Calibration of a Stock's Beta Using Option Prices

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Abstract In this paper, we present a continuous time Capital Asset Pricing Model where the volatilities of the market index and the stock are both stochastic. Using a singular perturbation technique, we provide approximations for the prices of European options on both the stock and the index. We derive then an estimator of the parameter beta under the risk-neutral pricing measure $\mathbb{P}^*$ using option and underlying prices. Following that, we study empirically the discrepancy between the implied value of the parameter β under $\mathbb{P}^*$ and its realized value under the real-world probability measure $\mathbb{P}$. Finally, we show that the parameter β is crucial for the hedging of stock options using instruments on the index, and we study numerically the performance of the proposed hedging strategies.

1 Introduction

The capital asset pricing model is an econometric model that provides an estimation of an asset's return in function of the systematic risk or the market risk. This model is an expansion of an earlier work of Markowitz on portfolio construction (see [1]). The CAPM model was considered to be an original and innovative model because it introduced he concept of systematic and specific risk. The parameter β, which is a key parameter in this model, enables to separate the stock risk into two parts: the first part represents the systematic risk contained in the market index, while the second part is the idiosyncratic risk that reflects the specific performance of the stock. The parameter beta in the capital asset pricing model is very useful for portfolio construction purposes (see [2–4]), thus its estimation is a matter of interest and has been a subject of study for several authors in the last decades. This parameter

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was traditionally estimated using historical data of daily returns of the stock and the market index where it is obtained as the slope of the linear regression of stock returns on market index returns (see [5]). This approach is backward-looking as it estimates the realized value of the parameter in the past using historical data. The authors in [6, 7] showed that the parameter β is not constant but rather time-varying. Thus, the value of the realized beta in the future can be remarkably different from its value in the past, and the backward-looking estimation may be inefficient.

In the recent literature, different authors have focused on the estimation of the beta coefficient using option data, which provides a different estimation method for this parameter. Indeed, whereas classical methods allow an historical estimation of this parameter, the option based estimation method enables to obtain a "forward looking" measure of this parameter. Thus, the obtained estimator represents the information contained in derivatives prices and then summarizes the expectation of market participants for the forward realization of this parameter.

In [8], Christoffersen, Jacobs and Vainberg provided an estimation of this parameter using the risk-neutral variance and skewness of the stock and the index. More recently, Fouque and Kollman proposed in [9] a continuous-time CAPM model in which the market index has a stochastic volatility driven by a fast mean-reverting process. Using a singular perturbation method, they managed to obtain an approximation of the beta parameter depending on the skews of implied volatilities of both the stock and the index. Fouque and Tashman introduced in [10] a "Stressed-Beta model" in which the parameter β can take two values depending on the market regime. Using this model, Fouque et al. provided a method to price options on the index and the stock. This method enables also to estimate the parameter β based on options data. In [11], Carr and Madan used the CAPM model to price options on the stock when options on the index are liquid. Their approach didn't aim to estimate the parameter beta using option prices, but to price options on the stock given the parameter beta and options prices on the market index.

This work deals with the estimation of the coefficient beta under the risk-neutral measure using options prices and highlights the utility of the obtained estimator for diverse applications. The paper is organized as follows. In the second section, we present the capital asset pricing model with constant idiosyncratic volatility for the stock. We recall briefly the method presented in [9], which allows to estimate the parameter beta using implied volatility data. In the third section, we consider a new model in which the stock's idiosyncratic volatility is stochastic. This choice was motivated by several empirical studies which confirm the random character of the idiosyncratic volatility process, thus the new model is more adapted to describe the joint dynamics of the stock and the market index. In the setting of the new model, we provide approximations for European option prices on both the stock and the index through the use of a singular perturbation technique. Afterward, we deduce an estimator of the parameter beta using option and underlying prices, and we call it the implied beta. In the fourth section, we present some possible applications which emphasize the utility of the beta estimator. First, we investigate the capacity of the implied beta to predict the forward realized beta under the historic probability measure. Then, in a second application, we show that the parameter beta can be used for the purpose of hedging stock options using instruments on the index. We run Monte-Carlo simulations to support the theoretical study and test numerically the hedging methods.

