$

which is the restriction of ρ (16) to $H^{-1}(h)$. From the identity

$$\sum_{1 \le i < j \le 4} (x_i y_j - x_j y_i)^2 = \langle x, x\rangle \langle y, y\rangle - \langle x, y\rangle^2 \tag{19}$$

we see that the image of ρ_h is contained in the submanifold C_h of C defined by $\sum_{1 \le i < j \le 4} S_{ij}^2$
▷ $= 2h$. C_h is diffeomorphic to $S^2_{\sqrt{h/2}} \times S^2_{\sqrt{h/2}}$.

(2.8) **Proof**: Adding and subtracting one half times (17) from one quarter times the defining equation of C_h, and using the variables

$$\begin{aligned} \xi_1 &= \tfrac{1}{2}(S_{12} + S_{34}) \qquad & \eta_1 &= \tfrac{1}{2}(S_{12} - S_{34}) \\ \xi_2 &= \tfrac{1}{2}(S_{13} - S_{24}) \qquad & \eta_2 &= \tfrac{1}{2}(S_{13} + S_{24}) \\ \xi_3 &= \tfrac{1}{2}(S_{14} + S_{23}) \qquad & \eta_3 &= \tfrac{1}{2}(S_{14} - S_{23}), \end{aligned} \tag{20}$$

we obtain $\xi_1^2 + \xi_2^2 + \xi_3^2 = h/2$ and $\eta_1^2 + \eta_2^2 + \eta_3^2 = h/2$. $\square$

We now investigate the geometry of the map ρ_h.

Claim: For every $h > 0$, the map $\rho_h : H^{-1}(h) \to C_h$ (18) is a surjective submersion each of whose fibers is a single oriented orbit of the geodesic vector field X_H of energy h.

(2.9) **Proof**: To show that ρ_h is surjective, suppose that $S = (S_{ij}) \in C_h$. Since C_h is contained in $C = \rho(TS^3)$, there is an $(x,y) \in TS^3$ such that $\rho(x,y) = S$. But $2h = \sum_{1 \le i < j \le 4} S_{ij}^2$ since $S \in C_h$. From (19), the definition of S_{ij} (14), and the fact that $(x,y) \in TS^3$, we find that $\frac{1}{2}\langle y, y\rangle = h$. Hence $(x,y) \in H^{-1}(h)$.

To show that ρ_h is a submersion, we must verify that the rank of $T_{(x,y)}\rho_h$ is 4 for every $(x,y) \in H^{-1}(h)$, because C_h is 4-dimensional. Towards this goal, let $V_{(x,y)}$ be the space spanned by the Hamiltonian vector fields $X_{S_{ij}}$, $1 \le i < j \le 4$, on $(T\mathbf{R}^4, \omega_4)$ corresponding to the Hamiltonian function $S_{ij} : T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto x_i y_j - x_j y_i$. Since $S_{ij}|TS^3$ is an integral of of the geodesic vector field X_H on TS^3, it follows that $V_{(x,y)} \subseteq \ker \mathrm{d}H(x,y) = T_{(x,y)}H^{-1}(h)$ for every $(x,y) \in H^{-1}(h)$. Now

$$\big(T_{(x,y)}\rho_h\big)|V_{(x,y)} = \big(\mathrm{d}S_{ij}(x,y) X_{S_{\ell k}}(x,y)\big) = \big(\{S_{ij}, S_{\ell k}\}_{\mathscr{S}}\big) = \widetilde{P}. \tag{21}$$

Using (20) we see that 6×6 matrix $\widetilde{P}$ is conjugate to the matrix

$$P=\begin{pmatrix}(\{\xi_i,\xi_j\}_{\mathscr{S}}) & 0\\ 0 & (\{\eta_i,\eta_j\}_{\mathscr{S}})\end{pmatrix}=\begin{pmatrix}(\sum_k \varepsilon_{ijk}\xi_k) & 0\\ 0 & (\sum_k \varepsilon_{ijk}\eta_k)\end{pmatrix}.$$

The last equality follows using table 2.1. But $S=(S_{ij})=\rho_h(x,y)\in C_h$. Because ξ and η lie in $S^2_{\sqrt{h/2}}$, each of the 3×3 skew symmetric matrices $(\sum_k \varepsilon_{ijk}\,\xi_k)$ and $(\sum_k \varepsilon_{ijk}\,\eta_k)$ is nonzero. Thus each of these matrices has rank 2. Therefore, the rank of $T_{(x,y)}\rho_h$ is 4 for every $(x,y)\in H^{-1}(h)$. Thus ρ_h is a submersion.

Given $S=(S_{ij})\in C_h$, the fiber $W=\rho_h^{-1}(S)$ is a union of orbits of the geodesic vector field X_H of energy h because $S_{ij}|TS^3$ are integrals of X_H. By definition of ρ_h (18), W is the set of all ordered pairs $\{x,y\}$ of orthogonal vectors in $\mathbf{R}^4$ such that $\langle x,x\rangle=1$, $\langle y,y\rangle=2h$ and the 2-plane Π spanned by $\{x,y\}$ has Plücker coordinates (S_{ij}). Since any two such bases of Π are related by a *counterclockwise* rotation in Π, we find that

$$W=\{(x\cos\theta-y\sin\theta,x\sin\theta+y\cos\theta)\in H^{-1}(h)\,\big|\,\theta\in[0,2\pi]\}.$$

Therefore W is a unique oriented orbit of X_H traced out by an integral curve of X_H. □

Corollary: C_h is the space of orbits of positive energy h of the geodesic vector field X_H on TS^3 with orbit mapping $\rho_h: H^{-1}(h)\to C_h$.

(2.10) **Proof**: The corollary follows immediately from the claim and the definition of orbit space, see chapter VII §2. □

The goal of the following discussion is to construct a symplectic form on C_h. We begin by defining a Poisson bracket $\{\,,\,\}$ on the space $C^\infty(\mathscr{S})$ of smooth functions on the Lie algebra $(\mathscr{S},\{\,,\,\}_{\mathscr{S}})$. For $f,g\in C^\infty(\mathscr{S})$ let

$$\{f,g\}=\sum_{\substack{1\le i<j\le 4\\ 1\le \ell<k\le 4}}\frac{\partial f}{\partial S_{ij}}\frac{\partial g}{\partial S_{\ell k}}\{S_{ij},S_{\ell k}\}_{\mathscr{S}}. \tag{22}$$

As is shown in example 1 of chapter VI §4, $(C^\infty(\mathscr{S}),\{\,,\,\}_{\mathscr{S}})$ is a Lie algebra. On $C^\infty(\mathscr{S})$ define a multiplication $\cdot$ by $(f\cdot g)(s)=f(s)g(s)$ for every $s\in\mathscr{S}$. Then $(C^\infty(\mathscr{S}),\cdot)$ is a commutative ring with unit. Using (22) it is straightforward to check that Leibniz' rule holds, namely $\{f,g\cdot h\}=\{f,g\}\cdot h+\{f,h\}\cdot g$, for every $f,g,h\in C^\infty(\mathscr{S})$. Therefore $\mathscr{A}=(C^\infty(\mathscr{S}),\{\,,\,\}_{\mathscr{S}},\cdot)$ is a Poisson algebra. The functions

$$C_1=\sum_{1\le i<j\le 4}S_{ij}^2-2h \quad\text{and}\quad C_2=S_{12}S_{34}-S_{13}S_{24}+S_{14}S_{23}$$

are *Casimirs* for $\mathscr{A}$. In other words, $\{C_1,f\}=\{C_2,f\}=0$ for every $f\in C^\infty(\mathscr{S})$. From (22) it is enough to show that $\{C_1,S_{ij}\}=\{C_2,S_{ij}\}=0$ for $1\le i<j\le 4$. This is a direct verification using table 2.1. Let $\mathscr{I}$ be the ideal in $(C^\infty(\mathscr{S}),\cdot)$ which is generated by C_1
▷ and C_2. Then $\mathscr{I}$ is a *Poisson ideal* in $\mathscr{A}$, that is, if $g\in\mathscr{I}$, then $\{f,g\}\in\mathscr{I}$ for every $f\in C^\infty(\mathscr{S})$.

(2.11) **Proof**: Since $f\in\mathscr{I}$ there are $f_1,f_2\in C^\infty(\mathscr{S})$ such that $f=f_1C_1+f_2C_2$. Now

$$\begin{aligned}\{f,g\} &= \{f_1,g\}\cdot C_1 + f_1\cdot\{C_1,g\} + \{f_2,g\}\cdot C_2 + f_2\cdot\{C_2,g\}, \text{ by Leibniz' rule}\\ &= \{f_1,g\}\cdot C_1 + \{f_2,g\}\cdot C_2 \in \mathscr{I},\end{aligned}$$

where the equality above follows because C_1 and C_2 are Casimirs. Therefore we can define a Poisson bracket $\{\,,\,\}_{C_h}$ on $C^\infty(\mathscr{S})/\mathscr{I}$ by $\{f+\mathscr{I}, g+\mathscr{I}\}_{C_h} = \{f,g\}$. In order to be able to identify the space $C^\infty(\mathscr{S})/\mathscr{I}$ with the space $C^\infty(C_h)$ of smooth functions on C_h, we need to know that $\mathscr{I}$ is the set of smooth functions vanishing identically on C_h. This is a consequence of the following general

Fact: Suppose that 0 is a regular value of the smooth map $F : \mathbf{R}^n \to \mathbf{R}^k : z \mapsto \big(F_1(z), \ldots, F_k(z)\big)$. Then $M = F^{-1}(0)$ is a smooth submanifold of $\mathbf{R}^n$ defined by $F_1(z) = \cdots = F_k(z) = 0$. If $G : \mathbf{R}^n \to \mathbf{R}$ is a smooth function, which vanishes identically on M, then there are smooth functions $g_i : \mathbf{R}^n \to \mathbf{R}$, $1 \le i \le k$, such that $G = \sum_{i=1}^k g_i F_i$.

(2.12) **Proof**: Locally the fact follows using Taylor's formula with integral remainder. The global result is obtained by piecing together the local results using a partition of unity. We leave the details to the reader. □

Consequently, we may define the quotient Poisson algebra $\mathscr{B} = \mathscr{A}/\mathscr{I} = (C^\infty(C_h), \{\,,\,\}_{C_h}, \cdot)$. Because $\{S_{ij}+\mathscr{I}, S_{\ell k}+\mathscr{I}\}_{C_h} = \{S_{ij}, S_{\ell k}\}_{\mathscr{S}}$, the matrix of Poisson brackets $(\{S_{ij}+\mathscr{I}, S_{\ell k}+\mathscr{I}\}_{C_h})$ has rank 4. Therefore C_h is a *cosymplectic manifold*. In other words, the Poisson bracket $\{\,,\,\}_{C_h}$ is *nondegenerate* and hence defines a symplectic form ω_h on
▷ C_h, see chapter VI §4. Moreover, ω_h satisfies $\rho_h^*\omega_h = \Omega_4|H^{-1}(h)$.

(2.13) **Proof**: For every $(x,y) \in H^{-1}(h)$ we know that $T_{(x,y)}H^{-1}(h)$ is spanned by the vectors $\{X_{S_{ij}}(x,y)\}_{1\le i<j\le 4}$. Since (TS^3, Ω_4) is a cosymplectic submanifold of $(T\mathbf{R}^4, \omega_4)$, we have

$$\begin{aligned}\Omega_4(x,y)\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big) &= \omega_4\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big) = \{S_{ij}, S_{\ell k}\}_{\mathscr{S}}(x,y)\\ &= \{S_{ij}, S_{\ell k}\}_{C_h}(\rho_h(x,y)) = \omega_h(\rho_h(x,y))\big(T_{(x,y)}\rho_h X_{S_{ij}}(x,y), T_{(x,y)}\rho_h X_{S_{\ell k}}(x,y)\big)\\ &= (\rho_h^*\omega_h)(x,y)\big(X_{S_{ij}}(x,y), X_{S_{\ell k}}(x,y)\big).\end{aligned}$$ □

We now prove the main result of this section, which describes the geometry of the mapping ρ (16). As a consequence, we know the geometry of the SO(4)-momentum mapping $\mathscr{J}$ (12) of the geodesic vector field X_H on (TS^3, Ω_4).

Claim: The mapping $\rho : TS^3 \subseteq T\mathbf{R}^4 \to C \subseteq \bigwedge^2\mathbf{R}^4 : (x,y) \mapsto x \wedge y$ is a surjective submersion, each of whose fibers is a unique oriented orbit of the geodesic vector field X_H on (TS^3, Ω_4).

(2.14) **Proof**: We have already shown that ρ is surjective ((2.7)). To show that each of its fibers is a unique oriented orbit of X_H we argue as follows. Suppose that $S = (S_{ij}) \in C$. Because S is nonzero, $\sum_{1\le i<j\le 4} S_{ij}^2 = 2h$ for some $h > 0$. Therefore $S \in C_h$. Since the fiber $\rho_h^{-1}(S)$ of ρ_h is a unique oriented orbit of X_H of energy h ((2.9)), so is the fiber $\rho^{-1}(S)$ of ρ because $\rho = \rho_h$ on $H^{-1}(h)$.

To show that ρ is a submersion, first note that by ((2.9)) the map $\rho_h : H^{-1}(h) \to C_h$ is a submersion. Note that $H^{-1}(h)$ and C_h are codimension 1 submanifolds of TS^3 and C, respectively. Since a normal direction to $H^{-1}(h)$ at $(x,y) \in TS^3$ and a normal direction to C_h at $\rho(x,y) \in C$ is spanned by $\operatorname{grad} H(x,y)$ and $\operatorname{grad} F(\rho(x,y))$ respectively, where $F(S_{ij}) = \sum_{1\le i<j\le 4} S_{ij}^2 - 2h = 0$ defines C_h as a submanifold of C, it suffices to show that $\langle T_{(x,y)}\rho \operatorname{grad} H(x,y), \operatorname{grad} F(\rho(x,y))\rangle$ is nonzero. We compute. Clearly $\operatorname{grad} H(x,y) = (0,y)$. Hence $T_{(x,y)}\rho\big(\operatorname{grad} H(x,y)\big) = x\wedge y = \big(S_{ij}(x,y)\big)$. But $\operatorname{grad} F(\rho(x,y)) = 2\big(S_{ij}(x,y)\big)$. Therefore

$$\big\langle T_{(x,y)}\rho\big(\operatorname{grad} H(x,y)\big), \operatorname{grad} F(\rho(x,y))\big\rangle = 2 \sum_{1\le i<j\le 4} S_{ij}^2(x,y) = 4h > 0. \qquad \square$$

This claim has some interesting consequences.

Corollary 1: The *space of orbits* of the geodesic vector field with positive energy is the manifold C. The orbit map is $\rho : TS^3 \to C$, see (16).

(2.15) **Proof**: This follows immediately from the claim and the definition of orbit space, see chapter VII §2. $\square$

Observe that every smooth integral of the geodesic vector field on TS^3 is a smooth function of the integrals S_{ij}. More precisely we prove

Corollary 2: Suppose that $G : TS^3 \subseteq T\mathbf{R}^4 \to \mathbf{R}$ is a smooth integral of the geodesic vector field X_H on (TS^3, Ω_4). Then there is a smooth function $\widehat{G} : C \subseteq \bigwedge^2 \mathbf{R}^4 \to \mathbf{R}$ such that $G = \rho^*\widehat{G}$.

(2.16) **Proof**: Since G is a smooth integral of X_H on TS^3, it is constant on every orbit of X_H on TS^3. Because each fiber of ρ is a unique orbit of X_H on TS^3, G descends to a smooth function $\widehat{G}$ on the orbit space C. But $\rho : TS^3 \to C$ (16) is the orbit map, so $G = \rho^*\widehat{G}$. $\square$

3 The Kepler problem

We investigate the bounded motion of a particle in $\mathbf{R}^3$ which is under the influence of a gravitational field of a second particle fixed at the origin. This is *Kepler's problem*.

3.1 The Kepler vector field

In this subsection we define the Kepler Hamiltonian system $(H, T_0\mathbf{R}^3, \omega_3)$. We then show that the Kepler Hamiltonian vector field X_H conserves energy H, angular momentum $\mathbf{J}$, and the eccentricity vector $\mathbf{e}$. On the set Σ_- of positions and momenta where the values of H are negative, the orbits of X_H are bounded, yet the flow of X_H is incomplete.

On the phase space $T_0\mathbf{R}^3 = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3$ with coordinates (q,p) and symplectic form $\omega_3 = \sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i$, consider the *Kepler Hamiltonian*

$$H : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \tfrac{1}{2}\langle p,p\rangle - \mu\|q\|^{-1}. \tag{23}$$

Here $\langle\, ,\, \rangle$ is the Euclidean inner product on $\mathbf{R}^3$ and $\|q\|$ is the length of the vector q. The integral curves of the Hamiltonian vector field X_H on $T_0\mathbf{R}^3$ satisfy the equations

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\mu \, \|q\|^{-3} q, \end{aligned} \tag{24}$$

which describe the motion of a particle of mass 1 about the origin under the influence of an inverse $|q|^2$ force — such as Newtonian gravity. We consider the case where the force is attractive, that is, $\mu > 0$. However, much of the following analysis can be carried out without change for $\mu < 0$.

The *Kepler vector field* X_H has some obvious integrals: the *total energy*

$$h = \tfrac{1}{2}\langle p, p\rangle - \mu \|q\|^{-1}, \tag{25}$$

which is nothing but the Hamiltonian H, and the *angular momentum*

$$\mathbf{J} = (J_1, J_2, J_3) = q \times p. \tag{26}$$

Here $\times$ is the *vector product* on $\mathbf{R}^3$.

(3.1) **Proof**: A direct way to see that $\mathbf{J}$ is an integral is to compute

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t} \times p + q \times \frac{\mathrm{d}p}{\mathrm{d}t} = p \times p - \mu \|q\|^{-3} q \times q = 0,$$

where the second to last equality follows using (24).

A more sophisticated way to see this is to note that the SO(3)-action $\mathrm{SO}(3) \times \mathbf{R}^3 \to \mathbf{R}^3 : (O, q) \mapsto Oq$ lifts to a Hamiltonian action

$$\mathrm{SO}(3) \times T_0\mathbf{R}^3 \to T_0\mathbf{R}^3 : \big(O, (q,p)\big) \mapsto (Oq, Op).$$

This latter action has the momentum mapping

$$\widetilde{J} : T_0\mathbf{R}^3 \to \mathrm{so}(3)^* : (q,p) \mapsto \begin{pmatrix} 0 & J_3 & -J_2 \\ -J_3 & 0 & J_1 \\ J_2 & -J_1 & 0 \end{pmatrix},$$

defined by $\widetilde{J}(q,p)X = \langle p, X(q)\rangle$ where $X \in \mathrm{so}(3)$. Now use the map $k^\flat$ associated to the Killing metric $k : \mathrm{so}(3) \times \mathrm{so}(3) \to \mathbf{R} : (X,Y) \mapsto \frac{1}{2}\mathrm{tr}\, XY^t$ to identify $\mathrm{so}(3)^*$ with $\mathrm{so}(3)$. This identification boils down to taking transposes. Follow this by the map

$$i : \mathrm{so}(3) \to \mathbf{R}^3 : X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \mapsto \mathbf{x} = (x_1, x_2, x_3),$$

which identifies $\mathrm{so}(3)$ with $\mathbf{R}^3$, see chapter III §1. Then $\widetilde{J}$ becomes the usual angular momentum $\mathbf{J} : T_0\mathbf{R}^3 \to \mathbf{R}^3 : (q,p) \mapsto q \times p$. Since the SO(3) action on $(T_0\mathbf{R}^3, \omega_3)$ leaves the Kepler Hamiltonian H (23) invariant, every component of the angular momentum $\mathbf{J}$ is constant on the integral curves of X_H. □

▷ There is another integral of the Kepler vector field, called the *eccentricity vector*:

$$\mathbf{e} = (e_1, e_2, e_3) = -\|q\|^{-1}q + \mu^{-1}\, p \times (q \times p). \tag{27}$$

(3.2) **Proof**: To see this we calculate

$$\begin{aligned}\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} &= -\frac{\mathrm{d}}{\mathrm{d}t}(\|q\|^{-1}q) + \mu^{-1}\frac{\mathrm{d}p}{\mathrm{d}t} \times \mathbf{J} = \|q\|^{-3}\langle \frac{\mathrm{d}q}{\mathrm{d}t}, q\rangle q - \|q\|^{-1}\frac{\mathrm{d}q}{\mathrm{d}t} + \mu^{-1}\frac{\mathrm{d}p}{\mathrm{d}t} \times \mathbf{J} \\ &= \|q\|^{-3}\big(\langle q,p\rangle q - \langle q,q\rangle p\big) - \|q\|^{-3}q \times \mathbf{J}, \quad \text{using (24)} \\ &= \|q\|^{-3}\big(q \times (q \times p) - q \times (q \times p)\big) = 0. \qquad \square\end{aligned}$$

We now prove some properties of the flow of the Kepler vector field X_H.

Claim: If the energy h is negative, then the image of every integral curve of the Kepler vector field under the bundle projection $\tau : T_0\mathbf{R}^3 \to \mathbf{R}^3 : (q,p) \mapsto q$ is bounded.

(3.3) **Proof**:

CASE 1. $\mathbf{J} = 0$. Since $\mathbf{e}$ is an integral of X_H and $\mathbf{J} = 0$, the direction $\mathbf{e} = -q\|q\|^{-1}$ of the motion is constant. Therefore the motion takes place on the line $q(t) = r(t)\,\mathbf{e}$. From conservation of energy we obtain $h + \mu r^{-1} = \frac{1}{2}\dot{r}^2 \ge 0$. Therefore $\|q(t)\| \le \mu(-h)^{-1}$.

CASE 2. $\mathbf{J} \ne 0$. Since $J^2 = \|q \times p\|^2 = \|q\|^2\|p\|^2 - \langle q,p\rangle^2$, we have

$$h = \tfrac{1}{2}\langle p,p\rangle - \mu\|q\|^{-1} = \tfrac{1}{2}\langle q,p\rangle^2\|q\|^{-2} + \tfrac{1}{2}J^2\|q\|^{-2} - \mu\|q\|^{-1} \ge \tfrac{1}{2}J^2\|q\|^{-2} - \mu\|q\|^{-1}.$$

Now the function $V_J(\|q\|) = \frac{1}{2}J^2\|q\|^{-2} - \mu\|q\|^{-1}$ has a unique nondegenerate minimum at $\|q\| = J^2/\mu$ corresponding to the critical value $-\mu^2/(2J^2)$. Since $\lim_{\|q\|\searrow 0} V_J(\|q\|) \nearrow \infty$ and $\lim_{\|q\|\nearrow\infty} V_J(\|q\|) \nearrow 0$, the function V_J is proper on the set where it has negative values. Therefore $V_J^{-1}([-\mu^2/(2J^2), h])$ is compact. Thus the length of $q(t)$ is bounded, when $h < 0$. $\square$

Claim: The flow of the Kepler vector field X_H is *not complete*.

