Distortion Risk Measure or the Transformation of Unimodal Distributions into Multimodal Functions

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1 Introduction

A commonly used risk metrics is the standard deviation. For examples mean-variance portfolio selection maximises the expected utility of an investor if the utility is quadratic or if the returns are jointly normal. Mean-variance portfolio selection using quadratic optimisation was introduced by Markowitz (1959) and became the standard model. This approach was generalized for symmetrical and elliptical portfolio (Ingersoll 1987; Huang and Litzenberger 1988). However, the assumption of elliptically symmetric return distributions became increasingly doubtful (Bookstaber and Clarke 1984; Chamberlain 1983) to characterize the returns distributions making standard deviation an intuitively inadequate risk measure.

Recently the financial industry has extensively used quantile-based downside risk measures based on the Value-at-Risk (VaR_α for confidence level α). While the VaR_α measures the losses that may be expected for a given probability it does not address how large these losses can be expected when tail events occur. To address this issue the mean excess function has been introduced, (Rockafellar and Uryasev 2000; Embrechts et al. 2005; Artzner et al. 1999) and Delbaen (2000) describe the properties that risk measures should satisfy including their coherence in particular the VaR is not a coherent risk measure, failing to be sub-additive.

When we use a sub-additive measure the diversification of the portfolio always leads to risk reduction while if we use measures violating this axiom the diversification benefit may be lost even if partial risks are triggered by mutually exclusive events.

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The sub-additive property is required for capital adequacy purposes in banking supervision: for instance if we consider a financial institution made of several subsidiaries or business units, if the capital requirement of each of them is dimensioned to its own risk profile authorities. Consequently it has appeared relevant to construct a more flexible risk measure which is sub-additive.

Nevertheless, the VaR remains preeminent even though it suffers from the theoretical deficiency of not being sub-additive. The problem of sub-additivity violations is not as important for assets verifying the regularity conditions[1] than for those which do not and for most assets these violations are not expected. Indeed, in most practical applications the VaR_α can have the property of sub-additivity. For instance, when the return of an asset is heavy tailed, the VaR_α is sub-additive in the tail region for high level of confidence if it is computed with the heavy tail distribution (Ingersoll 1987; Danielson et al. 2005; Embrechts et al. 2005). Non sub-additivity of the VaR_α is highlighted when assets have very skewed return distributions. When the distributions are smooth and symmetric, when assets dependency is highly asymmetric, and when underlying risk factors are dependent but heavy-tailed, it is necessary to consider other risks measures.

Unfortunately, non sub-additivity is not the only problem characterizing the VaR. First VaR only measures distribution percentiles and thus disregards any loss beyond its confidence level. Due to combined effects of this limitation and the occurrence of extreme losses there is a growing interest for risk managers to focus on the tail behavior and its Expected Shortfall[2] (ES_α) since it shares properties that are considered desirable and applicable in a variety of situations. Indeed, expected shortfall considers the loss beyond the VaR_α confidence level and is sub-additive and therefore it ensures the coherence of the risk measure (Rockafellar and Uryasev 2000).

Since using expected utility, the axiomatic approach to risk theory has expanded dramatically as illustrated by (Yaari 1987; Panjer et al. 1997; Artzner et al. 1999; De Giorgi 2005; Embrechts et al. 2005; Denuit et al. 2006) among others. Thus other classes of risk measures were proposed each with their own properties including convexity (Follmer and Shied 2004), spectral properties (Acerbi and Tasche 2002), notion of deviation (Rockafellar et al. 2006) or distortion (Wang et al. 1997). Acerbi and Tasche (2002) studied spectral risk measures which involve a weighted average of expected shortfalls at different levels. Then, the dual theory of choice under risk leads to the class of distortion risk measures developed by Yaari (1987) and Wang (2000), which transforms the probability distribution shifting it in order to better quantify the risk in the tails instead of modifying returns as in the expected utility framework.

