Chapter 2
Variational Principles for Canonical Profiles in a Tokamak

Abstract This Chapter is devoted to the variational formulation for the "canonical profiles" of the plasma temperature and pressure. The basis for the variational description is the functional for the magnetic energy associated with the plasma current together with the conditions for the conservation of the total plasma current and total magnetic flux. The variation of this functional leads to the Euler equation that defines the canonical profile. We start with a cylindrical plasma with circular cross section and then generalize to a toroidal plasma with arbitrary cross section. We also derive the variational formulation for the canonical profile of the toroidal rotation.

2.1 The Principle of Total Energy Minimum by Hsu and Chu

According to Hsu and Chu [1], let's introduce the polar coordinates system r, φ, z with z-axis which coincides with the main axis of the torus. The general form for magnetic field in a tokamak (axially symmetric torus) is as follows:

$$r\mathbf{B} = [\nabla\psi][\nabla\varphi][\nabla\varphi] \tag{2.1}$$

where ψ is a potential of poloidal magnetic field, $F = F(\psi)$ is a diamagnetic function ($F = rB_\varphi$). Grad-Shafranov two-dimensional equilibrium equation

$$\Delta^*\psi = -rj_\varphi = -(FF' + r^2 p') \tag{2.2}$$

determines the potential ψ distribution in space. Here $j_\varphi = j_\varphi(\psi, r)$ is a toroidal current density, $p = p(\psi)$ is a plasma pressure, $p' = dp/d\psi$,

$$\Delta^*\psi \equiv r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2}. \tag{2.3}$$

The equation $\psi = \text{const}$ defines the magnetic surfaces.

It is assumed in this model that the relaxed equilibrium state of the plasma is determined by the minimum of the total plasma energy W (including magnetic and thermal energy)

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$$W = \int_V dV \left\{ \left[F^2 + (\nabla \psi)^2 \right] / (2r^2) + \frac{3}{2} p \right\}, \tag{2.4}$$

while maintaining the total toroidal current

$$I = \frac{1}{2\pi} \int_V \frac{(F\,F' + r^2 p')}{r^2} dV \tag{2.5}$$

and the equilibrium condition (2.2). The independent variable in the variation problem is the variable ψ.

We will solve the variation problem (2.2, 2.4, 2.5) with Lagrange method. Let's introduce the extended functional

$$W_{ex} = W - 2\pi\lambda I \tag{2.6}$$

and consider the problem of the unconditional extremum of the functional (2.6). The first variation of (2.6) must be equal to zero in the extremum point:

$$\delta W_{ex} = \delta W - 2\pi\lambda\delta I = \int_V dV\,\delta\psi \left[\frac{FF' - \Delta^* \psi}{r^2} + \frac{3}{2} p' - \lambda \left(\frac{F'F' + FF''}{r^2} + p'' \right) \right] = 0. \tag{2.7}$$

As ψ is the independent variable and $\delta\psi$ is an arbitrary increment, so we obtain a two-dimensional Euler equation

$$\frac{FF' - \Delta^* \psi}{r^2} + \frac{3}{2} p' - \lambda \left(\frac{F'F' + F\,F''}{r^2} + p'' \right) = 0 \tag{2.8}$$

Let's substitute the equilibrium Eq. (2.2) to Eq. (2.8); then we reduce the Euler equation to the form

$$\left[2FF' - \lambda (F\,F')' \right] / r^2 + \left[\frac{5}{2} p' - \lambda p'' \right] = 0. \tag{2.9}$$

The second term in square brackets in (2.9) is permanent on the magnetic surface. The first term is permanent only if it is equal to zero. As a result, two-dimensional Eq. (2.9) is divided into two independent one-dimensional equations

$$2FF' - \lambda (F\,F')' = 0, \quad \frac{5}{2} p' - \lambda p'' = 0. \tag{2.10}$$

For future convenience we denote $\frac{5}{2\lambda}$ through λ. Then the solutions of Eq. (2.10) take the form

$$FF' = C_F \exp\left(\frac{4}{5} \lambda \psi \right), \quad p' = C_p \exp(\lambda \psi). \tag{2.11}$$

After substitution (2.11) into (2.2), we obtain the canonical equilibrium equation

$$\Delta^* \psi = -rj_\varphi = C_F \exp\left(\frac{4}{5}\lambda\psi\right) - C_p r^2 \exp(\lambda\psi). \tag{2.12}$$

The boundary conditions should be formulated to determine the constants C_F, C_p and λ. For example the potentials on the magnetic axis O_1 and on the plasma boundary S and the total plasma current can be set: $\psi(O_1) = 0$, $\psi(S) = \psi_a$, $I=I_p$. Sometimes it is more convenient to set the values $I=I_p$, $\beta_p = \beta_p^0$, $q(0)=q_0$. Here β_p is the ratio of plasma pressure to the poloidal magnetic field pressure, $q=\delta\Phi/\delta\psi$ is the plasma safety factor, Φ is the toroidal magnetic flux inside the magnetic surface.

Later on we will need a canonical pressure profile $p_c=p_c(\psi)$. This profile, which determines the pressure profile in the relaxed state, should weakly depend on the boundary conditions. To do this, we require that $p_c(\psi)$ as $p_c'(\psi)$ in (2.11) exponentially dependent on ψ

$$p_c(\psi) = p_0 \exp(\lambda\psi). \tag{2.13}$$

Here, p_0 is the pressure on the magnetic axis, where $\psi=0$.

If $\beta_p \sim 1$, the first term on the right side (2.2) is small compared to the second one and it can be omitted. After averaging the remaining part over the magnetic surface, we obtain

$$\langle j_\varphi \rangle = \langle r \rangle p_c' \propto \langle r \rangle p_c. \tag{2.14}$$

Here, the angle brackets denote the averaging over the magnetic surface S

$$\langle f \rangle = \int_S f\, dS \Big/ \int_S dS,$$

Taking the left hand side of (2.14) as a definition of the canonical current profile j_c, we obtain the connection between the canonical profiles of current and pressure

$$j_c \propto \langle r \rangle p_c. \tag{2.15}$$

Since $\langle r \rangle \approx R_0 + \Delta_S(\psi)$, where R_0 is a major radius of the torus, $\Delta_S(\psi)$ is a Shafranov shift and $\Delta_S(\psi) \ll R_0$, then $\langle r \rangle$ is little different from constant R_0, and the profiles j_c and p_c are close to each other.

In the approximation of a straight cylinder with a circular cross-section $\langle r \rangle = \text{const}$ and canonical profiles of current and pressure are the same: $j_c \propto p_c$. In this approximation, the canonical equilibrium equation (2.12) at $\beta_p \sim 1$ is as follows

$$\Delta\psi \equiv \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi}{d\rho}\right) = -C_p \exp(\lambda\psi), \tag{2.16}$$

where ρ is the radial coordinate in the cylinder. Its solution is the function

$$\psi = -\ln(1+A_1\rho^2)^2, \tag{2.17}$$

so, the canonical profiles of current density and pressure are as follows

$$\frac{j_c(\rho)}{j_c(0)} = \frac{p_c(\rho)}{p_c(0)} = \frac{1}{(1+A_2\rho^2)^2}, \tag{2.18}$$

where A_1 and A_2 are the constants determined by boundary conditions. It is convenient to define the boundary conditions for the function $\mu=1/q=R_0B_\theta/\rho B_0$, where B_θ and B_0 are the poloidal and toroidal magnetic fields. If $\mu_0=\mu(0)$ and $\mu_a=\mu(a)$, then $A_2=\mu_0/\mu_a-1$. The function $\mu_c(\rho)$ itself has the form

$$\mu_c(\rho) = \mu_0(1+A_2\rho^2)^{-1}. \tag{2.19}$$

The comparison of (2.18) and (2.19) shows that

$$\frac{j_c(\rho)}{j_c(0)} = \left(\frac{\mu_c(\rho)}{\mu_0}\right)^2, \tag{2.20}$$

Thus, the variation problem with the functional (2.4) allows us to construct the canonical pressure profile (2.13), which coincides at $\beta_p\sim 1$ with canonical current profile.

Difficulties arise during the construction of transport model. Experiment shows that the characteristic relaxation times of temperature and current are very different. In a tokamak, the characteristic time of current profile relaxation may be ten times higher than the plasma energy confinement time. During the transition process of the plasma current evolution the current profile could be far from the canonical current profile and the equilibrium equation cannot be written in the form (2.12). As a result one has to abandon the use of a simple variational principle (2.4−2.5) and look for other variation approach for the design of transport models. This does not exclude the possibility that some of the conclusions of this section will be used further in this book.