(3.4) **Proof**: Consider a bounded motion with $\mathbf{J} = 0$ and $h < 0$ which starts at $(r(0), \dot{r}(0)) = (\mu/(-h), 0)$. The time it takes to reach the origin is $T = \int_0^{\mu/(-h)} \frac{\mathrm{d}r}{\sqrt{2\mu r^{-1} + 2h}}$. This is obtained by separating variables in $\frac{1}{2}\dot{r}^2 = h + \mu r^{-1}$ and integrating. Performing the integral gives $T = \frac{\pi}{2}\,\mu(-2h)^{-3/2}$, which is finite. $\square$

3.2 The so(4)-momentum map

Let Σ_- be the open subset of $T_0\mathbf{R}^3$ where the energy H is negative. In this subsection we show that on Σ_- the components of the angular momentum $\mathbf{J}$ and the *modified eccentricity vector* $\widetilde{\mathbf{e}} = -\nu\,\mathbf{e}$, where $\nu = \mu/\sqrt{-2H}$, form a Lie algebra under Poisson bracket which is isomorphic to so(4). This defines a representation of so(4) on the space of Hamiltonian vector fields on $(\Sigma_-, \widetilde{\omega}_3 = \omega_3|\Sigma_-)$ which has a momentum mapping $\widetilde{\mathscr{J}}$. In fact $\widetilde{\mathscr{J}}$ is a surjective submersion from Σ_- to

$$C = \{(\mathbf{J}, \widetilde{\mathbf{e}}) \in \mathbf{R}^6 \,\big|\, \langle \mathbf{J} + \widetilde{\mathbf{e}}, \mathbf{J} + \widetilde{\mathbf{e}}\rangle = \langle \mathbf{J} - \widetilde{\mathbf{e}}, \mathbf{J} - \widetilde{\mathbf{e}}\rangle > 0\} \tag{28}$$

each of whose nonempty fibers is a unique oriented bounded orbit of X_H.

▷ First we show that on $(\Sigma_-, \widetilde{\omega}_3)$ the components of the angular momentum $\mathbf{J}$ and the modified eccentricity vector $\widetilde{\mathbf{e}}$ satisfy the Poisson bracket relations

$$\{J_i, J_j\} = \sum_k \varepsilon_{ijk} J_k, \ \{J_i, \widetilde{e}_j\} = \sum_k \varepsilon_{ijk} \widetilde{e}_k, \ \text{and} \ \{\widetilde{e}_i, \widetilde{e}_j\} = \sum_k \varepsilon_{ijk} J_k. \tag{29}$$

(3.5) **Proof**: We verify only the third equality in (29). Let $\mathbf{A} = \mu\, \mathbf{e}$. Since $\{q_\ell, p_m\} = \delta_{\ell m}$, $\{q_i, q_j\} = 0$, and $\{p_i, p_j\} = 0$, we have

$$\{J_a, \|q\|\} = 0, \ \{q_a, J_b\} = \sum_c \varepsilon_{abc} q_c, \ \text{and} \ \{p_a, J_b\} = \sum_c \varepsilon_{abc} p_c.$$

Using bilinearity and the derivation property of Poisson bracket, expand

$$\{A_i, A_j\} = \big\{ \sum_{j,k} \varepsilon_{ijk} p_j J_k - \mu \|q\|^{-1} q_i, \sum_{m,n} \varepsilon_{\ell mn} p_m J_n - \mu \|q\|^{-1} q_\ell \big\}$$

to obtain $\{A_i, A_j\} = -2H \sum_k \varepsilon_{ijk} J_k$. Recall the identity $\sum_i \varepsilon_{ijk} \varepsilon_{i\ell m} = \delta_{mk}\delta_{\ell j} - \delta_{jm}\delta_{\ell k}$. □

▷ The bracket relations (29) define a Lie algebra which is isomorphic to $\mathrm{so}(4)$.

(3.6) **Proof**: For $i = 1,2,3$ define $\xi_i = \frac{1}{2}(J_i + \widetilde{e}_i)$ and $\eta_i = \frac{1}{2}(J_i - \widetilde{e}_i)$. In terms of ξ_i and η_i the bracket relations (29) become

$$\{\xi_i, \xi_j\} = \sum_k \varepsilon_{ijk} \xi_k, \ \{\eta_i, \eta_j\} = \sum_k \varepsilon_{ijk} \eta_k, \ \text{and} \ \{\xi_i, \eta_j\} = 0. \tag{30}$$

These relations define the Lie algebra $\mathrm{so}(3) \times \mathrm{so}(3)$, which is isomorphic to $\mathrm{so}(4)$. □

The mappings $J_i \mapsto \mathrm{ad}_{J_i} = -X_{J_i}$ and $\widetilde{e}_i \mapsto \mathrm{ad}_{\widetilde{e}_i} = -X_{\widetilde{e}_i}$ define a representation of $\mathrm{so}(4)$ on the space of Hamiltonian vector fields on $(\Sigma_-, \widetilde{\omega}_3)$. In other words, we have a Hamiltonian action of the *Lie algebra* $\mathrm{so}(4)$ on $(\Sigma_-, \widetilde{\omega}_3)$. Associated to this Lie algebra action is the mapping

$$\widetilde{\mathscr{J}} : \Sigma_- \to \mathbf{R}^6 : (q,p) \mapsto (\mathbf{J}, \widetilde{\mathbf{e}}) = \big(q \times p, \nu \big(\|q\|^{-1} q - \mu^{-1}\, p \times (q \times p) \big) \big). \tag{31}$$

Here we have chosen $\{\varepsilon_i\}_{1 \le i \le 6} = \{J_1, J_2, J_3, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_2\}$ as a basis for $\mathrm{so}(4)$ with Lie bracket $\{\,,\}$. Let $\widetilde{\mathscr{J}}^{\varepsilon_i}$ be the i^{th} component of the mapping $\widetilde{\mathscr{J}}$. Then the bracket relations (29) may be written as $\{\widetilde{\mathscr{J}}^{\varepsilon_i}, \widetilde{\mathscr{J}}^{\varepsilon_j}\} = \widetilde{\mathscr{J}}^{\{\varepsilon_i, \varepsilon_j\}}$. Therefore we say that the map $\widetilde{\mathscr{J}}$ is the *momentum map* of the $\mathrm{so}(4)$-action on $(\Sigma_-, \widetilde{\omega}_3)$.

We now investigate the geometric properties of the mapping $\widetilde{\mathscr{J}}$ (31). We begin by noting that the vectors $\mathbf{J}$ and $\widetilde{\mathbf{e}}$ satisfy

$$\begin{aligned} \langle \mathbf{J}, \widetilde{\mathbf{e}} \rangle &= 0 \\ \langle \mathbf{J}, \mathbf{J} \rangle + \langle \widetilde{\mathbf{e}}, \widetilde{\mathbf{e}} \rangle &= \nu^2 > 0. \end{aligned} \tag{32}$$

The verification of the first equation in (32) is a straightforward. For the second, see (37) below. These relations define a smooth 4-dimensional manifold C_ν, which is diffeomorphic to $S^2_\nu \times S^2_\nu$ because (32) is equivalent to

$$\langle \mathbf{J} + \widetilde{\mathbf{e}}, \mathbf{J} + \widetilde{\mathbf{e}} \rangle = \langle \mathbf{J} - \widetilde{\mathbf{e}}, \mathbf{J} - \widetilde{\mathbf{e}} \rangle \ = \ \nu^2 > 0. \tag{33}$$

Write $\nu = \mu/\sqrt{-2h}$ for some $h < 0$ and consider the map

$$\widetilde{\mathscr{J}_h} = \widetilde{\mathscr{J}}|H^{-1}(h) : H^{-1}(h) \subseteq \Sigma_- \to C_\nu \subseteq \mathbf{R}^6. \tag{34}$$

Claim: $\widetilde{\mathscr{J}_h}$ is a surjective submersion.

(3.7) **Proof**: Let $(q,p) \in H^{-1}(h)$ and let $V_{(q,p)} = \mathrm{span}\,\{X_{J_j}(q,p), X_{\widetilde{e}_j}(q,p)\}_{1\le j\le 3}$. Since $\mathbf{J}$ and $\widetilde{\mathbf{e}}$ are integrals of X_H, it follows that $V_{(q,p)} \subseteq \ker \mathrm{d}H(q,p)$ which is $T_{(q,p)}H^{-1}(h)$. Therefore

$$D\widetilde{\mathscr{J}_h}(q,p)|V_{(q,p)} = \begin{pmatrix} \mathrm{d}J_j(q,p) \\ \mathrm{d}\widetilde{e}_j(q,p) \end{pmatrix}\Bigg| V_{(q,p)} = \begin{pmatrix} (\{J_i,J_j\}(q,p)) & (\{J_i,\widetilde{e}_j\}(q,p)) \\ (\{\widetilde{e}_i,J_j\}(q,p)) & (\{\widetilde{e}_i,\widetilde{e}_j\}(q,p)) \end{pmatrix} = P.$$

On C_ν (32) the rank of P is 4, because P is conjugate to the matrix

$$\begin{pmatrix} (\{\xi_i,\xi_j\}) & 0 \\ 0 & (\{\eta_i,\eta_j\}) \end{pmatrix} = \begin{pmatrix} 2\sum_k \varepsilon_{ijk}\,(J_k+\widetilde{e}_k) & 0 \\ 0 & 2\sum_k \varepsilon_{ijk}\,(J_k-\widetilde{e}_k) \end{pmatrix},$$

see (30) and (33). Therefore $\widetilde{\mathscr{J}_h}$ is a submersion.

To show that $\widetilde{\mathscr{J}_h}$ is surjective, let $(\mathbf{J},\widetilde{\mathbf{e}}) \in C_\nu$. Then $e = \|\mathbf{e}\| = \nu^{-1}\,\|\widetilde{\mathbf{e}}\| \in [0,1]$ because $\nu^2 = \langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle \ge \nu^2\langle \mathbf{e},\mathbf{e}\rangle$. Choose

$$(q,p) = \begin{cases} \big(-\nu\mu^{-1}\,(1-e)e^{-1}\widetilde{\mathbf{e}}, -\mu\nu^{-3}\,(e(1-e))^{-1}\,\mathbf{J}\times\widetilde{\mathbf{e}}\big), \text{ when } e \in (0,1) \text{ and } \mathbf{J} \ne 0 \\ \big(-\mu^{-2}\nu^2\, p\times\mathbf{J}, p\big), \text{ when } e = 0 \text{ and } \mathbf{J} \ne 0. \text{ Here } \langle p,\mathbf{J}\rangle = 0,\ \|p\| = \mu\nu^{-1} \\ \big(-\nu\mu^{-1}\widetilde{\mathbf{e}}, -\nu\mu^{-2}\widetilde{\mathbf{e}}\big), \text{ when } \mathbf{J} = 0. \text{ Here } e = 1. \end{cases}$$

A straightforward calculation shows that $(q,p) \in H^{-1}(h)$ and $\widetilde{\mathscr{J}_h}(q,p) = (\mathbf{J},\widetilde{\mathbf{e}})$. □

Corollary: For every $c \in C_\nu$ the fiber $\widetilde{\mathscr{J}_h}^{-1}(c)$ is a union of bounded Keplerian orbits.

(3.8) **Proof**: From the fact that $\widetilde{\mathscr{J}_h}$ is a submersion, it follows that

$$\dim\ker D\widetilde{\mathscr{J}_h}(q,p) = \dim T_{(q,p)}H^{-1}(h) - \dim\mathrm{im}\, D\widetilde{\mathscr{J}_h}(q,p) = 5-4 = 1.$$

But $X_H(q,p) \in \ker D\widetilde{\mathscr{J}_h}(q,p)$. Hence for every $c \in C_\nu$

$$T_{(q,p)}\widetilde{\mathscr{J}_h}^{-1}(c) = \ker D\widetilde{\mathscr{J}_h}(q,p) = \mathrm{span}\{X_H(q,p)\}.$$

Therefore $\widetilde{\mathscr{J}_h}^{-1}(c)$ is a union of bounded Keplerian orbits. □

The following claim is a substantial sharpening of the above corollary.

Claim: For every $c \in C_\nu$ the fiber $\widetilde{\mathscr{J}_h}^{-1}(c)$ is

1. an oriented ellipse, when $c \notin C_\nu \cap \{\mathbf{J} = 0\}$;
2. a line which is the union of two half open line segments

$$\Big\{\big(\sigma\nu^{-1}\widetilde{\mathbf{e}}, \pm\nu^{-1}\big(\sqrt{2h+2\mu\sigma^{-1}}\big)\widetilde{\mathbf{e}}\big) \in T_0\mathbf{R}^3 \,\big|\, \sigma \in (0,\mu/(-h)]\Big\},$$

that join smoothly at $(\mu/(-\nu h)\widetilde{\mathbf{e}}, 0)$, when $c \in C_\nu \cap \{\mathbf{J} = 0\}$.

(3.9) **Proof**:

CASE 1. $c = (\mathbf{J}, \widetilde{\mathbf{e}}) \in C_\nu \setminus (C_\nu \cap \{\mathbf{J} = 0\})$. Let $h = -\mu^2/2\nu^2$. We have to show that the data $h < 0$, $\mathbf{J} \neq 0$, and $\mathbf{e} = -\nu^{-1}\widetilde{\mathbf{e}}$ determine a unique oriented ellipse which is traced out by the projection $t \mapsto q(t)$ of an integral curve $t \mapsto (q(t), p(t))$ of X_H. Because $\mathbf{J} \neq 0$ and $\langle q(t), \mathbf{J}\rangle = \langle p(t), \mathbf{J}\rangle = 0$, the curves $t \mapsto q(t)$ and $t \mapsto p(t)$ lie in a plane $\Pi \subseteq \mathbf{R}^3$ which is perpendicular to $\mathbf{J}$. Since $\langle \mathbf{J}, \mathbf{e}\rangle = \mathbf{0}$, the eccentricity vector $\mathbf{e}$ also lies in Π. Therefore we may write $\langle q, \mathbf{e}\rangle = \|q\| e \cos f$, where f is the *true anomaly*, namely, the angle $\angle AOP$. From the definition of the eccentricity vector $\mathbf{e}$ (27) it follows that $\langle q, \mathbf{e}\rangle = -\|q\| + \mu^{-1} J^2$. Therefore

$$\|q\| e \cos f = -\|q\| + \mu^{-1} J^2. \tag{35}$$

Suppose that $e = 0$. Then (35) becomes $\|q\| = \mu^{-1} J^2$, which defines a circle $\mathscr{C}$ in Π with center at the origin. Since $0 = \frac{\mathrm{d}\|q(t)\|^2}{\mathrm{d}t} = \langle q, \frac{\mathrm{d}q}{\mathrm{d}t}\rangle = \langle q, p\rangle$, the tangent vector $p(t)$ to $\mathscr{C}$ at

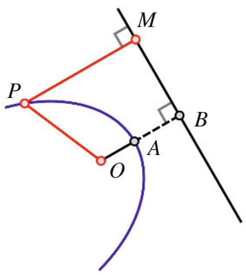

Figure 3.1. Ellipse in the plane Π.

$q(t)$ is perpendicular to $q(t)$. Because $\{q, p, p \times q\}$ is a positively oriented basis of $\mathbf{R}^3$, $\{q, p\}$ is a positively oriented basis for Π. Hence the circle traced out by $t \mapsto q(t)$ is positively oriented. Suppose that $e \neq 0$. Then equation (35) may be written as

$$e\big((\mu e)^{-1} J^2 - \|q\| \cos f\big) = \|q\|. \tag{36}$$

Equation (36) describes the locus of points P in the plane Π for which the ratio of the distance $\overline{OP}$ to the origin to the distance $\overline{PM}$ to the line MB, where $\overline{OB} = (\mu e)^{-1} J^2$, is a constant e, see figure 3.1. Thus the locus is a conic section. To see which conic it is, we calculate the size of e.

$$\begin{aligned} e^2 = \|\mathbf{e}\|^2 &= 1 - 2\mu^{-1}\|q\|^{-1}\|q \times p\|^2 + \mu^{-2}\|p \times (q \times p)\|^2, \quad \text{using (27)} \\ &= 1 - 2\mu^{-1}\|q\|^{-1} J^2 + \mu^{-2}(\|p\|^2 J^2 - \langle p, \mathbf{J}\rangle^2), \quad \text{using } \mathbf{J} = q \times p \\ &= 1 + 2\mu^{-2} J^2 h, \quad \text{since } \langle p, \mathbf{J}\rangle = 0 \text{ and } h = \tfrac{1}{2}\langle p, p\rangle - \mu\|q\|^{-1}. \end{aligned} \tag{37}$$

Since $h < 0$, it follows that $e \in [0, 1)$. Therefore the locus

$$\|q\| = J^2 \mu^{-1}(1 + e \cos f)^{-1} \tag{38}$$

is an *ellipse* in Π with *eccentricity* e and *major semiaxis* lying along $\mathbf{e}$, which is directed from the *center of attraction* O, that is also a *focus*, to the *periapse* A, of length $a =$

▷ $J^2\mu^{-1}(1-e^2)^{-1} = \mu/(-2h)$. When traced out by $t \mapsto q(t)$, this ellipse is oriented in the direction of increasing true anomaly f.

(3.10) **Proof**: From the fact that $\{q,p\}$ is a positively oriented basis of the plane Π, we obtain

$$\begin{aligned} J &= \|q\times p\|, \text{ which is the area of the positively oriented parallogram spanned by } \{q,p\}. \\ &= \det\begin{pmatrix} \langle q, e^{-1}\mathbf{e}\rangle & \langle q,(Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle \\ \langle p, e^{-1}\mathbf{e}\rangle & \langle p,(Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle \end{pmatrix}, \\ &\quad \text{since } \{e^{-1}\mathbf{e},(Je)^{-1}\mathbf{J}\times\mathbf{e}\} \text{ is a positively oriented orthonormal basis of } \Pi \\ &= \|q\|^2\frac{\mathrm{d}f}{\mathrm{d}t}. \end{aligned} \tag{39}$$

Equation (39) follows by first differentiating $\langle q, e^{-1}\mathbf{e}\rangle = \|q\|\cos f$ and $\langle q,(Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle = \|q\|\sin f$ along an integral curve of X_H and then using the fact that $p = \frac{\mathrm{d}q}{\mathrm{d}t}$ and $\dot{\mathbf{e}} = \dot{\mathbf{J}} = 0$ to obtain $\langle p, e^{-1}\mathbf{e}\rangle = \frac{\mathrm{d}\|q\|}{\mathrm{d}t}\cos f - \|q\|\sin f\frac{\mathrm{d}f}{\mathrm{d}t}$ and $\langle p,(Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle = \frac{\mathrm{d}\|q\|}{\mathrm{d}t}\sin f + \|q\|\cos f\frac{\mathrm{d}f}{\mathrm{d}t}$. From (39) we see that $\frac{\mathrm{d}f}{\mathrm{d}t} > 0$. □

CASE 2. $c = (\mathbf{J},\widetilde{\mathbf{e}}) \in C_\nu \cap \{\mathbf{J} = 0\}$. Since $\mathbf{J} = 0$, the modified eccentricity vector $\widetilde{\mathbf{e}} = \nu\|q\|^{-1}q$. Because $\widetilde{\mathbf{e}}$ is constant along any integral curve $t \mapsto (q(t),p(t))$ of X_H and $h<0$, the image of $t\mapsto q(t)$ lies along $\widetilde{\mathbf{e}}$ and is the half open line segment $\{\sigma\nu^{-1}\widetilde{\mathbf{e}} \in \Pi \,\big|\, \sigma \in (0,\mu/(-h)]\}$. From $\mathbf{J}=0$ it follows that $p = \lambda\widetilde{\mathbf{e}}$ for some $\lambda \in \mathbf{R}$. In order that $(\sigma\nu^{-1}\widetilde{\mathbf{e}},p) \in H^{-1}(h)$, where $h = -\mu^2/(2\nu)^2$, we must have $\lambda^2\nu^2 = \langle p,p\rangle = 2h + 2\mu\sigma^{-1}$. Therefore $\widetilde{\mathscr{J}}_h^{-1}(c)$ is the line which is the union of the two half open line segments $\{\big(\sigma\nu^{-1}\widetilde{\mathbf{e}}, \pm\nu^{-1}\big(\sqrt{2h+2\mu\sigma^{-1}}\big)\widetilde{\mathbf{e}}\big) \in T_0\mathbf{R}^3 \,\big|\, \sigma \in (0,\mu/(-h)]\}$, which join smoothly at $(\mu/(-h\nu)\widetilde{\mathbf{e}},0)$. □

It is not hard to show that on $\widetilde{\mathscr{J}}_h^{-1}\big(C_\nu \setminus (\{\mathbf{J}=0\}\cap C_\nu)\big)$ the mapping $\widetilde{\mathscr{J}}_h$ is proper, whereas on $\widetilde{\mathscr{J}}_h^{-1}(\{\mathbf{J}=0\}\cap C_\nu)$ it is not.

We now turn to examining the so(4)-momentum mapping $\widetilde{\mathscr{J}}$ (31). Let C be the submanifold of $\mathbf{R}^3\times(\mathbf{R}^3\setminus\{(0,0)\})$ defined by $\langle \mathbf{J},\widetilde{\mathbf{e}}\rangle = 0$.

Claim: The map

$$\widetilde{\mathscr{J}} : \Sigma_- \to C \subseteq \mathbf{R}^6 : (q,p) \mapsto \big(q\times p, \nu(\|q\|^{-1}q - \mu^{-1}p\times(q\times p))\big) = (\mathbf{J},\widetilde{\mathbf{e}})$$

is a surjective submersion, each of whose fibers is a unique bounded orbit of the Kepler vector field X_H.

(3.11) **Proof**: First we show that $\widetilde{\mathscr{J}}$ is surjective. Supppose that $c = (\mathbf{J},\widetilde{\mathbf{e}}) \in C$. Then $\|\mathbf{J}+\widetilde{\mathbf{e}}\|^2 = \|\mathbf{J}-\widetilde{\mathbf{e}}\|^2 = \nu^2$ for some $\nu>0$. Hence $(\mathbf{J},\widetilde{\mathbf{e}}) \in C_\nu$. Let $h = -\mu^2/(2\nu^2)$. From ((3.7)) it follows that $\widetilde{\mathscr{J}}_h^{-1}(c)$ is nonempty. Hence $\widetilde{\mathscr{J}}^{-1}(c)$ is nonempty. Because $\widetilde{\mathscr{J}}_h^{-1}(c)$ is a unique oriented bounded orbit of the Kepler vector field, $\widetilde{\mathscr{J}}^{-1}(c)$ is as well.

