Chapter 2
The Phenomenological Stochastic Approach: A Short Route to Quantum Mechanics

Some physicists, among them myself, cannot believe that we must abandon, actually and forever, the idea of direct representation of physical reality in space and time...
A. Einstein (1954)

2.1 Why a Phenomenological Approach to Quantum Mechanics?

Independently of the interpretation embraced, randomness enters into the quantum description as a central and ever present ingredient.[1] Therefore, and leaving aside for the moment the problem of identifying the source of the randomness, a direct analysis of QM as a stochastic theory seems befitting as an introduction to the subject and in preparation for the chapters that follow. The phenomenological approach presented in this chapter is particularly suitable for such purposes, and in spite of its limitations, it reinforces the notion that a stochastic process underlies QM.[2]

The natural procedure to deal with a stochastic problem in physics involves a statistical treatment. However, any direct stochastic interpretation of QM faces immediately a fundamental difficulty, since QM is not a genuine statistical theory, as mentioned

[1] Not *quite independently* of the interpretation, strictly speaking. For example, Bell (1987, article 19) argues that "...the reversibility of the Schrödinger equation strongly suggests that quantum mechanics is not fundamentally stochastic in nature." This sentence sounds tempting... At this stage, how would you respond to it? (An answer is given at the end of the chapter.)

[2] Randomness plays an important role in several interpretations of QM, in addition to the one developed in this book. A case of major interest is Griffiths' theory of consistent histories, according to which most of the evolution is due to randomness. See Omnès (1994, 1999a, b), Griffiths (1996), Griffiths and Omnés (1999).

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in Chap. 1. This presents us with a quite discomforting alternative: (i) either a true statistical description of quantum systems is achieved, but then it cannot faithfully reproduce all of QM, or (ii) the stochastic model is constructed so as to reproduce the quantum behavior in detail, but then it will be endowed with some bizarre properties. The derivation of standard QM from a genuine stochastic theory, as developed in later chapters, will allow us to fully appreciate the importance of this assertion, since several approximations have to be made along the way, which result in a sui generis statistical description.

The stochastic theory presented in this chapter is based on a phenomenological model that goes along the second alternative and hence is completely in line with QM. As will become clear, this model has the advantage of its intuitive appeal and its simplicity. The fact that it is expressed in terms of dynamical variables that are not part of the standard quantum formalism, contributes to enrich the description by looking at the quantum phenomenon from a different angle, yet the phenomenological model also has some important shortcomings.

In developing the stochastic approach to QM a stochastic physical source with certain simple statistical properties is normally assumed to exist, but not identified. This lack of definition has given way to a variety of most dissimilar proposals, and even to the assumption that the hidden source does not exist at all and that the stochasticity is spontaneous, which of course leaves things as noncausal as they are without the need of another theory. In the context of the present chapter, it is a small sin to leave the source of stochasticity unspecified, since the rest of the book is devoted, to a large extent, to identify such source and to extract the consequences of its presence.

An important point to be made here is that nearly all forms of phenomenological approach to QM consider the stochastic process as a kind of Brownian motion. As will become evident, this is incorrect. Quantum stochasticity does *not* mean Brownian motion; at the phenomenologic level quantum and classical stochastic particles follow their own dynamical rules.

2.2 The Stochastic Description of Quantum Mechanics

Among the profuse arguments given in support of the notion of a stochastic process underlying QM, a couple of them go as follows (see also Comisar 1965; Hall and Collins 1971; Vasudevan et al. 2008). The first one, merely formal, is based on the analogy first observed by Schrödinger (1931, 1932) (and later by 1933) between his equation and the diffusion equation, which are related with one another by analytical continuation into imaginary times(see Sect. 4.4.5). The seed planted by Schrödinger expanded much later into a fuller theory in terms of Bernstein processes (Blanchard and Garbaczewski 1994; Jamison 1974; Zambrini 1986; see also Cramer 1986; Garbaczewski 1990, 1992, 1992, 1993a, 1993b, 1994, 1995.

A second interesting argument is that if a quantum particle is considered to follow a stochastic process in configuration space, the resulting Hausdorff fractal dimension

of such process turns out to be the same as that of a Brownian particle (Abbott and Wise 1981). This fact is in itself interesting, although one can think with no less legitimacy of a phase-space description, and then the analogy breaks down. Many other particular reasons have been given in support of a stochastic approach to the quantum problem; for example, de Broglie (1967) felt compelled to introduce it in order to make the particle switch at random from one guiding wave to another (the guiding wave proposed by de Broglie is touched upon in Chap. 8).

The first relatively accomplished stochastic theory of the quantum process was proposed by the Hungarian physicist Fényes (1946, 1952) [which was strongly criticized by Nicholson (1954)], and further developed by Kershaw (1964) and Bess (1973), among others.[3] Féynes' theory is based on an ad hoc Lagrangian within a Brownian context; the author went as far as to recover a good part of the Hilbert-space formalism and concluded that QM describes an inherently stochastic phenomenon. But perhaps the most widely known theory of this kind, based on a non dissipative Markov process, is *stochastic mechanics*, initiated by Nelson in (1966) (and subsequent papers 1967–2013), and further developed by Guerra (1981, 1984, 1985, 1988), Guerra and Marra (1983), Guerra and Morato (1983), and Davidson (1978, 1979a, b, c, 1981, 2007). A common characteristic of this collection of works, at least during the earlier stages of development, was the identification of the underlying process as classical and of a Brownian nature. This led Jammer (1974, p. 418) to the statement: "The main objective of the stochastic interpretation of quantum mechanics has been to show that quantum theory is fundamentally a classical theory of probabilities or stochastic processes, and as such conceptually of the same structure as, say, the Einstein-von Smoluchowski theory of Brownian motion. . ." in our own words (de la Peña and Cetto 1982), "this sounds as astonishing and implausible as the complementary assertion would sound, namely, that Brownian motion is fundamentally a quantum theory conceptually of the same structure as the Schrödinger theory of the electron". And indeed, the need for a clear conceptual distinction between these two stochastic processes gave rise to a different branch of research, whose scope was also the development of a possible stochastic interpretation of QM, but on the basis that the quantum stochasticity is distinctly nonclassical, i.e., essentially different from Brownian motion. This theory, which has been called *stochastic quantum mechanics*, is the one that will be presented here.[4]

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[3] At least two other stochastic proposals were made almost simultaneously to Féynes' work, by Novobátzky (1951) and Takabayasi (1952). A detailed account of the first developments of the stochastic approach to QM can be seen in Jammer (1974), Chap. 9.

[4] As noted earlier, Nelson calls the theory simply *stochastic mechanics*. His work is that of a mathematician and should be of major interest to the more mathematically inclined readers. There is another entirely different theory that goes under the same name *stochastic quantum mechanics*, pioneered by Prugovečki (1984, 1995) [see also Ali and Engliš (2005); Ali and Prugovečki (1986)]. It represents an attempt to unify physics into a rigorous quantum structure that considers a quantum spacetime and a universe which on the microscopic level follows a stochastic rather than deterministic evolution. Further, it should be noted that some authors speak of stochastic *quantum mechanics* while referring to Bohm's theory (see e.g. Feligioni et al. 2005).

A general feature of all these stochastic theories is their phenomenological nature; since they are aimed at reproducing QM, whether the process is considered classical or not, they are in principle unable to go *beyond* QM. In general, no specific assumption is made about the nature of the stochastic force, although, as indicated above, one can find the most varied suggestions in this regard, ranging from collisions with vacuum particles or *zerons*, interactions with a diversity of vacuum fields or even neutrinos, or a universal action reservoir (Lisi 2006), to fluctuations of the space-time metric (Santos 2006). The description may even be made compatible with the idea of an indeterministic electron, which is far from the realistic and causal persuasion that inspires the whole enterprise. Their phenomenological character is perhaps the strongest objection that can be made to these models, but taken at their face value they can be and indeed have been useful, at least because of the picture they suggest, and for several other reasons that will become evident in what follows.

2.3 Stochastic Quantum Mechanics

Our first task is to construct a theory of stochastic processes in configuration space that is sufficiently general (within the proper limits of the theory), so as to accommodate the quantum processes, assuming such a description is feasible. The (rather informal) exposition that follows, essentially based on de la Peña (1969), de la Peña and Cetto (1975, 1982, 1991, 1996) (we follow more closely the exposition in this last work), and Santos (1973), starts with the formulation of the appropriate kinematics. Different or complementary discussions can be seen in Nelson (1966, 2012), Guerra (1981), Blanchard et al. (1987), Kyprianidis (1992) and references therein.

2.3.1 Kinematics

Consider a particle undergoing a stochastic motion, so that its position $\boldsymbol{x}(t)$ constitutes a stochastic process. Thus, for each possible event (or rather, for each realization of the source of randomness, if any) a specific trajectory is followed, starting from the initial conditions. Assume that at a certain time t the particle is located at a point $\boldsymbol{x}$; at a slightly earlier time $t' = t - \Delta t$ it had a different position denoted by $\boldsymbol{x}'$, and similarly, at a slightly later time $t'' = t + \Delta t$ it will occupy the position $\boldsymbol{x}''$. For an arbitrary C_∞-function g of the stochastic variable $\boldsymbol{x}$ a Taylor series expansion, with

$$\Delta_+ \boldsymbol{x} = \boldsymbol{x}'' - \boldsymbol{x}, \quad \Delta_- \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}', \tag{2.1}$$

gives the expression (a sum over repeated indices is understood)

$$\frac{g(\boldsymbol{x}'') - g(\boldsymbol{x}')}{2\Delta t} = \frac{\partial g}{\partial x_i}\frac{\Delta_+ x_i + \Delta_- x_i}{2\Delta t} + \frac{\partial^2 g}{\partial x_i \partial x_j}\frac{\Delta_+ x_i \Delta_+ x_j - \Delta_- x_i \Delta_- x_j}{4\Delta t} + \cdots \tag{2.2}$$

For a smooth (sure or nonstochastic) motion we can take the limit $\Delta t \to 0$, when this expression reduces to $dg(\boldsymbol{x})/dt = (\nabla g)\cdot(d\boldsymbol{x}/dt)$.[5] However, in the presence of stochasticity Δt cannot be taken arbitrarily small. The reason is that at the time scale of the 'instantaneous' description (i.e., according to the available experimental time resolution) the components $\Delta_\pm x_i$ for a given member of the ensemble may happen to be non differentiable (or changing very fast), due to abrupt kicks impressed by the random source. On the other hand, since $\Delta_\pm x_i(t)$ refers also to a stochastic variable, also the 'derivative' defined above becomes a random function. Nevertheless, it is possible to construct an approximate or phenomenological derivative, for small Δt, as follows.