2.2 The Principle of Minimum of the Plasma Current Magnetic Energy for a Circular Plasma Cylinder (Kadomtsev)

2.2.1 The Natural but Contradictive Statement of the Variational Problem

As before, we assume that the canonical profiles are the goals to the relaxation of plasma parameters. The toroidal magnetic field in a tokamak stabilizes the large-scale MHD instabilities, which characteristic times are much smaller than the relaxation times. Plasma, in turn, has little effect on the toroidal field, so this field can be removed from the energy reservoir defining the transport in plasma. In the previous section we see that the canonical pressure profiles associate with the canonical profiles of current by the equilibrium equation. Therefore, the thermal energy can also be excluded from the consideration of the problem of canonical profiles. As a result, we come to the following variation principle for a circular cylindrical plasma [2]: the relaxed plasma state is defined by the minimum of the magnetic energy of the toroidal current:

$$W_m = 2\pi \int_0^a \frac{B_\theta^2}{8\pi} \rho d\rho \tag{2.21}$$

provided that the current magnitude

$$I = 2\pi \int_0^a j\rho d\rho \tag{2.22}$$

and the magnetic flux

$$\Psi = 2\pi \int_0^a B_\theta d\rho \tag{2.23}$$

are conserved. Here ρ and θ are the radial and poloidal coordinates, a is a radius of plasma cylinder, $B_\theta = B_\theta(\rho)$ is the poloidal magnetic field, $j = j(\rho)$ is a current density.

For convenience let's introduce the dimensionless quantity

$$\mu = 1/q = R_0 B_\theta / (\rho B_0) \tag{2.24}$$

and accept it as an independent variable. Here B_0 is the magnitude of the toroidal magnetic field, R_0 is the equivalent of a major radius of the torus $(R_0 \gg a)$. The variation problem (2.21–2.23) will be solved by Lagrange method, for which we introduce the extended functional

$$W_{m,ex} = W_m + \lambda I + C\Psi. \tag{2.25}$$

For this functional the variation problem is reduced to the problem of the unconditional minimum. The additional assumption is introduced in [2] that in the vicinity of the extremum of the functional (2.25) the current density depends on μ only:

$$j = j(\mu). \tag{2.26}$$

What really lies behind this assumption will be discussed in the next Sect. 2.2.2.

Now we can find the first variation of the functional (2.25) and set it equals to zero:

$$\delta W_{m,ex} = 2\pi \int_0^a \rho d\rho \left(\frac{B_0^2}{4\pi R_0^2} \rho^2 \mu + \lambda \frac{dj}{d\mu} + \frac{CB_0}{R_0} \right) \delta\mu = 0. \tag{2.27}$$

Hence we obtain the Euler equation

$$\frac{B_0^2}{4\pi R_0^2} \rho^2 \mu + \lambda \frac{dj}{d\mu} + \frac{CB_0}{R_0} = 0. \tag{2.28}$$

We assume that the desired solution $\mu(\rho)$ of equation (2.28) is monotonic along the radius, i.e. $\mu'(\rho) \neq 0$ at $\rho \neq 0$. Then

$$\frac{dj}{d\mu} = \frac{dj}{d\rho} \left(\frac{d\mu}{d\rho} \right)^{-1}. \tag{2.29}$$

We also assume that at the extremal function (on the solutions of the Euler equation (2.28)) the Maxwell equation is satisfied. Its projection on the z-axis is given by

$$j = \frac{B_0}{\mu_{00} R_0} \frac{1}{\rho} \frac{d}{d\rho} (\rho^2 \mu) \tag{2.30}$$

Then the Euler equation (2.28) is transformed as follows:

$$\rho^2 \frac{d\mu^2}{d\rho} + \lambda_1 \frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} (\rho^2 \mu) \right] + C_1 \frac{d\mu}{d\rho} = 0. \tag{2.31}$$

The equation (2.31) is the equation of the second order with two, while non-defined, Lagrange parameters λ_1 and C_1. It can be reduced to the third order equation with one uncertain parameter

$$\frac{d}{d\rho} \left[\frac{\rho^2}{\mu'} \frac{d}{d\rho} \left(\mu^2 + \lambda_2 \frac{d\mu}{d\rho^2} \right) \right] = 0 \quad (\mu' \equiv d\mu/d\rho) \tag{2.32}$$

In this form the equation can be easily integrated.

We need four boundary conditions to obtain a unique solution of (2.32). Symmetry condition at the magnetic axis leads to $\mu'(0) = 0$. The requirement of the current conservation means that $\mu(a) \equiv \mu_a = 0.2 I_p R_0 / (a^2 B_0)$. The practical units that are used here: the plasma current I_p in MA, the length in m, the magnetic field B_0 in T. The rest two boundary conditions were chosen in [2] as follows: $\mu(0) = \mu_0 \sim 1$, $\mu(\rho) \to 0$ at $\rho \to \infty$. We will follow these conditions. We obtain the following set of four boundary conditions, collecting all the terms together:

$$\mu(0) = \mu_0 \sim 1, \quad \mu'(0) = 0, \quad \mu(a) \equiv \mu_a = 0.2 I_p R_0 / (a^2 B_0),$$

$$\mu(\rho) \to 0 \quad \text{at} \quad \rho \to \infty. \tag{2.33}$$

The solution of (2.32) satisfying the boundary conditions (2.33) will be called the canonical profile of the function μ and will be denoted as $\mu_c(\rho)$. After integration of (2.32), we obtain

$$\frac{\rho^2}{\mu'} \frac{d}{d\rho} \left(\mu^2 + \lambda_2 \frac{d\mu}{d\rho^2} \right) = C_2. \tag{2.34}$$

To determine the constant C_2, let us consider the behavior of the regular solutions of the equation (2.34) in the environment of the point $\rho = 0$. Under the first two conditions (2.33) in this environment

$$\mu = \mu_0 (1 + \alpha_2 \rho^2 + \alpha^4 \rho^4 + \cdots). \tag{2.35}$$

For solutions of the type (2.35), the left side of (2.34) tends to zero as $\rho \to 0$. Hence, $C_2 = 0$. As a result, the Euler equation is now can be written as

$$\frac{d}{d\rho} \left(\mu^2 + \lambda_2 \frac{d\mu}{d\rho^2} \right) = 0. \tag{2.36}$$

The second of the boundary conditions (2.33) holds for solutions of (2.36) automatically. Therefore, there are only three essential boundary conditions:

$$\mu(0) = \mu_0 \sim 1, \quad \mu(a) \equiv \mu_a = 0.2 I_p R_0 / (a^2 B_0),$$

$$\mu(\rho) \to 0 \quad \text{at} \quad \rho \to \infty. \tag{2.37}$$

After integration (2.36), we obtain:

$$\mu^2 + \lambda_2 \frac{d\mu}{d\rho^2} = C_3. \tag{2.38}$$

At $\rho \to \infty$ the left side of (2.38) tends to zero by virtue of the third boundary condition (2.37). Hence, $C_3 = 0$. As a result, the Euler equation (2.38) becomes:

$$\lambda_2 \frac{d\mu}{\mu^2} = -d\rho^2 \tag{2.39}$$

Its solution is the function

$$\mu_c(\rho) = \frac{\mu_0}{1 + \rho^2 / a_j^2} \tag{2.40}$$

where $a_j = a\,[\mu_a/(\mu_0 - \mu_a)]^{1/2}$ is the so called *plasma current radius*, $\lambda_2 = \mu_0 a_j^2$. The function (2.40) will be called *Kadomtsev canonical profile* and will be denoted as $\mu_c^K(\rho)$. Using (2.30), we find the Kadomtsev canonical profile for the current density

$$j_c^K = j_0 \left(\frac{\mu_c^K}{\mu_0} \right)^2 = \frac{j_0}{(1+\rho^2/a^2)^2}. \tag{2.41}$$

By (2.15), for a circular cylinder the canonical profiles of current and pressure are the same. Therefore

$$p_c^K = p_0 \left(\frac{j_c^K}{j_0} \right) = \frac{p_0}{(1+\rho^2/a^2)^2}. \tag{2.42}$$

In subsequent chapters, we will need a dimensionless relative gradient of $\mu_c(\rho)$

$$-R_0 \frac{d\mu_c^K/d\rho}{\mu_c^K} = 2\frac{R_0\rho}{a_j^2}\frac{1}{1+\rho^2/a_j^2}. \tag{2.43}$$

Since the solution of (2.36) − (2.37) is found, it is easy to reformulate the last of boundary conditions (2.37), replacing it with the boundary conditions on the surface of the plasma. It is convenient to introduce the surface impedance in the form $X = i_a / 2\mu_a$, where i is the dimensionless current

$$i = \frac{1}{\rho}\frac{d}{d\rho}(\rho^2\mu) = 2\mu + \rho\mu', \quad i_a = i(a). \tag{2.44}$$

Using (2.40, 2.44), it is easy to find the impedance for the Kadomtsev canonical profile

$$X^K = \frac{\mu_a}{\mu_0}. \tag{2.45}$$

Thus, in the Kadomtsev problem for the equation (2.36) the equivalent boundary conditions can be used instead of the boundary conditions (2.37)

$$\mu(0) = \mu_0 \sim 1, \quad \mu(a) \equiv \mu_a = \frac{0.2 I_p R_0}{a^2 B_0}, \quad \frac{i_a}{2\mu_a} = \frac{\mu_a}{\mu_0}. \tag{2.46}$$

The last of the boundary conditions is a condition of the third kind, as it contains the unknown function μ and its derivative μ'. It is also unusual as it contains both the values of unknown function at the magnetic axis and at the plasma boundary. The boundary conditions (2.37) and the equivalent conditions (2.46) are naturally called "soft", as one of the boundary conditions (2.37) is stated at infinity, and it does not reflect the physical processes on the surface of the plasma.