2 Model with Constant Idiosyncratic Volatility

2.1 *Presentation of the Model*

Consider a financial market living on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = \{\mathscr{F}_t\}$ satisfies the usual conditions, and where $\mathbb{P}$ is the objective probability measure. The authors in [9] proposed a continuous time capital asset pricing model where the market index has a stochastic volatility. Under the probability measure P, the dynamics of the stock S and the index I are described as follows:

$$\begin{aligned}\frac{dI_t}{I_t} &= \mu_I dt + f(Y_t)dW_t^{(1)},\\ \frac{dS_t}{S_t} &= \mu_S dt + \beta\frac{dI_t}{I_t} + \sigma dW_t^{(2)},\\ dY_t &= \frac{1}{\varepsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}dW_t^{(3)},\end{aligned}$$

where $W_t^{(3)} = \rho W_t^{(1)} + \sqrt{1-\rho^2}W_t^{(4)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(4)} \end{pmatrix}$ is a Wiener process under $\mathbb{P}$

The authors made the assumption that $0 < \varepsilon \ll 1$ which implies that the process (Y), which drives the index volatility, is a fast mean-reverting Ornstein-Uhlenbeck process.

Let $\lambda_t = \begin{pmatrix} \frac{\mu_I - r}{f(Y_t)} \\ \frac{\mu_S + r(\beta-1)}{\sigma} \\ \gamma(Y_t) \end{pmatrix}$ where the function γ denotes the volatility risk-premium. The probability measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, is defined in the following way:

$$\frac{dP^*}{dP}_{|\mathscr{F}_t} = exp(-\int_0^t \lambda_u' dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du),$$

Let $W^* = \begin{pmatrix} W^{*,(1)} \\ W^{*,(2)} \\ W^{*,(4)} \end{pmatrix}$ such that: $W_t^* = W_t + \int_0^t \lambda_u du$. Using Girsanov's theorem, it follows that W^* is a $(\mathbb{P}^*, \{\mathscr{F}_t\})$ Brownian motion.

Under continuity and boundedness conditions on the function γ, $\mathbb{P}^*$ is a risk-neutral probability measure under which the index and the stock have the following dynamics:

$$\frac{dI_t}{I_t} = rdt + f(Y_t)dW_t^{*,(1)},$$
$$\frac{dS_t}{S_t} = rdt + \beta f(Y_t)dW_t^{*,(1)} + \sigma dW_t^{*,(2)},$$
$$dY_t = (\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\chi(Y_t))dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}dW_t^{*,(3)},$$

where $\chi(Y_t) = \rho\frac{\mu_I - r}{f(Y_t)} + \sqrt{1-\rho^2}\gamma(Y_t)$ and $W_t^{*,(3)} = \rho W_t^{*,(1)} + \sqrt{1-\rho^2}W_t^{*,(4)}$.

2.2 Calibration of Implied Beta

Using a singular perturbation method with respect to the small parameter ε, the authors in [9] obtained an approximation $\tilde{P}^{I,\varepsilon}(K_I, T)$ for the price of an European call on the index with strike K_I and maturity T, and an approximation $\tilde{P}^{S,\varepsilon}(K_S, T)$ for the price of an European call on the stock with strike K_S and maturity T. Afterward, through the use of a Taylor expansion in $\sqrt{\varepsilon}$ for the implied volatility of the stock and the index, they provided an approximation for the implied volatilities $\Sigma_I(K_I, T)$ and $\Sigma_S(K_S, T)$ of the index and the stock respectively:

$$\Sigma_I(K_I, T) = b_I + a_I\frac{\log(\frac{F_I}{K_I})}{T},$$
$$\Sigma_S(K_S, T) = b_S + a_S\frac{\log(\frac{F_S}{K_S})}{T}.$$