Since C is a 5-dimensional smooth manifold, the map $\widetilde{\mathscr{J}}$ is a submersion if for every

$(q,p) \in \Sigma_-$ the rank of $D\widetilde{\mathscr{J}}(q,p)$ is 5. Actually it suffices to show that for every $(q,p) \in H^{-1}(h)$ the vector $D\widetilde{\mathscr{J}}(q,p)\,\mathrm{grad}\,H(q,p)$ is normal to C_ν at $\widetilde{\mathscr{J}}(q,p)$, because

1. by ((3.7)), $D\widetilde{\mathscr{J}}(q,p)T_{(q,p)}H^{-1}(h) = T_{\widetilde{\mathscr{J}}(q,p)}C_\nu$;
2. a normal space to $H^{-1}(h)$ in Σ_- at (q,p) is spanned by $\mathrm{grad}\,H(q,p)$;
3. as a submanifold of C the manifold C_ν is defined by

$$F(\mathbf{J},\widetilde{\mathbf{e}}) = \langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle - \nu^2 = 0, \qquad (40)$$

where $\nu = \mu(\sqrt{-2H})^{-1/2}$.

Since the normal space to C_ν at $\widetilde{\mathscr{J}}(q,p) = (\mathbf{J},\widetilde{\mathbf{e}}) \in C$ is spanned by $\mathrm{grad}\,F(\mathbf{J},\widetilde{\mathbf{e}}) = 2(\mathbf{J},\widetilde{\mathbf{e}})$, it suffices to check that $\langle D\widetilde{\mathscr{J}}(q,p)\,\mathrm{grad}\,H(q,p),\, \mathrm{grad}\,F(\mathscr{J}(q,p))\rangle$ is nonzero. The following calculation does this.

$$\begin{aligned}
0 \neq \langle \mathrm{grad}\,H(q,p), \mathrm{grad}\,H(q,p)\rangle &= DH(q,p)\,\mathrm{grad}\,H(q,p)\\
&= D\big(-\tfrac{1}{2}\mu^2\,(\langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle)^{-1}\big)(q,p)\,\mathrm{grad}\,H(q,p),\\
&\qquad \text{using } H = -\mu^2/(2\nu^2) \text{ and (40)}\\
&= \tfrac{1}{2}\mu^2\,\big(\langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle\big)^{-2}\big(\langle \mathbf{J}, D\mathbf{J}(q,p)\,\mathrm{grad}\,H(q,p)\rangle\\
&\qquad + \langle \widetilde{\mathbf{e}}, D\widetilde{\mathbf{e}}(q,p)\,\mathrm{grad}\,H(q,p)\rangle\big)\\
&= \mu^{-2}H(q,p)^2\,\big\langle D\widetilde{\mathscr{J}}(q,p)\,\mathrm{grad}\,H(q,p), \mathrm{grad}\,F(\widetilde{\mathscr{J}}(q,p))\big\rangle.
\end{aligned}$$

□

The above result has several useful consequences.

Corollary 1. The smooth manifold C (28) is the *space of orbits* of negative energy of the Kepler vector field X_H and the momentum map $\widetilde{\mathscr{J}} : \Sigma_- \to C$ (31) is the orbit map.

(3.12) **Proof**: The corollary follows from ((3.11)) and the definition of orbit space. □

The next corollary says that every *smooth* integral of the Kepler vector field on Σ_- is a smooth function of the components of angular momentum $\mathbf{J}$ and the modified eccentricity vector $\widetilde{\mathbf{e}}$. More precisely,

Corollary 2. Suppose that $G : \Sigma_- \subseteq T_0\mathbf{R}^3 \to \mathbf{R}$ is a smooth integral of the Kepler vector field X_H. Then there is a smooth function $\widehat{G} : C \subseteq \mathbf{R}^6 \to \mathbf{R}$ such that $G = \widetilde{\mathscr{J}}^*\widehat{G}$.

(3.13) **Proof**: Since G is an integral of X_H on Σ_-, it is constant on each bounded orbit of X_H and hence is constant on the fibers of the momentum map $\widetilde{\mathscr{J}}$. Because C is smooth and is the space of orbits of X_H on Σ_- with orbit mapping $\widetilde{\mathscr{J}}$, G descends to a smooth function $\widehat{G} : C \subseteq \mathbf{R}^6 \to \mathbf{R}$. In other words, $G = \widetilde{\mathscr{J}}^*\widehat{G}$. □

3.3 Kepler's equation

So far we have only used the constants of motion to describe the orbits of the Kepler vector field X_H of negative energy. This means that we cannot tell where on the orbit the particle is at a given time.

In order to give a time parametrization of a bounded Keplerian orbit, we define a new time scale, the *eccentric anomaly* s, by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{-2h}\,\|q\|^{-1}. \tag{41}$$

Before finding a differential equation for $\|q(s)\|$, we use the integrals of energy and angular momentum to find a differential equation for $\|q(t)\|$. Multiplying the energy integral $h = \frac{1}{2}\langle p,p\rangle - \mu\|q\|^{-1}$ by $2\|q\|^2$ gives $\|q\|^2\|p\|^2 = 2\mu\,\|q\| + 2h\,\|q\|^2$. But $\|q\|^2\|p\|^2 = \|q\times p\|^2 + \langle q,p\rangle^2 = J^2 + \langle p,q\rangle^2$. In other words,

$$\|q\|^2\left(\frac{\mathrm{d}\|q\|}{\mathrm{d}t}\right)^2 + J^2 = 2\mu\,\|q\| + 2h\,\|q\|^2. \tag{42}$$

Using (41) to change to the time variable s and dividing by $-2h$ gives

$$\left(\frac{\mathrm{d}\|q\|}{\mathrm{d}s}\right)^2 + a^2(1-e^2) = 2a\,\|q\| - \|q\|^2, \tag{43}$$

since $a = \mu/(-2h) = J^2\mu^{-1}(1-e^2)^{-1}$. Instead of separating variables and immediately integrating (43) we first change variables by $ea\,\rho = a - \|q\|$. Then (43) simplifies to

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + \rho^2 = 1. \tag{44}$$

Since $\|q(0)\| = a(1-e)$, from the definition of ρ we obtain $\rho(0) = 1$. Therefore

$$\|q(s)\| = a - ae\cos s. \tag{45}$$

To find the relation between the eccentric anomaly time scale s and the physical time scale t, we substitute (45) into (41) and integrate to obtain

$$\sqrt{-2h}(t-\tau) = \sqrt{-2h}\int_\tau^t \mathrm{d}t = \int_0^s (a - ae\cos s)\,\mathrm{d}s = as - ae\sin s. \tag{46}$$

Here τ is a time related to the time of periapse passage. Its precise definition is given below. Dividing (46) by a and using $a = \mu/(-2h) = \nu^2/\mu$ gives *Kepler's equation*

$$s - e\sin s = \mu^2\nu^{-3}(t-\tau) = n\ell, \tag{47}$$

where ℓ is the *mean anomaly* and $n = \mu^2\nu^{-3}$ is the *mean motion*. Note that

$$\begin{aligned}\langle q,p\rangle = \langle q, \frac{\mathrm{d}q}{\mathrm{d}t}\rangle &= \|q\|\frac{\mathrm{d}\|q\|}{\mathrm{d}t} = \|q\|\,\frac{\mathrm{d}\|q\|}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\\ &= \sqrt{-2h}\,ae\,\sin s, \quad \text{using (41) and (45)}\\ &= \nu\, e\sin s.\end{aligned} \tag{48}$$

When $t = \tau$ from Kepler's equation it follows that $s = 0$. Let τ' be the physical time corresponding to $s = 2\pi$ in (47). Then $\tau - \tau'$ is the period of elliptical motion, which according to Kepler's equation, is $2\pi n^{-1} = 2\pi\nu^3\mu^{-2} = 2\pi\mu^{-1/2}a^{3/2}$. This is *Kepler's third law of motion*.

During elliptical motion the particle goes through the periapse periodically. Therefore the time τ in (47) is *not* uniquely determined by the initial condition $(q(0), p(0))$, which defines the integral curve of X_H. We will define τ as follows. In the interval $[-\pi, \pi]$ there are precisely *two* values $\varepsilon\widehat{s_0}$ (with $\varepsilon^2 = 1$) which satisfy $\|q(0)\| = a(1 - e\cos\varepsilon\widehat{s_0})$. To fix the choice of ε note that from (48) we have $\varepsilon = \langle q(0), p(0)\rangle/(\nu e \sin\widehat{s_0})$, unless $\widehat{s_0} = 0$ in which case ε is irrelevant. Set $s_0 = \varepsilon\widehat{s_0}$ and let $\tau = -n^{-1}(s_0 - e\sin s_0)$. In words, we define τ as follows. If at $t = 0$ the particle is in the upper half of the ellipse, then τ is the first time *before* $t = 0$ when the particle passed through the periapse; otherwise it is the first time *on* or *after* $t = 0$ when the particle passes through the periapse.

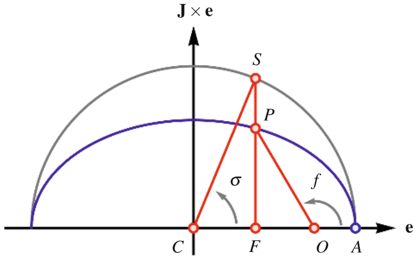

Figure 3.2. The eccentric anomaly.

To describe the classic geometric meaning of the eccentric anomaly s, consider the figure 3.2. Let O be the center of attraction, A the periapse and C the *center of the ellipse* of eccentricity e. The arrow on the ellipse indicates the direction of motion and P is the position of the particle on the ellipse with true anomaly f. Construct a line SP through P which is perpendicular to the line CA. Project P parallel along SP to the point S on the circle $\mathscr{C}$ with center C and radius equal to the distance $\overline{CA}$.

Claim: The *eccentric anomaly* s is the angle $\angle ACS$.

(3.14) **Proof**: Let $\sigma = \angle ACS$. From figure 3.2 we obtain $\overline{CS} = a$ and $\overline{CO} = ae$. Since $\overline{CF} = \overline{CO} + \overline{OF}$, we find that $a\cos\sigma = ae + \|q\|\cos f$. As the orbit is elliptical, we have $\|q\| = a(1-e^2)(1+e\cos f)^{-1}$. This may be rewritten as

$$\|q\| = a - e(ae + \|q\|\cos f) = a - ae\cos\sigma.$$

But $\|q\| = a - ae\cos s$. Hence $\cos s = \cos\sigma$. Since $s = 0$ when $\sigma = 0$, we obtain $\sigma = s$. $\square$

As the point S traces out the circle $\mathscr{C}$ uniformly with speed n, the point P on the ellipse traces out the projection of an integral curve of the Kepler vector field in configuration space.

4 Regularization

In this section we remove the incompleteness of the flow of the Kepler vector field by embedding it into a complete flow. This process is called regularization. We regularize the Kepler problem in two ways: one, called *Moser's regularization*, works on a fixed negative energy level; while the other, called *Ligon-Schaaf regularization*, works on all negative energy level sets at once.

4.1 Moser's regularization

We begin by discussing Moser's regularization. On the phase space $\big(T_0\mathbf{R}^3 = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3, \widetilde{\omega}_3 = \big(\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i\big)|T_0\mathbf{R}^3\big)$ with coordinates (q,p) consider the Kepler Hamiltonian

$$H(q,p) = \tfrac{1}{2}\|p\|^2 - \|q\|^{-1}. \tag{49}$$

Here $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbf{R}^3$ with associated norm $\|\ \|$. We have chosen physical units so that $\mu = 1$. The integral curves of the Hamiltonian vector field X_H associated to H satisfy

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= p = \frac{\partial H}{\partial p} \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\|q\|^{-3}q = -\frac{\partial H}{\partial q}. \end{aligned} \tag{50}$$

Let $\mathbf{R}_{>0}$ be the multiplicative group of positive real numbers. On $\mathbf{R} \times T_0\mathbf{R}^3$ define an $\mathbf{R}_{>0}$-action by

$$\widetilde{\Psi}_V : \mathbf{R}_{>0} \times (\mathbf{R} \times T_0\mathbf{R}^3) \to \mathbf{R} \times T_0\mathbf{R}^3 : \big(\rho, (t,q,p)\big) \mapsto (\rho^3 t, \rho^2 q, \rho^{-1} p). \tag{51}$$

▷ The equations of motion (50) of the Kepler problem are invariant under the action (51) of the *virial group*.

(4.1) **Proof**: We check this as follows.

$$\frac{\mathrm{d}(\rho^2 q)}{\mathrm{d}(\rho^3 t)} = \rho^{-1} p \quad \text{and} \quad \frac{\mathrm{d}(\rho^{-1} p)}{\mathrm{d}(\rho^3 t)} = -\rho^{-4}\|q\|^{-3} q = -\|\rho^2 q\|^{-3}\rho^2 q. \qquad \square$$

Under the virial group the Kepler Hamiltonian H (49) transforms as $H \mapsto \rho^{-2} H$ and the symplectic form transforms as $\widetilde{\omega}_3 \mapsto \rho\,\widetilde{\omega}_3$.

We now regularize the bounded orbits of the Kepler problem of fixed negative energy. Using the virial group we reduce our considerations to the level set $H^{-1}(-\frac{1}{2})$. First we introduce a new time scale s by $\frac{\mathrm{d}s}{\mathrm{d}t} = \|q\|^{-1}$. Consider the new Hamiltonian

$$\widetilde{F}(q,p) = \|q\|\big(H(q,p) + \tfrac{1}{2}\big) + 1 = \tfrac{1}{2}\|q\|(\|p\|^2 + 1). \tag{52}$$

▷ The integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$ are integral curves of X_H on $H^{-1}(-\frac{1}{2})$, using the time parameter s.

(4.2) **Proof**: Using the time parameter s the integral curves of X_H satisfy

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}s} &= \frac{\mathrm{d}q}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = p\|q\| = \|q\|\frac{\partial}{\partial p}\big(H(q,p)+\tfrac{1}{2}\big) \\ \frac{\mathrm{d}p}{\mathrm{d}s} &= \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = -\|q\|^{-2}q = -\|q\|\frac{\partial}{\partial q}\big(H(q,p)+\tfrac{1}{2}\big). \end{aligned} \tag{53}$$

On $H^{-1}(-\frac{1}{2})$ we have $H(q,p)+\frac{1}{2}=0$. So $\|q\|\frac{\partial}{\partial z}\big(H(q,p)+\frac{1}{2}\big) = \frac{\partial}{\partial z}\big(\|q\|\big(H(q,p)+\frac{1}{2}\big)\big)$ for $z=q$ or p. Therefore on $\widetilde{F}^{-1}(1)=H^{-1}(-\frac{1}{2})$ equation (53) is in Hamiltonian form

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}s} &= \frac{\partial\widetilde{F}}{\partial p} \\ \frac{\mathrm{d}p}{\mathrm{d}s} &= -\frac{\partial\widetilde{F}}{\partial q}. \end{aligned} \tag{54}$$

Hence the integral curves of X_H on $H^{-1}(-\frac{1}{2})$ are the same as the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$, using the time parameter s. □

Let $\widetilde{K}: T_0\mathbf{R}^3 \to \mathbf{R}$, where

$$\widetilde{K}(q,p) = \tfrac{1}{2}\widetilde{F}(q,p)^2 = \tfrac{1}{8}\|q\|^2(\|p\|^2+1)^2. \tag{55}$$

▷ The integral curves of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(\frac{1}{2})$ are the same as the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$.

(4.3) **Proof**: This follows because on $\widetilde{F}^{-1}(1)$ we have

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}s} &= \tfrac{1}{2}\frac{\partial\widetilde{F}^2}{\partial p} = \widetilde{F}\frac{\partial\widetilde{F}}{\partial p} = \frac{\partial\widetilde{K}}{\partial p} \\ \frac{\mathrm{d}p}{\mathrm{d}s} &= -\tfrac{1}{2}\frac{\partial\widetilde{F}^2}{\partial q} = -\widetilde{F}\frac{\partial\widetilde{F}}{\partial q} = -\frac{\partial\widetilde{K}}{\partial q}. \end{aligned}$$ □

On our way toward defining Moser's regularization map consider stereographic projection $\varphi: S^3_{\mathrm{np}} = S^3\setminus\{\mathrm{np}\} \to \mathbf{R}^3$ from the north pole $\mathrm{np}=(0,0,0,1)$ of the 3-sphere $S^3=\{u\in\mathbf{R}^4 \,\big|\, \langle u,u\rangle = \sum_{j=1}^4 u_j^2\}$ to the 3-plane $\mathbf{R}^3=\mathbf{R}^3\times\{0\}$ in $\mathbf{R}^4$. For each $u\in S^3_{\mathrm{np}}$ let $q=\varphi(u)$ be the point of intersection of the line joining np to u with $\mathbf{R}^3$. A short calculation shows that $q_i = \frac{u_i}{1-u_4}$ for $i=1,2,3$. Let $T^+S^3_{\mathrm{np}} = \{(u,v)\in T\mathbf{R}^3 \,\big|\, u\in S^3_{\mathrm{np}},\ 0=\langle u,v\rangle = \sum_{j=1}^4 u_jv_j,\ v\neq 0\}$. Consider the mapping

$$\Phi_M^{-1}: T^+S^3_{\mathrm{np}} \to T_0\mathbf{R}^3 : (u,v)\mapsto (q,p) = \big(-(1-u_4)\widetilde{v} - v_4\widetilde{u}, (1-u_4)^{-1}\widetilde{u}\big), \tag{56}$$

where $u=(\widetilde{u},u_4)$ and $v=(\widetilde{v},v_4)$. Then Φ_M^{-1} is the composition of a lift of stereographic projection

$$\widehat{\varphi}: T^+S^3_{\mathrm{np}} \to T_0\mathbf{R}^3 : (u,v)\mapsto (q,p) = \big((1-u_4)^{-1}\widetilde{u}, (1-u_4)\widetilde{v}+v_4\widetilde{u}\big),$$

followed by the momentum reversal $\psi: T_0\mathbf{R}^3 \to T_0\mathbf{R}^3 : (q,p)\mapsto(-p,q)$. When $(u,v)\in$
▷ $T^+S^3_{\mathrm{np}}$ we have the identities

$$\|q\|^2 = \|v\|^2(1-u_4)^2 \tag{57a}$$

$$\|p\|^2 + 1 = 2(1-u_4)^{-1} \tag{57b}$$
$$\langle q, p\rangle = -v_4. \tag{57c}$$

(4.4) **Proof**: Using $\langle u,u\rangle = 1$ and $\langle u,v\rangle = 0$, from (56) we get

$$\begin{aligned}
\|q\|^2 &= \sum_{i=1}^{3} q_i^2 = \sum_{i=1}^{3} \big(v_i(1-u_4) + u_i v_4\big)^2 \\
&= (1-u_4)^2(\|v\|^2 - v_4^2) - 2u_4 v_4^2(1-u_4) + v_4^2(1-u_4^2) \\
&= \|v\|^2(1-u_4)^2;
\end{aligned}$$

$$\|p\|^2 + 1 = (1-u_4)^{-2}\sum_{i=1}^{3} u_i^2 + 1 = (1-u_4)^{-2}(1-u_4^2) + 1 = 2(1-u_4)^{-1};$$

$$\begin{aligned}
\langle q,p\rangle &= \sum_{i=1}^{3} q_i p_i = -\sum_{i=1}^{3}\big((v_i(1-u_4) + u_i v_4)(1-u_4)^{-1}u_i\big) \\
&= u_4 v_4 - v_4(1-u_4)^{-1}(1-u_4^2) = -v_4.
\end{aligned}$$

□

Define *Moser's mapping*

$$\Phi_M : T_0\mathbf{R}^3 \to T^+S^3_{\mathrm{np}} : (q,p) \mapsto (u,v) = \big((\widetilde{u}, u_4), (\widetilde{v}, v_4)\big)$$

by

$$\begin{aligned}
\widetilde{u} &= 2p(\|p\|^2+1)^{-1} \quad \text{and} \quad u_4 = (\|p\|^2-1)(\|p\|^2+1)^{-1} \\
\widetilde{v} &= -\tfrac{1}{2}(\|p\|^2+1)q + \langle q,p\rangle p \quad \text{and} \quad v_4 = -\langle q,p\rangle.
\end{aligned} \tag{58}$$

▷ We now show that Φ_M^{-1} (56) is the inverse of Moser's mapping Φ_M (58).

(4.5) **Proof**: Suppose that $(u,v) \in T^+S^3$. Then

$$\begin{aligned}
\Phi_M(\Phi_M^{-1}(u,v)) &= \Phi_M(q,p), \quad \text{where } q = -(1-u_4)\widetilde{v} - v_4\widetilde{u} \text{ and } p = (1-u_4)^{-1}\widetilde{u} \\
&= \big(((1-u_4)(1-u_4)^{-1}\widetilde{u}, (2(1-u_4)^{-1}-2)\tfrac{1}{2}(1-u_4)), \\
&\qquad ((1-u_4)^{-1}[(1-u_4)\widetilde{v} + v_4\widetilde{u}] - v_4\widetilde{u}(1-u_4)^{-1}, v_4)\big) \\
&\qquad \text{using (58) the identities (57a) – (57c)} \\
&= \big((\widetilde{u}, 1-(1-u_4)), (\widetilde{v} + (1-u_4)^{-1}v_4\widetilde{u} - (1-u_4)^{-1}v_4\widetilde{u}, v_4)\big) = (u,v).
\end{aligned}$$

Now suppose that $(q,p) \in T_0\mathbf{R}^3$. Then

$$\begin{aligned}
\Phi_M^{-1}(\Phi_M(q,p)) &= \Phi_M^{-1}(u,v) = \big((-(1-u_4)\widetilde{v} - v_4\widetilde{u}, (1-u_4)^{-1}\widetilde{u}\big), \\
&\qquad \text{where } u, v \text{ are given by (58)} \\
&= \big(-2(\|p\|^2+1)^{-1}[-\tfrac{1}{2}(\|p\|^2+1)q + \langle q,p\rangle p] + 2\langle q,p\rangle(\|p\|^2+1)^{-1}p, \\
&\qquad 2(\|p\|^2+1)^{-1}\tfrac{1}{2}(\|p\|^2+1)p\big) = (q,p).
\end{aligned}$$

□

▷ The restriction $\widetilde{\Phi}$ of Moser's mapping Φ_M (58) to $H^{-1}(-\frac{1}{2})$ is a diffeomorphism of $H^{-1}(-\frac{1}{2})$ onto $T_1S^3_{\mathrm{np}} = \{(u,v) \in T^+S^3 \,\big|\, \|u\|^2 = 1\}$ with inverse $\Phi = \Phi_M^{-1}|T_1S^3_{\mathrm{np}}$.