The first change consists in averaging over all the possible events (or realizations of the background randomness), or rather over the ensemble of particles that reproduce all the possible trajectories. This operation is denoted by $\langle\cdot\rangle$, so that instead of $g(\boldsymbol{x}'') - g(\boldsymbol{x}')$ we will consider $\langle g(\boldsymbol{x}'') - g(\boldsymbol{x}')\rangle$. Next, the problem of taking the limit $\Delta t \to 0$ is solved by performing a moving averaging of the function $\boldsymbol{x}(t)$ during a 'small' time Δt,[6] much smaller than the characteristic time T_0 of the systematic (relevant) motions, but long enough for the particle to feel the effects of many blows from the stochastic source, so as to effectively smoothen out the most rapid changes in the instantaneous position. For example, in the case of Brownian motion the particle is so large compared with the solvent molecules that it receives a large number of molecular impacts during the time interval Δt, thus effectively averaging them into a (much) smoother function of time. These averaged quantities are the ones that obey the diffusion laws. Thus we choose

$$T_0 \gg \Delta t \gg t_c, \tag{2.3}$$

where t_c is appropiately selected so as to embrace many of the most closely spaced violent changes in each particular 'instantaneous' $\boldsymbol{x}(t)$. The resulting (coarse-time-scale) average time derivative or *systematic derivative* is denoted by the symbol $\mathcal{D}_c$; hence,

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[5] By writing

$$\frac{g(\boldsymbol{x}; t'') - g(\boldsymbol{x}; t')}{2\Delta t} = \frac{1}{2\Delta t}\int_{t'}^{t'+2\Delta t} \frac{\partial g(\boldsymbol{x}; s)}{\partial s} ds$$

it becomes clear that this expression is a coarse-grained time-derivative obtained by time-averaging the derivative $\partial g/\partial t$. This procedure mimicks the time smoothing produced by an observation, which is always extended in time. Such smoothing is particularly appropriate to deal with highly irregular (and even non-differentiable) functions.

[6] The moving average $x_{\Delta t}(t)$ of $x(t)$ is defined as $x_{\Delta t}(t) = (1/\Delta t)\int_t^{t+\Delta t} x(\tau)d\tau$.

$$\mathcal{D}_c g(\boldsymbol{x}) = \frac{\langle g(\boldsymbol{x}'') - g(\boldsymbol{x}') \rangle}{2\Delta t}, \tag{2.4}$$

with Δt such that Eq. (2.3) holds.

At this point it is necessary to make several assumptions about the properties of the stochastic motion. Considering the desired generality of the treatment and the lack of a specific model, these properties are unknown in principle; but for the cases of interest here it proves sufficient to assume that the stochasticity is due to a stationary, isotropic and homogeneous source; the second moments of $\Delta_{\pm}\boldsymbol{x}$ are then independent of the sign for equal signs, $\langle \Delta_+ x_i \Delta_+ x_j \rangle = \langle \Delta_- x_i \Delta_- x_j \rangle$ (up to terms of order Δt). Further, the fluctuations are assumed to be statistically independent, $\langle \Delta_+ x_i \Delta_- x_j \rangle = 0$ for all i and j. Each surviving second moment may have a contribution of order Δt due to the randomness of the motion, plus higher-order contributions,

$$\langle \Delta_+ x_i \Delta_+ x_j \rangle = 2D_{ij}(\boldsymbol{x}, t)\Delta t \ + \cdots, \quad \langle \Delta_- x_i \Delta_- x_j \rangle = 2D_{ij}(\boldsymbol{x}, t)\Delta t \ + \cdots \tag{2.5}$$

These expressions define (to zero order in Δt) the elements D_{ij} of the diffusion tensor as

$$D_{ij}(\boldsymbol{x}, t) = \frac{\langle \Delta_{\pm} x_i \Delta_{\pm} x_j \rangle}{2\Delta t}. \tag{2.6}$$

The difference $\langle \Delta_+ x_i \Delta_+ x_j \rangle - \langle \Delta_- x_i \Delta_- x_j \rangle$ is therefore of order higher than the first in Δt, and from Eqs. (2.2) and (2.4) we have to zero order in Δt (adding the contribution that may come from a possible explicit time dependence of g)

$$\mathcal{D}_c g(\boldsymbol{x}, t) = \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \right) g(\boldsymbol{x}, t), \tag{2.7}$$

with the *flux* or *systematic* (*convective*) *velocity* $\boldsymbol{v}(\boldsymbol{x}, t)$ given by

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{\langle \Delta_+ \boldsymbol{x} + \Delta_- \boldsymbol{x} \rangle}{2\Delta t} = \frac{\langle \boldsymbol{x}'' - \boldsymbol{x}' \rangle}{2\Delta t}. \tag{2.8}$$

This $\mathcal{D}_c$ coincides with the total time derivative of hydrodynamics. Note that application of Eq. (2.7) to each of the components x_i gives

$$\boldsymbol{v}(\boldsymbol{x}, t) = \mathcal{D}_c \boldsymbol{x}. \tag{2.9}$$

The systematic derivative defined above is only one of an infinite number of possible derivatives. Another one, equally important for what follows, is obtained by considering not the difference in (2.2), but the sum

$$\frac{g(\boldsymbol{x}'')+g(\boldsymbol{x}')}{2\Delta t}=\frac{2g(\boldsymbol{x})}{2\Delta t}+\frac{\partial g}{\partial x_i}\frac{\Delta_+x_i-\Delta_-x_i}{2\Delta t}+\frac{\partial^2 g}{\partial x_i\partial x_j}\frac{\Delta_+x_i\Delta_+x_j+\Delta_-x_i\Delta_-x_j}{4\Delta t}+\cdots \tag{2.10}$$

This leads to the definition of the *stochastic derivative* of the function $g(\boldsymbol{x},t)$, namely,

$$\mathcal{D}_s g(\boldsymbol{x},t)=\frac{\left\langle g(\boldsymbol{x}'')+g(\boldsymbol{x}')-2g(\boldsymbol{x})\right\rangle}{2\Delta t} \tag{2.11}$$

or

$$\mathcal{D}_s g(\boldsymbol{x},t)=\frac{\partial g}{\partial x_i}\frac{\langle\Delta_+x_i-\Delta_-x_i\rangle}{2\Delta t}+\frac{\partial^2 g}{\partial x_i\partial x_j}\frac{\left\langle\Delta_+x_i\Delta_+x_j+\Delta_-x_i\Delta_-x_j\right\rangle}{4\Delta t}+\cdots. \tag{2.12}$$

As for the first moments of the deviations of the coordinates entering into the expression for $\mathcal{D}_s g$, note that for a smooth motion the difference $(\Delta_+\boldsymbol{x}-\Delta_-\boldsymbol{x})$ is of order $(\Delta t)^2$; however, if there is a 'diffusion pressure', i.e., if the distribution of the instantaneous motions is inhomogeneous (there are more impacts per unit time from one side than from the other), the average of this difference may contain a term of order Δt. Therefore, we write

$$\boldsymbol{u}=\frac{\langle\Delta_+\boldsymbol{x}-\Delta_-\boldsymbol{x}\rangle}{2\Delta t}=\frac{\left\langle\boldsymbol{x}''+\boldsymbol{x}'-2\boldsymbol{x}\right\rangle}{2\Delta t} \tag{2.13}$$

and call $\boldsymbol{u}$ the *diffusive*, *stochastic*, or *osmotic velocity*. Collecting results, and neglecting again all higher-order terms, we get

$$\mathcal{D}_s g(\boldsymbol{x},t)=u_i\frac{\partial g}{\partial x_i}+D_{ij}\frac{\partial^2 g}{\partial x_i\partial x_j}, \tag{2.14}$$

an equation that applied to $\boldsymbol{x}$ gives

$$\boldsymbol{u}=\mathcal{D}_s\boldsymbol{x}. \tag{2.15}$$

In what follows we consider the simple case of a diagonal, isotropic and uniform diffusion tensor $D_{ij}=D\delta_{ij}$, with D constant, so that Eq. (2.14) reduces to

$$\mathcal{D}_s g(\boldsymbol{x},t)=\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}+D\boldsymbol{\nabla}^2\right)g(\boldsymbol{x},t). \tag{2.16}$$

The Markovian approximation made above (here *Markovian* means only retention of terms up to and including second-order moments) is by no means trivial and in each specific application its validity should be verified. However, it will prove sufficient and appropriate for the reproduction of the quantum description.

