

On the Theories of Plates and Shells at the Nanoscale

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Abstract During the last 50 years the nanotechnology is established as one of the advanced technologies manipulating matter on an atomic and molecular scale. New materials, devices or other structures possessing at least one dimension sized from 1–100 nm are developed. The question arises how structures composed of nanomaterials should be modeled. In addition, if it is necessary to perform a structural analysis what kind of theory should be used. Two approaches are suggested—theories which take into account quantum mechanical effects since they are important at the quantum-realm scale and theories which are based on the classical continuum mechanics adapted to nanoscale problems. Here the second approach will be discussed in detail. It will be shown that the classical continuum mechanics is enough for a sufficient description of the mechanical behavior of nanomaterials and-structures if surface stresses are taken into account. There are also other approaches in the literature, but they are not discussed in detail in this paper. The basic equations for nanosized plates and shells will be discussed. It is shown that for this class of objects with the help of suggested equations such effects like the size-dependent changes of the stiffness parameters can be described in a proper manner. In contrast to the results for the size-dependence of the Young's modulus (the Young's modulus increases when the specimen diameter becomes very thin) the plate stiffness parameters can increase or decrease when the plate thickness is in the range of several nm. Finally, the theory of plates with surface stresses will be compared with the theory of three-layered plates.

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1 Introduction and Historical Remarks

Structural mechanics is the computation of strains, deflections, and internal forces or stresses (stress resultants) within structures. The aim of the analysis is the safe design of new structures or the evaluation of existing structures. The starting point for any structural analysis are the following input data: loads (mechanical, thermal, electrical, etc.), the structure's geometry, support and contact conditions, and the material behavior information. Finally, the stresses, the strains and displacements are estimated for the whole structure. The classical analysis is limited by consideration of linear-elastic behavior and the description of the structure within the reference configuration. Advanced structural mechanics may include such effects like stability and non-linear behavior.

There are several approaches to the analysis. In the general case this results in solving a system of three-dimensional coupled partial differential equations. Analytical solution can be obtained only in exceptional cases. Thus during the last 50 years the finite element method was developed as a powerful tool for approximate solutions of three-dimensional problems. At the same time lower-dimensional theories became a further development. Since the pioneering works of Euler or Bernoulli we know that considering the different order of the dimensions of a structure one can describe the structural mechanics problems by simplified equations. Well-known examples are the theories of rods, beams, plates and shells. The first two are based on the assumption that the characteristic cross-section dimension is much smaller than the length, the last two assume that the thickness is much smaller in comparison with other dimensions. Below we focus our attention on plates and shells only.

Shell- and plate-like structures are used in civil and aerospace engineering as basic elements of constructions. Such structures are applied as a model of analysis in other branches, e.g. mechanical engineering, but also in new one like medical engineering. New applications are primarily related to new materials—instead of steel or concrete now one has to analyze sandwiches, laminates, foams, nano-films, biological membranes, etc. The new trends in applications demand improvements of the theoretical foundations of the plate and shell theory, since new effects must be taken into account. For example, in the case of small-size plate- or shell-like structures (for example, nano-tubes) the surface effect plays an increasing role in the mechanical analysis of these structural elements if the size decreases.

Let us make a brief overview on some important steps in the development of the theory of plates and shells. One of the first researcher in the field of plates and shells was Ernst Florens Friedrich Chladni (* November 30, 1756 in Wittenberg; † April 3, 1827 in Breslau). He was a physicist, astronomer and musician. His most important contribution included research on vibrating plates showing various modes of vibration (Chladni's nodal patterns). In this sense he was the founder of acoustics. Marie-Sophie Germain (* April 1, 1776 in Paris; † June 27, 1831 in Paris) paid attention on Chladni's experimental works. She took part in a contest organized by the French Academy of Sciences. The aim of the contest was to give "the mathematical theory of the vibration of an elastic surface and to compare the theory to experimental

evidence". She submitted a paper on this topic, but failed. Joseph-Louis Lagrange (* January 25th, 1736 in Turin; † April 10th, 1813 in Paris) derived an equation based on Germain's paper that was correct under special assumptions. Later in 1816 Germain published her third paper on the above mentioned topic and she was awarded the *prix extraordinaire*. She derived the correct vibration equation

$$N^2 \left(\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \right) + \frac{\partial^2 z}{\partial t^2} = 0 \quad (N = \text{const}),$$

but the experimental results were not very accurately predicted. In addition, she had trouble with the boundary conditions.

The next big step in the development of the plate theory was done by Gustav Robert Kirchhoff (* March 12, 1824 in Königsberg; † October 17, 1887 in Berlin). The Kirchhoff plate theory is an extension of the Euler-Bernoulli beam theory to thin plates and based on the assumption that a mid-surface plane can be used to represent the three-dimensional plate in two-dimensional form. He introduced following kinematic assumptions: straight lines normal to the mid-surface remain straight and normal after deformation and the thickness of the plate does not change during the deformation [75]. The governing equation of Kirchhoff's plate theory is

$$\nabla^2 \nabla^2 w = \frac{q}{D} \quad \text{with} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

Similar kinematical assumptions were introduced by Augustus Edward Hough Love (* April 17, 1863 in Weston-super-Mare; † June 5, 1940 in Oxford) in the case of shells.

The main disadvantage of Kirchhoff's plate theory is that the governing equation is of 4th order—but in some cases one has to satisfy three boundary conditions. In addition, the transverse flexibility is not presented in a satisfying manner. This was the starting point for several improvements of the Kirchhoff theory. Theodore von Kármán (* May 11, 1881 in Budapest; † May 6, 1963 in Aachen) presented at the beginning of the last century [38], which can be related to the reference configuration (in-plane behavior) and to the actual configuration (out-of-plane behavior) at the same time. Finally, he obtained the Föppl-von Kármán equations describing large deflections of thin plates. Another improvement was given by Eric Reissner (* January 5, 1913 in Aachen; † November 1, 1996 in La Jolla, CA). He has been accounted the stresses ignored in the Kirchhoff theory. In this sense Reissner's theory is named first order shear-deformation plate theory. A similar theory was presented by Raymond David Mindlin (* September 17, 1906 in New York; † November 22, 1987 in Hanover, New Hampshire) in 1951, but in contrast to Reissner he used Poisson's approach of dimension reduction applying power series.

Later in the former Soviet Union various plate and shell theories were suggested. Ilia Vekua (* April 23, 1907 in Sheshelety; † December 2, 1977 in Tbilisi) was a Georgian mathematician, which proposed a shell theory using generalized analytic functions [88]. Khamid Mushtari (* July 22, 1900 in Orenburg; † January 23, 1981

in Kazan) was a Tatarian scientist in the field of solid mechanics and mathematics, presenting a theory with large rotations which was successfully used in aerospace industries [60]. This theory was similar to Donnell's and Vlasovs approach and so the theory is usually named Mushtari-Donnell-Vlasov theory [22, 93]. Sergei Ambarcumyan (* March 17, 1922 in Gumry) suggested a first-order shear deformable theory with a special distribution law for the transverse stresses [13].

Another approach was used by Paul Mansur Naghdi (* March 29, 1924 in Teheran; † July 9, 1994 in Berkeley, CA) which was based on the direct approach introduced by Euler and developed by the Cosserat brothers. A summary concerning this approach one can find in [61]. Further historical remarks concerning the theory of plates and shell can be found, for example, in [12].

Relatively recently the developments of nanotechnologies lead to the derivation of models of plates and shells which can be used at the nano-scale. At this level the surface phenomena are important. The classical theory of capillarity was established by Young and Laplace and then was extended for the case of solids by Gibbs. Gurtin [32] and Steigmann [84] proposed models of surface stresses describing the surface elastic properties. The model by [32] found many applications for materials at the micro- and nanoscales, see the reviews [23, 36, 69, 90, 91]. In particular, the surface effects are used for explanation of deviation of the properties of nanosized specimens from the ones of bulk materials. The enhancements of the theory of plates and shells taking into account of surface stresses are discussed in [3, 9–11, 24] and reference therein.

After the work by [58, 59] the surface stresses can be modeled within the framework of the second-gradient theory of elasticity. Non-local and gradient type models of plates and shells are presented, for example, by [14, 18, 34, 45–48, 56, 62, 71–73, 80, 83, 87, 89]. Below only the first approach based on the introduction of equations for the bulk behavior and the surface behavior is used.

2 Materials and Structures Under Consideration

Material science classifies structural materials into three categories

- metals,
- ceramics, and
- polymers

It is difficult to give an exact assessment of the advantages and disadvantages of these three basic material classes, because each category covers whole groups of materials within which the range of properties is often as broad as the differences between the three material classes.

The following characteristic properties can be established:

- Mostly metallic materials are of medium to high density. They have good thermal stability and can be made corrosion-resistant by alloying. Metals have useful mechanical characteristics and it is moderately easy to shape and join. For this reason metals became the preferred structural engineering material.

- Ceramic materials have great thermal stability and are resistant to corrosion, abrasion and other forms of attack. They are very rigid but mostly brittle and can only be shaped with difficulty. In many cases they fail immediately beyond the elastic range.
- Polymer materials (plastics) are of low density, have good chemical resistance but lack thermal stability. They have poor mechanical properties, but are easily fabricated and joined. Their resistance to environmental degradation, e.g. the photochemical effects of sunlight, is moderate.

At present classical structural materials are more and more substituted by advanced materials. An example of advanced materials are composite materials which classification is given in Fig. 1. Another class of advanced materials are sandwich materials with solid and hollow cores (Fig. 2). Similar to the classical sandwiches short fibre reinforced composites have with respect to the technology (here injection molding) a layered sandwich-type microstructure (Fig. 3). Recently another type of advanced materials is used as a material for lightweight structures—plastic or metallic foams. Figure 4 shows closed-cell foams, Fig. 5—open-cell foams.

Classical laminates and sandwiches are modeled as usual as materials with piecewise constant properties. Now we have also materials with changing properties—so called functionally graded materials (FGM). Examples of FGM are shown on Fig. 6.

With the development of nanotechnology a new class of advanced materials was established: nanomaterials. These are materials with single units of which are sized (in at least one dimension) between 1 and 1,000 nm but is usually 1–100 nm (the usual definition of nanoscale, [15]). Examples of nanomaterials are presented on Fig. 7.