2.2.2 The Adjusted Statement of the Variational Problem

In setting up the variation problem in the previous Sect. 2.2.1, we, following the work of [2], have been forced to assume (2.26) that $j=j(\mu)$. However, if j is a local current density, it must be associated with μ by a relation

$$j \sim \frac{1}{\rho}\frac{d}{d\rho}(\rho^2\mu) = 2\mu + \rho\mu', \quad \mu' \equiv \frac{d\mu}{d\rho}. \tag{2.47}$$

We see that in this case, the current density j depends not only on μ, but also on the derivative μ'. Substituting (2.47) into the condition (2.22) and calculating the variation of the functional I, we see that it is equal to zero. Thus, this integral does not contribute to the Euler equation. This section is discussed how to remove the contradiction described.

Consider the following formulation of the variation problem for a circular plasma cylinder. Let $B_\theta = B_\theta(\rho)$ is the set of sufficiently smooth functions (admissible functions) vanished at $\rho = 0$ (these are the functions describing the poloidal magnetic field). In parallel, we also introduce the dimensionless admissible functions $\mu = \mu(\rho) = R_0 B_\theta/(B_0\rho)$. Consider the problem of minimizing of the poloidal magnetic energy functional

$$F_1 = \frac{1}{8\pi}\int_0^a B_\theta^2 \rho d\rho \sim \frac{1}{8\pi}\frac{B_0}{R_0}W_m, \quad W_m = \int_0^a \mu^2\rho^3 d\rho \tag{2.48}$$

with additional integral conditions [3]

$$J_1 = \int_0^a \mu^2 \rho d\rho = \text{const}, \tag{2.49}$$

$$J_2 = \int_0^a \mu\rho d\rho = \text{const} \tag{2.50}$$

and the boundary conditions

$$\mu(0) = \mu_0 \sim 1, \quad \mu(a) \equiv \mu_a = 0.2 I_p R_0/(a^2 B_0), \quad (0 < \mu_a < \mu_0). \tag{2.51}$$

Here I is a plasma current, and a is a plasma radius. To simplify the formulas below we omit the factor standing in F_1 (2.48) to W_m. Note that the integral that describes the plasma current is proportional to μ_a

$$I = \int_0^a j\rho d\rho \sim \mu_a. \tag{2.52}$$

and, in view of (2.46), is the same for all admissible functions. Therefore, the preservation of the total current is the result of the last of the boundary conditions (2.51). The meaning of (2.49) and (2.50) will be discussed below.

The problem (2.48–2.51) is equivalent to the problem of unconditional minimum of the extended functional

$$W_{m,ex} = \int_0^a \left(\mu^2\rho^2 + \lambda\mu^2 + C\mu\right)\rho d\rho, \tag{2.53}$$

where λ and C are Lagrange parameters. We find now the variation of the functional (2.53)

$$\delta W_{m,ex} = 2\int_0^a \delta\mu\left(\mu\rho^2 + \lambda\mu + \frac{C}{2}\right)\rho d\rho \tag{2.54}$$

and require that it vanish. Then we obtain the Euler equation

$$\mu\rho^2 + \lambda\mu + C/2 = 0. \tag{2.55}$$

As before, the solution of the Euler equation satisfying the boundary conditions (the canonical profile) will be denoted by the lower index "*c*". From (2.55) and (2.53), we obtain:

$$\mu_c = -\frac{C}{2(\rho^2 + \lambda)} = \frac{\mu_0}{(1+\rho^2/a_j^2)}. \tag{2.56}$$

The boundary conditions (2.51) allow us to determine the parameters λ and C

$$\lambda = a_j^2 = \frac{a^2}{\mu_0/\mu_a - 1} > 0, \quad C = -\frac{2\mu_0}{\lambda}. \tag{2.57}$$

The solution (2.56) coincides with the solution (2.40) of the previous section, which we called Kadomtsev canonical profile. This solution has a remarkable property: the dimensionless canonical current (2.44)

$$i_c \equiv \frac{1}{\rho}\frac{d}{d\rho}(\rho^2\mu_c) = \frac{2\mu_c^2}{\mu_0} \tag{2.58}$$

is proportional to the square of the function μ_c. This feature justifies the choice of a first additional condition in the form of (2.49). Note that the current density (2.58) is not equal to zero at the plasma boundary, but has a pedestal.

The considered variation problem is strongly degenerated in the sense that neither the functional (2.48), nor the additional conditions (2.49) − (2.50) do not depend on the derivative $d\mu/d\rho = \mu'$. As a result, the Euler equation (2.55) is an algebraic rather than a differential equation of the second order, as it could be obtained in a non-degenerate case. The solution of the Euler equation contains only two undefined parameters λ and C, but we have four conditions: the boundary conditions (2.51) and conditions (2.49) and (2.50). In order the posed variation problem to be solvable, the conditions (2.49) and (2.50) have to be consisted with the boundary conditions (2.51). We obtain such consistency conditions if we substitute the found solution (2.56) to the conditions (2.49) − (2.50):

$$J_1 = \int_0^a \mu_c^2 \rho d\rho = \mu_0 \mu_a \frac{a^2}{2}, \tag{2.59}$$

$$J_2 = \int_0^a \mu_c \rho d\rho = \mu_0 \frac{a_j^2}{2} \ln\left(\frac{\mu_0}{\mu_a}\right). \tag{2.60}$$

Thus, the problem (2.48) − (2.51) is solvable, if the values of the integrals (2.49) − (2.50) are defined by (2.59) − (2.60).

The meaning of (2.50) can be understood, if we introduce the usual poloidal flux ψ: $B_\theta \propto \rho\mu = d\psi/d\rho$. The condition (2.50) requires the preservation of difference $\psi(a)-\psi(0)$, which is equivalent to the flux conservation for all admissible functions.

The second variation of the functional (2.53)

$$\delta^2 W_{m,ex} = 2\int_0^a (\delta\mu)^2 (\rho^2 + \lambda)\rho d\rho \tag{2.61}$$

is positive in view of (2.57). Thus, the solution (2.56) of the Euler equation (2.55) realizes the minimum of the functional (2.53).

The Euler equation (2.55) can be rewritten as

$$2\mu\rho^2 + \lambda \frac{d\mu'^2}{d\mu} + C = 0 \tag{2.62}$$

or

$$2\mu\rho^2 + \lambda \frac{d\mu'^2}{d\rho}\left(\frac{d\mu}{d\rho}\right)^{-1} + C = 0. \tag{2.63}$$

In order to be able to go from (2.62) to (2.63), it needs to add the monotony condition of the admissible functions $\mu(\rho)$ to the limitations (2.49–2.50). That means the condition $d\mu/d\rho \neq 0$ has to be satisfied in the region $(0<\rho<a)$. It is evident that the function (2.56) satisfies this condition.