Note that neither $\boldsymbol{u}$ nor $\mathcal{D}_s$ exist in the *Newtonian limit*, i.e., for smooth motions in the absence of stochasticity. This allows us to define the Newtonian limit through

$$\text{Newtonian limit:} \quad \mathcal{D}_s \to 0 \quad \text{and} \quad \boldsymbol{u} \to 0. \tag{2.17}$$

In this limit of course $\mathcal{D}_c \to d/dt$, with the derivative taken along the flux of particles. As is now evident, by considering a sequence of time intervals previous to t' and following t'' it becomes possible to define as many different velocities as desired, and each additional one renders a more complete (but less local) statistical description of the motion. However, for the present purposes the two velocities $\boldsymbol{v}$ and $\boldsymbol{u}$ defined above happen to be sufficient. Yet certain linear combinations of them, as well as of the operators of time derivation $\mathcal{D}_c$ and $\mathcal{D}_s$, are particularly useful. Specifically, we have the *exit* and *access* combinations, denoted by the indices e and a, respectively [also called *forward* $(+)$ and *backward* $(-)$]. The velocities and operators of interest are summarized as follows,

$$\mathcal{D}_e = \mathcal{D}_c + \mathcal{D}_s, \quad \mathcal{D}_a = \mathcal{D}_c - \mathcal{D}_s; \tag{2.18a}$$

$$\mathcal{D}_c = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}, \quad \mathcal{D}_s = \boldsymbol{u}\cdot\boldsymbol{\nabla} + D\boldsymbol{\nabla}^2; \tag{2.18b}$$

$$\mathcal{D}_e = \frac{\partial}{\partial t} + \boldsymbol{v}_e\cdot\boldsymbol{\nabla} + D\boldsymbol{\nabla}^2, \quad \mathcal{D}_a = \frac{\partial}{\partial t} + \boldsymbol{v}_a\cdot\boldsymbol{\nabla} - D\boldsymbol{\nabla}^2; \tag{2.18c}$$

$$\boldsymbol{v}_e = \frac{\langle \boldsymbol{x}'' - \boldsymbol{x}\rangle}{\Delta t} = \boldsymbol{v} + \boldsymbol{u} = \mathcal{D}_e\boldsymbol{x}, \quad \boldsymbol{v}_a = \frac{\langle \boldsymbol{x} - \boldsymbol{x}'\rangle}{\Delta t} = \boldsymbol{v} - \boldsymbol{u} = \mathcal{D}_a\boldsymbol{x}; \tag{2.18d}$$

$$\boldsymbol{v} = \tfrac{1}{2}(\boldsymbol{v}_e + \boldsymbol{v}_a) = \mathcal{D}_c\boldsymbol{x}, \quad \boldsymbol{u} = \tfrac{1}{2}(\boldsymbol{v}_e - \boldsymbol{v}_a) = \mathcal{D}_s\boldsymbol{x}. \tag{2.18e}$$

Equation (2.18d) exhibits the access (exit) velocity $\boldsymbol{v}_a$ $(\boldsymbol{v}_e)$ as the local velocity of the particles reaching (leaving) point $\boldsymbol{x}$ at time t. However, these refer to the (local) average values of $\Delta\boldsymbol{x}_\pm$; to get instantaneous expressions it is required to add to each increment the corresponding instantaneous deviation from their respective local mean value, i.e.,

$$\Delta_+\boldsymbol{x} = \boldsymbol{v}_e\Delta t + \delta\boldsymbol{x}_+, \quad \Delta_-\boldsymbol{x}_i = \boldsymbol{v}_a\Delta t + \delta\boldsymbol{x}_-, \tag{2.19}$$

with $\delta\boldsymbol{x}_+$ and $\delta\boldsymbol{x}_-$ independent stochastic vector variables that average to zero.[7] In the absence of diffusion, $\boldsymbol{v}_a = \boldsymbol{v}_e$; but if there is diffusion, there may be more

[7] To reproduce the above results it is required that the second moment $\langle(\delta\boldsymbol{x}_+)^2\rangle$ be proportional to Δt, so that $\langle(\Delta_+\boldsymbol{x})^2\rangle/\Delta t$ acquires a finite value [as demanded by Eq. (2.6)]. This is a characteristic

(or fewer) particles leaving than entering the neighborhood of $\boldsymbol{x}$ in a given small time interval, the difference $2\boldsymbol{u}$ being then a measure of the intensity of the diffusion [see Eq. (2.27) below].

An important feature of the velocities is their different behavior with respect to time reversal. A time-reversal operation $\hat{T}$ interchanges t' and t'', and thus also the points $\boldsymbol{x}'$ and $\boldsymbol{x}''$:

$$\hat{T}\boldsymbol{x}'' = \boldsymbol{x}', \quad \hat{T}\boldsymbol{x}' = \boldsymbol{x}''. \tag{2.20}$$

It follows from Eqs. (2.18d) and (2.18e) that

$$\begin{array}{llll} \hat{T}\boldsymbol{v}_e = -\boldsymbol{v}_a; & \hat{T}\boldsymbol{v}_a = -\boldsymbol{v}_e, \\ \hat{T}\boldsymbol{v} = -\boldsymbol{v}; & \hat{T}\boldsymbol{u} = \boldsymbol{u}, \end{array} \tag{2.21}$$

and similarly for the derivative operators,

$$\begin{array}{llll} \hat{T}\mathcal{D}_e = -\mathcal{D}_a; & \hat{T}\mathcal{D}_a = -\mathcal{D}_e, \\ \hat{T}\mathcal{D}_c = -\mathcal{D}_c; & \hat{T}\mathcal{D}_s = \mathcal{D}_s. \end{array} \tag{2.22}$$

The next step is to construct appropriate expressions for the acceleration; this can be readily achieved by applying a time derivation to a velocity. We have at our disposal two velocities and two time derivatives, which can be combined into four different accelerations. These accelerations and their corresponding behavior under time reversal are

$$\begin{array}{ll} \boldsymbol{a}_{cc} = \mathcal{D}_c\boldsymbol{v} = \mathcal{D}_c\mathcal{D}_c\boldsymbol{x}; & \hat{T}\boldsymbol{a}_{cc} = +\boldsymbol{a}_{cc}; \\ \boldsymbol{a}_{ss} = \mathcal{D}_s\boldsymbol{u} = \mathcal{D}_s\mathcal{D}_s\boldsymbol{x}; & \hat{T}\boldsymbol{a}_{ss} = +\boldsymbol{a}_{ss}; \\ \boldsymbol{a}_{cs} = \mathcal{D}_c\boldsymbol{u} = \mathcal{D}_c\mathcal{D}_s\boldsymbol{x}; & \hat{T}\boldsymbol{a}_{cs} = -\boldsymbol{a}_{cs}; \\ \boldsymbol{a}_{sc} = \mathcal{D}_s\boldsymbol{v} = \mathcal{D}_s\mathcal{D}_c\boldsymbol{x}; & \hat{T}\boldsymbol{a}_{sc} = -\boldsymbol{a}_{sc}. \end{array} \tag{2.23}$$

2.3.2 Spatial Probability Density and Diffusive Velocity

Let $\rho(\boldsymbol{x}, t)$ denote the probability density of particles in configuration space. This function satisfies the forward (exit) Fokker-Planck Eq. (Risken 1984),

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\boldsymbol{v}_e - D\boldsymbol{\nabla}^2\rho = 0. \tag{2.24a}$$

(Footnote 7 continued)
feature of Brownian motion (or rather, of a white noise), and explains the extended reference to theories as the present one as 'Brownian-motion theories'.

The backward (access) Fokker-Planck equation can be obtained from (2.24a) by performing the substitution $t - \Delta t \to t + \Delta t$. This amounts to change the sign of the temporal derivate, and transforms $\boldsymbol{v}_e$ into $-\boldsymbol{v}_a$ [see the first line in Eq. (2.21)]. The resulting equation is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{v}_a + D\boldsymbol{\nabla}^2 \rho = 0. \tag{2.24b}$$

By combining Eqs. (2.24a) and (2.24b) and using (2.18e) one is led to

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{v} = 0, \tag{2.25}$$

$$\boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = D\boldsymbol{\nabla}^2 \rho. \tag{2.26}$$

The first of these equations is the continuity equation expressing the local conservation of particles. The second one can be rewritten as $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} - D\boldsymbol{\nabla}\rho) = 0$, which integrates into $\rho \boldsymbol{u} = D\boldsymbol{\nabla}\rho + \boldsymbol{\nabla} \times \boldsymbol{G}$, with $\boldsymbol{G}$ an arbitrary vector; however, by considering the balance of particles that go into and out of any small volume around a point $\boldsymbol{x}$ in space, it can be seen that one should take $\boldsymbol{G} = 0$ in general (de la Peña and Cetto 1969), which leads to the important formula[8]

$$\boldsymbol{u} = D\frac{\boldsymbol{\nabla}\rho}{\rho} = D\boldsymbol{\nabla} \ln \rho, \tag{2.27}$$

confirming that the motions described by $\boldsymbol{u}$ are due to diffusion, as discussed in relation with Eq. (2.13). This observation substantiates the selection $\boldsymbol{G} = 0$, since then $\boldsymbol{u}$ is due exclusively to the spatial changes in ρ. Notice that $\boldsymbol{u}$ can be rewritten alternatively in the form

$$\boldsymbol{u} = D\boldsymbol{\nabla} \ln \left(\frac{\rho}{\rho_{\text{ref}}} \right), \tag{2.28}$$

where ρ_{ref} is any arbitrary reference constant value. Thus, the diffusive velocity does not depend on the scale of the density $\rho(\boldsymbol{x}, t)$. In Chap. 4 we will relate the coefficient D in Eq. (2.27) to the source of the fluctuations (namely the zero-point radiation field) and in doing so we will endow $\boldsymbol{u}$ with a deeper physical meaning.

[8] In the literature it is possible to find the velocity $\boldsymbol{u}$ defined with the sign reversed. Equation (2.27) can be recast into the form $\boldsymbol{j}_{\text{diff}} \equiv \boldsymbol{u}\rho = D\boldsymbol{\nabla}\rho$, known as Fick's law (with due allowance for the reversed sign).

2.3.3 Dynamics

The lack of a specific model requires that we use a few basic arguments for the construction of the dynamics of stochastic mechanics. The best way is to opt for the most general law consistent with several obvious requirements. In the first place, one should expect the relationship between the (coarse-grained) accelerations and the forces to be linear. The acceleration $\boldsymbol{a}$ must then be expressible as a linear combination of the previous four accelerations, Eq. (2.23),

$$\boldsymbol{a} = \lambda_1 \boldsymbol{a}_{cc} + \lambda_2 \boldsymbol{a}_{ss} + \lambda_3 \boldsymbol{a}_{cs} + \lambda_4 \boldsymbol{a}_{sc}, \tag{2.29}$$

where the λ's are constant parameters to be determined. Notice that this expression is not time-reversal invariant, since upon time inversion the last two terms reverse their sign, whereas the first two remain unchanged.