Whatever the risk measures considered, the value associated to each measure is based and depends on the distribution fitted on the underlying data set by risk managers strategy. Mostly of the part the distributions belong to the elliptical domain,

[1] Regularly varying (heavy tailed distributions, fat tailed) non-degenerate tails with tail index $\eta > 1$ for more detail see Danielson et al. (2005).

[2] The terminology "Expected shortfall" was proposed by Acerbi and Tasche (2002) . A common alternative denotation is "Conditional Value at Risk" or CVaR that was suggested by Rockafellar and Uryasev (2002).

recently risk managers and researchers have focused on a class of distributions exhibiting asymmetry and producing heaving tails, All these distributions belong to the Generalized Hyperbolic class of distributions (Barndorff-Nielsen 1977), to the α-stable distributions (Samorodnitsky and Taqqu 1994) or the g- and -h distributions among others.

Nevertheless nearly all these distributions are unimodal. However, since the 2000s bubbles and financial crises and extreme events became more and more important, restricting unimodal distributions models for risk measures. Recently debates have been opened to convince economists to consider bimodal distributions instead of unimodal distributions to explain the evolution of the economy since the 2000s (Bhansali 2012). The debate about the choice of distributions characterized by several modes is timely. We propose an approach to build and fit these distributions on real data sets. An objective of this paper is to discuss this new approach and propose a theoretical framework to build multi-modal distributions to create new coherent risk measures.

The paper is organized as follows. In Section two we recall some principles and history of the risk measures: the VaR, the ES and the spectral measure. In Section three we discuss the notion of distortion to create new distributions. Section four proposes an application which illustrates the impact of the choice of unimodal or bimodal distribution associated to different risk measures to provide a value for the corresponding risk. Section five concludes.

2 Quantile-Based and Spectral Risk Measures

Traditional deviation risks measures such as the variance, the mean-variance analysis and the standard deviation, are not sufficient within the context of capital requirements. In this section we recall the definitions of several quantile-based risk measures:[3] the Value-at-Risk introduced in the 1980s, the Expected Shortfall proposed by Acerbi and Tasche (2002), the Tail Conditional Expectation suggested by Rockafellar and Uryasev (2002), and the spectral measure introduced by Acerbi and Tasche (2002).

Value at Risk initially used to measure financial institutions market risk, was mainly popularised by J.P. Morgan's RiskMetrics (1995). This measure indicates the maximum probable loss, given a confidence level and a time horizon. The VaR is sometimes referred as the "unexpected" loss.

Definition 1 Given a confidence level $\alpha \in (0, 1)$, the VaR is the relevant quantile[4] of the loss distribution: $VaR_\alpha(X) = \inf\{x \mid P[X > x] \leqslant 1-\alpha\} = \inf\{x \mid F_X(x) \geqslant \alpha\}$ where X is a risk factor admitting a loss distribution F_X .

[3] Artzner (2002) proposes a natural way to define a measure of risk as a mapping $\rho : L^\infty \to \mathbb{R} \cup \infty$.

[4] $VaR_\alpha(X) = q_{1-\alpha} = F_X^{-1}(\alpha)$

Table 1 Expression of ES_α as function of VaR_α for some usual distributions in finance