The above mentioned materials have a common feature concerning modeling: in all these cases multi-scale and homogenized models are suggested. On Figs. 8 and 9

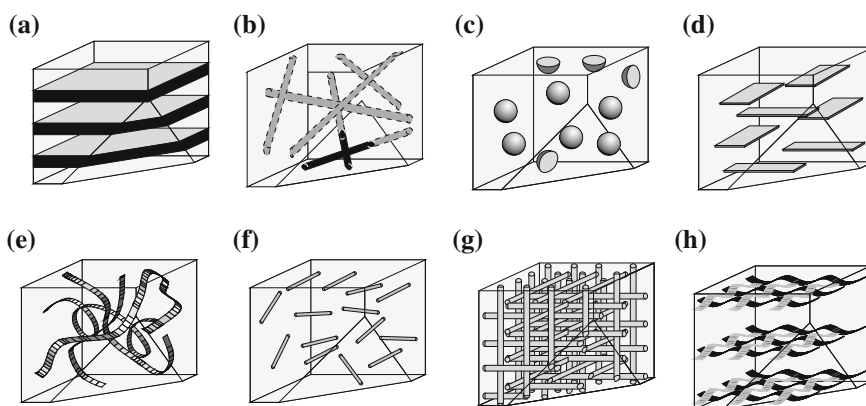


Fig. 1 Classification of composites: **a** laminate, **b** irregular reinforcement, **c** reinforcement with particles, **d** reinforcement with plate shaped particles, **e** random arrangement of continuous fibres, **f** irregular reinforcement with short fibres, **g** spatial reinforcement, **h** reinforcement with surface tissues (after [8])

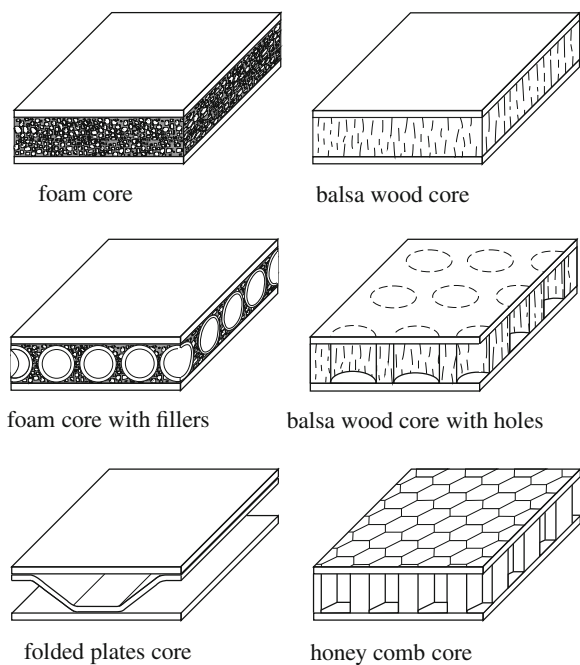


Fig. 2 Sandwich materials with solid and hollow cores [8]

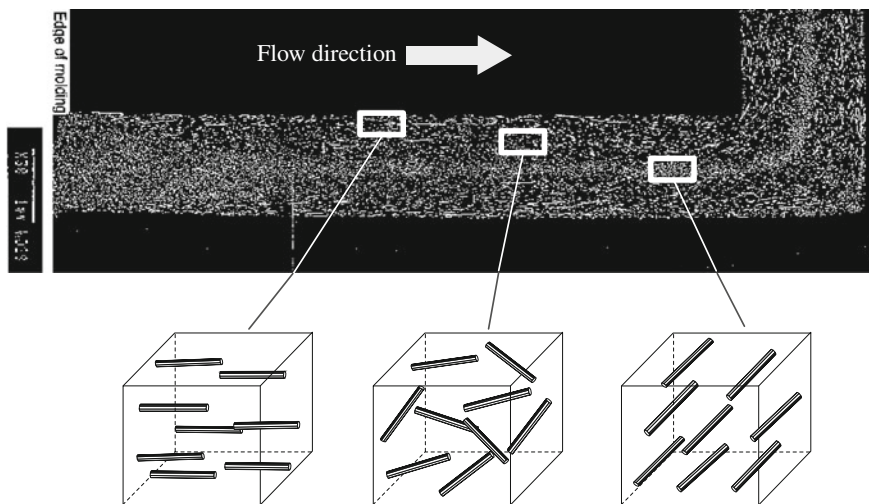


Fig. 3 Short fibre reinforced composite (after [81])

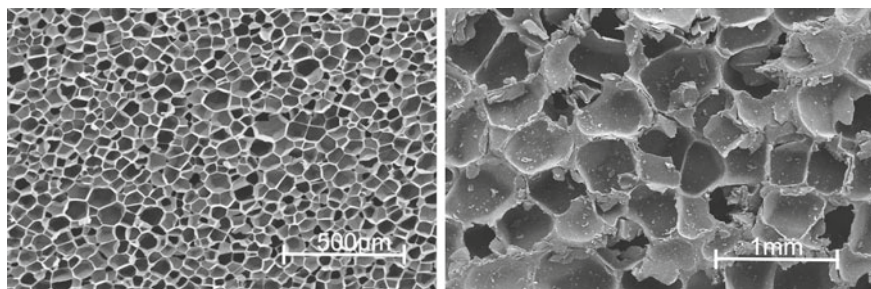


Fig. 4 Closed-cell polymeric foams with various densities after [43]

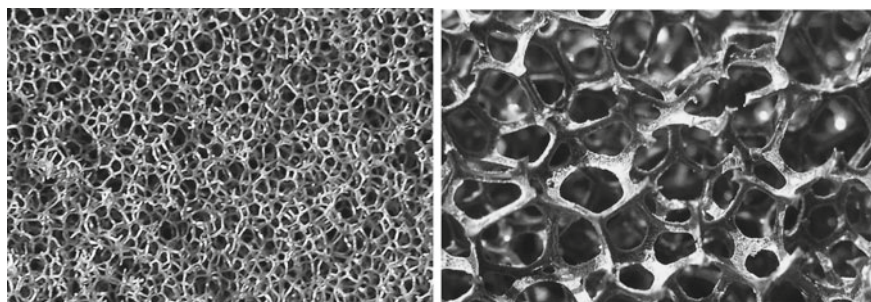


Fig. 5 Open-cell foams with various densities after [43]

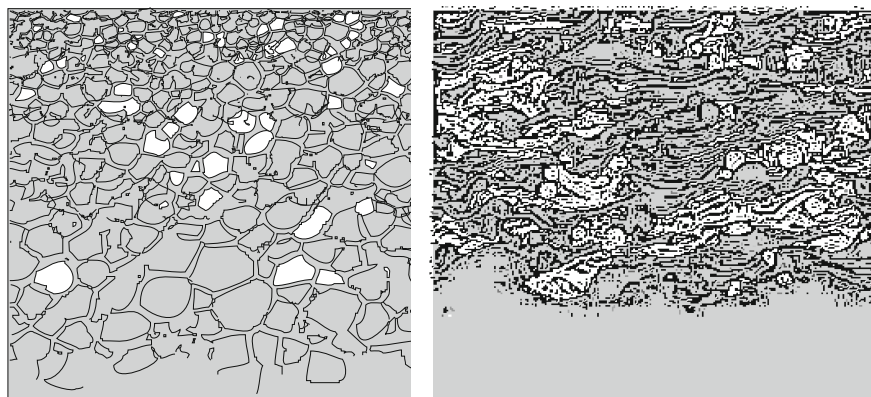


Fig. 6 Examples of FGMs with inhomogeneous microstructure: foam (*left*), thermal coating (*right*)

the model hierarchy in the case of laminated plates and open- and closed-cell foams is shown.

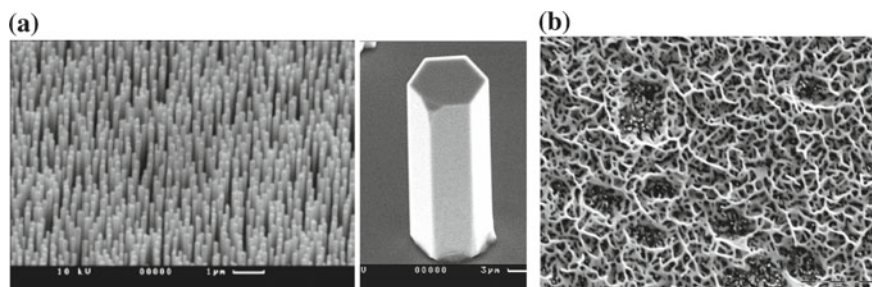


Fig. 7 Nanostructures: **a** ordered array of ZnO nanocrystals and single crystal, **b** ZnO nanofoam

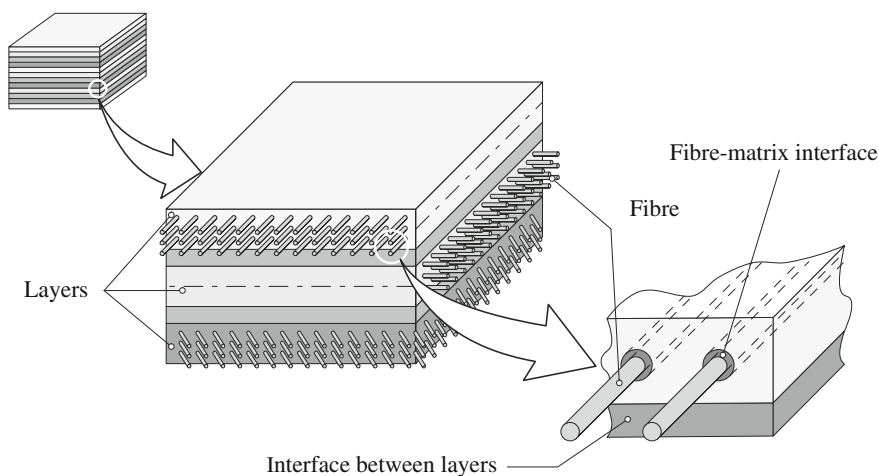


Fig. 8 Multi-scale modeling of laminated plates [8]

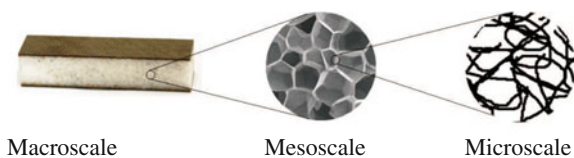


Fig. 9 Three scales for foam structures

3 Plate and Shell Theories: Fundamentals

The basic problem in civil and mechanical engineering is the analysis of the strength, the vibration behavior and the stability of structures with the help of structural models. The structural models can be classified by their

- spatial dimensions
- loadings
- kinematical and/or statical hypotheses

The starting point for any structural analysis is order of the dimensions in the three-dimensional space. We have to distinguish three basic models:

- The three spatial dimensions have the same order, no predominant direction for the dimensions exists. Typical examples of geometrical simple, compact structural elements in theory of elasticity are cube, prism, cylinder, sphere, etc.
- Two spatial dimensions have the same order, the third, which is related to the thickness is much smaller. Typical examples of surface structural elements in structural mechanics are: discs, plates, shells, folded structures, etc.
- Two spatial dimensions, which can be related to the cross-section, have the same order. The third dimension, which is related to the length of the structural element, has a much larger order in comparison with the cross-section dimensions. Typical examples in engineering mechanics are: rods, beams, torsion beam, etc.

Thin-walled structural elements (thin-walled light-weight profile structures) require an extension of the classical structural models. If the spatial dimensions are of significantly different order and the thickness of the profile is small in comparison to the other cross-section dimensions, and the cross-section dimensions are much smaller in comparison to the length of the structure one can introduce quasi-one-dimensional structural elements. Suitable theories are

- the thin-walled beam approach and
- the semi-membrane theory or generalized beam theory [93]

The further discussion are limited to structures subjected to the following definition.

Definition 2.1 (*Two-Dimensional Structure*) A two-dimensional load-barring structural element is a model for the analysis in engineering and structural mechanics with two geometrical dimensions, which are of the same order and which are significantly larger than the third (thickness) direction.