(4.6) **Proof**: Using $(u,v) \in T_1S^3_{\mathrm{np}}$ and the identities $\|q\|^2 = \|v\|^2(1-u_4)^2$ and $\|p\|^2+1 = 2(1-u_4)^{-1}$ we get $\frac{1}{2}\|p\|^2 - \|q\|^{-1} + \frac{1}{2} = (1-u_4)^{-1} - (\|v\|(1-u_4))^{-1} = 0$, that is, $(q,p) \in H^{-1}(-\frac{1}{2})$. So $\Phi = \Phi_M^{-1}|T_1S^3_{\mathrm{np}}$ maps $T_1S^3_{\mathrm{np}}$ into $H^{-1}(-\frac{1}{2})$. From the fact that $\nu(q,p) = (-2H(q,p))^{-1/2} = 1$ when $(q,p) \in H^{-1}(-\frac{1}{2})$, a straightforward calculation, given in (69a) – (69c), shows that $\widetilde{\Phi} = \Phi_M|H^{-1}(-\frac{1}{2})$ maps $H^{-1}(-\frac{1}{2})$ into $T_1S^3_{\mathrm{np}}$. Because Moser's mapping is a diffeomorphism of $T_0\mathbf{R}^3$ onto $T^+S^3_{\mathrm{np}}$, it follows that $\widetilde{\Phi}$ is a diffeomorphism of $H^{-1}(-\frac{1}{2})$ onto $T_1S^3_{\mathrm{np}}$ with inverse Φ. □

▷ The map Φ_M^{-1} pulls back the 1-form $\big(\sum_{i=1}^3 q_i\,\mathrm{d}p_i\big)|H^{-1}(-\frac{1}{2})$ on $H^{-1}(-\frac{1}{2})$ to the 1-form $-\big(\sum_{j=1}^4 v_j\,\mathrm{d}u_j\big)\,|T_1S^3$ on T_1S^3.

(4.7) **Proof**: On $H^{-1}(-\frac{1}{2})$ we have

$$\begin{aligned}
(\Phi_M^{-1})^*(\sum_{i=1}^3 q_i\,\mathrm{d}p_i) &= -\sum_{i=1}^3 \big((1-u_4)v_i + v_4u_i\big)\,\mathrm{d}\big((1-u_4)^{-1}u_i\big) = -\sum_{i=1}^3 v_i\,\mathrm{d}u_i \\
&\quad -\sum_{i=1}^3 u_iv_i(1-u_4)^{-1}\,\mathrm{d}u_4 - v_4(1-u_4)^{-1}\sum_{i=1}^3 u_i\,\mathrm{d}u_i - v_4(1-u_4)^{-2}\sum_{i=1}^3 u_i^2\,\mathrm{d}u_4 \\
&\qquad \text{since } 0 = \mathrm{d}(\textstyle\sum_{j=1}^4 u_j^2) \text{ gives } -u_4\,\mathrm{d}u_4 = \textstyle\sum_{i=1}^3 u_i\,\mathrm{d}u_i \\
&= -\sum_{i=1}^3 v_i\,\mathrm{d}u_i + v_4u_4(1-u_4)^{-1}\,\mathrm{d}u_4 + v_4u_4(1-u_4)^{-1}\,\mathrm{d}u_4 \\
&\quad -(1-u_4^2)(1-u_4)^{-2}\,v_4\,\mathrm{d}u_4, \quad \text{since } \textstyle\sum_{i=1}^3 u_iv_i = -u_4v_4 \text{ and } \sum_{i=1}^3 u_i^2 = 1-u_4^2 \\
&= -\sum_{j=1}^4 v_j\,\mathrm{d}u_j.
\end{aligned}$$ □

▷ The inverse Φ_M^{-1} (56) of Moser's mapping is symplectic,

(4.8) **Proof**: On $T_1S^3_{\mathrm{np}}$ we get

$$\begin{aligned}
(\Phi_M^{-1})^*(\sum_{i=1}^3 \mathrm{d}q_i\wedge\mathrm{d}p_i) &= (\Phi_M^{-1})^*(\mathrm{d}\sum_{i=1}^3 q_i\,\mathrm{d}p_i) = \mathrm{d}\big((\Phi_M^{-1})^*(\sum_{i=1}^3 q_i\,\mathrm{d}p_i)\big) \\
&= -\mathrm{d}\big(\sum_{j=1}^4 v_j\,\mathrm{d}u_j\big) = \sum_{j=1}^4 \mathrm{d}u_j\wedge\mathrm{d}v_j.
\end{aligned}$$ □

On $T^+S^3_{\mathrm{np}}$ let

$$K(u,v) = (\Phi_M^{-1})^*\widetilde{K}(u,v) = \tfrac{1}{2}\|v\|^2, \tag{59}$$

using (57a) and (57b). From ((4.7)) and ((4.8)) it follows that on the energy level $H^{-1}(-\frac{1}{2})$ the Hamiltonian system $(H, T_0\mathbf{R}^3, \omega)$ is equivalent to the Hamiltonian system $\big(K, T^+S^3_{\mathrm{np}}, (\sum_{j=1}^4 \mathrm{d}u_j\wedge\mathrm{d}v_j)\big|T^+S^3_{\mathrm{np}}\big)$ on $K^{-1}(\frac{1}{2})$ via Moser's mapping Φ_M (58). Clearly K extends to a smooth function on $TS^3 \cap K^{-1}(\frac{1}{2})$, whose Hamiltonian vector field X_K defines the geodesic flow on S^3 on $K^{-1}(\frac{1}{2})$. Hence Moser's mapping embeds the incomplete flow of the Kepler vector field X_H on $H^{-1}(-\frac{1}{2})$ into the complete geodesic flow on $K^{-1}(\frac{1}{2})$. Using the virial group we can use Moser's mapping to regularize the Kepler vector field a negative energy level at a time.

▷ We now see how the integrals of the geodesic flow pull back under the of Moser's mapping. We show that under Moser's mapping Φ_M the integral $S_{ij} = u_i v_j - u_j v_i$, $1 \le i < j \le 3$, of X_K on $K^{-1}(\frac{1}{2})$ pulls back to the k^{th} component J_k of the SO(3)-momentum $J = p \times q$ restricted to $H^{-1}(-\frac{1}{2})$. Here $\{i, j, k\} = \{1, 2, 3\}$. The integral $S_{i4} = u_i v_4 - v_i u_4$, $1 \le i \le 3$, of X_K on $K^{-1}(\frac{1}{2})$ pulls back to the the i^{th} component of the eccentricity vector $\mathbf{e} = -\|q\|^{-1} q + p \times (q \times p)$ restricted to $H^{-1}(-\frac{1}{2})$.

(4.9) **Proof**: When $1 \le i < j \le 3$ we get

$$\begin{aligned}
\Phi_M^*\big(S_{ij}|K^{-1}(\tfrac{1}{2})\big)(u,v) &= \big(\Phi_M^*(u_i v_j - u_j v_i)\big)|H^{-1}(-\tfrac{1}{2}) \\
&= \big[2p_i(\|p\|^2+1)^{-1}\big(-\tfrac{1}{2}(\|p\|^2+1)q_j + \langle q,p\rangle p_j\big) \\
&\qquad -\big(-\tfrac{1}{2}(\|p\|^2+1)q_i + \langle q,p\rangle p_i\big)2p_j(\|p\|^2+1)^{-1}\big], \text{ using (58)} \\
&= (q_i p_j - p_j q_i)|H^{-1}(-\tfrac{1}{2}) = J_k|H^{-1}(-\tfrac{1}{2}).
\end{aligned}$$

Also when $i = 1, 2, 3$ we have

$$\begin{aligned}
\Phi_M^*\big(S_{i4}|K^{-1}(\tfrac{1}{2})\big)(u,v) &= \big(\Phi_M^*(u_i v_4 - u_4 v_i)\big)|H^{-1}(-\tfrac{1}{2}) \\
&= \big[-2p_i(\|p\|^2+1)^{-1}\langle q,p\rangle + \tfrac{1}{2}q_i(\|p\|^2-1) \\
&\qquad -\langle q,p\rangle\|p\|^2(\|p\|^2+1)^{-1}p_i + \langle q,p\rangle(\|p\|^2+1)^{-1}p_i\big] \\
&= \big[-p_i\langle q,p\rangle + q_i\|p\|^2 - \tfrac{1}{2}(\|p\|^2+1)q_i\big] \\
&= \big[-\|q\|^{-1}q_i + q_i\langle p,p\rangle - p_i\langle q,p\rangle\big]\,|H^{-1}(-\tfrac{1}{2}) = e_i|H^{-1}(-\tfrac{1}{2}).
\end{aligned}$$

The second to last equality follows since $\frac{1}{2}(\|p\|^2+1) = \|q\|^{-1}$ defines $H^{-1}(-\frac{1}{2})$. □

▷ Let

$$\widetilde{J}: T^+S^3 \to \mathbf{R}: (u,v) \mapsto \sum_{1 \le i < j \le 3} (u_i v_j - u_j v_i)^2 = \|\widetilde{u} \times \widetilde{v}\|^2.$$

Then $\widetilde{J}^{-1}(0)$ is the set of all integral curves of the geodesic vector field X_K, which pass through the *collision set* $C = \{(u,v) \in T^+S^3 \,\big|\, u_4 = 1\}$ on T^+S^3.

(4.10) **Proof**: The image of each integral curve of X_K under the bundle projection map is a great circle on S^3. Each great circle intersects the equatorial 2-sphere $\{u_4 = 0\} \cap S^3$ at some point $P = (\widetilde{u}, 0, \widetilde{v}, v_4)$. Suppose that $\widetilde{J}(P) = 0$. Then $\widetilde{u} \times \widetilde{v} = 0$. If $\widetilde{v} \ne 0$, then $\widetilde{u} = \lambda \widetilde{v}$ for some nonzero $\lambda \in \mathbf{R}$. But $(u,v) \in T^+S^3$. So $0 = \langle u, v\rangle = \langle \widetilde{u}, \widetilde{v}\rangle + u_4 v_4 = \langle \widetilde{u}, \widetilde{v}\rangle$, since $u_4 = 0$. Consequently, $0 = \lambda\langle \widetilde{v}, \widetilde{v}\rangle$, which contradicts the fact that $\lambda \ne 0$ and $\widetilde{v} \ne 0$. Therefore $\widetilde{v} = 0$, that is, $P = (\widetilde{u}, 0, 0, v_4)$. For some $\tau > 0$ the integral curve $\gamma: \mathbf{R} \to T^+S^3: t \mapsto \varphi_t^K(P)$ of X_K passes through the collision set C. To see this we must find τ so that

$$\begin{pmatrix} 0 \\ 1 \\ \widetilde{v}' \\ v_4' \end{pmatrix} = \varphi_\tau^K \begin{pmatrix} \widetilde{u} \\ 0 \\ 0 \\ v_4 \end{pmatrix} = \begin{pmatrix} \cos(\tau\sqrt{2K}) \begin{pmatrix} \widetilde{u} \\ 0 \end{pmatrix} + \frac{\sin(\tau\sqrt{2K})}{\sqrt{2K}} \begin{pmatrix} 0 \\ v_4 \end{pmatrix} \\ -\sqrt{2K}\sin(\tau\sqrt{2K}) \begin{pmatrix} \widetilde{u} \\ 0 \end{pmatrix} + \cos(\tau\sqrt{2K}) \begin{pmatrix} 0 \\ v_4 \end{pmatrix} \end{pmatrix}, \qquad (60)$$

using (8). Noting that $K = K(P) = \frac{1}{2}v_4^2 > 0$, the fourth component of (60) reads $1 = v_4|v_4|^{-1}\sin(\tau\sqrt{2K})$, which gives $\tau = \pi/(2v_4)$, if $v_4 > 0$ or $\tau = 3\pi/(2|v_4|)$, if $v_4 < 0$. Thus the first three components of (60) read $0 = \cos(\tau\sqrt{2K})$, as desired. Consequently, at time $t = \tau$ the integral curve γ, which starts at $P = (\widetilde{u}, 0, 0, v_4)$ passes through the collision set C. Thus $\widetilde{J}^{-1}(0)$ is a subset of C. The collision set is clearly a subset of $\widetilde{J}^{-1}(0)$. □

Under Moser's mapping $\widetilde{J}^{-1}(0)$ corresponds to the set of bounded orbits of the Kepler problem with 0 angular momentum. This is precisely the set of bounded Keplerian orbits which reach the origin of $\mathbf{R}^3$ in finite time.

4.2 Ligon-Schaaf regularization

On the subset Σ_- of phase space $T_0\mathbf{R}^3$, where the Kepler Hamiltonian is *negative*, one can perform regularization in such a way that the embedding is symplectic and the resulting vector field is Hamiltonian with an $\mathrm{SO}(4)$ symmetry, which integrates the $\mathrm{so}(4)$ symmetry of the Kepler Hamiltonian. This symmetry does *not* arise from a lift of a symmetry on configuration space.

We regularize all *negative* energy Keplerian orbits at once using the *Ligon-Schaaf map*

$$\begin{aligned} &\Phi_{LS} : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T^+S^3_{\mathrm{np}} \subseteq T\mathbf{R}^4 : \\ &(q,p) \mapsto \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\nu(q,p)\sin v_4 & \nu(q,p)\cos v_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned} \tag{61}$$

where

$$\begin{aligned} u &= \big(\nu(q,p)^{-1}\|q\|p, \langle p,p\rangle\|q\| - 1\big) \\ v &= \big(-\|q\|q + \langle q,p\rangle p, -\nu(q,p)^{-1}\langle q,p\rangle\big), \end{aligned} \tag{62}$$

and $\nu(q,p) = (\frac{2}{\|q\|} - \|p\|^2)^{-1/2}$. We start by factoring Φ_{LS}.

Claim: Let

$$S : \Sigma_- \subseteq T_0\mathbf{R}^4 \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \subseteq T\mathbf{R}^4 \times \mathbf{R} : (q,p) \mapsto (u,v,\nu), \tag{63a}$$

where $\nu = \nu(q,p)$ and (u,v) is given by (62). Also let

$$L : T_1S^3 \times \mathbf{R}_{>0} \to T^+S^3 \subseteq T\mathbf{R}^4 : (u,v,\nu) \mapsto \begin{pmatrix} \widetilde{r} \\ \nu\widetilde{s} \end{pmatrix}, \tag{63b}$$

where $\begin{pmatrix} \widetilde{r} \\ \widetilde{s} \end{pmatrix} = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\sin v_4 & \cos v_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Then $\Phi_{LS} = L \circ S$.

(4.11) **Proof**: Before proving the claim we look at each factor of the Ligon-Schaaf map more carefully starting with the mapping S (63a). On Σ_- we have an $\mathbf{R}_{>0}$-action

$$\Psi^V : \mathbf{R}_{>0} \times \Sigma_- \to \Sigma_- : \big(\rho, (q,p)\big) \mapsto (\rho^2 q, \rho^{-1}p) \tag{64}$$

of the *scaling group*. To see that Ψ^V is well defined suppose that $\rho \in \mathbf{R}_{>0}$ and $(q,p) \in \Sigma_-$. Then at (q,p) the value of the Kepler Hamiltonian H (49) is negative. So

$$H(\rho^2 q, \rho^{-1}p) = \tfrac{1}{2}\|\rho^{-1}p\|^2 - \|\rho^2 q\|^{-1} = \rho^{-2}H(q,p) < 0.$$

▷ Thus for every $\rho \in \mathbf{R}_{>0}$ the map Ψ^V_ρ sends Σ_- into itself. The action Ψ^V is free for if
▷ $(\rho^2 q, \rho^{-1} p) = (q,p)$, then $\rho = 1$. Every orbit of the $\mathbf{R}_{>0}$-action (64) intersects the level set $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ in exactly one point.

(4.12) **Proof**: Suppose that $(q,p) \in \Sigma_-$. Then $m = \Psi^V_{\nu(q,p)^{-1}}(q,p) \in H^{-1}(-\frac{1}{2})$, since

$$H(m) = H(\nu(q,p)^{-2}q, \nu(q,p)p) = \nu(q,p)^2 H(q,p) = -\tfrac{1}{2}$$

for $\nu(q,p)^2 = (-2H(q,p))^{-1}$. Because $H(\Psi^V_\rho(q,p)) = \rho^{-2}H(q,p)$, the Ψ^V-orbit through (q,p) intersects $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ at m only when $\rho = \nu(q,p)^{-1}$. □

▷ The orbit space $\Sigma_-/\mathbf{R}_{>0}$ is diffeomorphic to $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ with orbit map

$$\pi_V : \Sigma_- \subseteq T_0\mathbf{R}^3 \to (H|\Sigma_-)^{-1}(-\tfrac{1}{2}) \subseteq \Sigma_- : (q,p) \mapsto (\widehat{q}, \widehat{p}) = \big(\nu(q,p)^{-2}q, \nu(q,p)p\big). \quad (65)$$

(4.13) **Proof**: We need only show that $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ is a smooth submanifold of $T_0\mathbf{R}^3$, since $\Sigma_- = H^{-1}((-\infty,0))$ is an open subset of $T_0\mathbf{R}^3$. Suppose that $(q,p) \in T_0\mathbf{R}^3$ is a critical point of H. Then $0 = \mathrm{d}H(q,p) = \langle p, \mathrm{d}p\rangle + \|q\|^{-3}\langle q, \mathrm{d}q\rangle$. This implies $q = 0 = p$, which contradicts the fact that $(q,p) \in T_0\mathbf{R}^3$. Thus every negative real number is a regular value of $H|\Sigma_-$. Consequently, $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ is a smooth manifold. □

Let $T_1S^3_{\mathrm{np}} = \{(u,v) \in T^+S^3_{\mathrm{np}} \,\big|\, \|v\|^2 = 1\}$. Define an $\mathbf{R}_{>0}$-action Ψ^T on $T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ by

$$\Psi^T : \mathbf{R}_{>0} \times (T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}) \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} : \big(\mu, ((u,v),\lambda)\big) \mapsto ((u,v), \mu\lambda). \qquad (66)$$

The action Ψ^T is free for if $\big((u,v),\mu\lambda\big) = \big((u,v),\lambda\big)$, then $\mu = 1$. The space $T_1S^3_{\mathrm{np}} \times \{1\}$
▷ is the orbit space $(T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0})/\mathbf{R}_{>0}$ of the action Ψ^V. The orbit map is

$$\pi_T : T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \to T_1S^3_{\mathrm{np}} \times \{1\} : \big((u,v),\mu\big) \mapsto \big((u,v),1\big). \qquad (67)$$

Claim: Using the restriction of Moser's mapping Φ_M (58) to $(H|\Sigma_-)^{-1}(-\frac{1}{2})$, given by

$$\begin{aligned}\Phi : (H|\Sigma_-)^{-1}(-\tfrac{1}{2}) \subseteq T_0\mathbf{R}^3 &\to TS^3_{\mathrm{np}} : (q,p) \mapsto (u,v) \\ &= \big((\|q\|p, \|p\|^2\|q\| - 1), (-\|q\|^{-1}q + \langle q,p\rangle p, -\langle q,p\rangle)\big),\end{aligned} \qquad (68a)$$

and *Moser's fibration*

$$\begin{aligned}&F_M : \Sigma_- \to T_1S^3_{\mathrm{np}} : (q,p) \mapsto \\ &\big((\nu(q,p)^{-1}\|q\|p, \|p\|^2\|q\| - 1), (-\|q\|^{-1}q + \langle q,p\rangle, -\nu(q,p)^{-1}\langle q,p\rangle)\big),\end{aligned} \qquad (68b)$$

where $\nu(q,p)^{-2} = \frac{2}{\|q\|} - \|p\|^2$, we obtain the following commutative diagram.

$$\begin{tikzcd}
\Sigma_- \arrow[r, "S"] \arrow[d, "\pi_V"'] & T_1 S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \arrow[d, "\pi_T"] & (q,p) \arrow[r] \arrow[d] & (F_M(q,p), \nu(q,p)) \arrow[d] \\
(H|\Sigma_-)^{-1}(-\frac{1}{2}) \arrow[r, "s"] & T_1 S^3_{\mathrm{np}} \times \{1\} & (\widehat{q},\widehat{p}) \arrow[r] & \big(\Phi(\widehat{q},\widehat{p}),1\big)
\end{tikzcd}$$

Diagram 4.2.1

Morover, the bundle mapping S is an $\mathbf{R}_{>0}$-bundle isomorphism.

(4.14) **Proof**: The next calculation shows that the image of Σ_- under Moser's fibration F_M is contained in $T_1 S^3_{\mathrm{np}}$. Using the definition of F_M (68b) and $\nu = \nu(q,p)$ we have

$$\begin{aligned}
\langle u,u\rangle &= \nu^{-2}\|q\|^2\|p\|^2 + (\|p\|^2\|q\|-1)^2 \\
&= (-\|p\|^2 + 2\|q\|^{-1})\|q\|^2\|p\|^2 + (\|p\|^2\|q\|-1)^2 = 1; && \text{(69a)}\\
\langle u,v\rangle &= \nu^{-1}\|q\|\langle q,p\rangle\|p\|^2 - \nu^{-1}\langle q,p\rangle + \nu^{-1}\langle q,p\rangle - \nu^{-1}\|q\|\langle q,p\rangle\|p\|^2 = 0; && \text{(69b)}\\
\langle v,v\rangle &= \|q\|^{-2}\|q\|^2 - 2\|q\|^{-1}\langle q,p\rangle^2 + \|p\|\langle q,p\rangle^2 + \nu^{-2}\langle q,p\rangle^2 \\
&= 1 - 2\|q\|^{-1}\langle q,p\rangle^2 + \|p\|^2\langle q,p\rangle^2 + 2\|q\|^{-1}\langle q,p\rangle^2 - \|p\|^2\langle q,p\rangle^2 = 1. && \text{(69c)}
\end{aligned}$$

Thus $F_M(\Sigma_-) \subseteq T_1 S^3$. Suppose that $u_4 = 1$. Then $\|q\|\|p\|^2 = 2$ using (62). So $-2H(q,p) = \|q\|^{-1}(2 - \|q\|\|p\|^2) = 0$, which contradicts the fact that $H(q,p) < 0$. Consequently, $S(\Sigma_-) \subseteq T_1 S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$, since $\nu(q,p) = (-2H(q,p))^{-1/2} > 0$ for $(q,p) \in \Sigma_-$. Diagram 4.2.1 is commutative because $\nu(\widehat{q},\widehat{p}) = 1$ and

$$\begin{aligned}
\Phi(\widehat{q},\widehat{p}) = \Phi(\nu^{-2}q, \nu p) &= \big((\|\nu^{-2}q\|\nu p, \|\nu p\|^2\|\nu^{-2}q\| - 1), \\
&\qquad (-\|\nu^{-2}q\|^{-1}\nu^{-2}q + \langle \nu^{-2}q, \nu p\rangle \nu p, -\langle \nu^{-2}q, \nu p\rangle)\big) \\
&= \big((\nu^{-1}\|q\|p, \|p\|^2\|q\| - 1), (-\|q\|^{-1}q, +\langle q,p\rangle p, \nu^{-1}\langle q,p\rangle)\big) = F_M(q,p).
\end{aligned}$$

From ((4.6)) it follows that the mapping s in diagram 4.2.1 is a diffeomorphism.