It should be precised that he quantities F_I and F_S denote the forward prices for maturity T of the index and the stock respectively, while the quantities b_I, a_I, b_S, a_S are functions of the model parameters. Thus, the parameter β can be approximated by $\hat{\beta}$ which is defined as :

$$\hat{\beta} = (\frac{a_S}{a_I})^{\frac{1}{3}}\frac{b_S}{b_I}. \tag{1}$$

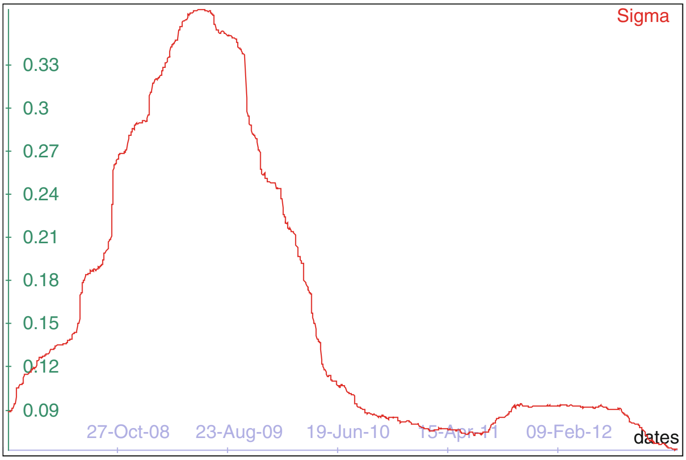

Fig. 1 Evolution of the idiosyncratic volatility of XLF with respect to the SPX index

2.3 Limits of the Model

In the model described so far, the stock's idiosyncratic volatility is supposed to be constant and equal to σ. This hypothesis can be considered too strong, indeed the authors in [12–15] conducted empirical studies on the idiosyncratic volatility and gave empirical evidence of the randomness of this process.

In order to have an idea about the magnitude of fluctuations of the idiosyncratic volatility, the graph of the parameter σ of the Financial Select Sector (named XLF) when projected on the SPX index is given below. The considered period ranges from 01/01/2008 to 31/12/2012. The parameter σ is obtained through the computation of the standard deviation of errors in the linear regression of the daily returns of the Financial Select Sector (XLF) on the daily returns of the SPX index using a sliding window of one month (Fig. 1).

The inspection of the graph above shows that the assumption of constant idiosyncratic volatility can be strong, and thus can induce a misleading understanding of the joint dynamics of the stock and the index. Consequently, a new model will be introduced in the next section in order to account for this characteristic.

3 Model with Stochastic Idiosyncratic Volatility

A new continuous-time capital asset pricing model is presented here. In this model setting, the stock's idiosyncratic volatility is driven by a fast mean-reverting Ornstein-Uhlenbeck process. The aim of this section is to derive approximations for European option prices on both the stock and the index, and to provide an estimator of the parameter β using option prices.

3.1 Presentation of the Model

Under the historic probability measure $\mathbb{P}$, the stock and the index have the following dynamics:

$$
\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f_1(Y_t)dW_t^{(1)},\\
\frac{dS_t}{S_t} &= \mu_S dt + \beta\frac{dI_t}{I_t} + f_2(Z_t)dW_t^{(2)},\\
dY_t &= \frac{1}{\varepsilon}(m_Y - Y_t)dt + \frac{\nu_Y\sqrt{2}}{\sqrt{\varepsilon}}dW_t^{(3)},\\
dZ_t &= \frac{\alpha}{\varepsilon}(m_Z - Z_t)dt + \frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\varepsilon}}dW_t^{(4)},
\end{aligned}
$$

where $W_t^{(3)} = \rho_Y W_t^{(1)} + \sqrt{1-\rho_Y^2}W_t^{(5)}$, $W_t^{(4)} = \rho_Z W_t^{(2)} + \sqrt{1-\rho_Z^2}W_t^{(6)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(5)} \\ W^{(6)} \end{pmatrix}$ is a $(\mathbb{P}, \{\mathscr{F}_t\})$ Wiener process. It is supposed that $0 < \varepsilon \ll 1$ and $0 < \frac{\varepsilon}{\alpha} \ll 1$. This hypothesis implies that the processes (Y) and (Z) are fast mean-reverting Ornstein-Uhlenbeck processes.