The mathematical consequence is that instead of a three-dimensional problem, which is presented by a system of partial differential equations, one can analyze a two-dimensional problem. The transition from the three-dimensional to the two-dimensional problem is not simple, but the solution effort decreases significantly. Within the definition the following model classes are included

- thin homogeneous plates,
- thin inhomogeneous plates (laminates, sandwiches),
- plates with structural anisotropy,
- moderately thick homogeneous plates,
- folded plates,
- membranes,
- biological membranes,
- nanotubes

Such models are applied in space and aircraft industries, automotive industries, shipbuilding industries, vehicle systems, civil engineering, medicine, etc. The developments in this field are widely discussed, for example, in [12, 30, 31, 61, 75, 77]. Recently several conferences were devoted to this topic: Euromech Colloquia 444 and 527 [4, 41], IUTAM Symposium on Relation of Shell, Plate, Beam and 3D Models [35], *Shell Structures Theory & Applications* conference [67] among others.

In the literature there are various formulation principles for two-dimensional theories. The corresponding equations can be deduced:

- starting from three-dimensional continuum mechanics equations or
- starting from two-dimensional equation describing the behavior of a deformable surface

If one starts from three-dimensional continuum mechanics equations there exist two possibilities to reduce the dimension

- the use of proper hypotheses or
- the use of mathematical approaches

All suggested methods have advantages and disadvantages, but hypotheses based theories are preferred by the engineers.

If the hypotheses based method is used we can introduce assumptions for the stress, strain or displacement states. For example, if we start with displacement approximations we can deduce the full set of governing equations and boundary conditions.

- Examples of displacement approximations for a plate (Fig. 10) are given as follows:

- Kirchhoff [42]

$$u_\alpha(x_\beta, z) = u_\alpha^0(x_\beta) - zw_{,\alpha}(x_\beta), \quad w(x_\beta, z) = w(x_\beta)$$
- Mindlin [57]

$$u_\alpha(x_\beta, z) = u_\alpha^0(x_\beta) + z\varphi_\alpha(x_\beta), \quad w(x_\beta, z) = w(x_\beta)$$
- Levinson [50] and Reddy [74]

$$u_\alpha(x_\beta, z) = u_\alpha^0(x_\beta) - [w_{,\alpha}(x_\beta) + \varphi_\alpha(x_\beta)] \frac{4z^3}{3h^2}, \quad w(x_\beta, z) = w(x_\beta)$$
- Meenen and Altenbach [55]

$$u_\alpha(x_\beta, z) = u_\alpha^q(x_\beta)\phi^q(z) + w_{,\alpha}^q(x_\beta)\psi^q(z), \quad w(x_\beta, z) = w(x_\beta)^q\chi^q(z)$$

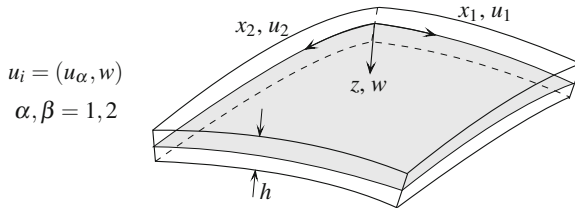


Fig. 10 Examples of displacement approximations

- Mathematical approaches are based on
 - Power series for the displacements, stresses and strains which were used by [17, 39, 40, 52, 55, 70] among others,
 - Special or trigonometric functions which were applied, for example, in [44, 86, 88],
 - Asymptotic integration which was used, for example, in [28]

Another approach is the so-called direct approach. The starting point is the *a priori* introduction of an two-dimensional deformable surface [2, 29, 61, 79, 94, 95]. This is a natural way of formulation the two-dimensional equations which was also mentioned indirectly by [82]: “...everyone, who is thinking about the foundations of Continuum Mechanics, will attend the world of images of the Cosserat brothers”. Firstly the direct approach was discussed by Leonard Euler introducing the moment vector as independent quantity in the theory of beams. Later this approach was extended to three-dimensional problems and a first summary was given by [20, 21]. Later this approach was discussed and extended by [16, 26, 27, 37, 63, 65]. Special contributions to the two-dimensional direct approach were made by [7, 29, 61, 78] among others. In all these papers the independence of translations and rotations/forces and moments is assumed and which corresponds to the undergraduate courses in Statics and Rigid Body Dynamics.

In the adjacent part the direct approach to the theory of plates and shells will be considered.

Definition 2.2 (*Simple Shell*) A simple shell is a 2D-continuum in which the interaction between neighboring parts is due to forces and moments [95].

In addition let us make the following two assumptions:

1. The plate or shell (homogeneous or inhomogeneous in thickness direction) will be represented by a deformable surface.
2. Each material point of the surface is an infinitesimal body with 5 degrees of freedom (3 translations and 2 rotations).

The last one assumption can be easily extended to 6 degrees of freedom [5].

The kinematical model of a simple shell can presented as usual in continuum mechanics by comparison of the reference and actual configurations. In the reference configuration (undeformed state) we have

$$\{\mathbf{r}(q^1, q^2); \mathbf{d}_k(q^1, q^2)\},$$

where $\mathbf{r}(q^1, q^2)$ is the position vector, $\mathbf{d}_k(q^1, q^2)$ are orthonormal vectors. The actual configuration (deformed state) is given by

$$\{\mathbf{R}(q^1, q^2, t); \mathbf{D}_k(q^1, q^2, t)\}, \quad \mathbf{D}_k \cdot \mathbf{D}_m = \delta_{km},$$

where the capital letters have the same meaning as in the reference configuration. The motion of the directed surface can be presented by

$$\mathbf{R}(q, t), \quad \mathbf{P}(q, t) \equiv \mathbf{D}^k(q, t) \otimes \mathbf{d}_k(q)$$

$\mathbf{P}(q, t) \equiv \mathbf{P}(q^1, q^2, t)$ is a rotation tensor, $\det \mathbf{P} = +1$. The linear and angular velocities $\mathbf{v}(q, t), \boldsymbol{\omega}(q, t)$ are defined as

$$\mathbf{v} = \dot{\mathbf{R}}, \quad \dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}, \quad \mathbf{P}(q^1, q^2, 0) = \mathbf{P}_0, \quad \dot{f} \equiv \frac{df}{dt}$$

and we obtain the local equations of motion:

- First Euler equation of motion

$$\tilde{\nabla} \cdot \mathbf{T} + \rho \mathbf{F}_* = \rho(\mathbf{v} + \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega})'$$

- Second Euler equation of motion

$$\tilde{\nabla} \cdot \mathbf{M} + \mathbf{T}_\times + \rho \mathbf{L} = \rho(\boldsymbol{\Theta}_1 \cdot \mathbf{v} + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega})' + \rho \mathbf{v} \times \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega}$$

$\mathbf{T} = \mathbf{R}_\alpha \otimes \mathbf{T}^\alpha, \mathbf{M} = \mathbf{R}_\alpha \otimes \mathbf{M}^\alpha$ are the force and the moment tensor, respectively, $\mathbf{T}_\times \equiv \mathbf{R}_\alpha \times \mathbf{T}^\alpha$ is the vector invariant of the force tensor, $\mathbf{F}_*, \mathbf{L}$ are the mass density of external forces and moments, $\rho, \rho \boldsymbol{\Theta}_1, \rho \boldsymbol{\Theta}_2$ are the density, the first and the second tensor of inertia, respectively, $\tilde{\nabla} \equiv \mathbf{R}^\alpha(q^1, q^2, t) \frac{\partial}{\partial q^\alpha}$ is the nabla operator.

Here and in the next following parts the direct tensor notation is used (see, for example, [49]).

The equation of the balance of energy can be formulated in the local form

$$\rho \dot{\mathcal{U}} = \mathbf{T}^T \cdot \tilde{\nabla} \mathbf{v} - \mathbf{T}_\times \cdot \boldsymbol{\omega} - \mathbf{M}^T \cdot \tilde{\nabla} \boldsymbol{\omega}$$

$\mathcal{U}$ is the mass density of the internal energy. Introducing the energetic tensors [54]

$$\mathbf{T}_e = (\tilde{\nabla} \mathbf{r})^T \cdot \mathbf{T} \cdot \mathbf{P}, \quad \mathbf{M}_e = (\tilde{\nabla} \mathbf{r})^T \cdot \mathbf{M} \cdot \mathbf{P}$$

Another form of the balance of energy is given by

$$\rho \dot{\mathcal{U}} = \mathbf{T}_e^T \cdot \dot{\mathbf{E}} + \mathbf{M}_e^T \cdot \dot{\mathbf{F}}$$

$\mathbf{E}, \mathbf{F}$ are the first and the second deformation tensors

$$\mathbf{E} = \tilde{\nabla} \mathbf{R} \cdot \mathbf{P} - \mathbf{a}, \quad \mathbf{F} = (\boldsymbol{\Phi}_\alpha \cdot \mathbf{D}_k) \mathbf{r}^\alpha \otimes \mathbf{d}^k$$

with $\partial_\alpha \mathbf{P} = \boldsymbol{\Phi}_\alpha \times \mathbf{P} \Rightarrow 2\boldsymbol{\Phi}_\alpha = -[\partial_\alpha \mathbf{P} \cdot \mathbf{P}^T]_\times$. $\mathbf{a}$ is the first two-dimensional metric tensor. Now we can define the reduced deformation tensors. The internal energy $\mathcal{U} = \mathcal{U}(\mathbf{E}, \mathbf{F})$ contains 12 scalar arguments. The number of arguments can be reduced due to some restrictions [95]:

- simple shells of constant thickness,

- non-polar materials,
- $\mathbf{L} \cdot \mathbf{D}_3 = \mathbf{0}$, $\mathbf{M} \cdot \mathbf{D}_3 = \mathbf{0}$
- $\mathbf{M}_c^T \cdot [(\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{c}] + \mathbf{T}_c^T \cdot [(\mathbf{E} + \mathbf{a}) \cdot \mathbf{c}] = \mathbf{0}$

the specific energy $\mathcal{U}$ must satisfy

$$\left(\frac{\partial \mathcal{U}}{\partial \mathbf{E}} \right)^T \cdot [(\mathbf{E} + \mathbf{a}) \cdot \mathbf{c}] + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{F}} \right)^T \cdot [(\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{c}] = 0, \quad \frac{\partial \rho_0 \mathcal{U}}{\partial (\mathbf{F} \cdot \mathbf{n})} = \mathbf{0}$$

$\mathbf{b}$ is the second two-dimensional metric tensor and $\mathbf{c} = -\mathbf{n} \times \mathbf{a}$ is the discriminant tensor. The characteristic system of the first one equation is a system of 12th order

$$\frac{d}{ds} \mathbf{E} = (\mathbf{E} + \mathbf{a}) \cdot \mathbf{c}, \quad \frac{d}{ds} \mathbf{F} = (\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{c}$$

which have 11 independent integrals—the strain measures

$$\begin{aligned} 2\boldsymbol{\varepsilon} &= [(\mathbf{E} + \mathbf{a}) \cdot \mathbf{a} \cdot (\mathbf{E} + \mathbf{a})^T - \mathbf{a}], \\ \boldsymbol{\Phi} &= (\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{a} \cdot (\mathbf{E} + \mathbf{a})^T + \mathbf{b} \cdot \mathbf{c} \cdot \boldsymbol{\varepsilon} + \mathbf{b} \cdot \mathbf{c}, \\ \boldsymbol{\gamma} &= \mathbf{E} \cdot \mathbf{n}, \\ \boldsymbol{\gamma}_* &= \mathbf{F} \cdot \mathbf{n} \end{aligned}$$

The arbitrary function $\mathcal{U}(\boldsymbol{\varepsilon}, \boldsymbol{\Phi}, \boldsymbol{\gamma}, \boldsymbol{\gamma}_*)$ satisfies the first equation of the characteristic system. From the second equation follows that $\mathcal{U}$ does not depend on $\boldsymbol{\gamma}_*$. The tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\Phi}$, $\boldsymbol{\gamma}$ are called reduced deformation tensors. $\boldsymbol{\varepsilon}$ represents the plane tensile and shear strains, $\boldsymbol{\Phi}$ are the bending and torsional strains and $\boldsymbol{\gamma}$ are the transverse shear strains.