▷ We have not yet shown that the mapping S is an $\mathbf{R}_{>0}$-bundle isomorphism. To do this, and thus finish proving the ((4.14)), we must show that the mapping S is a fiber preserving $\mathbf{R}_{>0}$-isomorphism and is a diffeomorphism. This assertion follows when we establish:

i) For every $(\widehat{q},\widehat{p}) \in (H|\Sigma_-)^{-1}(-\frac{1}{2})$ and for every $\rho > 0$ we have $S\big(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})\big) = \Psi^T_{\rho^{-1}}\big(s(\widehat{q},\widehat{p})\big)$.

ii) The mapping $S|\pi_V^{-1}(\widehat{q},\widehat{p})$ from the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ to the fiber $\pi_T^{-1}\big(s(\widehat{q},\widehat{p})\big)$ is one to one and onto.

(4.15) **Proof**:

i) We compute

$$\nu(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})) = \nu(\rho^{-2}\widehat{q}, \rho\widehat{p}) = \big(\rho^2(-2H(\widehat{q},\widehat{p}))\big)^{-1/2} = \rho^{-1};$$

while $F_M(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})) = F_M(\rho^{-2}\widehat{q}, \rho\widehat{p}) = \Phi(\widehat{q},\widehat{p})$. Therefore

$$S\big(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})\big) = \big(\Phi(\widehat{q},\widehat{p}), \rho^{-1}\big) = \Psi^T_{\rho^{-1}}\big(s(\widehat{q},\widehat{p})\big).$$

ii) Suppose that $S\big(\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p})\big) = S\big(\Psi^V_{\sigma^{-1}}(\widehat{q},\widehat{p})\big)$ for some ρ, $\sigma \in \mathbf{R}_{>0}$. Then $\rho = \sigma$, because $\big(\Phi(\widehat{q},\widehat{p}),\rho^{-1}\big) = \big(\Phi(\widehat{q},\widehat{p}),\sigma^{-1}\big)$ by hypothesis. Therefore we get $\Psi^V_{\rho^{-1}}(\widehat{q},\widehat{p}) = \Psi^V_{\sigma^{-1}}(\widehat{q},\widehat{p})$, which shows that $S|\pi_V^{-1}(\widehat{q},\widehat{p})$ is an injective map from the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ into the fiber $\pi_T^{-1}(s(\widehat{q},\widehat{p}))$. Suppose that $(\widehat{Q},\widehat{P},\lambda) \in \pi_T^{-1}(s(\widehat{q},\widehat{p}))$. Then

$$(\widehat{Q},\widehat{P},\lambda) = \Psi^V_\lambda\big(\Phi(\widehat{q},\widehat{p}),1\big) = \big(\Phi(\widehat{q},\widehat{p}),\lambda\big) = S(\lambda^2\widehat{q},\lambda^{-1}\widehat{p}).$$

Thus S maps the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ onto the fiber $\pi_T^{-1}\big(s(\widehat{q},\widehat{p})\big)$. □

▷ To show that the mapping $S : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ (63a) is a diffeomorphism we argue as follows.

(4.16) **Proof**: Suppose that $(u,v,\nu) \in T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$. Then $\pi_T(u,v,\nu) = (u,v,1) \in T_1S^3_{\mathrm{np}}$. Since s is a diffeomorphism, there is a unique $(\widehat{q},\widehat{p}) \in (H|\Sigma_-)^{-1}(-\frac{1}{2})$ such that $(\widehat{q},\widehat{p}) = s^{-1}(u,v,1)$. Because S maps the fiber $\pi_V^{-1}(\widehat{q},\widehat{p})$ one to one and onto the fiber $\pi_T^{-1}(u,v,1)$ and the $\mathbf{R}_{>0}$-action Ψ^T is free, there is a unique $\rho \in \mathbf{R}_{>0}$ such that $S(q,p) = S(\Psi^V_\rho(\widehat{q},\widehat{p})) = (u,v,\nu)$. Thus the mapping S is one to one and onto. The next argument shows that for every $(q,p) \in \Sigma_-$ the tangent $T_{(q,p)}S : T_{(q,p)}\Sigma_- \to T_{S(q,p)}(T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0})$ of the mapping S is surjective. Suppose that $\gamma : [0,1] \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ is a smooth curve, which passes through (u,v,ν) at $t = 0$. Then $\pi_T\circ\gamma$ is a smooth curve on $T_1S^3_{\mathrm{np}} \times \{1\}$, which passes through $(u,v,1)$ at $t = 0$. Since s is a diffeomorphism of $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ onto $T_1S^3_{\mathrm{np}} \times \{1\}$, the curve $s^{-1}\circ(\pi_T\circ\gamma)$ on $(H|\Sigma_-)^{-1}(-\frac{1}{2})$ is smooth and passes through $(\widehat{q},\widehat{p})$ at $t = 0$. Therefore $\Psi^V_\rho\circ(s^{-1}\circ(\pi_T\circ\gamma))$ is a smooth curve on Σ_- which passes through (q,p). Consequently, $T_{(q,p)}S$ is surjective. Because $\dim\Sigma_- = \dim(T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0})$, it follows that $T_{(q,p)}S$ is bijective. In other words, S is a local diffeomorphism. Thus S is a global diffeomorphism since it is injective. □

Thus S is an isomorphism of $\mathbf{R}_{>0}$-bundles and this completes the proof of ((4.14)). □

Claim: Consider the 1-form $\theta = \nu(\langle v, \mathrm{d}u\rangle + \mathrm{d}v_4)|M$ on $M = T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0} \subseteq T\mathbf{R}^4 \times \mathbf{R}$ with coordinates (u,v,ν). Then

$$S^*\big((\nu\langle v,\mathrm{d}u\rangle)|M\big) = \big(\nu(q,p)\langle q,p\rangle\,\mathrm{d}(\nu(q,p)^{-1}) - \langle q,\mathrm{d}p\rangle\big)|\Sigma_- \tag{70a}$$

$$S^*\big((\nu\,\mathrm{d}v_4)|M\big) = \big(\nu(q,p)\,\mathrm{d}(\nu(q,p)^{-1}\langle q,p\rangle)\big)|\Sigma_-, \tag{70b}$$

that is,

$$S^*\theta = -\big(\langle q,\mathrm{d}p\rangle + \mathrm{d}\langle q,p\rangle\big)|\Sigma_-. \tag{70c}$$

(4.17) **Proof**: Equation (70c) follows from equations (70a) and (70b) because

$$\begin{aligned} S^*(\nu\langle v,\mathrm{d}u\rangle + \nu\,\mathrm{d}v_4) &= \nu(q,p)\langle q,p\rangle\,\mathrm{d}\nu(q,p)^{-1} - \langle q,\mathrm{d}\,p\rangle \\ &\qquad - \nu(q,p)\langle q,p\rangle\,\mathrm{d}\nu(q,p)^{-1} - \mathrm{d}\langle q,p\rangle \\ &= -\langle q,\mathrm{d}p\rangle - \mathrm{d}\langle q,p\rangle. \end{aligned}$$

The following calculation verifies equation (70a). We have

$$S^*(\langle v,\mathrm{d}u\rangle) = \big\langle\big(-\|q\|^{-1}q + \langle q,p\rangle p, -\nu^{-1}\langle q,p\rangle\big),\big(\mathrm{d}(\nu^{-1}\|q\|p),\mathrm{d}(\|q\|\|p\|^2)\big)\big\rangle$$

$$
\begin{aligned}
&= \nu^{-2}\langle q,p\rangle \|q\| (\|q\|^{-1} - \|p\|^2)\, \mathrm{d}\nu - \nu^{-1}\langle q,p\rangle \|q\| (\langle p, \mathrm{d}p\rangle + \|q\|^{-3}\langle q, \mathrm{d}q\rangle) \\
&\qquad - \nu^{-1}\langle q, \mathrm{d}p\rangle \\
&= \nu^{-1}\langle q,p\rangle \|q\| \big(- \|q\|^{-1}\nu^{-1}\, \mathrm{d}\nu + (-2H)(-2H)^{-1}\, \mathrm{d}H\big) \\
&\qquad - \nu^{-1}\langle q,p\rangle \|q\|^{-1}\, \mathrm{d}H - \nu^{-1}\langle q, \mathrm{d}p\rangle \\
&\qquad\quad \text{since } \nu = (-2H)^{-1/2} \text{ implies } \mathrm{d}\nu = (-2H)^{-3/2}\, \mathrm{d}H = \\
&\qquad\quad \nu(-2H)^{-1}\, \mathrm{d}H \text{ and } \mathrm{d}H = \langle p, \mathrm{d}p\rangle + \|q\|^{-3}\langle q, \mathrm{d}q\rangle \\
&= \langle q,p\rangle\, \mathrm{d}\nu^{-1} - \nu^{-1}\langle q, \mathrm{d}p\rangle .
\end{aligned}
$$

On the right hand side of the above equations we have used the abbreviation ν for $\nu(q,p)$. Since $S^*\nu_4 = -\nu(q,p)^{-1}\langle q,p\rangle$, we obtain equation (70b). □

Corollary: The mapping S is a symplectic diffeomorphism sending Σ_- to M with $S^*(\mathrm{d}\theta) = (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|\Sigma_-$.

(4.18) **Proof**: Take the exterior derivative of both sides of (70c). □

We now look at the factor L (63b) of the Ligon-Schaaf mapping Φ_{LS} (61). On T^+S^3 we have an $\mathbf{R}_{>0}$-action

$$\Psi^D : \mathbf{R}_{>0} \times T^+S^3 \to T^+S^3 : \big(\mu, (r,s)\big) \mapsto (r, \mu s) \tag{71}$$

of a *scaling group*. The action Ψ^D is free, for if $(r,\mu s) = (r,s)$, then $\mu = 1$. Because every orbit of the action Ψ^D intersects T_1S^3 exactly once, the orbit space $T^+S^3/\mathbf{R}_{>0}$ is diffeomorphic to T_1S^3, the unit tangent sphere bundle to S^3, with orbit mapping

$$\pi_D : T^+S^3 \to T_1S^3 : (r,s) \mapsto (r, \|s\|^{-1}s).$$

Claim: Using the mapping

$$L : T_1S^3 \times \mathbf{R}_{>0} \to T^+S^3 : (u,v,\nu) \mapsto \big(r(u,v,\nu), s(u,v,\nu)\big),$$

where

$$
\begin{aligned}
r(u,v,\nu) &= \widetilde{r}(u,v) = \cos \nu_4\, u + \sin \nu_4\, v \\
s(u,v,\nu) &= \nu\widetilde{s}(u,v) = \nu(-\sin \nu_4\, u + \cos \nu_4\, v),
\end{aligned}
\tag{72}
$$

diagram 4.2.2 is commutative

$$
\begin{tikzcd}
T_1S^3 \times \mathbf{R}_{>0} \arrow[r, "L"] \arrow[d, "\pi_T"'] & T^+S^3 \arrow[d, "\pi_D"] & (u,v,\nu) \arrow[r] \arrow[d] & \big(r(u,v,\nu), s(u,v,\nu)\big) \arrow[d] \\
T_1S^3 \times \{1\} \arrow[r, "\ell"] & T_1S^3 & (u,v,1) \arrow[r] & \big(\widetilde{r}(u,v), \widetilde{s}(u,v)\big)
\end{tikzcd}
$$

Diagram 4.2.2

Moreover, the mapping L is an $\mathbf{R}_{>0}$-bundle isomorphism.

(4.19) **Proof**: The maps in diagram 4.2.2 are properly defined, because if $(u,v) \in T_1S^3$, then for $(\widetilde{r},\widetilde{s}) = \ell(u,v)$ we have $\|\widetilde{r}\|^2 = 1 = \|\widetilde{s}\|^2$ and $\langle \widetilde{r},\widetilde{s}\rangle = 0$. So $(r,s) = (\widetilde{r}, \nu\widetilde{s}) = L(u,v,\nu) \in T^+S^3$. From (72) it follows that $\widetilde{r}(u,v) = r(u,v,1)$ and $\widetilde{s}(u,v) = s(u,v,1)$. Consequently, diagram 4.2.2 is commutative.

As a first step toward verifying that the mapping L is an $\mathbf{R}_{>0}$-bundle isomorphism, we ▷ will show that the mapping

$$\ell : T_1S^3 \times \{1\} = T_1S^3 \to T_1S^3 : (u,v) \mapsto \begin{pmatrix} \widetilde{r} \\ \widetilde{s} \end{pmatrix} = T(v_4)\begin{pmatrix} u \\ v \end{pmatrix} \tag{73}$$

with $T(v_4) = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\sin v_4 & \cos v_4 \end{pmatrix}$ is a diffeomorphism.

(4.20) **Proof**: We start by showing that for every $(u,v) \in T_1S^3$ the tangent mapping $T_{(u,v)}\ell$ of ℓ is a bijective linear mapping of $T_{(u,v)}(T_1S^3)$ into itself.

(4.21) **Proof**: Differentiating (73) gives

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = T_{(u,v)}\ell\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = T(v_4)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \dot{v}_4 T(v_4)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = T(v_4)\left[\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \dot{v}_4\begin{pmatrix} v \\ -u \end{pmatrix}\right] \tag{74}$$

for $(\dot{u},\dot{v}) \in T_{(u,v)}(T_1S^3)$ and $\dot{v}_4 \in \mathbf{R}$. Since $T_{(u,v)}(T_1S^3) = \{(\dot{x},\dot{y}) \in T\mathbf{R}^4 \,\big|\, \langle u,\dot{x}\rangle = 0 = \langle v,\dot{y}\rangle \;\&\; \langle u,\dot{y}\rangle + \langle \dot{x},v\rangle = 0\}$, the next calculation shows that $(\dot{x},\dot{y}) = (v,-u) \in T_{(u,v)}(T_1S^3)$.

$$\langle u,\dot{x}\rangle = \langle u,v\rangle = 0,\ \langle v,\dot{y}\rangle = -\langle v,u\rangle = 0, \text{ and } \langle u,\dot{y}\rangle + \langle \dot{x},v\rangle = -\langle u,u\rangle + \langle v,v\rangle = 1-1 = 0.$$

Therefore $(\dot{w},\dot{z})$ given by (74) lies in $T_{(u,v)}(T_1S^3)$. Let $(\dot{x},\dot{y}) \in T_{(u,v)}(T_1S^3)$. Set $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = T(v_4)^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \dot{v}_4\begin{pmatrix} v \\ -u \end{pmatrix}$. Then $(\dot{u},\dot{v}) \in T_{(u,v)}(T_1S^3)$. Using (74) we get

$$T_{(u,v)}\ell\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = T(v_4)\left(T(v_4)^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \dot{v}_4\begin{pmatrix} v \\ -u \end{pmatrix}\right) + \dot{v}_4 T(v_4)\begin{pmatrix} v \\ -u \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

Thus $T_{(u,v)}\ell$ is a surjective, and so bijective, linear mapping of $T_{(u,v)}(T_1S^3)$ into itself. □

We now show that the mapping ℓ (73) is a diffeomorphism of T_1S^3 into itself. Since ℓ is smooth and its tangent map is bijective at every point of T_1S^3, from the inverse function theorem it follows that ℓ is a local diffeomorphism. To show that ℓ is a global diffeomorphism it suffices to demonstrate that it is injective. For $s \in [0,1]$ let $\ell^s : T_1S^3 \to T_1S^3 : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto T(sv_4)\begin{pmatrix} u \\ v \end{pmatrix}$. Since $\ell^0 = \mathrm{id}_{T_1S^3}$ and $\ell^1 = \ell$, it follows that ℓ is homotopic to $\mathrm{id}_{T_1S^3}$, whose degree is 1. Hence the degree $\deg \ell$ of ℓ is 1. Induce an orientation on T_1S^3 from the standard orientation of $T\mathbf{R}^4 = \mathbf{R}^8$. For every $(u,v) \in T_1S^3$ the map $T_{(u,v)}\ell$ is bijective and orientation preserving, because $T(v_4) \in \mathrm{SO}(4,\mathbf{R})$. Since T_1S^3 is connected and compact, the mapping ℓ is surjective, being an open mapping. Therefore, every $(x,y) \in T_1S^3$ is a regular value of ℓ. The fiber $F = \ell^{-1}(x,y)$ is a finite set, because it is a discrete closed subset of a compact set. From the definition of degree of smooth mapping we have $1 = \deg \ell = \sum_{p \in F} 1$, see exercise 4 of chapter I. Therefore F has only one element. In other words, the mapping ℓ is injective. This proves ((4.20)). □

To finish the proof that the mapping L (72) is an $\mathbf{R}_{>0}$-bundle isomorphism, we need to show that L is a diffeomorphism and a fiber preserving $\mathbf{R}_{>0}$-isomorphism. We establish the latter assertion by verifying

i) For every $(u,v,1) \in T_1S^3 \times \mathbf{R}_{>0}$ and every $\rho > 0$ we have $L(\Psi^T_\rho(u,v,1)) = \Psi^D_\rho\big(\ell(u,v,1)\big)$.

ii) The mapping $L|\pi_T^{-1}(u,v,1)$ from the fiber $\pi_T^{-1}(u,v,1)$ to the fiber $\pi_D^{-1}\big(\ell(u,v,1)\big)$ is one to one and onto.

(4.22) **Proof**:

i) We compute

$$\begin{aligned} L\big(\Psi^T_\rho(u,v,1)\big) &= L(u,v,\rho) = \big(r(u,v,\rho), s(u,v,\rho)\big) = \big(\widetilde{r}(u,v), \rho\widetilde{s}(u,v)\big) \\ &= \Psi^D_\rho\big(\widetilde{r}(u,v), \widetilde{s}(u,v)\big) = \Psi^D_\rho\big(\ell(u,v,1)\big). \end{aligned}$$

ii) Suppose that $L\big(\Psi^T_\rho(u,v,1)\big) = L\big(\Psi^T_\sigma(u,v,1)\big)$ for some ρ, $\sigma \in \mathbf{R}_{>0}$. Then $\big(\widetilde{r}(u,v), \rho\widetilde{s}(u,v)\big) = \big(\widetilde{r}(u,v), \sigma\widetilde{s}(u,v)\big)$, which implies $\rho = \sigma$. Therefore $\Psi^T_\rho(u,v,1) = \Psi^T_\sigma(u,v,1)$. So $L|\pi_T^{-1}(u,v,1)$ is an injective map from the fiber $\pi_T^{-1}(u,v,1)$ to the fiber $\pi_D^{-1}\big(\ell(u,v,1)\big)$. Suppose that $(\widetilde{R}, \widetilde{S}, \lambda) \in \pi_D^{-1}\big(\ell(u,v,1)\big)$. Then

$$(\widetilde{R}, \widetilde{S}, \lambda) = \Psi^D_\lambda\big(\ell(u,v,1)\big) = \big(\widetilde{r}(u,v), \lambda\widetilde{s}(u,v)\big) = \big(r(u,v,\lambda), s(u,v,\lambda)\big) = L(u,v,\lambda).$$

Thus $L|\pi_T^{-1}(u,v,1)$ maps the fiber $\pi_T^{-1}(u,v,1)$ onto the fiber $\pi_D^{-1}\big(\ell(u,v,1)\big)$. □

▷ The mapping L is a diffeomorphism.

(4.23) **Proof**: The argument is similar to the proof of ((4.16)). We include the details. Suppose that $(r,s) \in T^+S^3$. Then $\pi_D(r,s) = (\widetilde{r}, \widetilde{s}) \in T_1S^3$. Since the mapping ℓ is a diffeomorphism, there is a unique $(u,v,1) \in T_1S^3 \times \{1\}$ such that $\ell^{-1}(\widetilde{r}, \widetilde{s}) = (u,v,1)$. Because L maps the fiber $\pi_T^{-1}(u,v,1)$ one to one and onto the fiber $\pi_D^{-1}(\ell(u,v,1))$, there is a unique $\nu \in \mathbf{R}_{>0}$ such that $L(u,v,\nu) = (r,s)$, since the $\mathbf{R}_{>0}$-action Ψ^T is free. Consequently, the mapping L is one to one and onto. The next argument show that for every $(u,v,\nu) \in T_1S^3 \times \mathbf{R}_{>0}$ the tangent $T_{(u,v,\nu)} : T_{(u,v,\nu)}(T_1S^3 \times \mathbf{R}_{>0}) \to T_{L(u,v,\nu)}(T^+S^3)$ is surjective. Suppose that $\gamma : [0,1] \to T^+S^3 : t \mapsto \gamma(t)$ is a smooth curve in T^+S^3, which passes through $(r,s) = L(u,v,\nu)$. Then $\pi_D \circ \gamma$ is a smooth curve in T_1S^3 which passes through $(\widetilde{r}, \widetilde{s})$ at $t = 0$. Since $\ell : T_1S^3 \times \{1\} \to T_1S^3$ is a diffeomorphism, $\ell^{-1} \circ \pi_D \circ \gamma$ is a smooth curve on $T_1S^3 \times \mathbf{R}_{>0}$, which passes through $(u,v,1) = \ell^{-1}(\widetilde{r}, \widetilde{s})$ at $t = 0$. Now $L^{-1}(r,s) = (u,v,\nu)$ for some unique $\nu \in \mathbf{R}_{>0}$, since the action Ψ^T is free. So the smooth curve $\Psi^V_\nu \circ \ell^{-1} \circ \pi_D \circ \gamma$ passes through (u,v,ν) at time $t = 0$. Thus $T_{(u,v,\nu)}L$ is surjective. Because $\dim T_{(u,v,\nu)}(T_1S^3 \times \mathbf{R}_{>0}) = \dim T_{L(u,v,\nu)}(T^+S^3)$, the tangent map $T_{(u,v,\nu)}L$ is injective and hence is bijective. Therefore L is a local diffeomorphism. Because L is one to one, it is a global diffeomorphism. □

Thus L is an isomorphism of $\mathbf{R}_{>0}$-bundles and this finishes the proof of ((4.19)). □

To finish the proof of ((4.11)) we show that the image of the Ligon-Schaaf mapping Φ_{LS} (61) is $T^+S^3_{\mathrm{np}}$. To do this we need to show that L maps $T_1S^3_{\mathrm{np}}$ onto $T^+S^3_{\mathrm{np}}$.