The total force acting on the particles can be represented as the sum of the external force and a stochastic force. In its turn this latter can be decomposed into two terms, namely the dissipative force (which embodies the systematic effects of the stochasticity on the particle), and the purely random force. The effective (locally averaged) force is thus composed of the external force plus a coarse-grained friction term. A force that depends only on the position should remain invariant with respect to time reversal, whereas velocity-dependent forces may change their sign under such operation. Therefore, if we decompose the net force $\boldsymbol{f}$ in the general form $\boldsymbol{f} = \boldsymbol{f}_+ + \boldsymbol{f}_-$, where $\hat{T}\,\boldsymbol{f}_\pm = \pm \boldsymbol{f}_\pm$, and assume a linear relation between forces and accelerations, it follows that the most general equations of motion acquire the form

$$\begin{aligned} m\,(\lambda_1 \boldsymbol{a}_{cc} + \lambda_2 \boldsymbol{a}_{ss}) &= \boldsymbol{f}_+, \\ m\,(\lambda_3 \boldsymbol{a}_{cs} + \lambda_4 \boldsymbol{a}_{sc}) &= \boldsymbol{f}_-, \end{aligned} \tag{2.30}$$

with m the mass of the particle. The parameters in these equations can be selected so as to adjust the theory to different purposes (see, e.g., Davidson 1978, 1979b; Nassar 1986a). In particular, in order to reproduce the quantum-mechanical description any friction term must be taken as zero (or considered negligible), since the dynamics as described by the Schrödinger equation is reversible. This situation differs substantially from the corresponding one in classical stochastic problems (say, of the Brownian-motion family), where the dissipative effects never cease and the long-term motions are purely stochastic.[9] Therefore, in the quantum case the source of stochasticity must be different from a white noise, since the total lack of coherence of the latter makes it unsuitable to sustain a systematic (mean) motion.

Considering first the conservative problem $\boldsymbol{f} = -\boldsymbol{\nabla} V(\boldsymbol{x})$, for which $\boldsymbol{f}_- = 0$ (the case $\boldsymbol{f}_- \neq 0$ is straightforward and is considered in the next section), the second equation in (2.30) gives

[9] The hipothesis of a Brownian process without friction is just the most characteristic feature of Nelson's (1966, 1985a, 2012) theory.

$$\mathcal{D}_c \boldsymbol{u} + \gamma \mathcal{D}_s \boldsymbol{v} = 0, \tag{2.31}$$

where the two last expressions in Eq. (2.23) were used and we put $\gamma = \lambda_4/\lambda_3$. On the other hand, by taking the gradient of the continuity Eq. (2.25) and using Eqs. (2.7) and (2.16), one arrives after some simplifications at

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\left(\boldsymbol{u}\cdot\boldsymbol{v}\right) + D\boldsymbol{\nabla}^2\boldsymbol{v} = -D\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{v}\right). \tag{2.32}$$

From (2.27) it follows that $\boldsymbol{\nabla}\times\boldsymbol{u} = 0$; further, in the conservative problem it is reasonable to reduce the description to the case $\boldsymbol{\nabla}\times\boldsymbol{v} = 0$ (in the next section the more general problem with $\boldsymbol{\nabla}\times\boldsymbol{v} \neq 0$ is reviewed). The above equation transforms thus into

$$\frac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} + \left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} + D\boldsymbol{\nabla}^2\boldsymbol{v} = 0, \tag{2.33}$$

which can be rewritten as

$$\mathcal{D}_c \boldsymbol{u} + \mathcal{D}_s \boldsymbol{v} = 0. \tag{2.34}$$

This equation is an alternative form of the continuity equation for the conservative problem. Comparison with Eq. (2.31) gives

$$\gamma = \frac{\lambda_4}{\lambda_3} = 1. \tag{2.35}$$

The results allow us to identify the second of Eq. (2.30) as a constraint on the system rather than a dynamical relation. It follows also that for $\boldsymbol{\nabla}\times\boldsymbol{v} \neq 0$ the continuity equation will determine, via the second equation in (2.30), the appropriate expression for the force $\boldsymbol{f}_-$, which will no longer be null [see, for example, Eq. (2.50)].

The first of Eq. (2.30), on the other hand, is a true dynamical law, which can be recast into the form

$$m\lambda_1\left(\boldsymbol{a}_{cc} - \lambda\boldsymbol{a}_{ss}\right) = \boldsymbol{f}_+, \tag{2.36}$$

with $\lambda = -\lambda_2/\lambda_1$. In the Newtonian limit, $\boldsymbol{a}_{ss} \to 0$ and $\boldsymbol{a}_{cc} = \mathcal{D}_c^2\boldsymbol{x} \to d^2\boldsymbol{x}/dt^2$; therefore, to recover the correct classical limit one must take $\lambda_1 = 1$, and the equation of motion becomes

$$m\left(\mathcal{D}_c\boldsymbol{v} - \lambda\mathcal{D}_s\boldsymbol{u}\right) = \boldsymbol{f}_+. \tag{2.37}$$

This is the most general dynamical law allowed by the theory, under the principles adopted. Note that it contains a single free parameter, whose value will be discussed

below. Notice also that for $\lambda \neq 0$ Eq. (2.37) differs from the classical equation of motion.

With the derivatives given by (2.18b), Eq. (2.37) reads explicitly

$$m\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} - \lambda\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} - \lambda D\boldsymbol{\nabla}^2\boldsymbol{u}\right] = \boldsymbol{f}_+. \tag{2.38}$$

This equation was first proposed with the specific value $\lambda = 1$ in Nelson 1966. It can also be cast in the form

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{f}_+ + \boldsymbol{f}_{\text{diff}}, \tag{2.39}$$

with the time derivative taken along the mean motion (with respect to the flux velocity $\boldsymbol{v}$). The term $\boldsymbol{f}_{\text{diff}}$ stands for a force (additional to the external one) originating in the diffusive velocity,

$$\boldsymbol{f}_{\text{diff}} = m\lambda\boldsymbol{a}_{ss} = -\boldsymbol{\nabla}V_{\text{diff}}, \tag{2.40}$$

with

$$V_{\text{diff}} = -\lambda\left(\frac{1}{2}m\boldsymbol{u}^2 + mD\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) = -\lambda\left(2mD^2\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}}\right), \tag{2.41}$$

where Eq. (2.27) was used to write the second equality.

It should be borne in mind that the equation of motion (2.39) refers just to mean values in $\boldsymbol{x}$-space. For a description of the instantaneous motion of a (single) particle it is necessary to resort to the Langevin equation for the problem. In Nelson's (1967) theory such equation is proposed to correspond to a frictionless Brownian motion, which (in one dimension, for simplicity) reads

$$dx = (v+u)\,dt + \sqrt{2D}\Delta W(t), \tag{2.42}$$

where ΔW is taken as a Gaussian white noise (or Wiener process), $\langle \Delta W(t)\rangle = 0$, $\left\langle (\Delta W(t))^2\right\rangle = dt$ [compare with Eq. (2.19)]. The trajectories are now nowhere differentiable with probability one, so that some appropriate procedure is required to integrate the equation of motion. In Nelson's formulation the Ito definition of the time integral is used, which means that the increment $\Delta W(t)$ is interpreted as equal to $\Delta W(t) = W(t+dt) - W(t)$ for $dt > 0$. Details can be seen e.g. in Nelson's works; Gardiner (1983), and Vasudevan et al. (2008).

2.3.4 Integrating the Equation of Motion

By taking $\lambda_1 = 1$ and $\lambda_3 = \lambda_4$ in Eq. (2.30), and further observing that λ_3 can be absorbed in f_- by an appropriate redefinition (which is equivalent to taking $\lambda_3 = 1$), we write the law of motion and the constraint in the form

$$\begin{aligned} m\left(\mathcal{D}_c \boldsymbol{v} - \lambda \mathcal{D}_s \boldsymbol{u}\right) &= \boldsymbol{f}_+, \\ m\left(\mathcal{D}_c \boldsymbol{u} + \mathcal{D}_s \boldsymbol{v}\right) &= \boldsymbol{f}_-. \end{aligned} \tag{2.43}$$

Equations (2.43) with the derivatives given by (2.18b) look impressive: they form a system of coupled, nonlinear partial differential equations involving $\boldsymbol{v}$ and $\boldsymbol{u}$. However, this system has the remarkable property that it can be integrated (once), uncoupled, and linearized if expressed in terms of appropriate functions. This is achieved in several steps as follows.

From Eq. (2.38) and writing $\boldsymbol{f}_+$ generically as

$$\boldsymbol{f}_+ = -\nabla V + \boldsymbol{F}, \tag{2.44}$$

the first equation in (2.43) reads

$$m\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} - \lambda\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \lambda D\nabla^2\boldsymbol{u}\right] = -\nabla V + \boldsymbol{F}. \tag{2.45}$$

We now decompose $\boldsymbol{v}$ in the general form

$$\boldsymbol{v} = 2D\nabla S + \boldsymbol{b}, \tag{2.46}$$

with $S = S(\boldsymbol{x}, t)$ a (dimensionless) real function and $\boldsymbol{b}$ a vector containing any possible rotational contribution to $\boldsymbol{v}$. With the help of the identities

$$\frac{1}{2}\nabla \boldsymbol{w}^2 = (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{w} + \boldsymbol{w}\times(\nabla\times\boldsymbol{w}),$$

$$\nabla^2\boldsymbol{w} = \nabla(\nabla\cdot\boldsymbol{w}) - \nabla\times(\nabla\times\boldsymbol{w}),$$

Equation (2.45) rewrites as

$$\nabla\left[2mD\frac{\partial S}{\partial t} + \frac{1}{2}m\boldsymbol{v}^2 + V_{\text{diff}} + V\right] = \boldsymbol{F} - m\frac{\partial \boldsymbol{b}}{\partial t} + m\boldsymbol{v}\times(\nabla\times\boldsymbol{v}). \tag{2.47}$$