For a shell composed of a linear-elastic material with relatively small strains while the displacements and rotations can be relatively large the quadratic approximation for the strain energy can be introduced

$$\begin{aligned} 2\rho_0 \mathcal{U} &= 2\mathbf{T}_0 \cdot \boldsymbol{\varepsilon} + 2\mathbf{M}_0^T \cdot \boldsymbol{\Phi} + 2\mathbf{N}_0 \cdot \boldsymbol{\gamma} + \boldsymbol{\varepsilon} \cdot {}^{(4)}\mathbf{C}_1 \cdot \boldsymbol{\varepsilon} + 2\boldsymbol{\varepsilon} \cdot {}^{(4)}\mathbf{C}_2 \cdot \boldsymbol{\Phi} \\ &\quad + 2\boldsymbol{\Phi} \cdot {}^{(4)}\mathbf{C}_3 \cdot \boldsymbol{\Phi} + \boldsymbol{\gamma} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{\gamma} + 2\boldsymbol{\gamma} \cdot {}^{(3)}\boldsymbol{\Gamma}_1 \cdot \boldsymbol{\varepsilon} + {}^{(3)}\boldsymbol{\Gamma}_2 \cdot \boldsymbol{\Phi} \end{aligned} \quad (1)$$

$\mathbf{T}_0$, $\mathbf{M}_0$, $\mathbf{N}_0$, ${}^{(4)}\mathbf{C}_1$, ${}^{(4)}\mathbf{C}_2$, ${}^{(4)}\mathbf{C}_3$, ${}^{(3)}\boldsymbol{\Gamma}_1$, ${}^{(3)}\boldsymbol{\Gamma}_2$, $\boldsymbol{\Gamma}$ are stiffness tensors of different rank. They express the effective elastic properties of the simple shell. The differences between various classes of simple shells are connected with different expressions of the stiffness tensors. The stiffness tensors do not depend on the deformations. Thus they may be found from tests based on the linear shell theory.

The constitutive equations can be obtained as the derivatives of the strain energy by the strains. In the simplest case ignoring the eigenstresses we get

- in-plane forces

$$\mathbf{T} \cdot \mathbf{a} = \rho_0 \frac{\partial \mathcal{U}}{\partial \boldsymbol{\varepsilon}} = {}^{(4)}\mathbf{C}_1 \cdot \boldsymbol{\varepsilon} + {}^{(4)}\mathbf{C}_2 \cdot \boldsymbol{\Phi} + \boldsymbol{\gamma} \cdot \boldsymbol{\Gamma}_1, \quad (2)$$

- transverse shear forces

$$\mathbf{T} \cdot \mathbf{n} = \rho_0 \frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}} = \boldsymbol{\Gamma} \cdot \boldsymbol{\gamma} + \boldsymbol{\Gamma}_1 \cdot \boldsymbol{\varepsilon} + \boldsymbol{\Gamma}_2 \cdot \boldsymbol{\Phi}, \quad (3)$$

- moments

$$\mathbf{M}^T = \rho_0 \frac{\partial \mathcal{U}}{\partial \boldsymbol{\kappa}} = \boldsymbol{\varepsilon} \cdot {}^{(4)}\mathbf{C}_2 + {}^{(4)}\mathbf{C}_3 \cdot \boldsymbol{\Phi} + \boldsymbol{\gamma} \cdot \boldsymbol{\Gamma}_2 \quad (4)$$

After the formulation of the governing equations one open question exists—the identification of the effective properties (stiffness, etc.). Various solutions of this problem are existing. To find the general structure of stiffness tensors the theory of symmetry must be applied. The classical theory of symmetry is not sufficient because it is valid for Euclidean tensors only. In the shell theory Euclidean and non-Euclidean tensors are involved. Details of the application of the theory of symmetry are presented in [95].

In the thickness direction of plates and shells we can assume homogeneous or inhomogeneous behavior. The second case is obtained if we have sandwiches and laminates (piecewise constant properties) or functionally graded materials (continuous distributed properties). Both particular cases can be modeled like a “microstructure”. The question arises how the symmetries of the “microstructure” do affect the physical properties? The answer comes from the Curie-Neumann’s principle [64, 66] in the physics of crystals: *The symmetry group of the reason belongs to the symmetry group of the consequence.*¹ The symmetry group of the reasons for the plates and shells is the intersection of

- symmetry of the material (fibre-reinforced material, rolled sheets, ...),
- symmetry of the surface shape (shell or plate),
- symmetry of the internal structure of the plate (for example, symmetry of the laminate stacking sequence with respect to the mid-surface, ...)

It is obvious that symmetry discussions for shells are more complex since the curvature has an influence.

Symmetries can be described in terms of the geometric operations which produce identical configurations. The set of symmetry operations and results of their combinations define a mathematical structure called a group. The symmetry operations which involve only rotations, reflection and inversion define the point group. The symmetries are described by orthogonal tensors

¹ Other formulations are:

- Any type of symmetry exhibited by the point group of a crystal is possessed by every physical property of the crystal.
- For a material element and for any of its physical properties, every material symmetry transformation of the material element is a physical symmetry transformation of the physical property.

- reflection ($\mathbf{n}$ is the unit normal to the mirror plane)

$$\mathbf{Q} = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}, \quad \det \mathbf{Q} = -1,$$

- rotation ($\mathbf{m}$ represents the axis and ψ is the angle of rotation)

$$\mathbf{Q}(\psi \mathbf{m}) = \mathbf{m} \otimes \mathbf{m} + \cos \psi (\mathbf{I} - \mathbf{m} \otimes \mathbf{m}) + \sin \psi \mathbf{m} \times \mathbf{I}, \quad \det \mathbf{Q} = 1, \quad |\psi| < \pi,$$

- inversion

$$-\mathbf{I}$$

The identification procedure is related to solving boundary value problems or eigenvalue problems. Comparing similar two-dimensional and three-dimensional quantities like stresses and forces, etc. one can obtain the effective properties of the plate or shell. The following relations between 2D and 3D properties are assumed:

- forces and moments

$$\mathbf{T} = \langle \boldsymbol{\mu}^{-1} \cdot \boldsymbol{\sigma} \rangle, \quad \mathbf{M} = \langle \boldsymbol{\mu}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{c}z \rangle, \quad \langle \dots \rangle = \int_{-h/2}^{h/2} (\dots) dz,$$

$\boldsymbol{\sigma}$ is the symmetric stress tensor of the classical theory of elasticity, $\boldsymbol{\mu}$ is the shifter tensor (see [61] among others) and $\mathbf{c}$ is the discriminant tensor.

- displacements and rotations

$$\rho_0(\mathbf{u} + \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\varphi}) = \langle \rho_0^* \mathbf{u}_* \rangle, \quad \rho_0(\boldsymbol{\Theta}_1 \cdot \mathbf{u} + \boldsymbol{\Theta}_2^T \cdot \boldsymbol{\varphi}) = \langle \rho_0^* \mathbf{u}_* \cdot \mathbf{c}z \rangle$$

$\mathbf{u}_*$ is the three-dimensional displacement vector.

- external force and moment

$$\rho_0 \mathbf{F}_* = \langle \rho_0^* \mathbf{F}^* \rangle + \mu^+ \boldsymbol{\sigma}_n^+ + \mu^- \boldsymbol{\sigma}_n^-,$$

$$\rho_0 \mathbf{L} = \mathbf{n} \times \langle \rho_0^* \mathbf{F}^* z \rangle + (h/2) \mathbf{n} \times (\mu^+ \boldsymbol{\sigma}_n^+ - \mu^- \boldsymbol{\sigma}_n^-)$$

$\mu^{+(-)} = 1 - (+)hH + (h^2/4)G$, $\boldsymbol{\sigma}_n^{+(-)}$ are stress vectors on the upper and lower face surfaces of the plate or shell. H and G denotes the mean and Gaussian curvature, respectively.

Symmetry considerations (orthotropic material behavior, plane mid-surface) result in the following representation of the stiffness tensors

$$\begin{aligned}
\mathbf{A} &= A_{11}\mathbf{a}_1\mathbf{a}_1 + A_{12}(\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1) + A_{22}\mathbf{a}_2\mathbf{a}_2 + A_{44}\mathbf{a}_4\mathbf{a}_4, \\
\mathbf{B} &= B_{13}\mathbf{a}_1\mathbf{a}_3 + B_{14}\mathbf{a}_1\mathbf{a}_4 + B_{23}\mathbf{a}_2\mathbf{a}_3 + B_{24}\mathbf{a}_2\mathbf{a}_4 + B_{42}\mathbf{a}_4\mathbf{a}_2, \\
\mathbf{C} &= C_{22}\mathbf{a}_2\mathbf{a}_2 + C_{33}\mathbf{a}_3\mathbf{a}_3 + C_{34}(\mathbf{a}_3\mathbf{a}_4 + \mathbf{a}_4\mathbf{a}_3) + C_{44}\mathbf{a}_4\mathbf{a}_4, \\
\mathbf{\Gamma} &= \Gamma_1\mathbf{a}_1 + \Gamma_2\mathbf{a}_2, \quad \mathbf{\Gamma}_1 = \mathbf{0}, \quad \mathbf{\Gamma}_2 = \mathbf{0}
\end{aligned}$$

with $\mathbf{a}_1 = \mathbf{a} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2$, $\mathbf{a}_2 = \mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2\mathbf{e}_2$, $\mathbf{a}_3 = \mathbf{c} = \mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1$, $\mathbf{a}_4 = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1$, $\mathbf{e}_1, \mathbf{e}_2$ are unit basic vectors. In addition, the orthogonality condition for $\mathbf{a}_i$ ($i = 1, 2, 3, 4$) is fulfilled

$$\frac{1}{2}\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$$

Let us assume for the elastic orthotropic law the following cases:

Case 1: Homogeneous plates—all properties are constant with respect to z .

Case 2: Inhomogeneous plates—all properties are functions of z .

The identification of the effective properties can be performed with the help of static boundary value problems (two-dimensional, three-dimensional) and the comparison of the forces and moments (in the sense of averaged stresses or stress resultants)

$$\mathbf{T} = \langle \mathbf{a} \cdot \boldsymbol{\sigma} \rangle, \quad \mathbf{M} = \langle \mathbf{a} \cdot \boldsymbol{\sigma} z \cdot \mathbf{c} \rangle$$

Here the simplest case (plates) is considered. Details for homogeneous shells are given in [95], solutions for eigenvibrations are presented in [94].