Let us substitute (2.56) and (2.58) into (2.63) and multiply it by $d\mu_c/d\rho$. Then we obtain the identity

$$\rho^2 \frac{d\mu_c^2}{d\rho} + \frac{\lambda\mu_0}{2} \frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho^2 \mu_c)\right] + C\frac{d\mu_c}{d\rho} = 0$$

We omit index "c" now, redefine the parameter λ and consider this expression as the equation concerning $\mu(\rho)$

$$\rho^2 \frac{d\mu^2}{d\rho} + \lambda \frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho^2\mu)\right] + C\frac{d\mu}{d\rho} = 0 \tag{2.64}$$

Now we will try to put the boundary conditions for the equation (2.64) to solve the problem with function (2.56). Equation (2.64) is a nonlinear second-order equation containing two parameters, so to obtain a unique solution four boundary conditions are required. Two of them should be taken from the boundary conditions (2.51), the third is a condition of the axial symmetry $\mu'(0) = 0$, and the fourth condition is the property (2.58) at the border. Thus, the total set of the necessary boundary conditions is as follows:

$$\mu_c(0) = \mu_0 > 0, \quad \mu'(0) = 0, \quad \mu_c(a) = \mu_a = 0.2I_p R_0/(a^2 B_0),$$

$$\frac{\mu_0 i_c(a)}{2\mu_c^2(a)} = 1 \tag{2.65}$$

The fourth of the boundary conditions, in view of (2.47) is the boundary condition of the third kind, containing a derivative of function μ at the edge of the plasma. The solution of (2.64) − (2.65) is the function (2.56). In the future, the equation (2.64) will be called the modified Euler equation. This equation has an important property: it does not contain any information about the integrand in (2.49), i.e. the function μ^2.

Let us make a few comments:

Remark 1 The Euler equation of the form (2.64) coincides with (2.31). This result justifies not quite consistent statement of variation problem in the previous section and the assumption (2.26). After comparison of Kadomtsev additional condition (2.22) with the condition (2.49), we can say that, instead of the current density j in (2.22) it is sufficient to use such a function of μ (in this case μ^2), which is close to j, *b*ut coincides with it only at the extremal function of the functional (2.25).This important result will be used below.

Remark 2 The additional condition (2.49) has the following remarkable property. On the one hand, it is really the restriction on the class of admissible functions and contributes to the Euler equation. On the other hand, according to (2.58), this condition for the extremal function μ_c has a form of the current conservation

$$J_1 = \int_0^a \mu_c^2 \rho d\rho = \frac{\mu_0}{2}\int_0^a i_c \rho d\rho = \text{const.} \tag{2.66}$$

However, such a condition is already used in the boundary conditions (2.51). It follows that the condition (2.49) is degenerated on the extremal, and so it is weak in the environment of the extremal. This property of the problem (2.48) − (2.51) makes it selected of other possible variation problems for the functional (2.48) with integral constraints. One can expect that it defines the lower boundary of the functional (2.48) with respect to other reasonable integral limitations that may be used instead of (2.49).

Remark 3 In this section, we consider the problem of minimizing the magnetic energy of the plasma. Thermal energy is not part of this task, so the results of Sect. 2.1 can be adopted for it (for the plasma pressure), and, in particular, the formula (2.18). For our purposes, it can be written as follows:

$$\frac{p_c(\rho)}{p_c(0)} = \frac{i_c(\rho)}{i_c(0)} = \frac{1}{\left(1+\rho^2/a_j^2\right)^2} = \left(\frac{\mu_c(\rho)}{\mu_c(0)}\right)^2. \tag{2.67}$$

Let us consider now the total energy functional

$$E = 2\pi\left[\frac{1}{8\pi}\int_0^a B_\theta^2 \rho d\rho + \frac{3}{2}\int_0^a p\rho d\rho\right] \tag{2.68}$$

If we state the variation problem of the type (2.48 – 2.50) for this functional then the Euler equation will contain two unknown functions $\mu(\rho)$ and $p(\rho)$. The closure of the formulation of the problem will have to make an additional assumption about the relationship of these functions. We assume that the admissible functions are connected by the relation using as a basis the expression (2.67)

$$p(\rho) \propto \mu^2(\rho). \tag{2.69}$$

Then the second term in (2.68) takes the form of the integral (2.49). As a result, the Euler equation (2.55) and formulas (2.56 – 2.58) are saved.

A minimum of the total energy functional E (2.68) is determined by its value on the canonical profiles

$$E(\mu_c, p_c) = \left(\frac{B_0}{2R_0}\right)^2 \left[W_m(\mu_c) + \frac{3}{4}a^4\frac{\mu_a^3}{\mu_0}\beta_\theta\right] \tag{2.70}$$

Here

$$W_m(\mu_c) = \int_0^a \mu_c^2 \rho^3 d\rho = \lambda(\mu_0 J_2 - J_1) > 0 \tag{2.71}$$

is the value of the magnetic energy over the canonical profile,

$$\beta_\theta = \frac{8\pi p_0}{B_{\theta a}^2}, \tag{2.72}$$

$B_{\theta a} = aB_0\mu_a/R_0$, p_0 is the pressure in the plasma centre, J_1 and J_2 are determined by formulas (2.59)−(2.60). The first term in (2.70) describes the minimum of magnetic energy, the second term is the heat energy. The magnetic energy in absolute value is limited by a given total current. In the adopted statement of the problem there is no limit on the absolute value of the thermal energy, so a minimum of heat energy is determined by the free parameter β_θ. The possibility of such a formulation of the problem is determined by the openness of the system, and the reasonableness of the statement that is by the existence of self organization plasma effects. The concept of openness means that the system is continuously absorbing and throwing out the heat energy and particles. The bound energy remaining in the plasma is determined by the dissipative properties of the system (thermal conductivity, radiation), so it cannot be described by the variation principle. To determine the evolution of the system one has to build the transport models, which are discussed in the following chapters.

Remark 4 It was shown above that for circular cylindrical plasma the Euler equation with the necessary boundary conditions can be represented in five different forms. First, it is an algebraic equation (2.55), which contains two parameters, with the boundary conditions (2.51). Second, it is the first order differential equation with a single parameter (2.39) and with the same boundary conditions. Third, it is the second order differential equation with a single parameter (2.36) and three boundary conditions (2.46). Fourth, it is a second-order differential equation (2.64) with two parameters and with four boundary conditions (2.65). Fifth and finally, it is the third-order differential equation (2.32) with one parameter and four boundary conditions (2.33) or (2.65). All listed forms of problem setting are equivalent in the sense that they have the same unique solution in the form of the canonical profile (2.56).

Remark 5 In subsequent chapters, we will need a canonical profile for temperature. It can be obtained on the basis of the following reasons [2]. In the relaxed quasi-stationary state the current profile $j(\rho)$ and electron temperature profile $T_e(\rho)$ are close to the canonical profiles $j_c(\rho)$ and $T_c(\rho)$

$$j(\rho) \approx j_c(\rho), \quad T_e(\rho) \approx T_c(\rho). \tag{2.73}$$

At the same time the profiles $j(\rho)$ and $T_e(\rho)$ are related by Ohm's law

$$j(\rho) \propto T_e^{3/2}(\rho). \tag{2.74}$$

According to (2.67), (2.73) and (2.74), we have

$$p_c(\rho) = n_c(\rho)T_c(\rho) \propto j_c(\rho) \propto T_c^{3/2}(\rho). \tag{2.75}$$

It follows that

$$T_c \propto j_c^{2/3} \propto p_c^{2/3}, \quad n_c \propto T_c^{1/2} \propto j_c^{1/3}. \tag{2.76}$$

For the relative dimensionless gradients we obtain

$$\frac{R_0 T_c'}{T_c} = \frac{2}{3}\frac{R_0 p_c'}{p_c}, \quad \frac{R_0 n_c'}{n_c} = \frac{1}{3}\frac{R_0 p_c'}{p_c}, \quad T' = \frac{dT}{d\rho}. \tag{2.77}$$

Thus, the canonical profile of the electron temperature for a circular cylindrical plasma has the form

$$T_c(\rho) = \frac{T_c(0)}{(1+\rho^2/a_j^2)^{4/3}}. \tag{2.78}$$

We accept below that the canonical profiles for ions and electrons are the same.

2.3 Canonical Profiles for Toroidal Plasma with Arbitrary Cross-Section

Suppose that for a given distributions of current and pressure in a toroidal plasma with arbitrary cross section the problem of equilibrium with the given boundary conditions (the Grad-Shafranov equation for the function of poloidal magnetic flux ψ) is solved [4]. Then the equation $\psi = \text{const}$ determines the magnetic surfaces.