(4.24) **Proof**: Suppose that for some $(u,v,\nu) \in T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ we have $L(u,v,\nu) = (\mathrm{np}, s) \in T^+S^3 \subseteq T\mathbf{R}^4$. Here $\mathrm{np} = (0,0,0,1) \in \mathbf{R}^4$. Then $0 = \langle \mathrm{np}, s\rangle = s_4$. So $L(u,v,\nu) = T_\nu(v_4)\begin{pmatrix} u \\ v \end{pmatrix} = \big(\mathrm{np}, (\widehat{s},0)\big)^t$, where $T_\nu(v_4) = \begin{pmatrix} \cos v_4 & \sin v_4 \\ -\nu \sin v_4 & \nu\cos v_4 \end{pmatrix}$. Set $\widehat{u} = \mathrm{np}$, $\widehat{v} = (\widetilde{v}, v_4) = (\widehat{s}, 0)$, and $\widehat{\nu} = 1$. Then $L(\widehat{u},\widehat{v},\widehat{\nu}) = T_1(0)\begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = \big(\mathrm{np}, (\widehat{s},0)\big)^t$. But the mapping L is one to one. Thus $(\widehat{u},\widehat{v},1) = (\mathrm{np}, (\widehat{s},0), 1)$ is the only point of $T_1S^3 \times \mathbf{R}_{>0}$ which maps to $(\mathrm{np}, (\widehat{s},0))$ in T^+S^3 under L. This proves the assertion. □

▷ Consider the 1-form $\langle r, \mathrm{d}s\rangle | T^+S^3$ on $T^+S^3 \subseteq T\mathbf{R}^4$ with coordinates (r,s). Then

$$L^*\big(\langle r, \mathrm{d}s\rangle | T^+S^3\big) = -\nu\big(\nu\,\mathrm{d}v_4 + \nu\langle v, \mathrm{d}u\rangle\big) | (T_1S^3 \times \mathbf{R}_{>0}). \tag{75}$$

(4.25) **Proof**: Using the definition of the mapping L (72) we get $\mathrm{d}(L^*s) =$

$$= \nu\,[(-\cos v_4\,\mathrm{d}v_4)u - \sin v_4\,\mathrm{d}u - (\sin v_4\,\mathrm{d}v_4)v + \cos v_4\,\mathrm{d}v] + \big(-\sin v_4\,u + \cos v_4, v\big)\,\mathrm{d}\nu.$$

So

$$\begin{aligned} L^*(\langle r, \mathrm{d}s\rangle) &= \langle L^*r, \mathrm{d}(L^*s)\rangle \\ &= \big(-\sin v_4\cos v_4\langle u,u\rangle + \cos^2 v_4\langle u,v\rangle - \sin^2 v_4\langle v,u\rangle + \sin v_4\cos v_4\langle v,v\rangle\big)\,\mathrm{d}\nu \\ &\qquad + \nu\big(-\cos^2 v_4\langle u,u\rangle - \cos v_4\sin v_4\langle u,v\rangle - \cos v_4\sin v_4\langle v,u\rangle - \sin^2 v_4\langle v,v\rangle\big)\,\mathrm{d}v_4 \\ &\qquad\quad - \cos v_4\sin v_4\langle u,\mathrm{d}u\rangle + \cos^2 v_4\langle u,\mathrm{d}v\rangle - \sin^2 v_4\langle v,\mathrm{d}u\rangle + \cos v_4\sin v_4\langle v,\mathrm{d}v\rangle \\ &= -\nu\,\mathrm{d}v_4 - \langle v, \mathrm{d}u\rangle. \end{aligned}$$

The last equality above follows because $\langle u,u\rangle = \langle v,v\rangle = 1$ and $\langle u,v\rangle = 0$, which implies $\langle u,\mathrm{d}u\rangle = \langle v,\mathrm{d}v\rangle = 0$ and $\langle u,\mathrm{d}v\rangle + \langle v,\mathrm{d}u\rangle = 0$. □

Corollary: L is a symplectic diffeomorphism sending $M = T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ to $T^+S^3_{\mathrm{np}}$ with $L^*\big((\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j) | T^+S^3_{\mathrm{np}}\big) = \mathrm{d}\theta$, where $\theta = \nu\big(\langle v, \mathrm{d}u\rangle + \mathrm{d}v_4\big) | M$.

(4.26) **Proof**: Take the exterior derivative of both sides of (75). □

Claim: The Ligon-Schaaf map $\Phi_{LS} : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T^+S^3_{\mathrm{np}} \subseteq T\mathbf{R}^4$ (61) has the following properties.

1. It is a symplectic diffeomorphism of $\big(\Sigma_-, (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i) | \Sigma_-\big)$ onto $\big(T^+S^3_{\mathrm{np}}, (\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j) | T^+S^3_{\mathrm{np}}\big)$.
2. It pulls back the *Delaunay Hamiltonian* $\mathscr{H} : T^+S^3_{\mathrm{np}} \subseteq T\mathbf{R}^4 \to \mathbf{R} : (r,s) \mapsto -\frac{1}{2}\|s\|^{-2}$ to the Kepler Hamiltonian $H : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \frac{1}{2}\|p\|^2 - \|q\|^{-1}$.
3. It pulls back the *Delaunay vector field* $X_{\mathscr{H}}$ on $T^+S^3_{\mathrm{np}}$, whose integral curves satisfy

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \|s\|^{-4}s \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= -\|s\|^{-2}r, \end{aligned} \tag{76}$$

and whose flow is

$$\varphi_t^{\mathscr{H}}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos \nu^{-3}t & \nu^{-1}\sin \nu^{-3}t \\ -\nu \sin \nu^{-3}t & \cos \nu^{-3}t \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} \tag{77}$$

with $\nu = \|s\|$, to the Kepler vector field X_H on Σ_-.

4. It intertwines the SO(3)-momentum mapping

$$J : T_0\mathbf{R}^3 \to \mathbf{R}^3 : (q,p) \mapsto p \times q, \tag{78}$$

with the SO(4)-momentum mapping

$$\mathscr{J} : T^+S^3 \subseteq T\mathbf{R}^4 \to \bigwedge^2 \mathbf{R}^4 : (r,s) \mapsto r \wedge s = \sum_{1 \le i < j \le 4} (r_i s_j - r_j s_i) e_i \wedge e_j, \tag{79}$$

that is, $J = \Phi_{LS}^* \mathscr{J}$. Here $\bigwedge^2 \mathbf{R}^4$ is identified with so(4) via the mapping which sends $e_i \wedge e_j$ to the 4×4 skew symmetric matrix e_{ij}, whose ij^{th} entry is 1, whose ji^{th} entry is -1, and whose other entries are 0.

(4.27) **Proof**:

1. As Φ_{LS} is the composition of the mappings $S : T_0\mathbf{R}^3 \to T_1S^3_{\mathrm{np}} \times \mathbf{R}_{>0}$ and $L : T_1S^3 \times \mathbf{R}_{>0} \to T^+S^3$, each of which are symplectic diffeomorphisms with $S^*(\mathrm{d}\theta) = (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|T_0\mathbf{R}^3$ ((4.18)) and $L^*\big((\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big) = \mathrm{d}\,\theta$ ((4.25)), it follows that Φ_{LS} is a symplectic diffeomorphism from $\big(T_0\mathbf{R}^3, (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|T_0\mathbf{R}^3\big)$ onto $\big(T^+S^3_{\mathrm{np}}, (\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big)$.

2. We compute. From the definition of the mapping S (63a) we have $S^*(-\frac{1}{2}\nu^{-2}) = -\frac{1}{2}\nu(q,p)^{-2} = H(q,p)$; while from the definition of the mapping L (63b) we have $L^*(-\frac{1}{2}\|s\|^{-2}) = -\frac{1}{2}\nu^{-2}$. Therefore $\Phi_{LS}^*\mathscr{H} = H$.

3. Since the Ligon-Schaaf mapping Φ_{LS} exhibits an equivalence between the Kepler Hamiltonian system $\big(H, T_0\mathbf{R}^3, (\sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i)|T_0\mathbf{R}^3\big)$ and the Delaunay Hamiltonian system $\big(\mathscr{H}, T^+S^3_{\mathrm{np}}, (\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|T^+S^3_{\mathrm{np}}\big)$, it pulls back the Delaunay vector field $X_{\mathscr{H}}$ on $T^+S^3_{\mathrm{np}}$ to the Kepler vector field X_H on Σ_-, that is, $\Phi_{LS}^* X_{\mathscr{H}} = X_H$. To find formula (76) for the Delaunay vector field $X_{\mathscr{H}}$ on T^+S^3, we look at the Hamiltonian system $\big(\widetilde{\mathscr{H}}, T\mathbf{R}^4, \sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j\big)$ with Hamiltonian $\widetilde{\mathscr{H}}(r,s) = -\frac{1}{2}\langle s,s\rangle^{-1}$ constrained to TS^3 with constraint functions $c_1(r,s) = \frac{1}{2}(\langle r,r\rangle - 1)$ and $c_2(r,s) = \langle r,s\rangle$. Since the matrix $(\{c_i,c_j\})$ of Poisson brackets is invertible with inverse given by $(C_{ij}) = \langle r,r\rangle^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the manifold TS^3 is cosymplectic with symplectic form $(\sum_{j=1}^4 \mathrm{d}r_j \wedge \mathrm{d}s_j)|TS^3$. We compute the constrained equations of motion using the modified Dirac bracket procedure. Let

$$H^* = \widetilde{\mathscr{H}} - \sum_{i,j=1}^{2} (\{\widetilde{\mathscr{H}}, c_i\} + \widetilde{\mathscr{H}_i})C_{ij}c_j,$$

where $\widetilde{\mathscr{H}_1} = \langle r,s\rangle(\langle s,s\rangle^{-2} - \frac{1}{2}\langle r,r\rangle)$, and $\widetilde{\mathscr{H}_2} = \langle s,s\rangle^{-1}(\langle r,r\rangle - 1)$. Then

$$\begin{aligned} H^* &= -\tfrac{1}{2}\langle s,s\rangle^{-1} - \langle r,r\rangle^{-1}\big\langle\big(-\langle r,r\rangle\langle s,s\rangle^{-1} + \widetilde{\mathscr{H}_1}, \langle s,s\rangle^{-1} + \widetilde{\mathscr{H}_2}\big), \big(-\langle r,s\rangle, \tfrac{1}{2}(\langle r,r\rangle - 1)\big)\big\rangle \\ &= -\tfrac{1}{2}\langle s,s\rangle^{-1} - \langle r,s\rangle^2 + \tfrac{1}{2}\langle s,s\rangle^{-1}(\langle r,r\rangle - 1). \end{aligned}$$

So

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\partial H^*}{\partial s} = \langle s,s\rangle^{-2} s - \langle r,s\rangle s + \langle s,s\rangle^{-1}(\langle r,r\rangle - 1) \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= -\frac{\partial H^*}{\partial r} = -\langle s,s\rangle^{-1} r + \langle r,s\rangle s. \end{aligned}$$

Therefore the Delaunay vector field $X_{\mathscr{H}} = X_{H^*}|T^+S^3$ has integral curves which satisfy (76). It is straightforward to verify that $\varphi_t^{\mathscr{H}}$ given by (77) is the flow of $X_{\mathscr{H}}$. Comparing (77) with the geodesic flow (8) in §1, one sees that the Delaunay flow is the same as the geodesic flow with time parameter $t = \nu^3 s$, where s is the geodesic time parameter.

4. Since

$$L^*(r \wedge s) = (\cos v_4\, u + \sin v_4\, v) \wedge (-\nu \sin v_4\, u + \nu \cos v_4\, v) = \nu\, u \wedge v,$$

we obtain $L^*\big((r \wedge s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) = (\nu\, u \wedge v)|K^{-1}(\tfrac{1}{2})$, see (59). Consequently,

$$\Phi^*\big((r \wedge s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) = S^*\big((\nu\, u \wedge v)|K^{-1}(\tfrac{1}{2})\big) = \big(\nu(q,p)J\big)|H^{-1}(-\tfrac{1}{2}) = J|H^{-1}(-\tfrac{1}{2}),$$

using $\nu(q,p) = 1$ if $(q,p) \in H^{-1}(-\tfrac{1}{2})$ and ((4.9)). We now use scaling to obtain the desired result. For every $d > 0$ we have $(r,s) \in \mathscr{H}^{-1}(-\tfrac{1}{2}d^{-2})$ if and only if $\|s\| = d$. Then $(r, d^{-1}s) \in \mathscr{H}^{-1}(-\tfrac{1}{2})$. So for every $d > 0$ we have

$$\begin{aligned} L^*\big((r \wedge s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) &= dL^*\big((r \wedge d^{-1}s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) \\ &= \big(d\nu(u \wedge d^{-1}v)\big)|K^{-1}(\tfrac{1}{2}) = \big(\nu(u \wedge v)\big)|K^{-1}(\tfrac{1}{2}d^2), \end{aligned}$$

So for every $d > 0$,

$$S^*L^*\big((r \wedge s)|\mathscr{H}^{-1}(-\tfrac{1}{2})\big) = \big(d^{-1}\nu(q,p)J\big)|H^{-1}(-\tfrac{1}{2}d^{-2}) = J|H^{-1}(-\tfrac{1}{2}d^{-2}),$$

which implies $\Phi_{LS}^*\mathscr{J} = J$ on Σ_- and thus $\Phi_{LS}^*\mathscr{J} = J$ on $T\mathbf{R}^4$. This follows because $\bigcup_{d>0} \mathscr{H}^{-1}(-\tfrac{1}{2}d^{-2}) = T^+S^3$, which is an open subset of $T\mathbf{R}^4$ and $\Sigma_- = \bigcup_{d>0} H^{-1}(-\tfrac{1}{2}d^{-2})$ is an open subset of $T_0\mathbf{R}^3$. Moreover, the components of the momentum mapping $\mathscr{J}$ and J are polynomials.

For every $1 \le i < j \le 4$ we have

$$\begin{aligned} L_{X_{\mathscr{H}}}(r_i s_j - r_j s_i) &= \dot{r}_i s_j + r_i \dot{s}_j - \dot{r}_j s_i - r_j \dot{s}_j \\ &= \|s\|^{-4} s_i s_j - \|s\|^{-2} r_i r_j - \|s\|^{-4} s_j s_i + \|s\|^{-2} r_j r_i, \quad \text{using (76)} \\ &= 0. \end{aligned}$$

Therefore the SO(4)-momentum mapping $\mathscr{J}$ is conserved by the flow of the Delaunay vector field $X_{\mathscr{H}}$. □

5 Exercises

1. (sl$(2,\mathbf{R})$ and the Kepler problem.) For the Kepler problem with rotationally symmetric Hamiltonian $H = \frac{1}{2} p \cdot p - \frac{1}{|q|}$ let $j = q \times p$, $x = q \cdot q$, $y = p \cdot p$, and $z = q \cdot p$.

   a) Show that the functions x, y, z Poisson commute with the components of j. Moreover, show that the Poisson brackets of x,y, and z define a representation of sl$(2,\mathbf{R})$. Conclude that so(3) and sl$(2,\mathbf{R})$ form a dual pair in the Lie algebra of homogeneous quadratic polynomials.

   b) In xyz-space draw that level sets of $j^2 = $ const. for different values of the constant including zero. These are models for the SO(3) reduced space.

   c) Draw the intersections of the $h = $ const. surfaces with a given $j = $ const. surface to see the integral curves of the reduced dynamics.

   d) Show that the level sets $j^2 = $ const. are symplectic leaves for the Poisson manifold sl$(2,\mathbf{R}) = \mathbf{R}^3$ with coordinates (x, y, z)

2. (Geodesics on a hyperboloid.) Consider $H^{3,1} = \{x \in \mathbf{R}^4 \,\big|\, \langle x, x \rangle = -1\}$, which is the set of vectors in $\mathbf{R}^4$ whose Lorentz length squared is -1. Here $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$ is the Lorentz inner product on $\mathbf{R}^4$. Geometrically, $H^{3,1}$ is a hyperboloid of two sheets. Its tangent bundle $TH^{3,1} = \{(x, y) \in T\mathbf{R}^4 \,\big|\, \langle x, x \rangle = -1 \;\&\; \langle x, y \rangle = 0\}$ is a symplectic manifold with symplectic form $\omega = \omega_4 | TH^{3,1}$. Here $\omega_4 = \sum_{i=1}^4 \mathrm{d}x_i \wedge \mathrm{d}y_i$ is the standard symplectic form on $T\mathbf{R}^4$.

   a) Consider the Hamiltonian system $(H, TH^{3,1}, \omega)$, where
   $$H : TH^{3,1} \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x, y) \mapsto \tfrac{1}{2} \langle y, y \rangle \tag{80}$$
   is the Hamiltonian. Show that $TH^{3,1}$ is an invariant manifold of the vector field on $T\mathbf{R}^4$ whose integral curves satisfy
   $$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \langle y, y \rangle\, x. \end{aligned} \tag{81}$$
   Show that an integral curve of (81), which starts on $T^+H^{3,1} = \{(x, y) \in TH^{3,1} \big| \langle y, y \rangle > 0\}$, is an integral curve of the Hamiltonian vector field X_H. Verify that the flow of X_H on $T^+H^{3,1}$ is given by
   $$\varphi^H : \mathbf{R} \times T^+H^{3,1} \to T^+H^{3,1} : \big(t, (x, y)\big) \mapsto \begin{pmatrix} \cosh t\sqrt{2H} & (\sinh t\sqrt{2H})/\sqrt{2H} \\ \sqrt{2H} \sinh t\sqrt{2H} & \cosh t\sqrt{2H} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
   Show that the image of an integral curve of X_H, under the bundle projection map $T^+H^{3,1} \subseteq T\mathbf{R}^4 \to H^{3,1} : (x, y) \mapsto x$, is a geodesic on $H^{3,1}$. Verify that every integral curve of X_H on $T^+H^{3,1}$ lies in the 2-plane in $\mathbf{R}^4$ spanned by its initial conditions.

   b) Let $\mathrm{O}(3,1) = \{O \in \mathrm{Gl}(4,\mathbf{R}) \,\big|\, \langle Ox, Oy \rangle = \langle x, y \rangle$ for all $x\ y \in \mathbf{R}^4\}$ be the Lorentz group. The Lie algebra of $\mathrm{O}(3,1)$ is $\mathrm{o}(3,1) = \{\xi \in \mathrm{gl}(4,\mathbf{R}) \,\big|\, \langle \xi x, y \rangle + \langle x, \xi y \rangle = 0$, for all $x, y \in \mathbf{R}^4\}$. Show that the Lie bracket on $\mathrm{o}(3,1)$ is given by
   $$[\xi, \eta] = \xi\,\eta - \eta\,\xi = \begin{pmatrix} i(\sigma \times \tau) + x \otimes y^t - y \otimes x^t & \sigma \times y - \tau \times x \\ (\sigma \times y - \tau \times x)^t & 0 \end{pmatrix},$$

where $\xi = \begin{pmatrix} i(\sigma) & x \\ x^t & 0 \end{pmatrix}$, $\eta = \begin{pmatrix} i(\tau) & y \\ y^t & 0 \end{pmatrix}$, and σ, τ, x, $y \in \mathbf{R}^3$. Here $i : \mathbf{R}^3 \to \mathrm{so}(3,\mathbf{R}) : x \mapsto \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$.

Verify that the $\mathrm{O}(3,1)$-action on $H^{3,1}$, given by $\varphi : \mathrm{O}(3,1) \times H^{3,1} \to H^{3,1} : (A,x) \mapsto Ax$, lifts to a Hamiltonian action $\Phi : \mathrm{O}(3,1) \times TH^{3,1} \to TH^{3,1} : \big(A,(x,y)\big) \mapsto (Ax, Ay)$ with coadjoint equivariant momentum mapping $J : TH^{3,1} \subseteq T\mathbf{R}^4 \to \mathrm{o}(3,1)^*$. Here $J(x,y)\xi = J^{\xi}(x,y)$ with $\xi \in \mathrm{o}(3,1)$ and

$$J^{\xi} : TH^{3,1} \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x,y) \mapsto \langle \xi x, y \rangle. \tag{82}$$

Observing that the Hamiltonian H (80) is invariant under the action Φ of $\mathrm{O}(3,1)$ on $TH^{3,1}$, deduce that J^{ξ} is an integral of X_H for every $\xi \in \mathrm{o}(3,1)$.

c) Define the mapping $\vartheta : \Lambda^2\mathbf{R}^4 \to \mathrm{o}(3,1) : v \wedge w \mapsto \ell_{v,w}$, where $\ell_{v,w} : \mathbf{R}^4 \to \mathbf{R}^4 : z \mapsto \langle z, v \rangle w - \langle z, w \rangle v$. Prove the following statements.