- Problem 1: Tension and Bending

The following two-dimensional kinematical field is given

$$\mathbf{u} = D_1 x_1 \mathbf{e}_1 + D_2 x_2 \mathbf{e}_2 - \frac{1}{2} \left(\frac{x_1^2}{R_1} + \frac{x_2^2}{R_2} \right) \mathbf{n}, \quad \boldsymbol{\varphi} = -\frac{x_2}{R_2} \mathbf{e}_1 + \frac{x_1}{R_1} \mathbf{e}_2$$

D_1, D_2, R_1 and R_2 are constants. Then the strains can be computed

$$\boldsymbol{\mu} = D_1 \mathbf{e}_1 \mathbf{e}_1 + D_2 \mathbf{e}_2 \mathbf{e}_2, \quad \boldsymbol{\gamma} = \mathbf{0}, \quad \boldsymbol{\kappa} = \frac{1}{R_1} \mathbf{e}_1 \mathbf{e}_2 - \frac{1}{R_2} \mathbf{e}_2 \mathbf{e}_1$$

and we obtain the forces and moments from the constitutive equations. The three-dimensional strain tensor components are

$$\varepsilon_1 = D_1 + \frac{z}{R_1}, \quad \varepsilon_2 = D_2 + \frac{z}{R_2}$$

Stress tensor components (plane stress state is assumed) are

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}\nu_{21}}(\varepsilon_1 + \nu_{21}\varepsilon_2), \quad \sigma_2 = \frac{E_2}{1 - \nu_{12}\nu_{21}}(\varepsilon_2 + \nu_{12}\varepsilon_1)$$

Finally we have to compare both solutions

$$\begin{aligned}
 A_{11} &= \frac{1}{4} \left\langle \frac{E_1 + E_2 + 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} \right\rangle, \quad A_{12} = \frac{1}{4} \left\langle \frac{E_1 - E_2}{1 - \nu_{12}\nu_{21}} \right\rangle, \\
 A_{22} &= \frac{1}{4} \left\langle \frac{E_1 + E_2 - 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} \right\rangle, \\
 B_{13} &= -\frac{1}{4} \left\langle \frac{E_1 + E_2 + 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} z \right\rangle, \quad -B_{23} = B_{14} = \frac{1}{4} \left\langle \frac{E_1 - E_2}{1 - \nu_{12}\nu_{21}} z \right\rangle, \quad (5) \\
 B_{24} &= \frac{1}{4} \left\langle \frac{E_1 + E_2 - 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} z \right\rangle, \\
 C_{33} &= \frac{1}{4} \left\langle \frac{E_1 + E_2 + 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} z^2 \right\rangle, \quad C_{34} = -\frac{1}{4} \left\langle \frac{E_1 - E_2}{1 - \nu_{12}\nu_{21}} z^2 \right\rangle, \\
 C_{44} &= \frac{1}{4} \left\langle \frac{E_1 + E_2 - 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} z^2 \right\rangle
 \end{aligned}$$

• Problem 2: Plane Shear

Let us assume the two-dimensional kinematical field

$$\mathbf{u} = S_2 x_2 \mathbf{e}_1 + S_1 x_1 \mathbf{e}_2 - S_2 x_1 x_2 \mathbf{n}, \quad \boldsymbol{\varphi} = -S_2 (x_1 \mathbf{e}_1 - x_2 \mathbf{e}_2)$$

S_1 and S_2 are constants. From the kinematical field the strains are computed. With the help of the constitutive equations the forces and moments are estimated.

The corresponding three-dimensional strain tensor component is

$$\gamma_{12} = u_{1,2}^* + u_{2,1}^* = S_1 + S_2 z$$

After calculation the stresses we can compare the results of the two-dimensional and three-dimensional solutions

$$A_{44} = \langle G_{12} \rangle, \quad B_{42} = -\langle G_{12} z \rangle, \quad C_{22} = \langle G_{12} z^2 \rangle \quad (6)$$

• Problem 3: Torsion

Let us introduce a deformable strip ($|x_1| \leq l_1, |x_2| < \infty$) under constant torsion moment at the boundaries $x_1 = \pm l_1$. Then one gets the two-dimensional kinematical field

$$\mathbf{u} = u_2(x_1) \mathbf{e}_2, \quad \boldsymbol{\varphi} = -\varphi_1(x_1) \mathbf{e}_1$$

and the force and moment tensors

$$\mathbf{T} = (A_{44} u_{2,1} - B_{42} \varphi_{2,1}) \mathbf{a}_4 + (\Gamma_1 - \Gamma_2) \varphi_2 \mathbf{e}_2 \mathbf{n}, \quad \mathbf{M} = (B_{42} u_{2,1} - C_{22} \varphi_{2,1}) \mathbf{a}_2$$

The dual three-dimensional problem (strip $|x_1| \leq l_1, |x_2| < \infty, |z| \leq h/2$) results in the displacements

$$u_1^* = w^* = 0, \quad u_2^* = u_2^*(x_1, z),$$

the stress tensor

$$\boldsymbol{\sigma} = G_{12} \frac{\partial u_2^*}{\partial x_1} \mathbf{a}_4 + G_{2n} \frac{\partial u_2^*}{\partial z} (\mathbf{e}_2 \mathbf{n} + \mathbf{n} \mathbf{e}_2)$$

and equilibrium equation

$$G_{12} \frac{\partial^2 u_2^*}{\partial x_1^2} + \frac{\partial}{\partial z} \left(G_{2n} \frac{\partial u_2^*}{\partial z} \right) = 0$$

The solution with respect to the boundary conditions $\sigma_n = \tau_{1n} = \tau_{2n} = 0$ at the top and the bottom surfaces $|z| = h/2$ can be obtained by the following Fourier's ansatz: $u_2^*(x_1, z) = X(x_1)Z(z)$ which yields a Sturm-Liouville problem

$$\frac{d}{dz} \left(G_{2n} \frac{dZ}{dz} \right) + \lambda_*^2 G_{12} Z = 0, \quad \frac{dZ}{dz} \Big|_{|z|=\frac{h}{2}} = 0$$

and

$$\frac{d^2 X}{dx_1^2} - \lambda_*^2 X = 0$$

The lowest non-trivial positive solution λ_* one obtains from

$$X(x_1) = B \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1} \text{ and } u_2^* = B Z(z) \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}$$

Finally, after comparison of the two-dimensional and the three-dimensional solutions one gets

$$\lambda = \lambda_* = \sqrt{\frac{(\Gamma_1 - \Gamma_2)A_{44}}{A_{44}C_{22} - B_{42}^2}}$$

T_{12} and M_{12} obtained by the two-dimensional and the three-dimensional approaches are in a full agreement. For the kinematical fields one gets

$$< G_{12}(u_2^* - u_2 - z\varphi_2)^2 > = \min(u_2, \varphi_2)$$

$$u_2 = \frac{M_{12}^* < G_{12} z >}{< G_{12} > < G_{12} z^2 > - < G_{12} z^2 > \lambda_* \cosh \lambda_* l_1} \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1},$$

$$\varphi_2 = - \frac{M_{12}^* < G_{12} >}{< G_{12} > < G_{12} z^2 > - < G_{12} z^2 > \lambda_* \cosh \lambda_* l_1} \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}$$

In addition, one has to analyze the similar problem for the second direction in the two-dimensional case ($|x_1| < \infty$, $|x_2| \leq l_2$) as in the three-dimensional case ($|x_1| < \infty$, $|x_2| \leq l_2$, $|z| \leq h/2$) with the constant torsion moment at the boundary $|x_2| \leq l_2$. The following final results can be obtained

$$\frac{d}{dz} \left(G_{1n} \frac{dZ^*}{dz} \right) + \eta^2 G_{12} Z^* = 0, \quad \left. \frac{dZ^*}{dz} \right|_{|z|=\frac{h}{2}} = 0, \quad \eta = \sqrt{\frac{(\Gamma_1 + \Gamma_2) A_{44}}{A_{44} C_{22} - B_{42}^2}}$$

Finally, we get the expressions for the transverse shear stiffness tensor components

$$\Gamma_1 = \frac{1}{2}(\lambda^2 + \eta^2) \frac{A_{44} C_{22} - B_{42}^2}{A_{44}}, \quad \Gamma_2 = \frac{1}{2}(\eta^2 - \lambda^2) \frac{A_{44} C_{22} - B_{42}^2}{A_{44}} \quad (7)$$

From the above mentioned stiffness values one gets the classical stiffness tensors for the isotropic homogeneous plate. The basic geometrical property is the thickness h , the plate is symmetrical with respect to the mid-plane which results in $\mathbf{B} \equiv \mathbf{0}$. Let us assume the following material data: the Young's modulus E and the shear modulus $G = E/2(1 + \nu)$, ν is the Poisson's ratio. All material properties are constant. The in-plane and out-of-plane stiffness parameters can be computed by

$$\begin{aligned} A_{11} &= \frac{Eh}{2(1 - \nu)}, \quad A_{22} = \frac{Eh}{2(1 + \nu)} = A_{44} = Gh, \\ C_{33} &= \frac{Eh^3}{24(1 - \nu)}, \quad C_{44} = \frac{Eh^3}{24(1 + \nu)} = C_{22} = \frac{Gh^3}{12} \end{aligned}$$

The classical plate (bending) stiffness [85] follows as

$$C_{33} + C_{44} = \frac{Eh^3}{12(1 - \nu^2)},$$

the transverse shear stiffness can be estimated by

$$\Gamma = \lambda^2 C_{22} \quad \text{with} \quad \frac{d^2 Z}{dz^2} + \lambda^2 Z = 0, \quad \left. \frac{dZ}{dz} \right|_{|z|=\frac{h}{2}} = 0$$

$\cos \lambda z = 0$ yields the smallest eigenvalue $\lambda = \frac{\pi}{h}$

$$\Gamma = \frac{\pi^2}{h^2} \frac{Gh^3}{12} = \frac{\pi^2}{12} Gh$$

The value $\pi^2/12$ is the same like the shear correction of [57], Reissner's estimate $5/6$ [76] is close to this value.

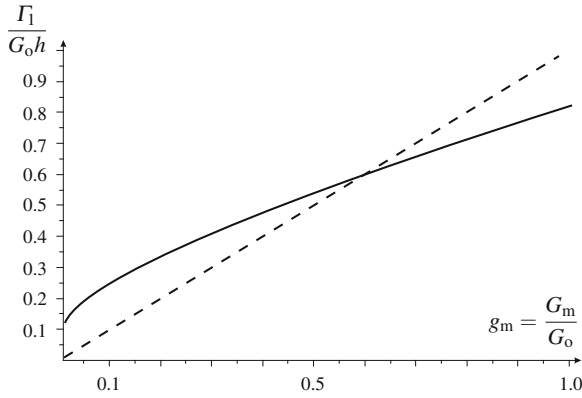


Fig. 11 Transverse shear stiffness of a FGM plate (Reissner's solution is presented by the *dashed* line, G_m , G_o shear modulus at the midplane, at outer faces, respectively)

The suggested approach can be applied in more general cases. As an example on Fig. 11 the transverse shear stiffness of a FGM plate is shown, for details see [1]. The following comments can be made

- Reissner's solution gives understated values of the transverse stiffness when the difference between elastic moduli is big enough.
- On the other hand, Reissner's solution gives overstated values when the elastic moduli do not differ.
- Reissner's formula gives us good coincidence with our results when the $g_m \sim 0.6$.

4 Nano-Sized Plates and Shells

The development of nanotechnologies extends the field of application of the classical or non-classical theories of plates and shells towards the new thin-walled structures. Nanomaterials have physical properties which are different from the bulk material. The classical linear elasticity can be extended to the nanoscale by taking into account the surface stresses. In particular, the surface stresses are responsible for the size-effect, that means the material properties of a specimen depend on its size. For example, Young's modulus of a cylindrical specimen increases significantly, when the cylinder diameter becomes very small. The surface stresses are the generalization of the scalar surface tension which is well-known phenomenon in the theory of capillarity.