Let the process λ be defined as $\lambda_t = \begin{pmatrix} \frac{\mu_I - r}{f_1(Y_t)} \\ \frac{\mu_S + r(\beta-1)}{f_2(Z_t)} \\ \gamma_1(Y_t) \\ \gamma_2(Z_t) \end{pmatrix}$ where the functions γ_1 and γ_2 denote the volatility risk premiums related to the processes Y and Z respectively. The probability measure P^*, equivalent to P, can then be defined as follows:

$$
\frac{dP^*}{dP} = exp(-\int_0^t \lambda_u' dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du).
$$

Let $W_t^* = W_t + \int_0^t \lambda_u du$. It can be deduced, through the use of Girsanov's theorem, that W^* is a $(\mathbb{P}^*, \{\mathscr{F}_t\})$ Brownian motion. Thus, the dynamics of the stock and the index, and under $\mathbb{P}^*$, can be described in the following way:

$$
\begin{aligned}
\frac{dI_t}{I_t} &= rdt + f_1(Y_t)dW_t^{*,(1)},\\
\frac{dS_t}{S_t} &= rdt + \beta f_1(Y_t)dW_t^{*,(1)} + f_2(Z_t)dW_t^{*,(2)},
\end{aligned}
$$

$$dY_t = \frac{1}{\varepsilon}(m_Y - Y_t)dt - \frac{\nu_Y\sqrt{2}}{\sqrt{\varepsilon}}\chi_1(Y_t)dt + \frac{\nu_Y\sqrt{2}}{\sqrt{\varepsilon}}dW_t^{*,(3)},$$

$$dZ_t = \frac{\alpha}{\varepsilon}(m_Z - Z_t)dt - \frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\varepsilon}}\chi_2(Z_t)dt + \frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\varepsilon}}dW_t^{*,(4)},$$

where: $\chi_1(Y_t) = \rho_Y \frac{\mu_I - r}{f_1(Y_t)} + \sqrt{1-\rho_Y^2}\gamma_1(Y_t)$ and $\chi_2(Z_t) = \rho_Z \frac{\mu_S + r(\beta-1)}{f_2(Z_t)} + \sqrt{1-\rho_Z^2}\gamma_2(Z_t)$.

The processes $W^{*,(3)}$ and $W^{*,(4)}$ are Brownian motions under $\mathbb{P}^*$ such that:

$$W^{*,(3)} = \rho_Y W^{*,(1)} + \sqrt{1-\rho_Y^2}W^{*,(5)},$$

$$W^{*,(4)} = \rho_Z W^{*,(2)} + \sqrt{1-\rho_Z^2}W^{*,(6)}.$$

3.2 Pricing Options on the Index and the Stock

3.2.1 Approximation Formula for Index Option Price

Let $P^{I,\varepsilon}(K_I, T) = E^{P^*}(e^{-r(T-t)}(I_T - K_I)^+|\mathscr{F}_t)$ be the price of an European call on the index with strike K_I and maturity T. The processes (I) and (Y) have the same dynamics as in the model with constant idiosyncratic volatility, which implies that the pricing of index options remains the same.

By means of a singular perturbation method with respect to the parameter ε, the authors obtained in [9] an approximation $\tilde{P}^{I,\varepsilon}(K_I, T)$ for the price $P^{I,\varepsilon}(K_I, T)$ at order 1 in ε. The proof of the approximation result is given in [9] and also in (Appendix 1) for completeness. The results are recalled here:

$$\tilde{P}^{I,\varepsilon} = \tilde{P}_0^{I,\varepsilon} - (T-t)V_3^{I,\varepsilon} I_t \frac{\partial}{\partial I_t}(I_t^2 \frac{\partial^2 \tilde{P}_0^{I,\varepsilon}}{\partial I_t^2}), \tag{2}$$