Distribution	$ES_\alpha = f(VaR_\alpha)$
Normal	$\mu + \frac{\sigma}{\sqrt{2\pi}} \frac{exp\left[-\frac{1}{2}\left(\frac{VaR_\alpha-\mu}{\sigma}\right)^2\right]}{1-\alpha}$
Student-t	$\mu + \frac{\sigma}{1-\alpha}\frac{\sqrt{\eta}}{(\eta-1)\sqrt{\pi}}\frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)}\left(1+\frac{(VaR_\alpha(X)-\mu)^2}{\sigma^2\eta}\right)^{-\frac{\eta+1}{2}}$
Logistic	$\mu + \frac{\sigma}{1-\alpha}\left(\ln\left(1+e^{VaR_\alpha(X)}\right) - VaR_\alpha(X)\left[1+e^{-(VaR_\alpha(X))}\right]^{-1}\right)$
Exponential power	$\mu + \frac{\sigma\beta^{\left(\frac{1}{\beta}-1\right)}}{2(1-\alpha)\Gamma(1+1/\beta)}\Gamma\left(\frac{2}{\beta},\frac{1}{\beta}\left(\frac{VaR_\alpha(X)-\mu}{\sigma}\right)^{\beta}\right)$
Generalized hyperbolic	$\mu+\beta\mathbb{E}(W)+\frac{\sigma}{\sqrt{2\pi}}\frac{exp\left[-\frac{1}{2}\left(\frac{VaR_\alpha-\mu}{\sigma}\right)^2\right]}{1-\alpha}\mathbb{E}(\sqrt{W})$
Generalized pareto	$\frac{VaR_\alpha(X)}{1-\xi}+\frac{\sigma-\xi u}{1-\xi}$
g and h	$\mu+\frac{\sigma}{g(1-\alpha)\sqrt{1-h}}\left[e^{(g^2/2(1-h))}\bar{\phi}\left(\sqrt{1-h}z_\alpha-\frac{g}{1-h}\right)-\bar{\phi}\left(\sqrt{1-h}z_\alpha\right)\right]$

As discussed in the Introduction, the VaR does not always appear sufficient. When a tail event occurs in a unimodal distribution, the loss in excess of the VaR is not captured. To avoid this problem we consider the expected shortfall (ES_α) proposed by Artzner et al. (1999). This measure is more conservative than the VaR_α as it captures the information contained in the tail. The expected shortfall is defined as follows:

Definition 2 The Expected Shortfall (ES_α) is defined as the average of all losses which are greater or equal than VaR_α:

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\alpha dp$$

The Expected Shortfall has a number of advantages over the VaR_α. Accordingly the ES takes accounts for the tail risk and fulfills the sub-additive property[5] (Acerbi and Tasche 2002)[6] . Table 1 summarizes the link between ES_α and VaR_α for some distributions given α.

Expected Shortfall is the smallest coherent risk measure that dominates the VaR. Acerbi and Tasche (2002) derived from this concept a more general class of coherent risk measures called spectral risk measures[7]. Spectral risk measures are a subset of coherent risk measures. Instead of averaging losses beyond the VaR, a weighted

[5] An extension can be found in Inui and Kijima (2005).

[6] In this last paper, the difference between ES and TCE is conceptual and is only related to the distributions. If the distribution is continuous then the expected shortfall is equivalent to the tail conditional expectation.

[7] If ρ_i is coherent risk measures for $i = 1 \ldots n$, then, any convex combination $\rho = \sum_1^n \beta_i \rho_i$ is a coherent risk measure (Acerbi and Tasche 2002).

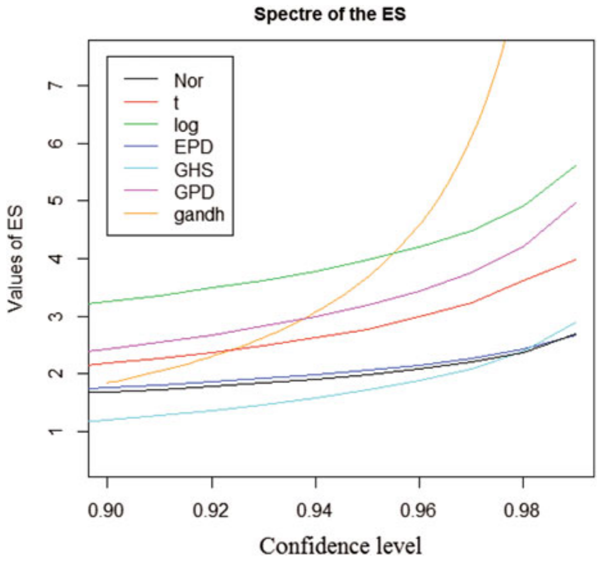

Fig. 1 Spectrum of the ES for some well known distributions for several $\alpha \in [0.9, 0.99]$. Each *line* corresponds to the graph of the ES as a function of α for each distribution introduced in Table 1

average of different levels of ES_α is used. These weights characterize risk aversion: different weights are assigned to different α levels of ES_α in the left tail. The associated spectral measure could be $\sum_\alpha w_\alpha ES_\alpha$, where $\sum_\alpha w_\alpha = 1$. In Fig. 1 we exhibit a spectrum corresponding to the sequence of ES_α for different α.