The investigations of the surface phenomena were initiated by Laplace, Young and Gibbs. A summary of the investigations are given in the reviews of [68] or [23]. In [32, 68, 84] the surface stresses are considered. Recently two-dimensional theories of nanosized plates and shells were suggested. The theory of elasticity with surface stresses is applied to the modifications of the two-dimensional theories of nanosized

plates in [33, 53] or [9]. Various theories of plates are formulated. The approaches can be classified, for example, by the starting point of the derivation. This can be the well-known three-dimensional continuum mechanics equations. In contrast, one can introduce à priori a two-dimensional deformable surface which is the basis for a more natural formulation of the two-dimensional governing equations. This so-called direct approach should be supplemented by the theoretical or experimental determination of the material parameters included in the constitutive equations.

4.1 Basic Equations of Linear Elasticity with Surface Stresses

Let V is a bounded domain in $\mathbb{R}^3$ with sufficiently regular boundary that a body occupies. Here we consider problems with mixed boundary conditions. Suppose Ω_1 , a nonempty part of the boundary surface Ω of V , to be fixed: $\mathbf{u}|_{\Omega_1} = \mathbf{0}$ ($\mathbf{u}$ is the three-dimensional displacement vector). On the rest part $\Omega_2 = \Omega \setminus \Omega_1$ it is defined the stress vector $\mathbf{t}$ expressed through a given load $\boldsymbol{\phi}$ and $\mathbf{t}_S$ (the stress vector due the surface stresses) by the formula

$$\mathbf{t} = \boldsymbol{\phi} + \mathbf{t}_S,$$

where $\mathbf{t}_S$ is determined through the surface stress tensor $\boldsymbol{\tau}$, see [32, 68]. As a result, we have

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \ddot{\mathbf{u}}, \quad \mathbf{x} \in V, \quad (8)$$

$$\mathbf{u}|_{\Omega_1} = \mathbf{0}, \quad \mathbf{n} \cdot \boldsymbol{\sigma}|_{\Omega_2} = \mathbf{t}, \quad \mathbf{x} \in \Omega, \quad (9)$$

where $\boldsymbol{\sigma}$ is the stress tensor, ∇ the 3D gradient operator (3D nabla operator), ρ the body density, $\mathbf{f}$ the density of the volume forces, $\mathbf{n}$ the external unit normal to Ω , and the dot over a quantity denotes its partial derivative with respect to the time t . The surface stress vector is defined by

$$\mathbf{t}_S = \tilde{\nabla} \cdot \boldsymbol{\tau}, \quad (10)$$

where $\boldsymbol{\tau}$ is the surface stress tensor on Ω and $\tilde{\nabla}$ is the nabla operator on the surface Ω that relates with ∇ by the formula

$$\tilde{\nabla} = \nabla - \mathbf{n} \frac{\partial}{\partial z},$$

and z is the coordinate along the normal to Ω .

Let us consider the special problem when the static conditions are given on the whole boundary

$$\mathbf{n} \cdot \boldsymbol{\sigma}|_{\Omega} = \mathbf{t}, \quad \mathbf{x} \in \Omega \quad (11)$$

For simplicity, we restrict ourselves to an isotropic material. The constitutive equation for the material is Hooke's law

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + \lambda\mathbf{I}\text{tr}\mathbf{e} \quad \text{with} \quad \mathbf{e} = \mathbf{e}(\mathbf{u}) \equiv \frac{1}{2} \left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T \right] \quad (12)$$

For the surface stresses we assume the following constitutive equation:

$$\boldsymbol{\tau} = \tau_0\mathbf{a} + 2\mu_S\boldsymbol{\varepsilon} + \lambda_S\mathbf{a}\text{tr}\boldsymbol{\varepsilon} \quad \text{with} \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{v}) \equiv \frac{1}{2} \left[\tilde{\nabla}\mathbf{v} \cdot \mathbf{a} + \mathbf{a} \cdot (\tilde{\nabla}\mathbf{v})^T \right], \quad (13)$$

where $\mathbf{v}$ is the displacement of the film point $\mathbf{x}$ of Ω_2 , τ_0 is the residual (initial) surface tension. Here $\mathbf{I}$ and $\mathbf{a} \equiv \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ are the three- and two-dimensional unit tensors, respectively, λ and μ are Lamé's coefficients of the bulk material whereas λ_S and μ_S are the elastic characteristics of the surface film Ω_2 (they are the surface analogues of Lamé's coefficients), $\mathbf{e}$ is the small strain tensor, and $\boldsymbol{\varepsilon}$ is the surface strain tensor. Following [32], we use the non-separation condition

$$\mathbf{u}|_{\Omega_2} = \mathbf{v}$$

This explicitly states that the displacements of the surface film Ω_2 coincide with the body displacements on the boundary. There are more general relations for the surface stresses that include residual stresses, anisotropy and other factors [6, 32, 69].

In equilibrium, the dynamic Eq. (8) changes to

$$\nabla \cdot \boldsymbol{\sigma} + \rho\mathbf{f} = \mathbf{0}. \quad (14)$$

Thus the equilibrium boundary value problem for an elastic body with surface stresses consists of Eq. (14) and the boundary conditions

$$\mathbf{u}|_{\Omega_1} = \mathbf{0}, \quad (\mathbf{n} \cdot \boldsymbol{\sigma} - \tilde{\nabla} \cdot \boldsymbol{\tau})|_{\Omega_2} = \boldsymbol{\phi}, \quad (15)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ satisfy relations (12) and (13), respectively. In Eq. (13) we set

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}) \equiv \frac{1}{2} \left[\tilde{\nabla}\mathbf{u} \cdot \mathbf{a} + \mathbf{a} \cdot (\tilde{\nabla}\mathbf{u})^T \right]|_{\Omega_2}$$

If $\Omega_2 = \Omega$ then part Ω_1 is absent.

4.2 Transition to the Theory of Plates and Shells

Let us consider thin-walled 3D solid called also shell-like body (Fig. 12). We assume that the volume of the shell-like body V is bounded by two faces $\Omega_{\pm}$ and lateral surface Ω_v . We also introduce the base (middle) surface ω . The radius-vector of

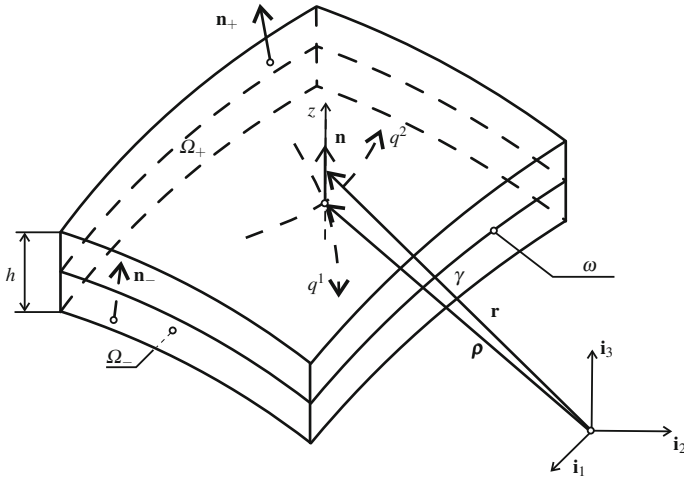


Fig. 12 Shell-like body

material points $\mathbf{r}$ is given as follows

$$\mathbf{r} = \boldsymbol{\rho}(q^1, q^2) + z\mathbf{n},$$

where $\boldsymbol{\rho}$ is the radius-vector of ω , $\mathbf{n}$ is the unit normal to ω , z is the transverse coordinate, $z \in [-h/2, h/2]$, h is the shell thickness, q^1, q^2 the coordinates on ω . Radius-vectors of $\Omega_{\pm}$ are given by $\mathbf{r}_{\pm} = \boldsymbol{\rho} \pm \mathbf{n}h/2$. For the sake of simplicity we assume that h is a constant.

Let us recall basic formulas used in the tensor calculus for description of tensor fields near ω , see [25, 49]. The basic and reciprocal bases on ω are given by

$$\boldsymbol{\rho}_{\alpha} = \frac{\partial \boldsymbol{\rho}}{\partial q^{\alpha}}, \quad \boldsymbol{\rho}_{\alpha} \cdot \boldsymbol{\rho}^{\beta} = \delta_{\alpha}^{\beta}, \quad \alpha, \beta = 1, 2,$$

where δ_{α}^{β} is the Kronecker symbol. The surface nabla-operator on ω is defined as

$$\tilde{\nabla} = \boldsymbol{\rho}^{\alpha} \frac{\partial}{\partial q^{\alpha}}$$

We use q^1, q^2, z as coordinates in the neighborhood of ω . Then we have

$$\begin{aligned} \mathbf{r}_{\alpha} &= \frac{\partial \mathbf{r}}{\partial q^{\alpha}} = \boldsymbol{\rho}_{\alpha} + z \frac{\partial \mathbf{n}}{\partial q^{\alpha}} = (\mathbf{a} - z\mathbf{b}) \cdot \boldsymbol{\rho}_{\alpha}, \quad \mathbf{r}_3 = \mathbf{r}^3 = \mathbf{n}, \\ \mathbf{r}^{\alpha} &= (\mathbf{a} - z\mathbf{b})^{-1} \cdot \boldsymbol{\rho}^{\alpha}, \quad \mathbf{r}_{\alpha} \cdot \mathbf{r}^{\beta} = \delta_{\alpha}^{\beta}, \quad \mathbf{b} = -\tilde{\nabla} \mathbf{n}, \end{aligned}$$

where $\mathbf{b}$ is the curvature tensor of ω . Operators ∇ and $\tilde{\nabla}^\pm$ are reduced to

$$\begin{aligned}\nabla &= \mathbf{r}^\alpha \frac{\partial}{\partial q^\alpha} + \mathbf{n} \frac{\partial}{\partial z} = (\mathbf{a} - z\mathbf{b})^{-1} \cdot \boldsymbol{\rho}^\alpha \frac{\partial}{\partial q^\alpha} + \mathbf{n} \frac{\partial}{\partial z} \\ &= (\mathbf{a} - z\mathbf{b})^{-1} \cdot \tilde{\nabla} + \mathbf{n} \frac{\partial}{\partial z},\end{aligned}$$

$$\tilde{\nabla}^\pm = \left(\mathbf{a} \mp \frac{h}{2} \mathbf{b} \right)^{-1} \cdot \boldsymbol{\rho}^\alpha \frac{\partial}{\partial q^\alpha} = \left(\mathbf{a} \mp \frac{h}{2} \mathbf{b} \right)^{-1} \cdot \tilde{\nabla}$$

Let us note that here the inverse tensor of $\mathbf{a} - z\mathbf{b}$ is defined according to the rule

$$(\mathbf{a} - z\mathbf{b})^{-1} \cdot (\mathbf{a} - z\mathbf{b}) = (\mathbf{a} - z\mathbf{b}) \cdot (\mathbf{a} - z\mathbf{b})^{-1} = \mathbf{a}$$

We assume that on $\Omega_\pm$ the surface stresses act, so $\Omega_2 = \Omega_+ \cup \Omega_-$. As a results, this leads to the following boundary conditions on $\Omega_\pm$

$$(\mathbf{n}_\pm \cdot \boldsymbol{\sigma} \mp \tilde{\nabla}^\pm \cdot \boldsymbol{\tau}_S^\pm) \big|_{\Omega_\pm} = \boldsymbol{\phi}_\pm \quad (16)$$

In (16) $\mathbf{n}_\pm$ are normals to $\Omega_\pm$ (see Fig. 12), $\boldsymbol{\tau}_S^\pm$ and $\boldsymbol{\phi}_\pm$ are surface stresses and loads on $\Omega_\pm$,

$$\boldsymbol{\tau}_\pm = \tau_0^\pm \mathbf{A} + \lambda_S^\pm \text{atr} \boldsymbol{\varepsilon}_\pm + 2\mu_S^\pm \boldsymbol{\varepsilon}_\pm, \quad 2\boldsymbol{\varepsilon}_\pm = (\tilde{\nabla}^\pm \mathbf{u}_S^\pm) \cdot \mathbf{a} + \mathbf{a} \cdot (\tilde{\nabla}^\pm \mathbf{u}_S^\pm)^T$$

Here $\mu_S^\pm$ $\lambda_S^\pm$ are the surface elastic moduli, and $\tau_0^\pm$ are residual surface stresses on $\Omega_\pm$. Let us note that the surface nabla operators $\tilde{\nabla}^\pm$ are $\Omega_\pm$ differ from each other, in general.