where the quantities $\tilde{P}_0^{I,\varepsilon}$ and $V_3^{I,\varepsilon}$ are defined as:

$$\tilde{P}_0^{I,\varepsilon} = P_{BS}^I(\bar{\sigma}_I^*), \tag{3}$$

$$V_2^{I,\varepsilon} = -\frac{\sqrt{\varepsilon}}{\sqrt{2}}\nu_Y \left\langle \phi_I' \chi_1 \right\rangle_1, \tag{4}$$

$$V_3^{I,\varepsilon} = \frac{\sqrt{\varepsilon}}{\sqrt{2}}\rho_Y \nu_Y \left\langle \phi_I' f_1 \right\rangle_1, \tag{5}$$

$$(\bar{\sigma}_I^*)^2 = \left\langle f_1^2 \right\rangle_1 - 2V_2^{I,\varepsilon}. \tag{6}$$

The operator $\langle . \rangle_1$ denotes the average with respect to the invariant distribution of the Ornstein-Uhlenbeck process (Y_1) whose dynamics are described as follows:

$$dY_{1,t} = (m_Y - Y_{1,t})dt + \nu_Y \sqrt{2} dW_t^{(3)}.$$

Besides, the function ϕ_I is defined as the solution of the following Poisson equation:

$$\mathscr{L}_0^I \phi_I(y) = f_1^2(y) - \left\langle f_1^2 \right\rangle_1, \tag{7}$$

where $\mathscr{L}_0^I$ is the infinitesimal generator of the process (Y_1) and can be written:

$$\mathscr{L}_0^I = \frac{\partial}{\partial y}(m_Y - y) + \nu_Y^2 \frac{\partial^2}{\partial y^2}.$$

3.2.2 Approximation Formula for Stock Option Price

Let $P_t^{S,\varepsilon}(K_S, T)$ be the price, at time t, of an European call on the stock with strike K_S and maturity T:

$$P_t^{S,\varepsilon}(K_S, T) = E^{P^*}(e^{-r(T-t)}(S_T - K_S)^+ | \mathscr{F}_t).$$

The notation $P_t^{S,\varepsilon}$ is used instead of $P_t^{S,\varepsilon}(K_S, T)$ for simplification purposes.

Using a singular perturbation technique on the parameter ε, an approximation $\tilde{P}^{S,\varepsilon}$ for the option's price $P_t^{S,\varepsilon}$ is obtained. The approximation error is at order 1 in ε. The results can be detailed as follows:

Proposition 3.1

$$\tilde{P}^{S,\varepsilon} = \tilde{P}_0^{S,\varepsilon} - (T-t) V_3^{S,\varepsilon} S_t \frac{\partial}{\partial S_t}(S_t^2 \frac{\partial^2 \tilde{P}_0^{S,\varepsilon}}{\partial S_t^2}), \tag{8}$$

where the quantities $\tilde{P}_0^{S,\varepsilon}$ and $V_3^{S,\varepsilon}$ are defined as:

$$\tilde{P}_0^{S,\varepsilon} = P_{BS}^S(t, S_t, \bar{\sigma}_S^*), \tag{9}$$

$$(\bar{\sigma}_S^*)^2 = \bar{\sigma}_S^2 - 2V_2^{S,\varepsilon}, \tag{10}$$

$$V_2^{S,\varepsilon} = -\frac{\sqrt{\varepsilon}}{\sqrt{2}}(\beta^2 \nu_Y \left\langle \phi_I' \chi_1 \right\rangle_{1,2} + \nu_Z \sqrt{\alpha} \left\langle \phi_{Idios}' \chi_2 \right\rangle_{1,2}), \tag{11}$$

$$V_3^{S,\varepsilon} = \frac{\sqrt{\varepsilon}}{\sqrt{2}}(\beta^3 \rho_Y \nu_Y \left\langle \phi_I' f_1 \right\rangle_{1,2} + \rho_Z \nu_Z \sqrt{\alpha} \left\langle \phi_{Idios}' f_2 \right\rangle_{1,2}). \tag{12}$$