Figure 1 points out that the spectrum of the ES is an increasing function of the confidence level α. It expresses the risk aversion as a weighted average for different level of ES_α to generate the spectral risk measure. This is one advantage when using a spectral risk measure. Moreover a spectral risk measure being a convex combination of ES_α for $\alpha \in [0.9, 0.99]$, it accounts for more information than only considering one value of α.

However the choice of weights is sensitive and need to be studied more carefully (Dowd et al. 2008). Finally, in practice the relation between spectral risk measure and risk aversion is not obvious depending on the choice of the weights.

3 Distortion Risk Measures

3.1 Notion of Distortion Risk Measures

Distortion risk measures have their origin in Yaari's (1987) dual theory of choice under risk that consists in measuring the risks by applying a distortion function g on the cumulative distribution function F_X. In order to transform a distribution into a new distribution we need to specify the property of the distorsion function g.

Definition 3 A function $g : [0, 1] \to [0; 1]$ is a distortion function if:

1. $g(0) = 0$ and $g(1) = 1$,
2. g is a continuous increasing function.

In order to quantify the risk instead of modifying the loss distribution (as with the expected utility framework), the distortion approach modifies the probability distribution. The risk measures (VaR and ES) derived from this transformation were originally applied to a wide variety of financial problems such as the determination of insurance premiums (Wang 2000), economic capital (Hürlimann 2004), and capital allocation (Tsanakas 2004). Acerbi (2002) suggests that they can be used to set capital requirements or obtain optimal risk-expected return trade-offs and could also be used by clearing-houses to set margin requirements that reflect their corporate risk aversion (Cotter and Dowd 2006).

One possibility is to shift the distribution function towards the left or the right sides to account for extreme values. Wang et al. (1997) developed the concept of distortion[8] risk measure by computing the expected loss from a non-linear transformation of the cumulative probability distribution of the risk factor. A formal definition of this risk measure computed from a distortion of the original distribution has been derived (Wang et al. 1997).

Definition 4 The distorted risk measure $\rho_g(X)$ for a risk factor X admitting a cumulative distribution $S_X(x) = \mathbb{P}(X > x)$, with a distortion function g, is defined[9] as:

$$\rho_g(x) = \int_{-\infty}^{0} [g(S_X(x)) - 1]dx + \int_{0}^{+\infty} g(S_X(x))dx. \tag{1}$$

Such a distortion risk measure corresponds to the expectation of a new variable whose probabilities have been re-weighted.

Finding appropriate distorted risk measures reduces to the choice of an appropriate distortion function g. Properties for the choice of a distortion function include continuity, concavity, and differentiability. Assuming g is differentiable on $[0, 1]$ and $S_X(x)$ is continuous, then a distortion risk measure can be re-written as:

$$\rho_g(X) = \mathbb{E}[Xg'(S_X(X))] = \int_0^1 F_X^{-1}(1-p)dg(p) = \mathbb{E}_g\left[F_X^{-1}\right]. \tag{2}$$

Distortion functions arose from empirical[10] observations that people do not evaluate risk as a linear function of the actual probabilities for different outcomes but

[8] The distortion risk measure is a special class of the so-called Choquet expected utility, i.e. the expected utility calculated under a modified probability measure.

[9] Both integrals in (1) are well defined and take a value in $[0, +\infty]$. Provided that at least one of the two integrals is finite, the distorted expectation $\rho_g(X)$ is well defined and takes a value in $[-\infty, +\infty]$.