For transition to the 2D equations of plates and shells we use the through-the-thickness integration procedure presented in, for example, [19, 49, 51]. Integrating (14) with respect to z and taking into account (16), we obtain

$$\tilde{\nabla} \cdot \mathbf{T} + G_+ \tilde{\nabla}^+ \cdot \boldsymbol{\tau}_+ + G_- \tilde{\nabla}^- \cdot \boldsymbol{\tau}_- + \mathbf{q} = \mathbf{0}, \quad (17)$$

where

$$\mathbf{T} = \langle (\mathbf{a} - z\mathbf{b})^{-1} \cdot \boldsymbol{\sigma} \rangle, \quad (18)$$

$$\mathbf{q} = G_+ \boldsymbol{\varphi}_+ - G_- \boldsymbol{\varphi}_- + \langle \mathbf{f} \rangle, \quad \langle (\dots) \rangle = \int_{-h/2}^{h/2} (\dots) G dz,$$

$$G = G(z) \equiv \det(\mathbf{a} - z\mathbf{b}), \quad G_\pm = G(\pm h/2).$$

$\mathbf{T}$ is the stress resultant tensor and $\mathbf{q}$ is the surface loads.

Cross-multiplying (14) by $z\mathbf{n}$ and integrating through the thickness using (16), we obtain the second equilibrium equation

$$\tilde{\nabla} \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{m} + \frac{h}{2} G_+ \mathbf{n} \times \tilde{\nabla}^+ \cdot \boldsymbol{\tau}_+ - \frac{h}{2} G_- \mathbf{n} \times \tilde{\nabla}^- \cdot \boldsymbol{\tau}_- = \mathbf{0}, \quad (19)$$

$$\mathbf{M} = -\langle (\mathbf{a} - z\mathbf{b})^{-1} \cdot z\boldsymbol{\sigma} \times \mathbf{n} \rangle, \quad (20)$$

$$\mathbf{m} = \frac{h}{2} G_+ \mathbf{n} \times \boldsymbol{\varphi}_+ + \frac{h}{2} G_- \mathbf{n} \times \boldsymbol{\varphi}_- + \langle z\mathbf{n} \times \mathbf{f} \rangle.$$

Here $\mathbf{M}$ is the moment stress tensor and $\mathbf{m}$ is the surface moments, the subindex $\times$ stands for vectorial invariant of stress tensor [92], see [49]. In particular, for a diad $\mathbf{a} \otimes \mathbf{b}$ it is given by $(\mathbf{a} \otimes \mathbf{b})_\times = \mathbf{a} \times \mathbf{b}$.

Omitting algebraic manipulations and assuming $h\|\mathbf{b}\| \ll 1$ we reduce with accuracy of $O(h\|\mathbf{b}\|)$ Eqs. (18) and (20) to more simple expressions

$$\mathbf{T} = \langle \mathbf{a} \cdot \boldsymbol{\sigma} \rangle, \quad \mathbf{M} = -\langle \mathbf{a} \cdot z\boldsymbol{\sigma} \times \mathbf{n} \rangle$$

Thus, (17) and (19) transform to

$$\tilde{\nabla} \cdot \mathbf{T} + \tilde{\nabla} \cdot (\boldsymbol{\tau}_+ + \boldsymbol{\tau}_-) + \mathbf{q} = \mathbf{0}, \quad (21)$$

$$\tilde{\nabla} \cdot \mathbf{M} - \frac{h}{2} \tilde{\nabla} \cdot [(\boldsymbol{\tau}_+ - \boldsymbol{\tau}_-) \times \mathbf{n}] + \mathbf{T}_\times + [(\boldsymbol{\tau}_+ - \boldsymbol{\tau}_-) \cdot \mathbf{b}]_\times + \mathbf{m} = \mathbf{0} \quad (22)$$

Since $\mathbf{m} \cdot \mathbf{n} = 0$, from (22) it follows the so-called sixth equilibrium equation in the form

$$\mathbf{M} \cdot \mathbf{b} + \mathbf{T}_\times \cdot \mathbf{n} = 0$$

Equations (21) and (22) dictate the form of effective stress resultants tensors $\mathbf{T}^*$ and $\mathbf{M}^*$ as follows

$$\mathbf{T}^* = \mathbf{T} + \mathbf{T}_S, \quad \mathbf{M}^* = \mathbf{M} + \mathbf{M}_S, \quad (23)$$

$$\mathbf{T}_S = \boldsymbol{\tau}_+ + \boldsymbol{\tau}_-, \quad \mathbf{M}_S = -\frac{h}{2}(\boldsymbol{\tau}_+ - \boldsymbol{\tau}_-) \times \mathbf{n}$$

For description of the shell deformations we use the standard approximation of the first order shear deformable plate and shell theory

$$\mathbf{u}(q^1, q^2, z) = \mathbf{w}(q^1, q^2) - z\boldsymbol{\vartheta}(q^1, q^2), \quad \mathbf{n} \cdot \boldsymbol{\vartheta} = 0. \quad (24)$$

Within the theory it is assumed that the rotation vector $\boldsymbol{\vartheta}$ is kinematically independent on the vector of translations $\mathbf{w}$ of the middle surface. From (24) we derive the following formulae

$$\mathbf{u}_S^\pm = \mathbf{w} \mp \frac{h}{2} \boldsymbol{\vartheta}, \quad \boldsymbol{\varepsilon}_\pm = \boldsymbol{\varepsilon} \mp \frac{h}{2} \boldsymbol{\kappa}, \quad (25)$$

where

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left(\tilde{\nabla} \mathbf{w} \cdot \mathbf{a} + \mathbf{a} \cdot (\tilde{\nabla} \mathbf{w})^T \right), \quad \boldsymbol{\kappa} = \frac{1}{2} \left(\tilde{\nabla} \boldsymbol{\vartheta} \cdot \mathbf{a} + \mathbf{a} \cdot (\tilde{\nabla} \boldsymbol{\vartheta})^T \right)$$

are 2D strain measures. Using (25), we obtain the following expressions of $\boldsymbol{\tau}_\pm$

$$\boldsymbol{\tau}_\pm = \tau_0^\pm \mathbf{a} + \lambda_S^\pm \text{atr} \boldsymbol{\varepsilon} + 2\mu_S^\pm \boldsymbol{\varepsilon} \mp \frac{h}{2} (\lambda_S^\pm \text{atr} \boldsymbol{\kappa} + 2\mu_S^\pm \boldsymbol{\kappa})$$

Thus, we have the formulae

$$\begin{aligned} \boldsymbol{\tau}_+ + \boldsymbol{\tau}_- &= (\tau_0^+ + \tau_0^-) \mathbf{a} + (\lambda_S^+ + \lambda_S^-) \text{atr} \boldsymbol{\varepsilon} + 2(\mu_S^+ + \mu_S^-) \boldsymbol{\varepsilon} \\ &\quad - \frac{h}{2} [(\lambda_S^+ - \lambda_S^-) \text{atr} \boldsymbol{\kappa} + 2(\mu_S^- - \mu_S^+) \boldsymbol{\kappa}], \\ \boldsymbol{\tau}_+ - \boldsymbol{\tau}_- &= (\tau_0^+ - \tau_0^-) \mathbf{a} + (\lambda_S^+ - \lambda_S^-) \text{atr} \boldsymbol{\varepsilon} + 2(\mu_S^+ - \mu_S^-) \boldsymbol{\varepsilon} \\ &\quad - \frac{h}{2} [(\lambda_S^+ + \lambda_S^-) \text{atr} \boldsymbol{\kappa} + 2(\mu_S^- + \mu_S^+) \boldsymbol{\kappa}] \end{aligned}$$

In what follows we consider the same surface properties for both faces, that is $\tau_0^+ = \tau_0^- = \tau_0$, $\mu_S^+ = \mu_S^- = \mu_S$, $\lambda_S^+ = \lambda_S^- = \lambda_S$. In this case the latter formulae can be simplified as follows

$$\boldsymbol{\tau}_+ + \boldsymbol{\tau}_- = 2\tau_0 \mathbf{a} + 2\lambda_S \text{atr} \boldsymbol{\varepsilon} + 4\mu_S \boldsymbol{\varepsilon}, \quad \boldsymbol{\tau}_+ - \boldsymbol{\tau}_- = -h (\lambda_S \text{atr} \boldsymbol{\kappa} + 2\mu_S \boldsymbol{\kappa})$$

This leads to the following expressions for stress resultants:

$$\begin{aligned} \mathbf{T}_S &= 2\tau_0 \mathbf{a} + C_1^S \boldsymbol{\varepsilon} + C_2^S \text{atr} \boldsymbol{\varepsilon}, \quad \mathbf{M}_S = - \left[D_1^S \boldsymbol{\kappa} + D_2^S \text{atr} \boldsymbol{\kappa} \right] \times \mathbf{n}, \\ C_1^S &= 4\mu_S, \quad C_2^S = 2\lambda_S, \quad D_1^S = h^2 \mu_S, \quad D_2^S = h^2 \lambda_S / 2 \end{aligned} \quad (26)$$

Taking into account (23) from (26) it follows that the surface stresses do not influence on the transverse shear forces and the transverse shear stiffness. Indeed, $\mathbf{T}_S \cdot \mathbf{n} = \mathbf{0}$.