Denote by r, φ, z the polar coordinates with axis coinciding with the symmetry axis of torus. Let us introduce the natural coordinates ρ, θ, ζ, where ρ is the coordinate of magnetic surface defined by the toroidal magnetic field flux Φ inside the magnetic surface with the cross section S.

$$\rho = \left(\frac{\Phi}{\pi B_0}\right)^{1/2}, \quad 0 < \rho < \rho_{max}, \quad \Phi = \int_S \mathbf{B} \cdot d\mathbf{S}, \quad \rho_{max} = \left(\frac{\Phi_{max}}{\pi B_0}\right)^{1/2} \tag{2.79}$$

Here θ is the poloidal angle, $\zeta = r\,\varphi$, the vector $\mathbf{dS}$ is perpendicular to the cross section S, Φ_{max} is the total toroidal flux through the plasma cross section. Below B_0 is the vacuum toroidal field in the center of the chamber, $A = R_0/a$ is the aspect ratio, R_0 and a are the major and minor plasma radii. We will use the notation $h(\rho_{max}) = h_a$ for the boundary values of surface function $h(\rho)$. The angle brackets $<\cdots>$ denote the averaging operation over the magnetic surface

$$< f >= \frac{2\pi}{V'} \int_0^{2\pi} \sqrt{g} f d\theta, \tag{2.80}$$

where g is the determinant of the metric tensor of the accepted system of coordinates:

$$\sqrt{g} = r\frac{D(r,z)}{D(\rho,\theta)}, \quad V' \equiv \frac{\partial V}{\partial \rho} = 2\pi \int_0^{2\pi} \sqrt{g} d\theta, \tag{2.81}$$

$V(\rho)$ is the volume of the plasma inside the magnetic surface. We also introduce a surface function

$$\mu = \frac{\partial \psi}{\partial \Phi} = \frac{1}{2\pi B_0 \rho} \frac{\partial \psi}{\partial \rho}. \tag{2.82}$$

As it is known, in a toroidal plasma the local poloidal magnetic field B_θ^l and local current density j_φ^l are not the surface functions. In particular,

$$B_\theta^l = \frac{|\nabla \rho|}{2\pi r} \frac{\partial \psi}{\partial \rho}. \tag{2.83}$$

Instead of the local poloidal magnetic field B_θ^l and local current density j_φ^l we introduce the averaged poloidal field $B_\theta = \left\langle B_\theta^l \right\rangle$ and averaged current density $j = \left\langle j_\varphi^l \right\rangle$ using (2.80). The surface functions $\mu \propto B_\theta/\rho$ and j are connected by the Maxwell equations. It is convenient to introduce the dimensionless current density i, proportional to the averaged current density j and similar to (2.44). In the adopted coordinates the relationship between the dimensionless current density and function μ is as follows

$$i = \mathrm{rot}_\varphi(\rho\mu) = \frac{1}{V'} \frac{\partial}{\partial \rho}(V' G \rho \mu), \tag{2.84}$$

where V defined by (2.81) and

$$G = G(\rho) = R_0^2 \left\langle \frac{(\nabla \rho)^2}{r^2} \right\rangle \tag{2.85}$$

are the metric coefficients. Note that the coefficient G is dimensionless. In a circular cylindrical plasma $G \equiv 1$, $V' = (2\pi R)\cdot(2\pi\rho)$.

The expression for the magnetic energy of the poloidal magnetic field has the form:

$$W_m = 2\pi R_0 \frac{1}{8\pi} \int_{S_{max}} B_\theta^2 dS. \tag{2.86}$$

Here S_{max} is the total cross-section of the plasma. Below the multiplier $2\pi R_0$, which appears due to integration over the toroidal coordinate, is omitted. Using relations (2.82)−(2.83), (2.85), it is easy to lead the functional (2.86) to the one-dimensional integral

$$W_m = \frac{B_0^2}{2\pi R_0^2} \int_0^{\rho_{\max}} V' G \mu^2 \rho^2 d\rho \tag{2.87}$$

We turn now to the formulation of the variation problem. Let $\mu(\rho)$ be a set of sufficiently smooth functions on the interval $(0<\rho<\rho_{max})$, satisfying the boundary conditions

$$\mu(0) = \mu_0 \sim 1, \quad \mu(a) = \mu_a, \quad (0 < \mu_a < \mu_0). \tag{2.88}$$

Here μ_a is defined by the equilibrium solution of the problem mentioned at the beginning of the section. As in the previous section, we assume that the canonical profile in a tokamak plasma is determined by the minimum of the functional of the poloidal field magnetic energy (2.87) with additional conditions

$$J_1 = \int_0^{\rho_{\max}} V' i(\mu) d\rho = \text{const}, \tag{2.89}$$

$$J_2 = \int_0^{\rho_{max}} \mu\rho d\rho = \text{const}. \tag{2.90}$$

According to Remark 1 of Sect. 2.2.2, $i(\mu)$ is a smooth function of μ, satisfying on the extremal function the expression (2.84)

$$i(\mu_c) = i_c \equiv \frac{1}{V'}\frac{\partial}{\partial\rho}\left(V' G\rho\mu_c\right). \tag{2.91}$$

Condition (2.90) describes the conservation of the poloidal magnetic flux for admissible functions. In fact, substituting (2.82) into (2.90), we obtain

$$J_2 = \frac{1}{2\pi B_0}\int_0^{\rho_{max}} \frac{d\psi}{d\rho} d\rho = \frac{1}{2\pi B_0}\left[\psi(\rho_{max}) - \psi(0)\right] = \text{const}. \tag{2.92}$$

We apply Lagrange method to the problem of (2.87)−(2.90). For this we introduce the extended functional

$$W_{m,ex} = \frac{B_0^2}{2\pi R_0^2}\int_0^{\rho_{max}} V'\left[G\mu^2\rho^2 + \lambda i(\mu) + C\frac{\mu\rho}{V'}\right] d\rho \tag{2.93}$$

and consider the problem of the unconditional extremum for it. For this we find the first variation of the functional (2.93) with respect to variable μ and put it to zero

$$\delta W_{m,ex} = \frac{B_0^2}{2\pi R_0^2}\int_0^{\rho_{max}} V'\delta\mu\left[2G\mu\rho^2 + \lambda\frac{\partial i(\mu)}{\partial\mu} + C\frac{\rho}{V'}\right] d\rho = 0. \tag{2.94}$$

Hence we obtain the Euler equation

$$2G\mu\rho^2 + \lambda\frac{\partial i(\mu)}{\partial\mu} + C\frac{\rho}{V'} = 0. \tag{2.95}$$

We now require in addition that the admissible functions $\mu(\rho)$ are monotonic. It's enough that they satisfy a condition

$$\mu'(\rho) \neq 0 \text{ at } 0 < \rho < \rho_{max} \tag{2.96}$$

Then

$$\frac{\partial i(\mu)}{\partial\mu} = \left(\frac{\partial i(\mu)}{\partial\rho}\right)\left(\frac{\partial\mu}{\partial\rho}\right)^{-1}. \tag{2.97}$$

Substituting (2.97) into (2.95), using (2.91) and carrying out the transformations, we obtain instead of (2.95)

$$\rho^2 G\frac{\partial\mu^2}{\partial\rho} + \lambda\frac{\partial}{\partial\rho}\left[\frac{1}{V'}\frac{\partial}{\partial\rho}(V'G\rho\mu)\right] + C\frac{\rho}{V'}\frac{\partial\mu}{\partial\rho} = 0. \tag{2.98}$$

Equation (2.98) generalizes the modified Euler equation (2.64) in the case of the toroidal plasma with arbitrary cross section. This equation is a nonlinear second-order equation containing two parameters C and λ. To determine the unique solution four boundary conditions are required, which are similar to conditions (2.65), but taking into account the toroidal geometry:

$$\mu(0) = \mu_0 > 0, \quad \mu'(0) = 0, \quad \mu(\rho_{max}) = \mu_a, \quad \frac{\mu_0 i_a}{2}\Big/G_a\mu_a^2 = U. \tag{2.99}$$