[10] This approach towards risk can be related to investor's psychology as in Kahneman and Tversky (1979).

rather as a non-linear distortion function. It is used to transform the probabilities of the loss distribution to another probability distribution by re-weighting the original distribution. This transformation increases the weight given to desirable events and deflates others. Different distortions g have been proposed in the literature. A wide range of parametric families of distortion functions is mentioned in Wang (2000), and Hardy and Wirch (2001). For well known utility functions we provide the function g in Table 2, where the parameters k and γ represent the confidence level corresponding and the level of risk aversion.

Table 2 Examples of utility functions with their associated convex spectrum

	Utility function	Parameters	Spectrum function
Exponential	$U_1(x) = -e^{-kx}$	$k > 0$	$g(p,k) = \frac{ke^{-k(1-p)}}{1-e^{-k}}$
Power	$U_2(x) = x^{1-\gamma}$	$\gamma \in (0,1)$	$g(p,\gamma) = \gamma(1-p)^{\gamma-1}$
Power	$U_3(x) = x^{1-\gamma}$	$\gamma > 1$	$g(p,\gamma) = \gamma(p)^{\gamma-1}$

When g is a concave function its first derivative g' is an increasing function, $g'(S_X(x))$ is a decreasing function[11] in x and $g'(S_X(x))$ represents a weighted coefficient which discounts the probability of desirable events while loading the probability of adverse events. Moreover, Hardy and Wirch (2001) have shown that distorted risk measure $\rho_g(X)$ introduced in (2) is sub-additive and coherent if and only if the distortion function is concave.

In his article, Wang (2000) specifies that the distortion operator g can be applied to any distribution. Nevertheless in applications due to technical practical reasons he restricts the illustration of his methodology to a function g defined as follows:

$$g_\alpha(u) = \Phi\left[\Phi^{-1}(u) + \alpha\right], \tag{3}$$

where Φ is the Gaussian cumulative distribution. In other words he applies the same perspective of preference to quantify the risk associated to gain and risk. Thus, a risk manager evaluates the risk associated to the upside and downside risks with the same function g implying a symmetric consideration for the two effects due to the distortion. Moreover it induces the same confidence level for the losses and the gain which implies the same level of risk aversion associated to the losses and the gains.

In Fig. 2 we illustrate the impact of the Wang (2000) distortion function introduced in Eq. (3) on the logistic distribution provided in Table 1. We can remark that the distorted distribution is always symmetrical under this kind of distortion function, and we observe a shift of the mode of the initial distribution towards the left.

To avoid the problem of symmetry in the previous distorsion, Sereda et al. (2010) propose to use two different functions issued from the same polynomial with different coefficients, say:

$$\rho_{g_i}(X) = \int_{-\infty}^{0} [g_1(S_X(x)) - 1]\, dx + \int_{0}^{+\infty} g_2(S_X(x)) dx. \tag{4}$$

[11] This property involves that $g'(S_X(x))$ becomes smaller for large values of the random variable X.

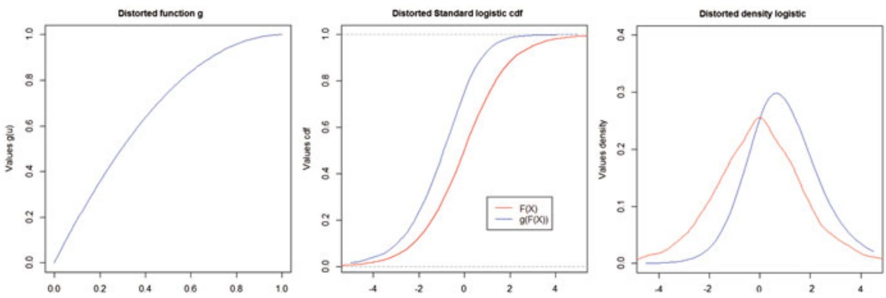

Fig. 2 Distortion of logistic distribution with mean 0 using a Wang distortion function with confidence level 0.65. It illustrates the effect of distortion

with $g_i(u) = u + k_i\left(u - u^2\right)$ for $k_i \in]0,1]$ et $\forall i \in \{1,2\}$. With this approach one models loss and gains differently relatively to the values of the parameters $k_i, i = 1,2$. Thus upside and downside risks are modeled in different ways. Nevertheless the calibration of the parameters $k_i, i = 1,2$ remains an open problem.