The operator $\langle . \rangle_{1,2}$ denotes the averaging with respect to the invariant distribution of $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_t$ which has the following dynamics:

$$d\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\left(\begin{pmatrix} m_Y \\ m_Z \end{pmatrix} - \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix}\right)dt + \sqrt{2}\begin{pmatrix} \nu_Y & 0 \\ 0 & \sqrt{\alpha}\nu_Z \end{pmatrix} d\begin{pmatrix} W_t^{(3)} \\ W_t^{(4)} \end{pmatrix}.$$

The function ϕ_{Idios} is the solution of the equation:

$$\mathcal{L}_0^S \phi_{Idios}(z) = f_2^2(z) - \left\langle f_2^2 \right\rangle_{1,2}.$$

where $\mathcal{L}_0^S$ is the infinitesimal generator of the two-dimensional Ornstein-Uhlenbeck process $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$.

Proof The price of an European call on the stock writes:

$$P_t^{S,\varepsilon} = E^{P^*}(e^{-r(T-t)}(S_T - K_S)^+ | S_t = x, Y_t = y, Z_t = z).$$

Since the process (S, Y, Z) is markovian, the Feynman-Kac theorem can be used to prove that $\mathcal{L}^S P_t^{S,\varepsilon} = 0$ where $\mathcal{L}^S$ is a differential operator expanded in powers of $\sqrt{\varepsilon}$:

$$\mathcal{L}^S = \mathcal{L}_2^S + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1^S + \frac{1}{\varepsilon}\mathcal{L}_0^S,$$

and $\mathcal{L}_0^S, \mathcal{L}_1^S, \mathcal{L}_2^S$ are defined as follows:

$$\begin{aligned}
\mathcal{L}_0^S &= (m_Y - y)\frac{\partial}{\partial y} + \nu_Y^2\frac{\partial^2}{\partial y^2} + \alpha(m_Z - z)\frac{\partial}{\partial z} + \alpha\nu_Z^2\frac{\partial^2}{\partial z^2}, \\
\mathcal{L}_1^S &= -\nu_Y\sqrt{2}\chi_1(y)\frac{\partial}{\partial y} + \beta S_t f_1(y)\sqrt{2}\rho_Y\nu_Y\frac{\partial^2}{\partial S\partial y} \\
&\quad - \nu_Z\sqrt{2\alpha}\chi_2(z)\frac{\partial}{\partial z} + S_t f_2(z)\sqrt{2\alpha}\rho_Z\nu_Z\frac{\partial^2}{\partial S\partial z}, \\
\mathcal{L}_2^S &= \frac{\partial}{\partial t} + r(\frac{\partial}{\partial S}S_t - .) + \frac{1}{2}\frac{\partial^2}{\partial S_t^2}S_t^2(\beta^2 f_1(y)^2 + f_2(z)^2).
\end{aligned}$$

The differential operator $\mathcal{L}_0^S$ is the infinitesimal generator of the two-dimensional Ornstein-Uhlenbeck process $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ which has the following dynamics:

$$d\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\left(\begin{pmatrix} m_Y \\ m_Z \end{pmatrix} - \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix}\right) dt + \sqrt{2}\begin{pmatrix} \nu_Y & 0 \\ 0 & \sqrt{\alpha}\nu_Z \end{pmatrix} d\begin{pmatrix} W_t^{(3)} \\ W_t^{(4)} \end{pmatrix}.$$

The following notations are used:

- The operator $\langle . \rangle_1$ denotes the averaging with respect to the invariant distribution of the process $(Y_{1,t})_t$.
- The operator $\langle . \rangle_2$ denotes the averaging with respect to the invariant distribution of the process $(Y_{2,t})_t$.
- The operator $\langle . \rangle_{1,2}$ denotes the averaging with respect to the invariant distribution of $\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix}_t$.