As for the second equation in (2.43), it reads explicitly

$$m\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v} + D\nabla^2\boldsymbol{v}\right] = \boldsymbol{f}_-. \tag{2.48}$$

The identity

$$\nabla (\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}),$$

together with Eq. (2.27), gives

$$mD\nabla \left[\frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} \right) \right] = \boldsymbol{f}_{-} + m\boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + mD\nabla \times (\nabla \times \boldsymbol{v}). \quad (2.49)$$

We thus see that, as expected, the equation $m\,(\mathcal{D}_c \boldsymbol{u} + \mathcal{D}_s \boldsymbol{v}) = \boldsymbol{f}_{-}$ imposes constraints via the continuity equation, which causes the term within square brackets to vanish. This fixes the force $\boldsymbol{f}_{-}$ as

$$\boldsymbol{f}_{-} = -m\boldsymbol{u} \times (\nabla \times \boldsymbol{v}) - mD\nabla \times (\nabla \times \boldsymbol{v}). \quad (2.50)$$

Now we come back to the dynamical Eq. (2.47) and assume that the additional force $\boldsymbol{F}$ in Eq. (2.44) refers to a Lorentz force due to an external electromagnetic potential $\boldsymbol{A}$,

$$\boldsymbol{F} = -\frac{e}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{e}{c}\boldsymbol{v} \times (\nabla \times \boldsymbol{A}). \quad (2.51)$$

In this case the flow velocity becomes

$$\boldsymbol{v} = 2D\nabla S - \frac{e}{mc}\boldsymbol{A}, \quad (2.52)$$

thus fixing $\boldsymbol{b} = -(e/mc)\boldsymbol{A}$. This value for $\boldsymbol{b}$, together with Eq. (2.51), implies that the right-hand side of Eq. (2.47) vanishes, so the equation reduces to

$$\nabla \left[2mD\frac{\partial S}{\partial t} + \frac{1}{2}m\boldsymbol{v}^2 + V_{\text{diff}} + V \right] = 0. \quad (2.53)$$

Further, introduction of Eq. (2.52) into (2.50) gives a diffusion-dependent Lorentz-force term that changes sign under time reversal. In the Coulomb gauge (i.e., taking $\nabla \cdot \boldsymbol{A} = 0$) this force is given by

$$\boldsymbol{f}_{-} = \frac{e}{c}\boldsymbol{u} \times (\nabla \times \boldsymbol{A}) + D\frac{e}{c}\nabla^2 \boldsymbol{A}. \quad (2.54)$$

Now we are in position to integrate the dynamical Eq. (2.53). The result, after absorbing into S the arbitrary function of time that arises from the integration, is

$$2mD\frac{\partial S}{\partial t} + \frac{1}{2}m\boldsymbol{v}^2 - \lambda 2mD^2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + V = 0. \quad (2.55a)$$

This, together with the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = 0, \tag{2.55b}$$

with $\boldsymbol{v}$ given by (2.52), constitutes a pair of differential equations for the variables ρ and S that is equivalent to the original pair (2.43), and bears the dynamical information for an ensemble of particles subject to the conservative force $-\nabla V$ and immersed in an external electromagnetic field represented by the vector potential $\boldsymbol{A}$. Equations (2.55a) and (2.55b) uncouple if we perform a transformation that takes ρ and S to new variables w_+ and w_-, such that

$$\begin{aligned} w_+ + w_- &= \ln \rho, \\ w_+ - w_- &= 2\frac{S}{\sqrt{-\lambda}}. \end{aligned} \tag{2.56}$$

Direct substitution leads, after some algebra, to the pair of separated equations

$$\begin{aligned} -2m\zeta\frac{\partial w_+}{\partial t} &= -2m\zeta^2\left[(\nabla w_+)^2 + \nabla^2 w_+\right] + V - \\ &\quad -\zeta\frac{e}{c}\left(2\boldsymbol{A}\cdot\nabla w_+ + \nabla\cdot\boldsymbol{A}\right) + \frac{e^2}{2mc^2}\boldsymbol{A}^2; \end{aligned} \tag{2.57}$$

$$\begin{aligned} +2m\zeta\frac{\partial w_-}{\partial t} &= -2m\zeta^2\left[(\nabla w_-)^2 + \nabla^2 w_-\right] + V + \\ &\quad +\zeta\frac{e}{c}\left(2\boldsymbol{A}\cdot\nabla w_- + \nabla\cdot\boldsymbol{A}\right) + \frac{e^2}{2mc^2}\boldsymbol{A}^2, \end{aligned} \tag{2.58}$$

with $\zeta = D\sqrt{-\lambda}$. As a final step, this system of equations can be linearized by introducing the further change of functions

$$\psi_+ = \exp w_+ = \exp\left(\ln\sqrt{\rho} + \frac{S}{\sqrt{-\lambda}}\right) = \sqrt{\rho}\exp\left(\frac{S}{\sqrt{-\lambda}}\right), \tag{2.59a}$$

$$\psi_- = \exp w_- = \exp\left(\ln\sqrt{\rho} - \frac{S}{\sqrt{-\lambda}}\right) = \sqrt{\rho}\exp\left(-\frac{S}{\sqrt{-\lambda}}\right), \tag{2.59b}$$

thus obtaining

$$-2mD\sqrt{-\lambda}\frac{\partial \psi_+}{\partial t} = \frac{1}{2m}\left(2mD\sqrt{-\lambda}\nabla - \frac{e}{c}\boldsymbol{A}\right)^2\psi_+ + V\psi_+, \tag{2.60a}$$

$$2mD\sqrt{-\lambda}\frac{\partial \psi_-}{\partial t} = \frac{1}{2m}\left(-2mD\sqrt{-\lambda}\nabla - \frac{e}{c}\boldsymbol{A}\right)^2\psi_- + V\psi_-. \tag{2.60b}$$

Further, Eqs. (2.59a, 2.59b) give

$$\rho = \psi_+ \psi_-, \tag{2.61}$$

and the velocities $\boldsymbol{v}$ and $\boldsymbol{u}$ rewrite as

$$\boldsymbol{v} = D\sqrt{-\lambda}\nabla \ln \frac{\psi_+}{\psi_-} - \frac{e}{mc}\boldsymbol{A} = D\sqrt{-\lambda}\left(\frac{\nabla\psi_+}{\psi_+} - \frac{\nabla\psi_-}{\psi_-}\right) - \frac{e}{mc}\boldsymbol{A}, \tag{2.62}$$

$$\boldsymbol{u} = D\nabla \ \ln \psi_+\psi_- = D\left(\frac{\nabla\psi_+}{\psi_+} + \frac{\nabla\psi_-}{\psi_-}\right). \tag{2.63}$$

In the absence of external electromagnetic field ($\boldsymbol{A} = 0$), $\boldsymbol{v}$ reduces to

$$\boldsymbol{v} = 2D\nabla S.$$

The function $S(\boldsymbol{x}, t)$ represents therefore a velocity potential, or a kind of statistical action function, the gradient of which gives the momentum associated with the systematic (mean local) velocity.

2.3.5 *Quantum and Classical Stochastic Processes*

Equations (2.60a, 2.60b) apply to any system that can be described by the present stochastic treatment, subject to the free (though nontrivial) choice of D and λ. This exhibits at once the strength and the weakness of the procedure. For on the one hand, an appropriate selection of the parameters leads to a Schrödinger-like description of the stochastic system; but on the other hand, the equation thus obtained is quite unspecific and the 'appropriate' selection of the parameters seems quite arbitrary. A complete theory should allow for an unambiguous derivation of both λ and the coefficients D_{ij} (not necessarily constant nor diagonal in the more general case), from first principles. This is beyond reach for the present phenomenological approach, due first and foremost to the nonspecificity of the random field. In Sect. 2.4 an argument is given that helps to perceive the generality (and arbitrariness) of Schrödinger-like equations, and to realize that the selection of the parameters is a matter of no minor importance. At this stage we just briefly explore the possible applications of the results just derived.[10]

Notice that in the integrated Eqs. (2.60a, 2.60b) the free parameter is the product $D\sqrt{-\lambda}$, not each factor separately. One may therefore consider that D takes care of

[10] In (Davidson 1979b and 2001) an interesting, slightly different selection of the parameters is discussed, which reproduces the *classical* nonlinear Schrödinger equation, derived in Sect. 4.5.5 of this book. See also (Bacciagaluppi 2011).

the scale, whereas the relevant property of λ is its sign, so that one can take $\lambda = \pm 1$. This leads to essentially two different theories, according to the sign of λ.

2.3.5.1 The Parabolic Solution

Take first $\lambda = -1$. In this case Eqs. (2.60a, 2.60b) are parabolic, the functions ψ_+ and ψ_- are both real and the process described by them is irreversible. This theory can be used to describe classical Markov processes, if due allowance is made for the unbalanced friction force, which can be introduced via an expression such as $\boldsymbol{f}_{\mathrm{f}} = -\beta \boldsymbol{v}_e$ with $\boldsymbol{v}_e = 2D(\nabla\psi_+/\psi_+)$, or the like (see e.g. Cetto 1972). However, with the introduction of such a term the theory ceases to be linear and it is then simpler to go back to the (linear) Fokker-Planck equation. Moreover, it happens that the values of both the friction parameter β and the diffusion coefficient D are problem-specific; the single way out of this situation is the fluctuation-dissipation relation, when it is at hand. As is well known, the presence of the friction force, together with the incoherence of the noisy background, leads to purely noisy solutions for $t \to \infty$. With all these drawbacks, a procedure as the present one seems to be of little help, if any, for such problems. Further elaborations can be seen in de la Peña and Cetto (1975), Skagerstam (1977), Nassar (1986a, b) and references therein.