To create bimodal or multi-modal distributions we have to impose other properties to the distortion function g. Indeed, transforming an unimodal distribution into a bimodal one provides different approaches to the risk aversion of losses and gains. This will allow us to introduce a new coherent risk measure in that latter case.

3.2 A New Coherent Risk Measure

We begin to discuss the choice of the function g to obtain a bimodal distribution. To do so we need to use a function g which creates saddle points. The saddle point generates a second hump in the new distribution which allows us to take into account different patterns located in the tails. The distortion function g fulfilling this objective is an inverse S-shaped polynomial function of degree 3 given by the following equation and characterized by two parameters δ and β:

$$g_\delta(x) = a\left[\frac{x^3}{6} - \frac{\delta}{2}x^2 + \left(\frac{\delta^2}{2} + \beta\right)x\right]. \tag{5}$$

We remark that $g_\delta(0) = 0$, and to get $g_\delta(1) = 1$ this implies that the coefficient of normalization is equal $a = \left(\frac{1}{6} - \frac{\delta}{2} + \frac{\delta^2}{2} + \beta\right)^{-1}$. The function g_δ will increase if $g'_\delta > 0$ requiring $0 < \delta < 1$. The parameter $\delta \in [0,1]$ allow us to locate the saddle point. The curve exhibits a concave part and a convex part. The parameter $\beta \in \mathbb{R}$ controls the information under each hump in the distorted distribution. To illustrate the role of δ on the location of the saddle points, we provide several simulations.

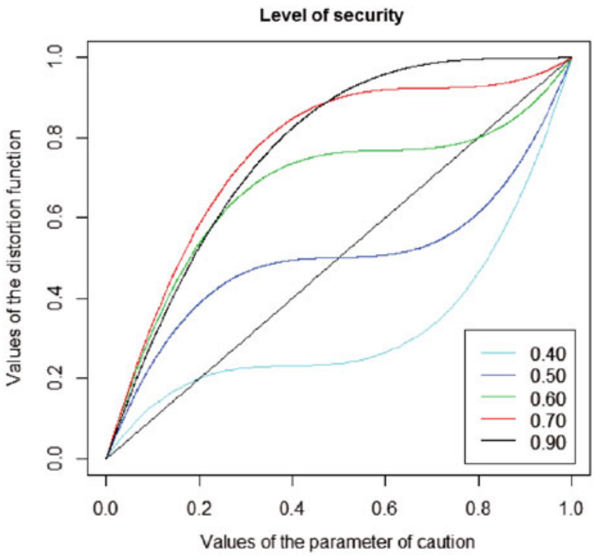

Fig. 3 Curves of the distortion function g_δ introduced in Eq. (5) for several value of δ and fixed values of $\beta = 0.001$

In Fig. 3, the value of the level of the discrimination of an event is given by $\beta = 0.001$ then we plot the function g_δ for different values of δ. This parameter β illustrates the fact that some events are discriminating more than others. Figure 3 shows the location of the saddle point creating convex and concave parts inside the domain $[0, 1]$. The convex part can be associated to the negative values of the returns associated to the losses and the concave part is associated to positive returns. We observe in this picture that for high values of δ the concave part diminishes and then the effect of saddle point decreases.

Variations in β in Fig. 4 exhibit different patterns for a fixed value of δ.

To understand the influence of the parameter β on the shape of the distortion function we use three graphs in Fig. 4. The two left graphs correspond to the same value of the parameters. The middle figure zooms on the x-axis from $[0, 1]$ to $[-4, 4]$. We show that the function g may not have a saddle point on $]0, 1[$ depending on the values of β. The right graph provides different representations of the distorsion function for several values of β. We observe that if β tends to 1 then the distortion function g tends to the identity mapping and when β tends to 0 the curve is more important and the effect of g on the distribution will be more important.