Here $U \sim 1$ is a constant. The value of U, equal to one in the case of a circular cylinder, is now to be determined by some additional considerations that help to highlight the Kadomtsev type solution, at least approximately satisfying the condition (2.58). To do this, let's consider the function

$$Z(\rho) = \frac{\mu_0 i_c(\rho)}{2}\Big/G(\rho)\mu_c^2(\rho). \tag{2.100}$$

Here $\mu_c(\rho)$ is the solution of (2.98)−(2.99) for some U and i_c is determined by (2.91). On the boundary of plasma $Z(\rho_{max}) = U$ due to (2.99). In the central part of the plasma the aspect ratio of magnetic surfaces is large, their cross section is close typically to the circle and $G \approx 1$. Therefore, and due to (2.58), $Z(0) \approx 1$. Calculations show that for $U = 1$ everywhere in the interval $(0, \rho_{max})$ $Z(\rho) \geq 1$. Therefore, it has a maximum inside this interval. At the increase of U this maximum moves outside and at some value of $U=U_{opt}>1$ the function $Z(\rho)$ becomes monotonic over the whole interval. In this case, the maximum reaches the boundary $\rho=\rho_{max}$, and the derivative $\delta Z(\rho)/\delta\rho$ becomes zero at the boundary. In the cylindrical case $Z(\rho)\equiv 1$. So it is natural to suppose that in the toroidal case the function $Z(\rho)$ has to be monotonic and close to unity for Kadomtsev type solutions. Therefore, the value of U must be equal to U_{opt} and to define it, we can use the following condition:

$$\frac{\partial Z}{\partial\rho}(\rho=\rho_{max})=0. \tag{2.101}$$

The condition (2.101) should be added to the conditions (2.99) to determine the parameter U. The canonical current profile $i_c(\rho)=j_c(\rho)/j_c(0)$ is determined through $\mu_c(\rho)$ by the expression (2.84). The canonical pressure profile links with canonical current profile by (2.15). As $<r>$ is slowly changing function of ρ we replace it by constant and will determine the pressure canonical profile as follows

$$\frac{p_c(\rho)}{p_c(0)}=\frac{j_c(\rho)}{j_c(0)}\propto i_c(\rho)=\frac{1}{V'}\frac{\partial}{\partial\rho}\left(V'G\rho\mu_c\right). \tag{2.102}$$

In this approximation, the canonical profiles of pressure and current coincide. Canonical profiles of temperature and density of the plasma are determined by the formulas (2.76).

2.4 Examples

To illustrate the change of the canonical profiles with changing of plasma parameters we choose the typical discharges for three tokamaks: T-10, JET and MAST. T-10 has a circular cross section of the plasma and a large aspect ratio $A=R_0/a$. Typical discharge parameters are as follows:

$$R_0=1.5\,\mathrm{m},\ a=0.3\,\mathrm{m}\ (A=5), B_0=2.5\,\mathrm{T}, I=0.25\ \mathrm{MA}. \tag{2.103}$$

With these parameters, the cylindrical safety factor is equal to

$$q_{cyl}=\frac{5a^2B_0}{IR_0}=\frac{1}{\mu_{cyl}}=3.$$

The JET device (the largest in the world) has the elongated cross section and a moderate aspect ratio. Below we will use, as an example, the parameters of the discharge # 26087, which are as follows:

$$R_0=2.94\,\mathrm{m},\ a=1.06\,\mathrm{m}\ (A=2.8), B_0=3\,\mathrm{T},$$

$$I = 3.2 \text{ MA}, \ k = 1.65, \delta = -0.34, \tag{2.104}$$

where k and δ are the elongation and triangularity of plasma cross-section, $q_{cyl} = 1.67$. The value of q at the plasma boundary, obtained by the solution of the equilibrium equation by the three-moments method, equals to $q_a = 1/\mu_a = 3.9$.

The MAST device has an elongated cross-section and low aspect ratio. The parameters of a typical discharge # 11438 are

$$R_0 = 0.8\,\text{m}, \ a = 0.6\,\text{m} \ (A = 1.35), B_0 = 0.47\,\text{T},$$

$$I = 0.62 \text{ MA}, \ k = 1.8, \delta = 0.36, \ q_{cyl} = 1.2, \ q_a = 1/\mu_a = 13.1 \tag{2.105}$$

Let us look first at the behaviour of the metric coefficients. Figure 2.1 shows the normalized profiles of the coefficient V' for the discharges (2.103)−(2.105). For a circular cylindrical plasma $V' = \rho$ (dashed line in Fig. 2.1). It can be seen that by decreasing the aspect ratio the curves are increasingly deviate from the linear function. For tight aspect ratio tokamak MAST the function V' becomes non-monotonic, but of course it is always positive.

Figure 2.2 shows the behavior of the function G which characterizes the density of magnetic surfaces on the outer side of the torus. In a cylindrical plasma $G = 1$ for the whole cross section. With the decrease in the aspect ratio Shafranov shift increases, and the magnetic surfaces on the outer side of the torus are sealed. In MAST near the external plasma boundary the values of G are high: $G > 10$.

The boundary values $G(a)$ versus triangularity δ for MAST are shown in Fig. 2.3. At $\delta < 0$ the density of magnetic surfaces at the outer side of the torus diminishes and the values of $G(a)$ are reduced correspondingly.

We now turn to the canonical profiles. First, we compare the canonical profiles obtained in Kadomtsev approximation for circular plasma cylinder (2.40), and ones obtained by the solution of the problem (2.98)−(2.99) for the Euler equation. As an example we take the discharge parameters (2.103) of T-10. Figure 2.4 shows the canonical profiles $\mu_c^K(\rho)$ (2. 40) and $\mu_c(\rho)$ at $\mu_0 = 1$. It can be seen that they are close to each other. Figure 2.5 shows the profiles of the dimensionless relative pressure gradient

$$R_0 / L_{pc} = -\frac{R_0 p_c'}{p_c} \tag{2.106}$$

for canonical profiles in Kadomtsev approximation and in the general case. It is seen that the pressure gradients differ much more than the functions $\mu_c(\rho)$ shown in Fig. 2.4. Note that the values of ratios R_0/L_{pc} will play an important role in the transport models.

The following figures for the JET discharge (2.104) show the impact of the choice of the parameter U to the behaviour of the canonical profiles. Figs. 2.6 and 2.7 shows the canonical profiles μ_c and j_c for three values of the parameter U:

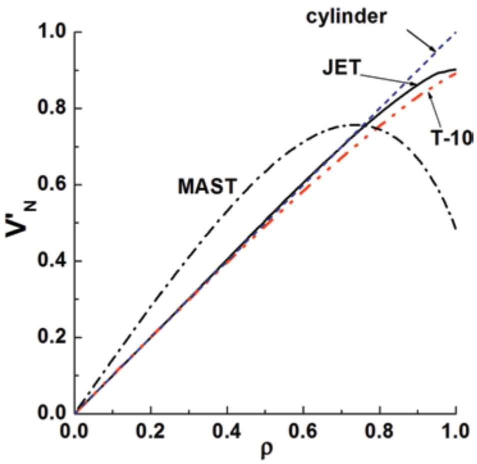

Fig. 2.1 The normalized profiles of the metric coefficient $V'_N = V'/(2\pi R_0 \cdot 2\pi\rho_{max})$ for discharges from T-10, JET and MAST devices with discharge parameters (1.103)–(1.105)

U=0.7, 1.3 (optimum) and 2. It is seen that the profiles of μ_c does not differ from each other radically at the change of U. They have the same boundary values and only derivatives are different on the boundary. The profiles of the canonical current j_c differ more because they have different boundary values. However, in all cases, they are monotonous. Figure 2.8 shows the profiles of the function $Z(\rho)$ for the same values of U. It can be seen that at U=0.7 the function $Z(\rho)$ has a maximum. In the second case (U=1.3) the condition (2.101) satisfies and in the third case (U=2) $Z(\rho)$ becomes monotonic and rapidly rises at the periphery. The profiles of critical gradients (2.106) for the same values of U are drawn in Fig. 2.9. The critical gradient for Kadomtsev canonical profile (2.42)

$$R_0 / L_{pc}^K = \frac{4R_0}{a_j^2} \frac{\rho}{1+\rho^2/a_j^2}. \tag{2.107}$$

is also shown here. It can be seen that at $U = U_{opt}$=1.3 the behaviour of R_0 / L_{pc}^K и R_0/L_{pc} is similar.

Let us turn to the canonical profiles of MAST. Figure 2.10 shows the profiles of the function $Z(\rho)$ for different values of U for the discharge # 11438 with parameters (2.105). In this case at U=1.5 the function $Z(\rho)$ has a maximum at $\rho \sim 0.8$, but at $U = U_{opt} \approx 2$ the condition (2.101) satisfies. Corresponding profiles of critical gradients R_0/L_{pc} are shown in Fig. 2.11. Here, as in Fig. 2.9, the curves have a maximum at the edge with the values $U \sim U_{opt}$. The performed calculations show that the values of U_{opt} increase with decreasing of the aspect ratio and increasing of the cross section triangularity.