2.3.5.2 The Hyperbolic Solution

Take now $\lambda = 1$. In this case Eqs. (2.59a, 2.59b) give $\psi_- = \sqrt{\rho}\exp(iS) = \psi_+^*$, and Eqs. (2.60a, 2.60b) become hyperbolic and each other's complex conjugate. The process is therefore reversible. The Schrödinger equation is obtained with the selection

$$D = \frac{\hbar}{2m} \tag{2.64}$$

for the diffusion coefficient. It acquires the nature of a wave equation thanks to the factor i in front of the first derivative with respect to time, which mimics a second-order time derivative, as discussed in Sect. 4.4.5.[11] It seems reasonable to ask why the theory should predict an undulatory behavior, when the whole treatment has been made in terms of particles that follow (deterministic) trajectories. These matters will be briefly discussed in Sect. 2.5.1 below (see also Chap. 9). In any case, we see that $\lambda = +1$ is the back door through which undulatory aspects enter the theory.

It should be stressed that the selection (2.64) is far from obvious or natural; there is not an a priori reason to assume that the tensor mD has a universal value, independent of the specific problem (this point is discussed in Sect. 2.4). Given the phenomenological character of the present theory, this formula enters as an empirical selection,

[11] Although giving rise to some bizarre wave phenomena; for comments and examples of this see e.g. Ballentine 1990, 1998.

although there have been of course some attempts to justify it from fundamental considerations (see e.g. de Broglie 1967) so as to ground the theory on a more solid basis. Since the problem of identifying the noise source behind the assumed stochastic process is left open in this kind of approach, that of *deriving* the detailed form and value of the diffusion tensor and the constant λ remains open as well.

A key point of the present stochastic approach is its adequacy to distinguish classical from quantum stochastic processes, which have become described by essentially different equations, even if at first sight the corresponding Langevin equations seem to be quite similar. One should therefore speak not of a Brownian analog of QM (as is so frequently done), but of a quantum stochastic process in itself. Insufficient attention to this crucial point is the cause of much confusion in the literature. More specifically, according to Eq. (2.37), the accelerations $\boldsymbol{a}_B$ for the classical (Brownian) case ($\lambda = -1$) and $\boldsymbol{a}_Q$ for the quantum system ($\lambda = 1$) are, respectively,

$$\boldsymbol{a}_B = \mathcal{D}_c \boldsymbol{v} + \mathcal{D}_s \boldsymbol{u}, \quad \boldsymbol{a}_Q = \mathcal{D}_c \boldsymbol{v} - \mathcal{D}_s \boldsymbol{u}. \tag{2.65}$$

Similarly, Eqs. (2.39)–(2.41) show that the sign of the extra potential V_{diff} is essential in determining the different dynamics. In the quantum case, V_{diff} (with $D = \hbar/2m$) becomes the so-called *quantum potential* V_Q, and $\boldsymbol{f}_{\text{diff}} = \boldsymbol{f}_Q$ can then be interpreted (in the language proper of Bohm's theory discussed in Chaps. 4 and 8) as a quantum force. From Eq. (2.41) we see that this extra potential is intimately related to the diffusion. Further, being due solely to the spatial variations in the density of particles ρ, it introduces a nonlocal ingredient into the description, since the probability density ρ contains information about the entire setup. This point will reappear in several of the following chapters, particularly in Chap. 8.

2.4 On Schrödinger-Like Equations

According to the exposition in Sect. 2.3.5, deriving a Schrödinger-like equation would seem to be quite an easy matter. However, as noted above, the proper selection of the parameters λ and D—which is crucial to obtain quantization—is by no means trivial. Thus a *true* derivation of a quantum equation of motion requires more than arriving at a Schrödinger-*like* equation, it requires also deriving the value of the parameters involved. To make this point clear, suffice it to recall the following alternative way of 'deriving' the Schrödinger equation, based merely on general arguments of a statistical nature. The sole intention of this example is to elaborate on the general relationship between the Schrödinger equation and a simple stochastic description in configuration space.

The starting point is the continuity Eq. (2.25) for the density of particles. Assume the flux to be laminar and write the drift (systematic) velocity $\boldsymbol{v}$ in terms of a velocity potential (an 'action' S) according to

$$\boldsymbol{v} = \frac{a}{m}\nabla S, \tag{2.66}$$

with the parameter a so selected as to make S a dimensionless function of $\boldsymbol{x}$ and t. A change of variables from ρ, S to a new complex pair ψ, ψ^*, defined as

$$\psi = \sqrt{\rho}e^{iS}, \quad \psi^* = \sqrt{\rho}e^{-iS}, \tag{2.67}$$

gives

$$\rho = \psi^*\psi, \quad \boldsymbol{v} = \frac{ia}{2m}\nabla\left(\ln\psi^* - \ln\psi\right). \tag{2.68}$$

The continuity equation thus transforms into

$$\psi^*\left(i\frac{\partial\psi}{\partial t} + \frac{a}{2m}\nabla^2\psi\right) - \psi\left(-i\frac{\partial\psi^*}{\partial t} + \frac{a}{2m}\nabla^2\psi^*\right) = 0. \tag{2.69}$$

At this point a separating real function U is introduced so that

$$\psi^*\left(i\frac{\partial\psi}{\partial t} + \frac{a}{2m}\nabla^2\psi\right) = \psi\left(-i\frac{\partial\psi^*}{\partial t} + \frac{a}{2m}\nabla^2\psi^*\right) = U\psi\psi^*. \tag{2.70}$$

As a result, Eq. (2.69) becomes separated into

$$i\frac{\partial\psi}{\partial t} = -\frac{a}{2m}\nabla^2\psi + U\psi \tag{2.71}$$

and its complex conjugate. The procedure can be applied to any mechanical system obeying the continuity equation (with a laminar flow), and hence to classical or quantum particles alike (de la Peña 1967). However, two problems remain, namely the determination of the parameter a and the function U, which may depend on $\boldsymbol{x}$, t *and even* on ψ and ψ^*. In what follows we proceed to determine the function U. See also Kracklauer (1992), and de la Peña and Cetto (1993) for other determinations of U.

We start by combining the expression (2.68) for $\boldsymbol{v}$ with Eq. (2.71), to obtain

$$\frac{\partial\boldsymbol{v}}{\partial t} = \frac{a^2}{4m^2}\nabla\left(\frac{1}{\psi^*}\nabla^2\psi^* + \frac{1}{\psi}\nabla^2\psi\right) - \frac{a}{m}\nabla U. \tag{2.72}$$

On the other hand, $\nabla\ln\psi = (1/2)\,\nabla\ln\rho + i\nabla S$, whence

$$\frac{\nabla^2\psi}{\psi} = \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + i\nabla^2 S + i\nabla S\cdot\frac{\nabla\rho}{\rho} - (\nabla S)^2. \tag{2.73}$$

This latter equation, when introduced into (2.72), leads to

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{\nabla}\left(\frac{a}{2m}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}^2\right) - \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{v}^2 - \frac{a}{m}\boldsymbol{\nabla}U, \tag{2.74}$$

with

$$\boldsymbol{u} = \frac{a}{2m}\boldsymbol{\nabla}\ln\rho. \tag{2.75}$$

Since $\boldsymbol{\nabla}\times\boldsymbol{v} = 0$, it follows that $\boldsymbol{\nabla}\boldsymbol{v}^2 = 2\,(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}$, which allows to rewrite Eq. (2.73) in terms of the total time derivative along the trajectory $d\boldsymbol{v}/dt = (\partial\boldsymbol{v}/\partial t) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}$, as

$$m\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}\left(-\frac{a}{2}\boldsymbol{\nabla}\cdot\boldsymbol{u} - \frac{1}{2}m\boldsymbol{u}^2\right) - a\boldsymbol{\nabla}U. \tag{2.76}$$

On the other hand, combining Eqs. (2.39)–(2.41) [with $D = a/2m$, in accordance with Eqs. (2.27) and (2.75)] one obtains

$$m\frac{d\boldsymbol{v}}{dt} = -\lambda\boldsymbol{\nabla}\left(-\frac{a}{2}\boldsymbol{\nabla}\cdot\boldsymbol{u} - \frac{1}{2}m\boldsymbol{u}^2\right) + \boldsymbol{f}_+. \tag{2.77}$$

For the conservative case ($\boldsymbol{f}_+ = -\boldsymbol{\nabla}V$), comparison of the last two equations implies that

$$(1-\lambda)\,\boldsymbol{\nabla}\left(-\frac{a}{2}\boldsymbol{\nabla}\cdot\boldsymbol{u} - \frac{1}{2}m\boldsymbol{u}^2\right) = \boldsymbol{\nabla}\,(V - aU)\,. \tag{2.78}$$

Integration of this expression gives the solution

$$aU = V + (1-\lambda)\,V_a + h(t), \tag{2.79}$$

with $h(t)$ an arbitrary function of time that can be taken as zero without loss of generality, and

$$V_a = \left(\frac{1}{2}m\boldsymbol{u}^2 + \frac{1}{2}a\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) = \frac{a^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}}. \tag{2.80}$$

Notice that according to Eq. (2.41), $V_{\text{diff}} = -\lambda V_a$, since $D = a/2m$.

We now introduce Eq. (2.79) into (2.71), thus obtaining

$$ia\frac{\partial\psi}{\partial t} = -\frac{a^2}{2m}\boldsymbol{\nabla}^2\psi + V\psi + (1-\lambda)\,V_a\psi. \tag{2.81}$$

The result just obtained evinces the distinctive nature of the case $\lambda = 1$: this is the single value of λ that linearizes Eq. (2.81), and transforms it into a Schrödinger equation,

$$ia\frac{\partial\psi}{\partial t} = -\frac{a^2}{2m}\nabla^2\psi + V\psi. \tag{2.82}$$

The value $\lambda = -1$, corresponding to the Brownian case as discussed above, leads to a Schrödinger-like equation but with a total potential $V + 2V_a$. Further, the value $\lambda = 0$ gives a net potential $V + V_a$. This case corresponds to a classical equation of motion devoid of stochasticity, but allows distributed velocities, since, according to the discussion following Eq. (2.36), the condition $\lambda = 0$ is equivalent to taking the Newtonian limit. As discussed in Sect. 4.5.5, Eq. (2.81) with $\lambda = 0$ is formally equivalent to a field theory for ψ with a classical Lagrangian.