1. $\ell_{v,w} \in \mathrm{o}(3,1)$ for every v, $w \in \mathbf{R}^4$.
2. ϑ is a bijective real linear mapping.
3. Consider the action

$$\delta : \mathrm{O}(3,1) \times \Lambda^2\mathbf{R}^4 \to \Lambda^2\mathbf{R}^4 : \big(O, v \wedge w\big) \mapsto Ov \wedge Ow. \tag{83}$$

The mapping ϑ intertwines the action δ with the adjoint action of $\mathrm{O}(3,1)$ on $\mathrm{o}(3,1)$, that is,

$$\vartheta \circ \delta_O = \mathrm{Ad}_O \circ \vartheta = O \circ \vartheta \circ O^{-1} \tag{84}$$

for every $O \in \mathrm{O}(3,1)$.

d) With $\langle x, y \rangle = \sum_{i=1}^{3} x_i y_i - x_4 y_4$ for every x, $y \in \mathbf{R}^4$ prove the identity

$$\langle x,x \rangle \langle y,y \rangle - \langle x,y \rangle^2 = \sum_{1 \le i < j \le 3} (x_i y_j - x_j y_i)^2 - \sum_{i=1}^{4} (x_i y_4 - x_4 y_i)^2. \tag{85}$$

Let

$$\mathrm{K} : \Lambda^2\mathbf{R}^4 \times \Lambda^2\mathbf{R}^4 \to \mathbf{R} : (v \wedge w, x \wedge y) \mapsto \det \begin{pmatrix} \langle v,x \rangle & \langle w,x \rangle \\ \langle w,x \rangle & \langle w,y \rangle \end{pmatrix}. \tag{86}$$

Show that K is a nondegenerate inner product on $\Lambda^2\mathbf{R}^4$ with $\{e_\ell \wedge e_k\}_{1 \le \ell < k \le 4}$ being an orthonormal basis with respect to which the matrix of K is diagonal. Show that the Morse index of K is 3. Verify that K is invariant under the action δ (83). Let $\mathrm{k} : \mathrm{o}(3,1) \times \mathrm{o}(3,1) \to \mathbf{R} : (\xi, \eta) \mapsto -\frac{1}{2}\mathrm{tr}\, \xi\eta$. Show that k is a nondegenerate inner product on $\mathrm{o}(3,1)$, which is invariant under the adjoint action Ad. With $\xi = \begin{pmatrix} i(\sigma) & x \\ x^t & 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} i(\tau) & y \\ y^t & 0 \end{pmatrix}$, where σ, τ, x, $y \in \mathbf{R}^3$, show that $\mathrm{k}(\xi, \eta) = (\sigma, \tau) - (x, y)$. Here $(\ ,\ )$ is the Euclidean inner product on $\mathbf{R}^3$. Verify that $\vartheta^*\mathrm{k} = \mathrm{K}$.

e) Let $\widetilde{J} : TH^{3,1} \subseteq T\mathbf{R}^4 \rightarrow \Lambda^2\mathbf{R}^4$ be the mapping $K^\flat \circ \vartheta^t \circ J$. Show that for every $(x,y) \in TH^{3,1}$

$$\widetilde{J}(x,y) = x \wedge y = \sum_{1 \le i < j \le 4} K(x \wedge y, e_i \wedge e_j)\, e_i \wedge e_j = \sum_{1 \le i < j \le 4} T_{ij}(x,y)\, e_i \wedge e_j, \quad (87)$$

where $\{e_i\}_{i=1}^4$ is the standard basis of $\mathbf{R}^4$ and $T_{ij}(x,y) = x_i y_j - x_j y_i$. Deduce that $\widetilde{J}$ intertwines the $\mathrm{O}(3,1)$-action Φ on $TH^{3,1}$ with the $\mathrm{O}(3,1)$-action δ on $\Lambda^2\mathbf{R}^4$. Let

$$T_{\sqrt{2h}}H^{3,1} = \{(x,y) \in T^+H^{3,1} \big| \langle x,x \rangle = -1,\ \langle x,y \rangle = 0, \text{ and } \langle y,y \rangle = 2h > 0\}$$

be the bundle of tangent vectors to $H^{3,1}$, whose squared Lorentz length is $2h > 0$. Then $T_{\sqrt{2h}}H^{3,1} = H^{-1}(h)$. Show that $\mathrm{O}(3,1)$ acts transitively on $T_{\sqrt{2h}}H^{3,1}$. Using Plücker coordinates $\{T_{ij}\}_{1 \le i < j \le 4}$ on $\Lambda^2(\mathbf{R}^4)$, show that the image of $T_{\sqrt{2h}}H^{3,1}$ under the mapping $\widetilde{J}$ (87) is the smooth submanifold V_h of $\Lambda^2(\mathbf{R}^4)$ defined by

$$\begin{aligned} T_{12}T_{34} - T_{13}T_{24} + T_{23}T_{14} &= 0 \\ T_{12}^2 + T_{13}^2 + T_{23}^2 - T_{34}^2 - T_{24}^2 - T_{14}^2 &= -2h \end{aligned} \quad (88)$$

Deduce that V_h is diffeomorphic to TS^2. Hint: use the diffeomorphism

$$V_h \subseteq \Lambda^2\mathbf{R}^4 \rightarrow \mathbf{R}^6 : (T_{ij})_{1 \le i < j \le 4} \mapsto \big(T_{12}, T_{13}, T_{23}, T_{34}X^{-1/2}, -T_{24}X^{-1/2}, T_{14}X^{-1/2}\big),$$

where $X = 2h + T_{12}^2 + T_{13}^2 + T_{23}^2 > 0$, since $2h > 0$. Show that $\mathrm{O}(3,1)$ acts transitively on V_h and that V_h is the space of orbits of the geodesic flow on $T^+H^{3,1}$ of energy $h > 0$.

f) Show that V_h is a symplectic manifold. For $u = e_4 \wedge \sqrt{2h}e_1 \in V_h$ let $\mu = \vartheta(u)$. The adjoint orbit $\mathscr{O}_\mu = \{\nu = \mathrm{Ad}_O \mu \in \mathrm{o}(3,1) \big| O \in \mathrm{O}(3,1)\}$ of $\mathrm{O}(3,1)$ through μ is a symplectic manifold with symplectic form $\omega_\nu(\mathrm{ad}_\nu \xi, \mathrm{ad}_\nu \eta) = \mathrm{k}(\nu, [\xi, \eta])$. Since $\vartheta|V_h : V_h \subseteq \Lambda^2(\mathbf{R}^4) \rightarrow \mathscr{O}_\mu \subseteq \mathrm{o}(3,1)$ is a diffeomorphism, $\Omega_h = (\vartheta|V_h)^*\omega_{\mathscr{O}_\mu}$ is a symplectic form on V_h.

We now find an explicit expression for the symplectic form Ω_h. For every $v \in V_h$ show that $\xi_v = T_e\delta_v\xi \in T_vV_h$ for every $\xi \in \mathrm{o}(3,1)$. In fact, $T_vV_h = \mathrm{span}_{\mathbf{R}}\{\xi_v \big| \xi \in \mathrm{o}(3,1)\}$. Infinitesimalizing (84) show that at every $v \in V_h$ we have $\mathrm{ad}_\xi \vartheta(v) = T_v\vartheta\xi_v$ for every $\xi \in \mathrm{o}(3,1)$. We have $\delta_O^*\Omega_h = \Omega_h$ for every $O \in \mathrm{O}(3,1)$. To see this justify each step of the following calculation.

$$\begin{aligned} \delta_O^*\Omega_h(u)(\xi_u, \eta_u) &= \Omega_h(\delta_O(u))(T_u\delta_O\xi_u, T_u\delta_O\eta_u) \\ &= \omega_\mu(\vartheta \circ \delta_O(u))(T_{Ou}\vartheta\, T_u\delta_O\xi_u, T_{Ou}\vartheta\, T_u\delta_O\eta_u) \\ &= \omega_\mu(\mathrm{Ad}_O\vartheta(u))\big(T_{\vartheta(u)}\mathrm{Ad}_O(\mathrm{ad}_\xi \vartheta(u)), T_{\vartheta(u)}\mathrm{Ad}_O(\mathrm{ad}_\xi \vartheta(u))\big) \\ &= \omega_\mu(\mathrm{Ad}_O\vartheta(u))\big(T_{\vartheta(u)}\mathrm{ad}_{\mathrm{Ad}_O\xi}\mathrm{Ad}_O\vartheta(u)), T_{\vartheta(u)}\mathrm{ad}_{\mathrm{Ad}_O\eta}\mathrm{Ad}_O\vartheta(u)) \\ &= \mathrm{k}(\mathrm{Ad}_O\vartheta(u), [\mathrm{Ad}_O\xi, \mathrm{Ad}_O\eta]) = \mathrm{k}(\mathrm{Ad}_O\vartheta(u), \mathrm{Ad}_O[\xi, \eta]) \\ &= \mathrm{k}(\vartheta(u), [\xi, \eta]) = \Omega_h(u)(\xi_u, \eta_u). \end{aligned}$$

Write $v = \delta_O u$. Then $T_u\delta_O\xi_u = \xi_v \in T_vV_h$. The above calculation shows that

$$\Omega_h(v)(\xi_v, \eta_v) = \mathrm{k}(\vartheta(u), [\xi, \eta]). \quad (89)$$

To make (89) explicit, show that

$$\xi_u = \xi e_4 \wedge \sqrt{2h} e_1 + e_4 \wedge \sqrt{2h}\,\xi e_1 = (-x_2, -x_3, 0, \sigma_2, -\sigma_3, 0) \in T_u V_h,$$

where $\xi = \begin{pmatrix} i(\sigma) & x \\ x^t & 0 \end{pmatrix}$, with σ, $x \in \mathbf{R}^3$ and we use $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4,$ $e_2 \wedge e_4, e_1 \wedge e_4\}$ as a basis for $\Lambda^2(\mathbf{R}^4)$. Let $\eta = \begin{pmatrix} i(\tau) & y \\ y^t & 0 \end{pmatrix}$, where τ, $y \in \mathbf{R}^3$. Note that $\vartheta(u) = -\sqrt{2h}\begin{pmatrix} 0 & e_1 \\ (e_1)^t & 0 \end{pmatrix}$. Justify each step of the following calculation.

$$\begin{aligned}
\mathrm{k}(\vartheta(u), [\xi, \eta]) &= -\tfrac{1}{2}\mathrm{tr}\big(\vartheta(u)\,[\xi, \eta]\big) \\
&= \tfrac{1}{2}\sqrt{2h}\,\mathrm{tr}\begin{pmatrix} 0 & e_1 \\ (e_1)^t & 0 \end{pmatrix}\begin{pmatrix} i(\sigma \times \tau) + x \otimes y^t - y \otimes x^t & \sigma \times y - \tau \times x \\ (\sigma \times y - \tau \times x)^t & 0 \end{pmatrix} \\
&= \tfrac{1}{2}\sqrt{2h}\,\mathrm{tr}\begin{pmatrix} e_1 \otimes (\sigma \times y - \tau \times x)^t & * \\ * & (e_1)^t(\sigma \times y - \tau \times x) \end{pmatrix} \\
&= \sqrt{2h}(\sigma \times y - \tau \times x)_1 = \sqrt{2h}(\sigma_2 y_3 - \sigma_3 y_2 - \tau_2 x_3 + \tau_3 x_2).
\end{aligned}$$

So

$$\begin{aligned}
\Omega_h(Ou)\big(T_u\delta_O(-x_2, -x_3, 0, \sigma_2, -\sigma_3, 0), T_u\delta_O(-y_2, -y_3, 0, \tau_2, -\tau_3, 0)\big) &= \\
= \sqrt{2h}(\sigma_2 y_3 - \sigma_3 y_2 - \tau_2 x_3 + \tau_3 x_2).&
\end{aligned}$$

3. (Positive energy Keplerian orbits.) This exercise deals with Keplerian orbits of positive energy. Specifically we discuss the changes that need to be made to the treatment of the Kepler problem in §3.2.

   a) First check that the arguments establishing the equation

$$\|q\| = \mu^{-1} J^2 (1 + e\cos f)^{-1} \tag{90}$$

   for the Keplerian orbit with angular momentum $\mathbf{J}$ and eccentricity vector $\mathbf{e}$ as well as the equation

$$e^2 = 1 + 2\mu^{-2} J^2 h \tag{91}$$

   for the magnitude squared of the eccentricity vector continue to hold for positive energy h. When $h > 0$ from (91) it follows that $e > 1$. Deduce that the Keplerian orbit (90) is one branch of a hyperbola. For (90) to hold show that $|f| < f_0 = \pi - \cos^{-1} e^{-1}$. Thus $(\cos f_0, \pm\sin f_0)$ are the directions of the asymptotes of the branch of the hyperbola. From (91) and the facts that $\langle q, \mathbf{J}\rangle = 0$ and $\langle p, \mathbf{J}\rangle = 0$ deduce that a Keplerian orbit of positive energy is a hyperbola, which lies in a 2-plane Π, which is perpendicular to $\mathbf{J}$. Show that $\{\mathbf{e}, \mathbf{J} \times \mathbf{e}\}$ is an orthogonal basis of Π. Let C be the center of the hyperbola, which is the origin of the $\mathbf{e}$-$(\mathbf{J} \times \mathbf{e})$ coordinate system. Let O be the center of attraction, which is a focus of the hyperbola. Show that the periapse A of the hyperbola lies on the $\mathbf{e}$-axis between C and O and that the major semi-axis of the hyperbola OA has length $a = J^2\mu^{-1}(e^2 - 1)^{-1} = \mu(2h)^{-1}$. For $u \in \mathbf{R}$ let $P = (a\cosh u, b\sinh u)$ be a point on the hyperbola. Let $\overline{OP} = \|q\|$ with f the true anomaly of P, that is, f is the angle between $\mathbf{e}$ and the line segment OP. Show that $\overline{CO} = ae = a\cosh u + \|q\|\cos f$. Deduce that the equation of the hyperbolic Keplerian orbit (90) can be written as $\frac{1}{a}\|q\| = e\cosh u - 1$. The minor semi-axis of the hyperbola lies on the $(\mathbf{J}\times\mathbf{e})$-axis. Show that its length is $b = a\sqrt{e^2-1}$ and that $\|q\|\sin f = b\sinh u$.

b) We now determine the analogue of Kepler's equation for a hyperbolic orbit. First we use

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\sqrt{2h}}{\|q\|} \tag{92}$$

to define the eccentric anomaly s. Following the derivation of equation (43) in §3.3 show that

$$\left(\frac{\mathrm{d}\|q\|}{\mathrm{d}s}\right)^2 + a^2(e^2-1) = 2a\|q\| + \|q\|^2$$

with $q(0) = a(e-1)$. Using the change of variable $ae\rho = \|q\| + a$, show that the above equation becomes

$$-\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + \rho^2 = 1$$

with $\rho(0) = 1$. Integrating, gives $\rho(s) = \cosh s$. Hence $\|q(s)\| = ae\cosh s - a$, which substituted into (92) and integrating gives the hyperbolic analogue of Kepler's equation

$$n\ell = e\sinh s - s, \tag{93}$$

where $n = \sqrt{2h}\,\mu^{-1} = \mu^{1/2}a^{-3/2}$ is the mean motion and $\ell = t-\tau$ is the mean anomaly. Here τ is the time at the passage of the periapse.

4. (Hamilton's theorem.) Hamilton's theorem states that the velocity of a particle of mass m subject to an attractive central force with potential $U(|\mathbf{x}|) = -k\frac{1}{|\mathbf{x}|}$, $k>0$ moves on a circle $\mathscr{C}$, which uniquely determines its Keplerian orbit. Here $|\mathbf{x}|$ is the length of a vector $\mathbf{x}\in\mathbf{R}^3\setminus\{0\}$ using the Euclidean inner product $\langle\ ,\ \rangle$. Assume that the conserved angular momentum $\mathbf{J} = \mathbf{x}\times m\mathbf{v}$ of the particle is nonzero. The argument outlined in sections a) – c) gives a proof of Hamilton's theorem.

a) Show that the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ of the particle at time t lies in a plane Π, which is perpendicular to $\mathbf{J}$, which we can assume to be the vector $(0,0,j)$, where $j = |\mathbf{J}| > 0$. Using polar coordinates (r,θ) in Π, show that $j = r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}$. Deduce that $\frac{\mathrm{d}\theta}{\mathrm{d}t} > 0$. Consequently, we can reparametrize the curves $t\mapsto\mathbf{x}(t)$ and $t\mapsto\mathbf{v}(t)$ using θ instead of t. Show that this reparametrization preserves the original positive of orientation of these curves given by increasing t.

b) Rewrite Newton's equations of motion

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -k\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{94}$$

using polar coordinates on Π and change the parametrization of the velocity in (94) to θ. Show that we obtain

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\theta} = (R\cos\theta, R\sin\theta, 0), \tag{95}$$

where $R = k/jm$. Integrating (95) gives

$$\mathbf{v}(\theta) = (-R\sin\theta, R\cos\theta, 0) + \mathbf{c}. \tag{96}$$

Deduce that $\mathbf{v}(\theta) - \mathbf{c}$ moves on a circle $\mathscr{C}$ in Π with center at $\mathbf{c}$ and radius R.

c) Choose coordinates on Π so that $\mathbf{c} = (0, c, 0)$, where $c = |\mathbf{c}| \ge 0$. Let $e = c/R$. Then $\mathbf{v}(\theta) - \mathbf{c} = \big(-R\sin\theta, R(e+\cos\theta), 0\big)$. Using

$$j = \langle J, (0,0,1)\rangle = \langle \mathbf{x}(\theta) \times m\mathbf{v}(\theta), (0,0,1)\rangle,$$

where $\mathbf{x}(\theta) = \big(r(\theta)\cos\theta, r(\theta)\sin\theta, 0\big)$, deduce that

$$r = r(\theta) = \Lambda(1 + e\cos\theta)^{-1}, \tag{97}$$

where $\Lambda = j/mR = j^2/k$. Equation (97) describes a conic section of eccentricity e with a focus at $O = (0,0,0)$.

d) When $0 \le e < 1$, show that $\theta \mapsto \mathbf{v}(\theta) - \mathbf{c}$ traces out the full velocity circle $\mathscr{C}$.

e) When $e > 1$ equation (97) describes a branch of a hyperbola. The following argument shows that $\theta \mapsto \mathbf{v}(\theta) - \mathbf{c}$ traces out a positively oriented arc of $\mathscr{C}$. This arc subtends a positive angle Θ, which is equal to the scattering angle of the hyperbola. Because $e > 1$ for equation (97) to hold $|\theta| < \theta_0 = \pi - \theta_*$, where $\theta_* = \cos^{-1} e^{-1}$. Using conservation of energy show that

$$|\mathbf{v}|^2 = \frac{2h}{m} + \frac{k}{m^2}\frac{1}{|\mathbf{x}|} > \frac{2h}{m}.$$

Hence the velocity of the particle lies outside of the closed 2-disk with center at O and radius $\sqrt{\frac{2h}{m}}$. Show that $\mathbf{v}(\theta) - \mathbf{c}$ lies on the velocity circle $\mathscr{C}$ and the energy circle $\partial\mathscr{E}$, given by $|\mathbf{v}| = \sqrt{\frac{2h}{m}}$, if and only if $0 = \frac{1}{r(\theta)} = \Lambda^{-1}(1 + e\cos\theta)$, that is, if and only if $\theta = \pm\theta_0 = \pm(\pi - \theta_*)$. Show that the velocity vectors $\mathbf{v}(\pm\theta_0) - \mathbf{c}$ are the end points of a closed arc $\mathscr{A}$ on $\mathscr{C}$ and that

$$\mathbf{v}(\pm\theta_0) - \mathbf{c} = \big(-R\sin(\pm\theta_0), R\cos(\pm\theta_0), 0\big) = (\mp R\sin\theta_*, -R\cos\theta_*, 0). \tag{98}$$

From (98) deduce that $\mathbf{v}(\theta_0) - \mathbf{c}$ lies the 3$^{\text{rd}}$ quadrant of Π; while $\mathbf{v}(-\theta_0) - \mathbf{c}$ lies the 4$^{\text{th}}$ quadrant of Π. Deduce that the positive arc $\mathscr{A}$, oriented so that θ increases, has an initial end point at $\mathbf{v}(-\theta_0) - \mathbf{c}$ and a final end point at $\mathbf{v}(\theta_0) - \mathbf{c}$. Show that

$$\mathbf{v}(\theta_0) = \big(R\cos(\tfrac{3}{2}\pi - \theta_*), R\sin(\tfrac{3}{2}\pi - \theta_*), 0\big);$$

while

$$\mathbf{v}(-\theta_0) = \big(R\cos(-(\tfrac{1}{2}\pi - \theta_*)), R\sin(-(\tfrac{1}{2}\pi - \theta_*)), 0\big).$$

Thus the positive angle Θ subtended by the positive arc $\mathscr{A}$ is equal to $2(\pi - \theta_*)$. When $-\theta_0 = -(\pi - \theta_*)$, then $\mathbf{d}_{-\theta_0} = \frac{\mathbf{x}(-\theta_0)}{|\mathbf{x}(-\theta_0)|}$ is the direction of the incoming asymptote of the branch of the hyperbola with center C at $(ae, 0, 0)$ in Π; while when $\theta_0 = \pi - \theta_*$, then $\mathbf{d}_{\theta_0} = \frac{\mathbf{x}(\theta_0)}{|\mathbf{x}(\theta_0)|}$ is the direction of the outgoing asymptote of the branch of the hyperbola. By definition, the scattering angle Ψ of the hyperbolic motion is the counterclockwise rotation about C, which sends $\mathbf{d}_{-\theta_0}$ into $\mathbf{d}_{\theta_0}$. Show that $\Psi = 2(\pi - \theta_*) = \Theta$.