For $\mathbf{T}$ and $\mathbf{M}$ we assume the following constitutive relations

$$\begin{aligned} \mathbf{T} \cdot \mathbf{a} - \frac{1}{2} (\mathbf{M} \cdot \cdot \mathbf{b}) \mathbf{a} \times \mathbf{n} &= \frac{\partial W_S}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{T} \cdot \mathbf{n} = \frac{\partial W_S}{\partial \boldsymbol{\gamma}}, \quad \mathbf{M} = \frac{\partial W_S}{\partial \boldsymbol{\kappa}}, \\ 2W_S &= \boldsymbol{\varepsilon} \cdot \cdot \mathbf{C} \cdot \cdot \boldsymbol{\varepsilon} + \boldsymbol{\kappa} \cdot \cdot \mathbf{D} \cdot \cdot \boldsymbol{\kappa} + \Gamma \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} \end{aligned}$$

Here W_S is the surface strain energy density, $\mathbf{C}$ and $\mathbf{D}$ are fourth-order tangential and bending stiffness tensors, respectively, $\boldsymbol{\gamma}$ is the vector of transverse shear deformations, $\boldsymbol{\gamma} = \tilde{\nabla}(\mathbf{w} \cdot \mathbf{n}) - \boldsymbol{\vartheta}$, and Γ is the transverse shear stiffness. In the case of an isotropic shells the tangential and bending stiffness tensors take the form

$$\mathbf{C} = C_{11}\mathbf{a}_1\mathbf{a}_1 + C_{22}(\mathbf{a}_2\mathbf{a}_2 + \mathbf{a}_4\mathbf{a}_4), \quad \mathbf{D} = D_{22}(\mathbf{a}_2\mathbf{a}_2 + \mathbf{a}_4\mathbf{a}_4) + D_{33}\mathbf{a}_3\mathbf{a}_3 \quad (27)$$

Components C_{11} , C_{22} , D_{22} , D_{33} and Γ are given by

$$\begin{aligned} C_{11} &= \frac{Eh}{2(1-\nu)}, & C_{22} &= \frac{Eh}{2(1+\nu)}, \\ D_{22} &= \frac{Eh^3}{24(1+\nu)}, & D_{33} &= \frac{Eh^3}{24(1-\nu)}, & \Gamma &= k\mu h, \\ E &= 2\mu(1+\nu), & \nu &= \frac{\lambda}{2(\lambda+\mu)}, \\ C \equiv C_{11} + C_{22} &= \frac{Eh}{1-\nu^2}, & D \equiv D_{11} + D_{22} &= \frac{Eh^3}{12(1-\nu^2)}, \end{aligned}$$

where C , D are tangential and bending stiffness of the shell, E and ν are the Young modulus and Poisson ratio of the shell material, and k is similar to the shear correction factor, see [31].

4.3 Theory of Plates

In the case of the plates the constitutive equations take more simple form. Indeed, here $\mathbf{b} = \mathbf{0}$, $\mathbf{n} = \mathbf{i}_3$. Effective stress resultants in the case of symmetric plates are given by [9, 24]

$$\begin{aligned} \mathbf{T}^* &= C_1\boldsymbol{\varepsilon} + C_2\text{atr}\boldsymbol{\varepsilon} + \Gamma\boldsymbol{\gamma} \otimes \mathbf{i}_3, & \mathbf{M}^* &= -[D_1\boldsymbol{\kappa} + D_2\text{atr}\boldsymbol{\kappa}] \times \mathbf{i}_3, \\ C_1 &= C(1-\nu) + 4\mu^S, & C_2 &= C\nu + 2\lambda^S, \\ D_1 &= D(1-\nu) + h^2\mu^S, & D_2 &= D\nu + h^2\lambda^S/2 \end{aligned} \quad (28)$$

The effective tangential and bending stiffness are

$$C_{\text{eff}} \equiv C_1 + C_2 = C + 4\mu_S + 2\lambda_S, \quad D_{\text{eff}} \equiv D_1 + D_2 = D + h^2\mu_S + h^2\lambda_S/2$$

Constitutive relations (28) with the equilibrium Eqs. (21) and (22) lead to equilibrium equations for translations $\mathbf{w}$ and rotations $\boldsymbol{\vartheta}$. In particular, the equation for deflection $w = \mathbf{w} \cdot \mathbf{i}_3$ can be transformed to

$$D_{\text{eff}}\tilde{\Delta}\tilde{\Delta}w = \tilde{\nabla} \cdot \mathbf{m} - \frac{D_{\text{eff}}}{\Gamma}\tilde{\Delta}q_n + q_n, \quad q_n = \mathbf{q} \cdot \mathbf{i}_3, \quad \tilde{\Delta} = \tilde{\nabla} \cdot \tilde{\nabla}$$

Let us consider dependence of D_{eff} , C_1 , C_2 , D_1 , D_2 on thickness h . The diagram D_{eff} vs. h is given in Fig. 13. Dimensionless parameters $\bar{C}_1 = C_1/C(1-\nu)$, $\bar{C}_2 = C_2/C\nu$, $\bar{D}_1 = D_1/D(1-\nu)$, $\bar{D}_2 = D_2/D\nu$ are shown in Fig. 14, [9, 24] for details. Here the material parameters take the values $\mu = 34.7$ GPa, $\nu = 0.3$, $\lambda_S = -3.48912$ N/m, $\mu_S = 6.2178$ N/m.

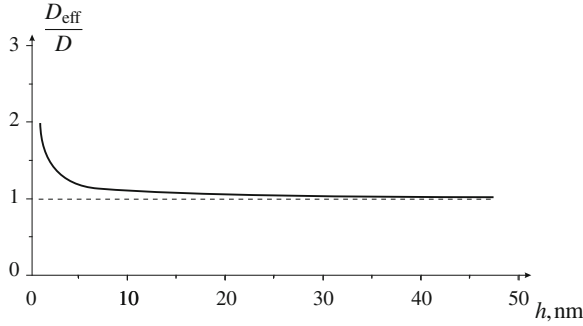


Fig. 13 Dependence of the bending stiffness on the thickness

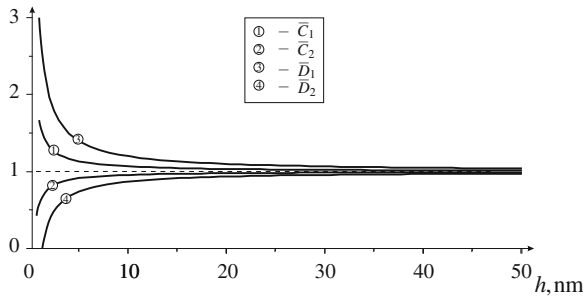


Fig. 14 Dependencies of the stiffness parameters on the thickness

4.4 Comparison with the Theory of Three-Layered Plates

Obviously, that there is similarity between the theory of plate with surface stresses and the theory of three-layered plates. Let us consider tension of a strip subjected by force P for two cases. The first is the tension of a strip with surface stresses while the second one is the tension of a sandwich plate, see Fig. 15. Let σ be stress in the bulk, τ be surface stress, and τ_f be the stress in the faces. Then we have elementary formulae

$$P = \sigma h + 2\tau, \quad P = \sigma(h - 2h_f) + 2\tau_f h_f,$$

where h and h_f are the thickness and thickness of surface layer, respectively. It is clear that

$$\tau = (\tau_f - \sigma)h_f$$

This gives one the possibility to interpret the surface stress τ as excess stress resultant for the surface layer of thickness h_f . As a result, the case of surface stresses can be obtained when $h_f \rightarrow 0$: $\tau = \lim_{h_f \rightarrow 0} \tau_f h_f$.

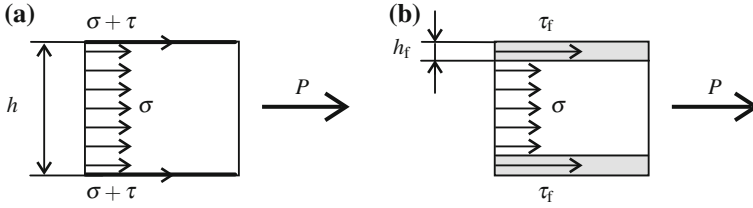


Fig. 15 Tension of **a** a strip with surface tension; **b** a three-layered strip

For detailed analysis we use the model of three-layered plates presented above. Let us consider the isotropic three-layered plate, see Fig. 15b. Here h_c is the thickness of the core, h_f the thickness of faces, and $h_c \gg h_f$. The thickness of the plate is $h = h_c + 2h_f$. The material properties are given by Young's moduli E_c , E_f and Poisson's ratio ν_c and ν_f of faces and core, respectively (or by shear moduli μ_c , μ_f).

Constitutive equations of three-layered plate are given by

$$\mathbf{T} = C_1 \boldsymbol{\varepsilon} + C_2 \text{atr} \boldsymbol{\varepsilon} + \Gamma \boldsymbol{\gamma} \otimes \mathbf{i}_3, \quad \mathbf{M} = -[D_1 \boldsymbol{\kappa} + D_2 \text{atr} \boldsymbol{\kappa}] \times \mathbf{i}_3, \quad (29)$$

where stiffness parameters can be computed by

$$\begin{aligned} C_1 &= 2C_{22}, \quad C_2 = C_{11} - C_{22}, \\ D_1 &= 2D_{22}, \quad D_2 = D_{33} - D_{22}, \quad \Gamma = \ell^2 D_{22}, \\ C_{11} &= \frac{1}{2} \left(\frac{2E_f h_f}{1 - \nu_f} + \frac{E_c h_c}{1 - \nu_c} \right), \quad C_{22} = \frac{1}{2} \left(\frac{2E_f h_f}{1 + \nu_f} + \frac{E_c h_c}{1 + \nu_c} \right), \\ D_{22} &= \frac{1}{24} \left[\frac{E_f (h^3 - h_c^3)}{1 + \nu_f} + \frac{E_c h_c^3}{1 + \nu_c} \right], \quad D_{33} = \frac{1}{24} \left[\frac{E_f (h^3 - h_c^3)}{1 - \nu_f} + \frac{E_c h_c^3}{1 - \nu_c} \right], \end{aligned}$$

where ℓ is the minimal non-zero root of the equation

$$\mu_0 \cos \ell \frac{h_f}{2} \cos \ell \frac{h_c}{2} - \sin \ell \frac{h_f}{2} \sin \ell \frac{h_c}{2} = 0, \quad \mu_0 = \mu_c / \mu_f$$

The bending stiffness of three-layered plate is

$$D_{\text{eff}} = D_{33} + D_{44} = \frac{1}{12} \left[\frac{E_f (h^3 - h_c^3)}{1 - \nu_f^2} + \frac{E_c h_c^3}{1 - \nu_c^2} \right]$$

Comparing (28) and (29), we can determine λ_S and μ_S within the parameters of the surface layers that is with parameters E_f , ν_f and h_f . Assuming $E_c = E$, $\nu_c = \nu$ and comparing tangential stiffness of three-layered plate and plate with surface stresses when $h_f \rightarrow 0$ with accuracy of $O(h_f^2)$, we obtain that

$$\mu_S \approx \frac{E_f h_f}{2(1 + \nu_f)} \equiv \mu_f h_f, \quad \lambda_S \approx \frac{\nu_f E_f h_f}{1 - \nu_f^2} \equiv \lambda_f h_f \frac{1 - 2\nu_f}{1 - \nu_f}, \quad (30)$$

where λ_f is the Lamé modulus of the surface layer. The same formulas follow from comparison of the bending stiffness when $h_f \rightarrow 0$. Thus, we derive

$$\mu_S = \lim_{h_f \rightarrow 0} \mu_f h_f, \quad \lambda_S = \lim_{h_f \rightarrow 0} \lambda_f \frac{1 - 2\nu_f}{1 - \nu_f} h_f$$

The latter relations determine the surface elastic moduli μ_S and λ_S through the elastic moduli of the surface layer and its thickness. Formulas (30) are exact $h_f \rightarrow 0$. For finite values of h_f the accuracy is $O(h_f^2)$. A more general model of plates with surface stresses is presented by [11].

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