Leaving aside the problem of justifying the selection $\lambda = 1$ to arrive at (2.82), we observe that the parameter a fixes the scale of the action aS, and is therefore, in principle, problem-dependent. Hence the a priori identification of Eq. (2.82) with the Schrödinger equation containing a universal constant a, is *not* warranted. This observation explains the singular role played by Schrödinger's equation in quantum theory, and therein resides a specific feature of quantum systems. Whereas in the classical case the value of the action integrals is determined by the initial conditions (whence a becomes highly arbitrary), in the quantum case this parameter becomes fixed in a more fundamental way: it is the initial conditions what are conditioned by the parameter, and determined so as to comply with an energy-balance condition, as will be shown in Chap. 4.[12]

Notwithstanding its importance, this discussion is frequently overlooked in the literature, characteristically in many published attempts to present variants of the above procedure as *bona fide* derivations of the Schrödinger equation from classical arguments. Still, doubts have been cast on the phenomenological stochastic theory, in particular on the legitimacy of the demand of single-valuedness on the wave solution, as discussed in Sect. 2.6.

Some 15 years ago, Yves Couder and his colleagues discovered the *bouncer*, a macroscopic particle (a small drop of silicon oil) that can be made to dance over the surface of a vertically vibrated bath of the same fluid. By increasing the peak acceleration of the vibrations, the droplet can be made to self-propel with constant speed. With this arrangement they have observed a variety of behaviors of the droplet that have a striking similarity with the wavelike behavior of quantum particles (see e.g. Couder and Fort 2006; Couder et al. 2005; Wind-Willassen et al. 2013). A detailed study by Brady and Anderson (2013) has revealed that this macroscopic hydrodynamic system can indeed be described by Eq. (2.82), with the parameter a appropriately selected for the specific system (and hence *not universal*), so that it can be taken as a close mechanical model of the quantum behavior. Here the vibrating oil bath is representing the substratum, which by its interplay with the particle generates a quantum-like dynamics, including quantization of orbital motions!

[12] For example, when solving the Heisenberg equations of motion for x and p, the initial conditions are given by matrices, which guarantees that the Heisenberg inequalities are satisfied starting from $t = 0$.

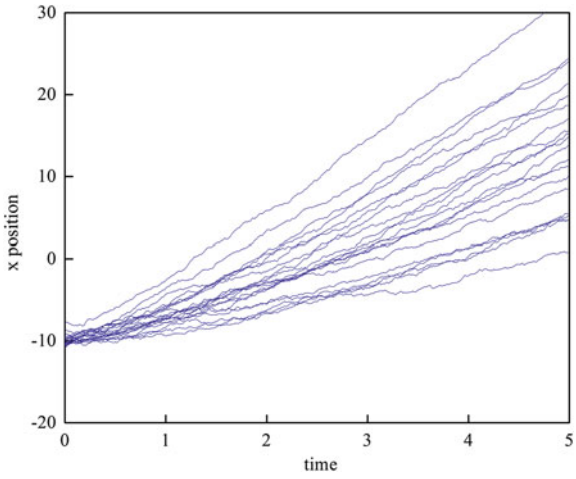

Fig. 2.1 Trajectories of a Gaussian wave packet made of 20 free particles with fixed energy, according to Eq. (2.42). Obtained by numerical simulation. Courtesy of Bárbara Moreno Munguía)

2.5 Stochastic Quantum Trajectories

Let us return to stochastic quantum mechanics. In this phenomenological approach, the ensemble is composed of particles (localized entities), although their statistical description appears encoded in Eq. (2.82), which has formally acquired the nature of a wave equation. One can therefore go back to the equation of motion (2.42) and use it to find individual trajectories, by means of numerical simulation. Compared to the corresponding calculations usually carried out in the framework of Bohm's theory (see Chap. 8), which resort to the *mean local* velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ (though in that approach they are seen as referring to the actual velocity of a single particle), these ones are somewhat more elaborate. The trajectories here obtained follow more closely the *instantaneous* motions, therefore they show more detail and provide extra information about the quantum dynamics. Figures 2.1 and 2.2 (similar to those in Moreno Murguía 2006), illustrate the results obtained for a Gaussian wave packet containing 20 particles, in the first case moving freely, in the second one impinging upon a narrow semitransparent barrier centered at the origin. The presence of rapid fluctuations—absent in Bohm's description and merely implicit in the quantum description—is conspicuous.

An analysis of Fig. 2.2 reveals several interesting aspects of the dynamics. The majority of the particles are reflected by the barrier, although an important fraction of them cross it and some remain inside for a relatively long time, going to and fro, until they escape in one direction or the other. This is particularly interesting because it shows that it is legitimate to speak of real particles in motion 'inside' the barrier. A most remarkable peculiarity displayed by the trajectories is the nonlocality of their behavior, as is further discussed in Chap. 8. It is clear from Fig. 2.2 that long before reaching the barrier, the particles already 'feel' its presence and start modifying their energy, either losing or gaining some, even enough to 'jump over' the barrier in some

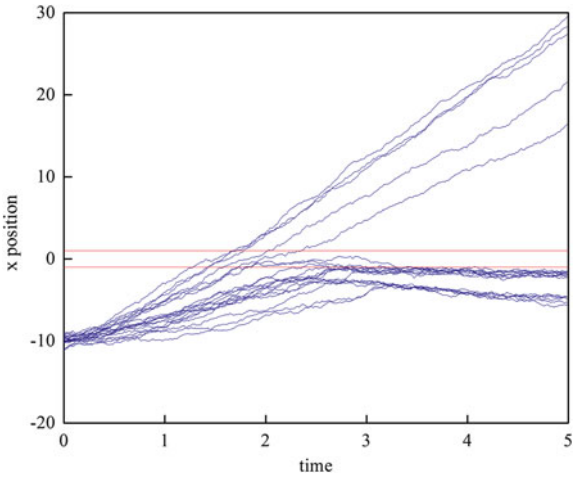

Fig. 2.2 The same packet as in Fig. 2.1, now in the presence of a barrier of width 1 and height 10 (arbitrary units), represented by a pair of fine *horizontal lines*. The energy of incidence is 1. Courtesy of Bárbara Moreno Murguía

cases (constant energy corresponds to a constant slope of the trajectories). Beyond the barrier the energy (hence the velocity) of the particles remains stable in statistical terms, and close to its original value. This applies also to the reflected particles, which rapidly tend to move as free particles. Both figures also clearly show that typically the trajectories intersect, due to the presence of stochasticity, in contrast with the prediction derived from Bohm's description (see e.g. Holland 1993). Figure 2.2 also hints at the effects of the interference of the incident and the reflected packets, which gives rise to diffraction.

2.5.1 Wavelike Patterns

A remarkable property of quantum particles is of course their wavelike behavior. At first it might seem counterintuitive to expect from a stochastic mechanical formulation as the one developed here, to reproduce the undulatory behavior of particles. On the other hand, as said below Eq. 2.64, we have arrived at a wave equation for describing the dynamics of the ensemble. In addition, a stochastic model that reproduces QM must account for the wavelike features. That this is so has been confirmed with the help of various numerical simulations, similar to the ones carried out for the semitransparent barrier mentioned in the previous section. One such example is presented in Fig. 2.3, taken from McClendon and Rabitz (1988). This figure shows the fringe pattern obtained for a 'wave packet' of several thousand particles emerging from two Gaussian slits, obtained by numerical integration within stochastic mechanics. The result compares well, statistically speaking, with that obtained from a quantum-mechanical calculation. A most important observation made with this

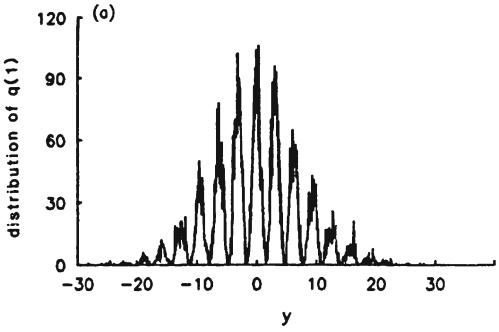

Fig. 2.3 Fringe pattern for a Gaussian wave packet containing 10,000 particles crossing a screen with two Gaussian slits. Obtained by numerical simulation. Reprinted with permission from McClendon and Rabitz (1988). Copyright 1988 by the American Physical Society

numerical experiment is that it demonstrates that each electron comes from just one slit.

The same conclusion is obtained in Webb (2011) using an event-based model for particles emitted one-at-a-time in the two-slit experiment. The results of the numerical simulations confirm that the fringe patterns should be interpreted in terms of the aggregate behavior of individual particles. Other interesting related quantum simulations are discussed in Michielsen et al. (2010) and De Raedt and Michielsen (2012). This gives a clear answer to a much raised question about the behavior of the electrons in such case. Of course such numerical calculations cannot give an explanation of the physics behind the diffraction pattern, since they are based on phenomenological models. In Chap. 9 we reconsider this problem from the point of view of the theory developed in the next chapters.

2.6 Extensions of the Theory, Some Brief Comments, and Assessment

Stochastic quantum mechanics can be extended along several important directions. For example—and this is perhaps one of its most remarkable outcomes—conditional *probabilities* (not amplitudes) have been constructed describing interference phenomena and the like (Petroni 1989). Variational methods and path-integral procedures have been introduced. In particular, by following a variational approach it has been possible to show that for stationary states $\boldsymbol{v}$ is irrotational wherever the density is different from zero, while in the nodes at $\rho = 0$ the vorticity tensor $\boldsymbol{\Omega}_{ij} = (\boldsymbol{\nabla} \times \boldsymbol{v})_{ij}$ can be different from zero. Using a generalization to mean velocities that are not irrotational, the process is shown to relax towards a standard (irrotational) solution, which can be seen as an attractor for the extended family of stochastic solutions.[13]

[13] The idea that the quantum stationary states are some kind of attractors within an appropriate set of solutions has been arrived at from other, complementary points of view; see e.g. de la Peña and Cetto (1995), 't Hooft (2006).