5. (Regularization of positive energy Keplerian orbits.) Let $(H, T_0\mathbf{R}^3, \omega_3)$ be the Kepler Hamiltonian system with $T_0\mathbf{R}^3 = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3$ having coordinates (q,p), symplectic form $\omega_3 = \sum_{i=1}^3 \mathrm{d}q_i \wedge \mathrm{d}p_i$, and Hamiltonian

$$H : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \tfrac{1}{2}(p,p) - \|q\|^{-1}. \tag{99}$$

Here $(\ ,\ )$ is the Euclidean inner product on $\mathbf{R}^3$ with $\|q\|$ being the length of the vector $q \in \mathbf{R}^3$. We look only at a positive energy Keplerian orbit, which in exercise 3 we have shown to be a branch of a hyperbola.

a) To regularize the positive energy Keplerian orbits, we will use an argument analogous to the one given in §4 for the negative energy orbits. Start by using the virial group to show that we may reduce our considerations to the level set $H^{-1}(\frac{1}{2})$. Next introduce a new time scale s by $\frac{\mathrm{d}s}{\mathrm{d}t} = \|q\|^{-1}$. Consider the rescaled Hamiltonian

$$\widetilde{F} : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \|q\|\big(H(q,p) - \tfrac{1}{2}\big) + 1 = \tfrac{1}{2}\|q\|(\|p\|^2 - 1). \tag{100}$$

Show that the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$ are the same as the integral curves of X_H on $H^{-1}(\frac{1}{2})$, using the time parameter s. Let

$$\widetilde{K} : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \mapsto \tfrac{1}{2}\widetilde{F}^2(q,p) = \tfrac{1}{8}\|q\|^2(\|p\|^2 - 1)^2 \tag{101}$$

be the regularized Hamiltonian. Show that the integral curves of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(\frac{1}{2})$ are the same as the integral curves of $X_{\widetilde{F}}$ on $\widetilde{F}^{-1}(1)$, using the time parameter s.

b) Let $\langle\ ,\ \rangle$ be the Lorentz inner product on $\mathbf{R}^4$ given by $\langle u,v\rangle = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4$. Let $H^{3,1} = \{u \in \mathbf{R}^3 \big| \langle u,u\rangle = -1\}$. Consider the stereographic projection map

$$\varphi^{-1} : H^{3,1} \subseteq \mathbf{R}^4 \to \mathbf{R}^3 : q \mapsto (1-q_4)^{-1}\widetilde{q} = (1-q_4)^{-1}(q_1,q_2,q_3)$$

from $(\widetilde{0},1)$ with inverse

$$\varphi : \mathbf{R}^3 \to H^{3,1} \subseteq \mathbf{R}^4 : \widetilde{q} \mapsto 2(1-\|\widetilde{q}\|^2)^{-1}\big(\widetilde{q}, -\tfrac{1}{2}(1+\|\widetilde{q}\|^2)\big).$$

The positive energy analogue of Moser's regularization map in §4 is

$$\begin{aligned}\Phi_M^{-1} : TH^{3,1} \subseteq T\mathbf{R}^4 \to T_0\mathbf{R}^3 :&\\ (u,v) \mapsto (q,p) = \big(-(1-u_4)\widetilde{v} - v_4\widetilde{u}, (1-u_4)^{-1}\widetilde{u}\big),&\end{aligned} \tag{102}$$

which is the composition of $T\varphi^{-1}$ followed by momentum reversal $(q,p) \mapsto (-p,q)$. For $(u,v) \in TH^{3,1} = \{(u,v) \in T\mathbf{R}^4 \big| \langle u,u\rangle = -1 \ \&\ \langle u,v\rangle = 0\}$ show that the following identities hold.

$$\|q\|^2 = \langle v,v\rangle(1-u_4)^2 \tag{103a}$$

$$\|p\|^2 - 1 = -2(1-u_4)^{-1} \tag{103b}$$

$$(q,p) = v_4. \tag{103c}$$

Using the above identities show that the inverse of the regularization mapping Φ_M^{-1} is given by

$$\Phi_M : T_0\mathbf{R}^3 \to TH^{3,1} \subseteq T\mathbf{R}^4 : (q,p) \mapsto \big((\widetilde{u},u_4),(\widetilde{v},v_4)\big),$$

where

$$\begin{cases} \widetilde{u} &= -(\|p\|^2-1)^{-1}(2p) \quad \text{and} \quad u_4 = (\|p\|^2-1)^{-1}(\|p\|^2+1) \\ \widetilde{v} &= \frac{1}{2}(\|p\|^2-1)q-(q,p)p \quad \text{and} \quad v_4 = (q,p). \end{cases} \tag{104}$$

c) Verify that the pull back by the regularization mapping Φ_M^{-1} (102) of the regularized Hamiltonian $\widetilde{K}$ (101) is the geodesic Hamiltonian

$$\mathscr{H} : TH^{3,1} \subseteq \mathbf{R}^4 \to \mathbf{R} : (u,v) \mapsto \tfrac{1}{2}\langle v,v\rangle. \tag{105}$$

Show that $(\Phi_M^{-1})^*\omega_3 = \omega_4|TH^{3,1}$. Deduce that the flow of the regularized Kepler vector field $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(\frac{1}{2})$ is the flow of the geodesic Hamiltonian vector field $X_{\mathscr{H}}$ on $\mathscr{H}^{-1}(\frac{1}{2})$.

d) Following the proof of ((4.9)) show that

$$\Phi_M^*\big((u_iv_j-v_iu_j)|\mathscr{H}^{-1}(\tfrac{1}{2})\big) = J_k|H^{-1}(\tfrac{1}{2}),$$

where $(i,j,k)=\{1,2,3\}$, and

$$\Phi_M^*\big((u_iv_4-v_iu_4)|\mathscr{H}^{-1}(\tfrac{1}{2})\big) = e_i|H^{-1}(\tfrac{1}{2}),$$

for $1 \le i \le 3$.

6. (Center of mass and the two body problem.)

   a) For the two body problem in space show that regular reduction by the translation group can be interpreted as passing to a center of mass frame. Do the reduction of the translation and rotational symmetries in one step by using the Euclidean group $E(3)$.

   b) Consider the spherical analogue of the planar two body problem. This is the motion of two particles connected by a spring constrained to move on the surface of a 2-sphere. The rotation group $\mathrm{SO}(3)$ is an obvious symmetry group of the problem, as compared to the Euclidean group $E(2)$ for the planar problem. Construct all the $\mathrm{SO}(3)$ reduced spaces. Show that there is *no notion* of a center of mass *frame*.

   c)* Is the spherical two body problem Liouville integrable?

7. a) Construct an isomorphism between the Lie algebra $\mathrm{so}(4)$ and $\mathrm{so}(3)\times\mathrm{so}(3)$.

   b) Show that the corresponding Lie-Poisson algebras are isomorphic.

   c) Write out Hamilton's equations on the Lie-Poisson algebra corresponding to $\mathrm{so}(3)\times\mathrm{so}(3)$.

8. (Souriau's linearization and regularization.) In the Kepler problem

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\tfrac{1}{r^3} q, \quad r = \|q\|, \end{aligned} \tag{106}$$

let $H = \frac{1}{2}\langle p, p\rangle - \frac{1}{r}$ be the Hamiltonian and define a new time variable s by

$$s = \langle q, p\rangle - 2ht.$$

a) Show that $\frac{ds}{dt} = \frac{1}{r}$. Thus s is the eccentric anomaly.

b) Define a 4-vector ξ by $\xi = \begin{pmatrix} t \\ q \end{pmatrix}$. Let $\Xi = \mathrm{col}(\xi, \xi', \xi'', \xi''')$ be a 4×4 matrix, where $'$ is differentiation with respect to s. Show that Ξ satisfies the *linear* differential equation

$$\Xi' = A\,\Xi \tag{107}$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2h & 0 & 0 \end{pmatrix}$.

c) Solve (107) and thus find $\xi(s)$. Note that because $\xi(s)$ is defined for *all* s and hence for all t by Kepler's equation, it follows that the Kepler problem has been regularized.

9. (Bacry-Györgyi variables and the conformal group.) Using the same notation in the Kepler problem as in exercise 8, set $\alpha = \sqrt{-2h}$, $P = \begin{pmatrix} t''' \\ \alpha^{-1} q''' \end{pmatrix}$ and $Q = \begin{pmatrix} \alpha t'' \\ q'' \end{pmatrix}$. Here we are confining ourselves to the case of bounded motions, namely, $h < 0$.

a) Show that $P^t P = Q^t Q = 1$ and $P^t Q = 0$.

b) Let ζ be the 6×6 matrix

$$\begin{pmatrix} QP^t - PQ^t & P & Q \\ P^t & 0 & 1 \\ Q^t & -1 & 0 \end{pmatrix}$$

c) Show that $\zeta^2 = 0$.

d) Show that the components of ζ satisfy the Poisson bracket relations for the Lie algebra $\mathrm{so}(4,2)$.

e) Show that the map from the regularized phase space of the negative energy orbits $(q, p, h) \to \zeta$ is a symplectic diffeomorphism if we equip the $\mathrm{SO}(4,2)$-coadjoint orbit through ζ with the symplectic structure given in chapter VI §2 example 3. The tricky part of this is deciding which component of the variety $\zeta^2 = 0, \zeta \neq 0$ in $\mathrm{so}(4,2)^*$ you need to map to.

10. (Levi-Civita regularization.)

a) Let $\mathbf{R}_0^2 = \mathbf{R}^2 - \{0\}$. On $T^*\mathbf{R}_0^2 = \mathbf{R}_0^2 \times \mathbf{R}^2$ with coordinates (x, y) and symplectic form $\omega = \sum_{i=1}^3 dx_i \wedge dy_i$ consider the Kepler Hamiltonian

$$H(x, y) = \tfrac{1}{2}(y_1^2 + y_2^2) - (x_1^2 + x_2^2)^{-1/2}. \tag{108}$$

Identify $\mathbf{R}_0^2$ with $\mathbf{C}_0 = \mathbf{C} - \{0\}$ and $T^*\mathbf{R}_0^2$ with $T^*\mathbf{C}_0 = \mathbf{C}_0 \times \mathbf{C}$. Introduce complex coordinates $q = x_1 + i x_2$ and $p = y_1 + i y_2$ on $T^*\mathbf{C}_0$. Show that $\omega = \mathrm{Re}(dq \wedge d\overline{p})$ and that the Kepler Hamiltonian becomes

$$H(q,p) = \tfrac{1}{2}\|p\|^2 - \|q\|^{-1}. \tag{109}$$

b) Using the time rescaling $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{k}{2|q|}$ show that the integral curves of X_H on the level set $H^{-1}(-k^2/2)$ are a time reparametrization of the integral curves of the vector field $X_{\widetilde{K}}$ on the level set $\widetilde{K}^{-1}(0)$ where

$$\widetilde{K}(q,p) = 2\,\|q\| k^{-1}\left(\tfrac{1}{2}\|p\|^2 - \|q\|^{-1} + \tfrac{1}{2}k^2\right) = k^{-1}\,\|q\|\,\|p\|^2 + k\,\|q\| - 2k^{-1}. \tag{110}$$

c) Define the Levi-Civita map

$$\mathscr{L} : T^*\mathbf{C}_0 \to T^*\mathbf{C}_0 : (u,v) \to (q,p) = \left((2k)^{-1}u^2,\, k v \overline{u}^{-1}\right). \tag{111}$$

Show that $\mathscr{L}$ has the following properties:

1) $\mathscr{L}$ is a smooth two to one surjective submersion with $\mathscr{L}(-u,-v) = \mathscr{L}(u,v)$.

2) $\mathscr{L}^*\left(\mathrm{Re}(q\,\mathrm{d}\overline{p})\right) = \mathrm{Re}(u\,\mathrm{d}\overline{v} - \overline{v}\,\mathrm{d}u)$. Hence $\mathscr{L}$ is symplectic.

3) The Hamiltonian

$$K(u,v) = (\mathscr{L}^*\widetilde{K})(u,v) = \tfrac{1}{2}(|v|^2 + |u|^2) - 2k^{-1} \tag{112}$$

is defined on $K^{-1}(0)$ which is a 3-sphere centered at the origin and having radius $2/\sqrt{k}$. Since $K^{-1}(0)$ is compact, all the integral curves of X_K on $K^{-1}(0)$ are defined for *all* time. Thus K is the Levi-Civita regularization of the Kepler Hamiltonian for negative energy orbits. Note that up to an additive constant, K is the harmonic oscillator Hamiltonian.

d) The Levi-Civita map $\mathscr{L}$ is *not* an equivalence between the Hamiltonian systems $(K, T^*\mathbf{C}_0, \omega)$ and $(\widetilde{K}, T^*\mathbf{C}_0, \omega)$, because it is *not* a diffeomorphism. Show that that vector fields X_K on $K^{-1}(0)$ and $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(0)$ are $\mathscr{L}$-related, that is, $T\mathscr{L} \circ X_K = X_{\widetilde{K}} \circ \mathscr{L}$. Thus the image of an integral curve of X_K on $K^{-1}(0)$ under the Levi-Civita map $\mathscr{L}$ is an integral curve of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(0)$.

e) On $T^*\mathbf{C}_0$ define a $\mathbf{Z}_2$-action generated by $(u,v) \to (-u,-v)$. Show that this action is free, preserves the symplectic form ω, and preserves the Hamiltonian K. Thus there is an induced Hamiltonian $\mathscr{K}$ on $(T^*\mathbf{C}_0/\mathbf{Z}_2, \omega)$. Since the map $\mathscr{L}$ is invariant under the $\mathbf{Z}_2$-action, it induces an equivalence between the Hamiltonian systems $(\mathscr{K}, T^*\mathbf{C}_0/\mathbf{Z}_2, \omega)$ and $(K, T^*\mathbf{C}_0, \omega)$. Thus the regularized energy surface $H^{-1}(-k^2/2)$ of the Kepler Hamiltonian is $\mathscr{K}^{-1}(0) = (S^3_{2/\sqrt{k}})/\mathbf{Z}_2$, which is real projective three space $\mathbf{RP}^3$.

11. (Kustaanheimo-Stiefel regularization.) Let $x = (x_1, x_2, x_3)$ be a vector in $\mathbf{R}_0^3 = \mathbf{R}^3 \setminus \{0\}$ and let $z = (z_1, z_2) \in \mathbf{C}_0^2 = \mathbf{C}^2 \setminus \{0\} = \mathbf{R}^4 \setminus \{0\}$. Define the 2×2 skew Hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\langle z, w \rangle = z_1\overline{w_1} + z_2\overline{w_2}$ be the standard Hermitian inner product on $\mathbf{C}^2$. Show that the mapping

$$\pi : \mathbf{C}^2_0 \to \mathbf{R}^3_0 : z \mapsto \big(\langle z, \sigma_1(z)\rangle, \langle z, \sigma_2(z)\rangle, \langle z, \sigma_3(z)\rangle\big), \tag{113}$$

is the Hopf map.

a) On $\mathbf{C}^2_0$ define an action

$$\varphi : U(1) \times \mathbf{C}^2_0 \to \mathbf{C}^2_0 : \big(\mathrm{e}^{is}, (z_1, z_2)\big) \mapsto (\mathrm{e}^{is}z_1, \mathrm{e}^{is}z_2).$$

Let $T^*\mathbf{C}^2_0 = (\mathbf{C}^2 - \{0\}) \times \mathbf{C}^2$. Lift φ to a $U(1)$-action

$$\Phi : S^1 \times T^*\mathbf{C}^2_0 \to T^*\mathbf{C}^2_0 : \big(\mathrm{e}^{is}, z, w\big) \mapsto \big(\mathrm{e}^{is}z, \mathrm{e}^{is}w\big).$$

Define a 1-form θ on $T^*\mathbf{C}^2_0$ by $\theta = -2i\,\mathrm{Im}\,\langle w, dz\rangle$. Show that $\Omega = -d\theta$ is a symplectic form on $T^*\mathbf{C}^2_0$ and that Φ is a Hamiltonian action with momentum map

$$\mathscr{I} : T^*\mathbf{C}^2_0 \to \mathbf{R} : (z, w) \mapsto 2\,\mathrm{Re}\,\langle w, z\rangle.$$

Let $\mathscr{I}_0 = \mathscr{I}^{-1}(0) \setminus \{0\}$.

b) The map π (113) lifts to the *Kustaanheimo-Stiefel* map

$$\begin{aligned}\mathscr{K}\mathscr{S} : T^*\mathbf{C}^2_0 \to T^*\mathbf{R}^3_0 : (z, w) \mapsto (x, y) = \\ \big((\langle z, \sigma_j(z)\rangle), \langle z, z\rangle^{-1}\,(\mathrm{Re}\,\langle w, \sigma_j(z)\rangle)\big), \quad \text{for } j = 1, 2, 3.\end{aligned}$$

The following calculation shows that

$$(\mathscr{K}\mathscr{S})^*(\vartheta|\mathscr{I}_0) = \theta|\mathscr{I}_0, \tag{114}$$

where $\vartheta = \langle y, \mathrm{d}x\rangle$ is the canonical 1-form on $T^*\mathbf{R}^3$. For every $u, w, z \in \mathbf{C}^2$

$$\sum_{j=1}^{3} \langle u, \sigma_j(z)\rangle\, \sigma_j(w) = 2\langle w, z\rangle\, u - \langle u, z\rangle\, w. \tag{115}$$

Interchanging u with z in (115) and subtracting the result from (115) gives

$$i\sum_{j=1}^{3} \mathrm{Im}\,\langle u, \sigma_j(z)\rangle\, \sigma_j(w) = \langle w, z\rangle\, u - \langle w, u\rangle\, z - i\,\mathrm{Im}\,\langle u, z\rangle\, w. \tag{116}$$

Taking the inner product of (116) with z and then adding the result to its complex conjugate gives

$$\sum_{j=1}^{3} \mathrm{Im}\,\langle u, \sigma_j(z)\rangle\, \mathrm{Im}\,\langle z, \sigma_j(w)\rangle = \mathrm{Re}\,\langle z, w\rangle\, \mathrm{Re}\,\langle u, z\rangle - \langle z, z\rangle\, \mathrm{Re}\,\langle u, w\rangle. \tag{117}$$

Replacing w in (117) with $-iw$ gives

$$\sum_{j=1}^{3} \langle u, \sigma_j(z)\rangle\, \langle z, \sigma_j(w)\rangle = \mathrm{Im}\,\langle z, u\rangle\, \mathrm{Re}\,\langle w, z\rangle - \langle z, z\rangle\, \mathrm{Im}\,\langle w, u\rangle. \tag{118}$$

Finally, replacing w by $\mathrm{d}z$ and u by w in (118) gives

$$\sum_{j=1}^{3} \mathrm{Re}\,\langle w, \sigma_j(z)\rangle\,\mathrm{Im}\,\langle z, \sigma_j(\mathrm{d}z)\rangle = \mathrm{Im}\,\langle z, \mathrm{d}z\rangle\,\mathrm{Re}\,\langle z, w\rangle - \langle z, z\rangle\,\mathrm{Im}\,\langle w, \mathrm{d}z\rangle. \tag{119}$$

Consequently

$$(\mathscr{K}\mathscr{S})^*\vartheta = 2i\left(\frac{\mathrm{Im}\langle z, \mathrm{d}z\rangle\,\mathrm{Re}\,\langle z, w\rangle - \langle z, z\rangle\,\mathrm{Im}\,\langle w, \mathrm{d}z\rangle}{\langle z, z\rangle}\right). \tag{120}$$

From (120) it follows that $(\mathscr{K}\mathscr{S})^*(\vartheta|\mathscr{I}_0) = \theta|\mathscr{I}_0$.

c) On $T^*\mathbf{R}^3_0$ with coordinates (x, y) and symplectic form $\omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i$, consider the time rescaled Kepler Hamiltonian

$$\widetilde{K}(x, y) = \tfrac{1}{2}k^{-1}\,\|x\|(\|y\|^2 + k^2)$$

whose μk^{-1}-level set corresponds to the $-k^2/2$-level set of the Kepler Hamiltonian. Setting $u = w$ in (117) show that on $\mathscr{I}_0$ $\|(\mathscr{K}\mathscr{S})^*\|y\|^2 = \langle w, w\rangle\langle z, z\rangle^{-1}$ and $\|(\mathscr{K}\mathscr{S})^*x\|^2 = \langle z, z\rangle$. Therefore on $\mathscr{I}_0$ we obtain the regularized Hamiltonian

$$K = (\mathscr{K}\mathscr{S})^*\widetilde{K} = \tfrac{1}{2}k^{-1}\left(\langle w, w\rangle + k^2\langle z, z\rangle\right). \tag{121}$$

When $k = 1$ the regularized Hamiltonian is the harmonic oscillator Hamiltonian on $(T^*\mathbf{C}^2, \Omega)$ restricted to the open cone $\mathscr{I}_0$. Show that the regularized Hamiltonian K (121) is invariant under the $U(1)$-action Φ. Since the mapping $\mathscr{K}\mathscr{S}$ is *not* a diffeomorphism, the harmonic oscillator vector field X_K is not equivalent to the Kepler vector field $X_{\widetilde{K}}$. Show that they are $\mathscr{K}\mathscr{S}$-related on $\mathscr{I}_0$, that is, on $\mathscr{I}_0$ we have $T(\mathscr{K}\mathscr{S})\circ X_K = X_{\widetilde{K}}\circ(\mathscr{K}\mathscr{S})$. Moreover, show that after dividing out the S^1-action Φ on $\mathscr{I}_0$ we obtain an equivalence of Hamiltonian systems. Show that the orbit space $\mathscr{I}_0/S^1$ is diffeomorphic to T^+S^3, the tangent bundle to S^3 less its zero section.

12. (Generalized Kepler equation.) Consider the Ligon-Schaaf map

$$LS: \Sigma_- \subseteq T_0\mathbf{R}^3 \rightarrow T^+S^3_{np} \subseteq T\mathbf{R}^4 : (q, p) \rightarrow (r, s),$$

with $\varphi = \nu^{-1}\langle q, p\rangle$. Show that its inverse is given by

$$\begin{aligned} q &= \mu^{-1}\langle s, s\rangle\big((\sin\varphi - \langle r, r\rangle^{-1/2}s_4)\widetilde{r} + \langle s, s\rangle^{-1/2}(r_4 - \cos\varphi)\widetilde{s}\big) \\ p &= \mu\langle s, s\rangle^{-1/2}\Big(\frac{\widetilde{r}\cos\varphi + \langle s, s\rangle^{-1/2}\widetilde{s}\sin\varphi}{1 - r_4\cos\varphi - \langle s, s\rangle^{-1/2}s_4\sin\varphi}\Big), \end{aligned}$$

where $r = (\widetilde{r}, r_4)$, $s = (\widetilde{s}, s_4)$ and φ is a smooth solution of

$$\varphi - r_4\sin\varphi - s_4\langle s, s\rangle^{-1/2}\cos\varphi = 0.$$

13. a) Show that SO(4)-action on an energy surface of the Delaunay vector field is transitive.

    b) Show that the mapping $\vartheta : \bigwedge^2 \mathbf{R}^4 \rightarrow \mathrm{so}(4)$, defined by $\varphi(u \wedge v)w = \langle v, w \rangle u - \langle v, u \rangle w$ for every $u, v, w \in \mathbf{R}^4$, intertwines the SO(4)-action $\mathrm{SO}(4) \times \bigwedge^2 \mathbf{R}^4 \rightarrow \bigwedge^2 \mathbf{R}^4 : (A, u \wedge v) \rightarrow Au \wedge Av$ with the adjoint action of SO(4) on so(4).

    c) Show that the orbit space (C_h, ω_h) of the flow of the Delaunay vector field on $\widetilde{\mathscr{H}}^{-1}(h)$ is symplectically diffeomorphic to the coadjoint orbit $\mathscr{O}_\mu$ through $\widetilde{\mathscr{J}}(e_1, he_2) = he^*_{12} = \mu \in \mathrm{so}(4)^*$ with its usual symplectic structure $\omega_{\mathscr{O}_\mu}$, see example 3 chapter VI §2.

14. Show that the Hamiltonian vector field X_{e_i} corresponding to the i$^{\mathrm{th}}$ component of the eccentricity vector (27) is incomplete. Give a geometric explanation of this incompleteness. State precisely where the flow of X_{e_i} is defined.

15. Given an initial position and momentum of a Keplerian elliptical orbit, determine the argument of the perihelion, that is, the angle between the line of nodes (= the line of intersection of the plane of the elliptical orbit and the equitorial plane of the celestial sphere) and the line joining the foci of the ellipse.

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