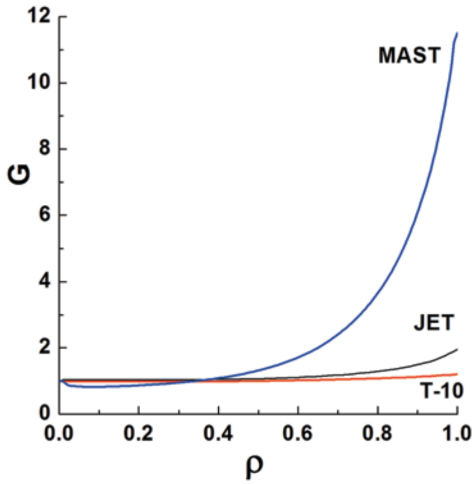

Fig. 2.2 The typical profiles of the metric coefficient G for T-10, JET and MAST devices with discharge parameters (1.103) − (1.105). This metric coefficient characterizes the density of the magnetic surfaces on the outer side of the torus

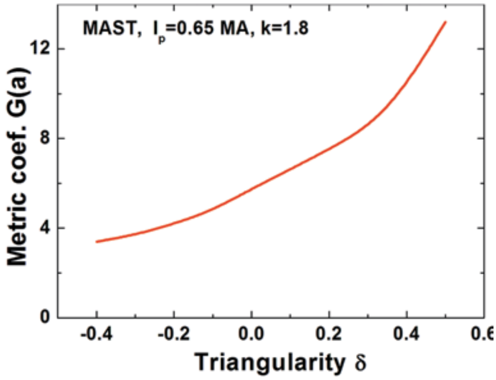

Fig. 2.3 The values of G at the MAST plasma boundary versus triangularity of plasma cross section at the elongation $k = 1.8$. The density of magnetic surfaces on the outer side of the plasma increases at the increase of triangularity

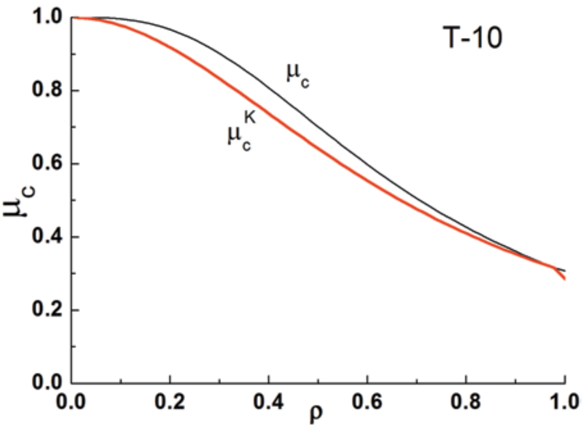

Fig. 2.4 The example of canonical profiles in general case and in Kadomtsev approximation for T-10 with plasma parameters (1.103)

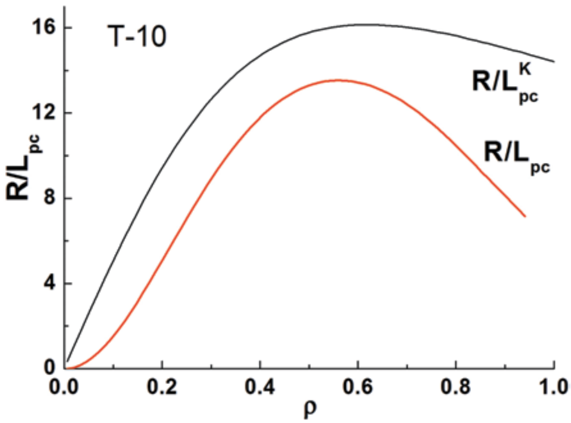

Fig. 2.5 The example of the dimensionless relative gradient profiles $R_0/L_{pc}=-R_0 p_c'/p_c$ in general case and in Kadomtsev approximation for T-10 with plasma parameters (1.103)

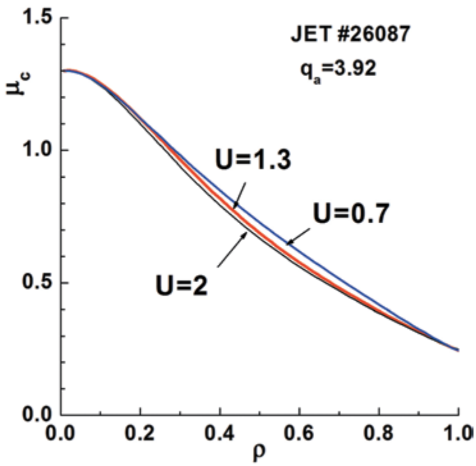

Fig. 2.6 The canonical profiles $\mu_c(r)$ for JET (discharge #26087) at different values of U

2.5 Canonical Profiles of the Toroidal Rotation

To determine the canonical profile of the plasma with toroidal rotation we use the procedure proposed in [1, 5] and discussed in Sect. 2.1. We first consider the equilibrium equation for plasma with rotation. For this we preset the toroidal rotation velocity υ_t as follows

$$\upsilon_t = \upsilon_t(\psi, r) = \omega r, \tag{2.108}$$

where $\omega=\omega(\psi)$ is the angular rotation frequency, r is a distance to the principal torus axis, ψ is the potential of the poloidal magnetic field, $\psi=$const at the magnetic surface. The equilibrium equation keeps the previous form (2.2)

$$\Delta^*\psi = -rj_\varphi = -(FF' + r^2 p'). \tag{2.109}$$

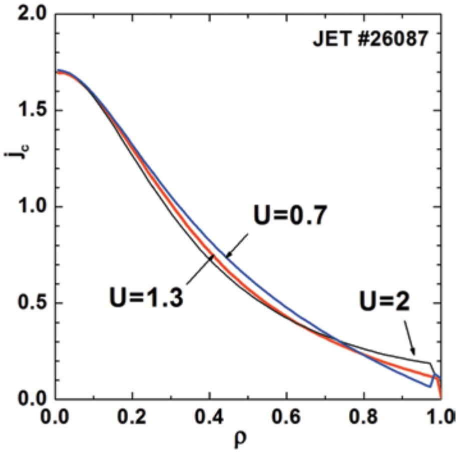

Fig. 2.7 The canonical profiles of current density $j_c(\rho)$ for JET at different values of U

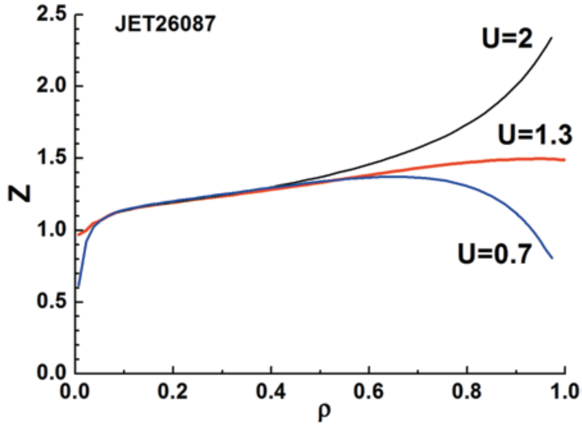

Fig. 2.8 Profiles of the function Z for JET at different values of U. It can be seen that the value of $U=1.3$ is optimal

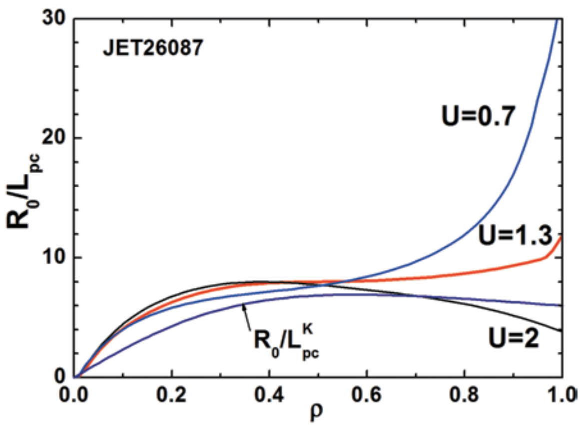

Fig. 2.9 Profiles of the relative pressure gradients R_0/L_{pc} for JET at different U. At the optimal value of $U=1.3$ the profile R_0/L_{pc} is almost flat at $\rho>0.3$. The profile R_0 / L_{pc}^K in Kadomtsev approximation is also shown

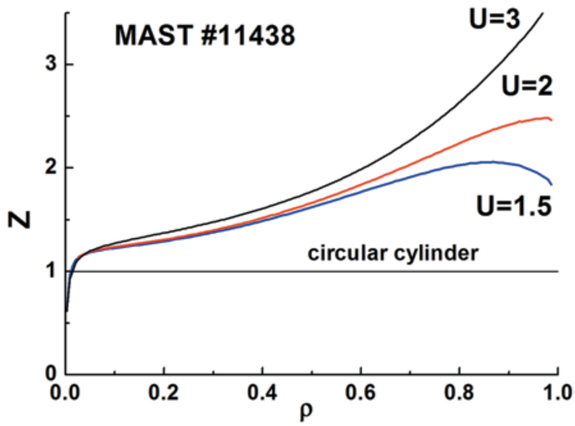

Fig. 2.10 Profiles of the function $Z(\rho)$ for MAST (discharge # 11438) at different values of U. It can be seen that the value of $U=2$ is close to the optimal one. In a circular cylindrical plasma $Z(\rho) \equiv 1$