Extensions of the theory to a wider range of problems include, among others, the electron spin (de la Peña 1971), the description of mixtures (Guerra 1984), radiative corrections (de la Peña and Cetto 1971), the relativistic case (de la Peña 1970; Hakim 1968; Morato 1992), and gravity (Smolin 1986). Spinning and relativistic particles have been studied by Dohrn et al. (1979), and a statistical description that can accommodate relativity and spin in a natural way has been proposed by Tiwari (1988). An independent, interesting development in a similar direction is the treatment of the Dirac equation in terms of a dichotomic (telegraph) stochastic process (Gaveau et al. 1984). Systems composed of several particles have been considered, dramatically exhibiting the characteristic nonlocalities of the description (de la Peña and Cetto 1969; Loffredo and Morato 2007). Further, a quantum field theory has been developed within Nelson's framework, as well as a procedure for stochastic quantization and a full study of quantum coherent states. The theory has also received close attention from the point of view of Bernstein processes (see e.g. the works of Garbaczewski). Dissipation in quantum systems is highly amenable to treatment with the stochastic methods (Marra 1987). Another noteworthy result is the sub-quantum H-theorem in Valentini (1991a, b). Further, stochastic quantum mechanics has been of some value in the study of stochastic chaos in Brownian systems obeying a Fokker-Planck equation that is formally analogous to the Schrödinger equation (see e.g. Alpatov and Reichl 1994).[14] A somewhat different and interesting realist and objective formulation of the stochastic approach to the quantum phenomenon has been developed in recent years by Budiyono (2012a, b, c, 2013a, b).

The stochastic theory also helps to gain some intuition on specific quantum problems, notably the (anti)symmetrization of the wave function (Nelson 1985a, Sect. 20; see also Loffredo and Morato 1987; Petroni and Morato 2000). Its application to the tunnel effect (Jona-Lasinio et al. 1981; Yasue 1981) is convenient for addressing aspects related to quantum trajectories, such as arrival times, first hitting time, sojourn times, and so on, and provides an illustration of typical (one-particle) quantum nonlocality, as shown in Sect. 2.5. The analysis of particle trajectories represents undoubtedly a valuable plus of the stochastic approach to QM (see McClendon and Rabitz 1988; Moreno Murguía 2006).

An additional contribution of the theory is that it discloses the link between the quantum potential V_Q [see paragraph following Eq. (2.65)] and the diffusive velocity $\boldsymbol{u}$, as shown in Eq. (2.41). This helps to assign a kinetic nature to V_Q, a point that will be revisited in detail in Chap. 8, where the relation between $\boldsymbol{u}$ and the nonlocal properties of the quantum system will become clear. An illustration of this can be seen in the expression for the acceleration $\boldsymbol{a}_Q$, Eq. (2.65), which is a function of the diffusive velocity and thus of the (changes of the) density of particles ρ: the essentially nonlocal nature of $\boldsymbol{u}$ is conveyed to the acceleration through $\lambda \neq 0$.[15]

[14] Some of these matters are discussed in Vasudevan et al. (2008). For the relativistic case see also Ramanathan (1997). Extensive and complementary lists of references to earlier work can be found in Jammer (1974), Guerra (1981, 1984, 1988), Blanchard et al. (1987), de la Peña and Cetto (1991), and *The Dice*.

[15] The formula for the acceleration $\boldsymbol{a}_B$ for classical (Brownian) particles is of course as nonlocal as the quantum acceleration, but nobody denies the usefulness of the Brownian-motion theory of

Despite its advantages in providing an alternative route for the understanding of QM, the theory has also its downsides, the most obvious one being its phenomenological nature, as has been stressed already. This may not perturb those who argue that thermodynamics is also a phenomenological theory and yet nobody quarrels about that. The point is that in the present description a most fundamental element is missing: the *physical* cause of the fluctuations, so the physical elements that determine the parameters in Schrödinger-like equations remains unidentified. This leaves things more or less as in QM itself: we face again the unexplained (noncausal) fluctuations, and the universality of a in Eq. (2.71) must be assumed a priori.

The theory has received further criticisms from a diversity of standpoints (see e.g. Ghirardi et al. 1978; Gillespie 1995; Grabert et al. 1979; Mielnik and Tengstrand 1980). One in particular, is that not a single stochastic process, but an infinity of them can be associated to a quantum state (Davidson 1979b). This is a peculiarity of the description in terms of a Schrödinger-type equation involving the sole product $D\sqrt{-\lambda}$, as discussed above, rather than a problem for its stochastic interpretation. This difficulty is solved by determining D on physical grounds, as is done in Chap. 4. It is further argued that, contrary to what happens with classical diffusions, the quantum stochastic process cannot be separated into 'subprocesses' satisfying a given set of initial conditions (Grabert et al. 1979); this means that the trajectory of a given particle depends nonlocally on all other trajectories that it *could* have followed, which is of course unrealistic and unacceptable. However, these (and other) bizarre peculiarities (see e.g. Ghirardi et al. 1978) are a manifestation of the quantum behavior; they constitute an integral part of quantum theory, even if some of them remain normally hidden. In other words, bizarre quantum properties manifest themselves as bizarre stochastic properties. Accepting quantum theory implies accepting them. We have become accostumed with time to accept the former, but are still very sensitive to the latter. What stochastic QM does is to expose them for further analysis.

The nonlocality problem in stochastic quantum mechanics has been strongly—and rightly—criticized by Nelson (1985a, b, 2005, Sect. 23) on the ground that any fundamental physical theory that violates locality is untenable. It is noteworthy that Nelson decided to abandon his succesful efforts in the development of his stochastic mechanics—which to a large extent is the one discussed in the present chapter—for a reason of principle, namely, when he discovered its nonlocal nature. Now, it is clear that a theory designed to reproduce QM will reproduce the niceties but also the quandaries of QM. And the nonlocality of Nelson's theory is a mere rebound of the quantum nonlocalities —yet nobody renounces QM by rejecting its nonlocalities. Quite the contrary: today it is fashionable to happily speak of quantum nonlocalities; a look at the literature around the Bell inequalities serves to attest this. It seems that the problem has two facets. For on the one hand it is important to understand why

(Footnote 15 continued)
Einstein and Smoluchowski within its domain of applicability. It even played a most important and historic role in the empirical demonstration of the reality of molecules at the beginning of the 20th century! Such description of the Brownian case is admittedly not a fundamental one. In the quantum case a problem arises when interpreting it as a fundamental theory, since a fundamental expression for the acceleration *must* be local.

QM implies a nonlocal description, and on the other hand, it is important to find the theory that supersedes this trait which, as Nelson put it, is untenable.[16]

Probably the most extended criticism towards stochastic quantum mechanics (or stochastic mechanics) is the one raised by Wallstrom (1989, 1994) in a frequently cited work (see also Goldstein 1987; Takabayasi 1952). In essence it asserts that the transition from the couple of Eq. (2.43) [or Eq. (2.39) and the continuity equation] to the Schrödinger equation may be unbefitting due to the fact that in the construction for $\psi \sim e^{iS}$ the function S may be many-valued, so that there is no reason to assume that ψ is single-valued, it being a mere mathematical object. Detailed rebuttals of Wallstrom's argument have been given in Smolin (2006) in a significant contribution to Nelson's theory, and by Fritsche and Haugk (2009) (and 2003), this latter offering a proof that the single-valuedness of the wave function ensues from the conservation of its normalization at all times. It is important to insist on these rebuttals because Wallstrom's work has been considered by many as the definitive blow against Nelson's and similar theories. An unfortunate example is Wick's (1995) book, an excellent and highly advisable book for the wide public, which contains a careful discussion of several of the conceptual problems of QM, particularly the 'infamous boundary' between the observed and the observer. Given the book's realistic and objective approach to the subject, one would expect it to pay serious attention to the stochastic theory—which it does not. In fact, the author confesses that he used to be appreciative of Nelson's theory, but was forced to change his point of view by Wallstrom's paper. The replies provided by Smolin and by Fritsche and Hangk hopefully help restore confidence in the stochastic theories of QM—within their natural limitations. In Chap. 4 we come back to this point.

To put things in the proper perspective we should bear in mind that as a phenomenological theory, stochastic quantum mechanics is not to be doomed for its properties or shortcomings. The error would lie in taking such a limited description as the accomplished theory. The parameter λ of the stochastic description of QM is *selected* so as to reproduce the latter, with all virtues and limitations of such selection. Difficulties appear due to the poorness of the configuration-space description: it is too restricted to hold the richness of the real stochastic phenomena.

Generally speaking, the critics of stochastic (quantum) mechanics are formally correct in their criticisms, although their objections normally relate in the last instance (and unknowingly to the critics) to the peculiarities of quantum systems rather than to the stochastic approach itself. What in reality many of the critics of the stochastic description of QM are doing is contribute to the catalog of the most relevant differences between classical and quantum stochastic processes. The differences are so substantial that one should not be surprised to find that the required stochastic quantum description falls far from the corresponding classical one.

[16] Recently, Nelson has attempted to apply stochastic mechanics to relativistic fields, hoping to avoid the above mentioned nonlocality features, and aiming to develop useful technical tools in constructive field theory (see Nelson 2013).

2.6.1 A Summing Up

The material of this chapter has hopefully served its purpose to assess the value of stochastic quantum mechanics in the present context. One first advantage of the theory is that it neatly discloses the stochastic nature of the quantum system, highlighting essential similarities and differences between QM and classical, Brownian-type stochastic processes. Another is that it leads to QM through a simple, phenomenological approach which, not being part of usual QM, enriches it by offering a complementary, intuitive picture of some important aspects of the theory.

The stochastic approach provides with relative simplicity a way to arrive at QM from a realist and objective physical picture; however, it leaves us with the feeling that the real thing continues to be hidden behind the phenomenological curtain. Something more fundamental is required. The search for it is the subject matter of the following chapters.

Answer to the quiz: The reversibility of the Schrödinger equation means that this equation describes an average behavior after any (subquantum) irreversible process that could exist, has ceased to be active. The meaning of this answer will become clearer as we proceed.

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