Figure 5 illustrates the effect of distortion of the Gaussian distribution for several values of β and fixed $\delta = 0.50$. We observe the same effects as in Fig. 4. For small values of the parameter β (0.00005 or 0.005) the distortion function has two distinct parts, one convex part for $x \in]0, 0.5[$ and one concave part for $x \in]0.5, 1[$. Moreover when β is close to 1 then the distorted cumulative distribution tends to the initial Gaussian variable.

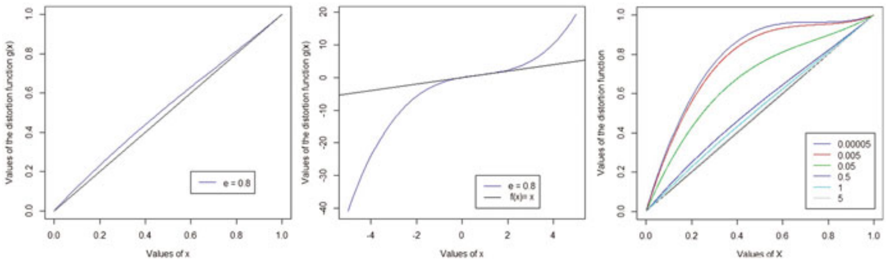

Fig. 4 The effect of β on the distortion function for a level of security $\delta = 0.75$ showing that if β tends to 1, the distortion function tends to the identity function

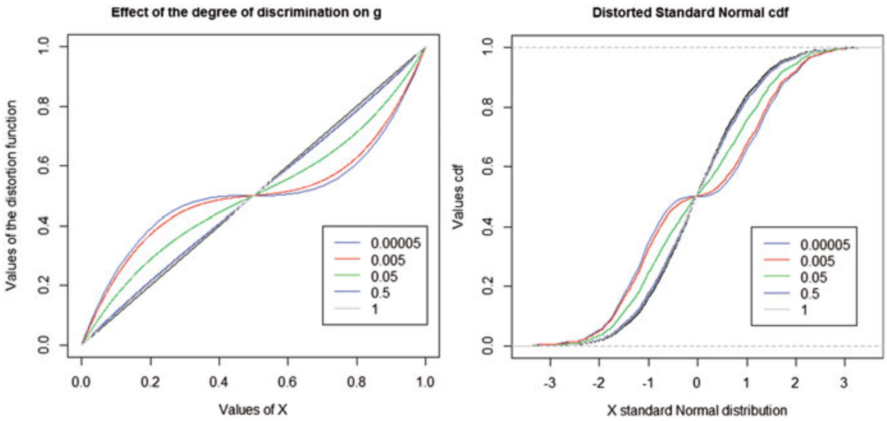

Fig. 5 The effect of β on the cumulative Gaussian distribution for $\delta = 0.50$

Figure 6 points out the effect of distortion on the density of the Gaussian distribution using the same values of the parameters than those used in Fig. 5. Again we generate a new distribution with two humps. Making both parameters varying permits to solve one of our objective: to create a asymmetrical distribution with more than one hump.

It is important to notice that the function g_δ creates a distorted density function which associates a small probability in the centre of the distribution and put greater weight in the tails. This phenomenon is illustrated in Fig. 7 where the derivative of g (density) indicates how weights on the tails can be increased.

Such discrimination is also illustrated in Fig. 8 which exhibits the particular effect of parameter β when δ is fixed to 0.75 for the creation of humps. From a Gaussian distribution, applying g_δ defined in (5), with $\delta = 0.75$ and $\beta = 0.48$ we create a distribution for which the probability of occurrences of the extremes in the right part is bigger than the probability of occurrence of the extremes in the left part which can be counter-intuitive for risk management but interesting from a theoretical point of view.

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