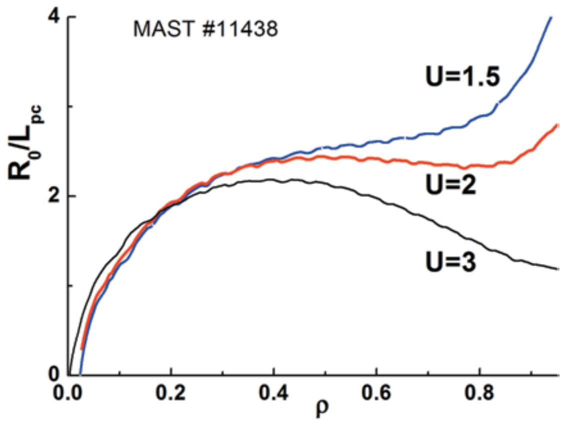

Fig. 2.11 Profiles of the relative pressure gradients R_0/L_{pc} for MAST (discharge # 11438) at different values of U

However, unlike the case of non rotated plasma, the plasma pressure p now depends on two variables and impermanent on the magnetic surface

$$p = p(\psi, r). \tag{2.110}$$

Let's recall that the bar (′) in (2.109) means the derivative with respect to ψ. The dependence of p on r must satisfy the condition (centrifugal force)

$$\frac{\partial p}{\partial r} = \frac{\rho_m v_t^2}{r} = \rho_m r \omega^2. \tag{2.111}$$

Here ρ_m is the hydrodynamic plasma density ($\rho_m = n\, m_i$, n is the plasma density, m_i is the ion mass). We assume that the kinetic energy of plasma rotation is much lower than the thermal energy

$$\frac{\rho_m \upsilon_t^2}{2} \ll p, \quad \text{or}\, \upsilon_t^2 \ll \upsilon_T^2, \quad M^2 = \frac{\upsilon_t^2}{\upsilon_T^2} \ll 1. \tag{2.112}$$

Here υ_T is the thermal velocity of the ions, M is Mach number. Now, as a function of $p(\psi, r)$, we can use a simple function that satisfies the condition (2.111)

$$p = p_0(\psi) + \frac{r^2}{2}\rho_m \omega^2 = p_0 + \frac{\rho_m \upsilon_t^2}{2}. \tag{2.113}$$

Let R_0 is a major radius of the plasma torus, then the function $(R_0{}^2/2)\,\rho_m \omega^2$ has the dimension of pressure. We denote it for the convenience through p_1. In this case

$$p = p_0 + \frac{r^2}{R_0^2} p_1, \quad p_1 = \frac{R_0^2}{2}\rho_m \omega^2. \tag{2.114}$$

By (2.112)

$$p_0 \gg p_1. \tag{2.115}$$

Therefore in the expression for p_1 (2.114) we can assume that $\rho_m = \rho_m(\psi)$. As a result, the equilibrium equation (2.109) takes the form

$$\Delta^* \psi = -rj_\varphi = -\left[FF' + r^2 \left(p_0 + \frac{r^2}{R_0^2} p_1 \right)' \right]. \tag{2.116}$$

We now turn to the variation problem. We define the canonical profiles of pressure, rotation, and function of the toroidal magnetic field F, *a*s profiles that minimize the total energy of the plasma while maintaining the toroidal current and equilibrium conditions. The total energy is given by (compare with (2.4))

$$W = \int_V dV \left[\frac{F^2 + (\nabla \psi)^2}{2r^2} + \frac{3}{2} p + \frac{\rho_m \upsilon_t^2}{2} \right] \tag{2.117}$$

The total current is

$$I = \int_V \frac{1}{2\pi r} \frac{FF' + r^2 p'}{r} dV. \tag{2.118}$$

Lagrange extended functional is as follows

$$\Phi = W - 2\pi\lambda I. \tag{2.119}$$

Its first variation in the extremal point must be equal to zero

$$\delta\Phi = \delta W - 2\pi\lambda\ \delta I = \int_V dV\ \delta\psi \left\{ \left[\frac{F\,F' - \Delta^* \psi}{r^2} + \frac{3}{2}\left(p_0 + \frac{r^2}{R_0^2} p_1 \right)' \right] - \right.$$
$$\left. -\lambda \left[\frac{F'F' + F\,F''}{r^2} + \left(p_0 + \frac{r^2}{R_0^2} p_1 \right)'' \right] \right\} = 0 \tag{2.120}$$

Hence we obtain two-dimensional Euler equation

$$\left[\frac{F\,F' - \Delta^* \psi}{r^2} + \frac{3}{2}\left(p_0 + \frac{r^2}{R_0^2} p_1 \right)' \right] - \lambda \left[\frac{F'F' + F\,F''}{r^2} + \left(p_0 + \frac{r^2}{R_0^2} p_1 \right)'' \right] = 0 \tag{2.121}$$

Using (2.116) it is converted to the form

$$\frac{2FF' - \lambda(FF')'}{r^2} + \left(\frac{5}{2} p_0' - \lambda p_0'' \right) + \frac{r^2}{R_0^2}\left(\frac{5}{2} p_1' - \lambda p_1'' \right) = 0. \tag{2.122}$$

The second term is permanent at a magnetic surface, but the remaining terms are variable with different dependencies on r. Hence we obtain three one-dimensional independent equations:

$$2FF' - \lambda(FF')' = 0, \quad \frac{5}{2} p_0' - \lambda p_0'' = 0, \quad \frac{5}{2} p_1' - \lambda p_1'' = 0. \tag{2.123}$$

Here, the first and second equations coincide with the corresponding equations for plasma without rotation (2.10). The third equation with respect to p_1 coincides with the equation for p_0. Thus, the canonical profile for the function $p_1 = \frac{R_0^2}{2} \rho_m \omega^2$ coincides with the canonical profile for the function p_0. Below, as in Sect. 1.1, we replace $5/(2\lambda)$ by λ. Then the solutions of equations (2.123) become as follows

$$F\,F' = C_F \exp(\tfrac{4}{5}\lambda\psi), \quad p_0' = C_{p0} \exp(\lambda\psi), \quad p_1' = C_{p1} \exp(\lambda\psi). \tag{2.124}$$

Substituting (2.124) into (2.116), we obtain the canonical equilibrium equation

$$\Delta^{*}\psi = -rj_{\varphi} = -\left[C_F \exp(\tfrac{4}{5}\lambda\psi) + C_p r^2 \exp(\lambda\psi)\left(p_0(0) + \tfrac{1}{2} r^2 \rho_{m0}\omega_0^2\right)\right]. \quad (2.125)$$

It is assumed that at the magnetic axis $\psi = 0$, $\rho_{m0} = \rho_m(0)$, $\omega_0 = \omega(0)$. The parameters C_F, C_p and λ are determined from additional conditions. For example

$$I = I_0, \quad \beta_p = \beta_p^0, \quad q(0) = q_0. \quad (2.126)$$

Integrating the last two Eq. (2.124), changing the notation of constants and leaving only the exponential parts of the solutions, we obtain, as in (2.13)

$$p_{c0} = C_p \exp(\lambda\psi), \quad p_{c1} = C_{p1} \exp(\lambda\psi). \quad (2.127)$$

Recalling that the canonical profiles are defined up to a multiplier, we obtain the following chain of equalities:

$$p_{c1} \propto p_{c0} \propto \rho_{mc}\omega_c^2 \propto n_c\omega_c^2 \propto n_c T_c. \quad (2.128)$$

Thus,

$$\omega_c^2 \propto T_c. \quad (2.129)$$

The canonical temperature profile is determined by (2.76), so

$$\omega_c \propto T_c^{1/2} \propto p_{c0}^{1/3}. \quad (2.130)$$

Later in the transport model the logarithmic derivatives of the canonical profiles will be used. By (2.130), they are linked by the following relations

$$\frac{\omega_c'}{\omega_c} = \frac{1}{3}\frac{p_{c0}'}{p_{c0}} = \frac{1}{2}\frac{T_c'}{T_c}. \quad (2.131)$$

Here the prime denotes the derivative with respect to the dimensional radial coordinate ρ, determining the magnetic surface ($0 < \rho < \rho_{max}$). The canonical current density and pressure profiles are calculated by the corresponding solutions of the problem (2.98, 2.99, 2.101) concerning $\mu = 1/q$ with one-dimensional Euler equation.

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