The option price $P^{S,\varepsilon}$ is expanded in powers of $\sqrt{\varepsilon}$:

$$P^{S,\varepsilon} = \sum_{i=0}^{\infty} (\sqrt{\varepsilon})^i P_i^{S,\varepsilon},$$

then, the expression $\mathscr{L}^S P_t^{S,\varepsilon}$ can be written as follows:

$$(\mathscr{L}_2^S + \frac{1}{\sqrt{\varepsilon}}\mathscr{L}_1^S + \frac{1}{\varepsilon}\mathscr{L}_0^S)(\sum_{i=0}^{\infty} (\sqrt{\varepsilon})^i P_i^{S,\varepsilon}) = 0.$$

By classifying the terms of the last equation by powers of $\sqrt{\varepsilon}$, the terms of orders $(-2), (-1), 0, 1, 2$ in $\sqrt{\varepsilon}$ are obtained:

$$(-2) : \mathscr{L}_0^S P_0^{S,\varepsilon} = 0, \tag{13}$$
$$(-1) : \mathscr{L}_1^S P_0^{S,\varepsilon} + \mathscr{L}_0^S P_1^{S,\varepsilon} = 0, \tag{14}$$
$$(0) : \mathscr{L}_2^S P_0^{S,\varepsilon} + \mathscr{L}_1^S P_1^{S,\varepsilon} + \mathscr{L}_0^S P_2^{S,\varepsilon} = 0, \tag{15}$$
$$(1) : \mathscr{L}_2^S P_1^{S,\varepsilon} + \mathscr{L}_1^S P_2^{S,\varepsilon} + \mathscr{L}_0^S P_3^{S,\varepsilon} = 0, \tag{16}$$
$$(2) : \mathscr{L}_2^S P_2^{S,\varepsilon} + \mathscr{L}_1^S P_3^{S,\varepsilon} + \mathscr{L}_0^S P_4^{S,\varepsilon} = 0. \tag{17}$$

The term of order (-2) in $\sqrt{\varepsilon}$ states that $\mathscr{L}_0^S P_0^{S,\varepsilon} = 0$. Since the operator $\mathscr{L}_0^S$ contains only derivatives with respect to y and z, this equation can be solved by choosing $P_0^{S,\varepsilon} = P_0^{S,\varepsilon}(t, S_t)$ independent of Y_t and Z_t.

The term of order (-1) in $\sqrt{\varepsilon}$ states that $\mathscr{L}_1^S P_0^{S,\varepsilon} + \mathscr{L}_0^S P_1^{S,\varepsilon} = 0$. The differential operator $\mathscr{L}_1^S$ contains first and second order derivatives with respect to y and z, thus $\mathscr{L}_1^S P_0^{S,\varepsilon} = 0$. The equation becomes then $\mathscr{L}_0^S P_1^{S,\varepsilon} = 0$. The equation is satisfied for $P_1^{S,\varepsilon} = P_1^{S,\varepsilon}(t, S_t)$ independent of Y_t and Z_t.

The quantities $P_0^{S,\varepsilon}$ and $P_1^{S,\varepsilon}$ being independent of Y_t and Z_t, it can be stated that:

$$\mathscr{L}_0^S P_0^{S,\varepsilon} = \mathscr{L}_1^S P_0^{S,\varepsilon} = \mathscr{L}_0^S P_1^{S,\varepsilon} = \mathscr{L}_1^S P_1^{S,\varepsilon} = 0.$$

Since $\mathscr{L}_1^S P_1^{S,\varepsilon} = 0$, the term of order 0 in $\sqrt{\varepsilon}$ becomes:

$$\mathscr{L}_2^S P_0^{S,\varepsilon} + \mathscr{L}_0^S P_2^{S,\varepsilon} = 0,$$

which is a Poisson equation for $P_2^{S,\varepsilon}$ with respect to $\mathscr{L}_0^S$. The solvability condition for this equation is:

$$\left\langle \mathscr{L}_2^S P_0^{S,\varepsilon} \right\rangle_{1,2} = \left\langle \mathscr{L}_2^S \right\rangle_{1,2} P_0^{S,\varepsilon} = 0.$$

The average $\langle \mathscr{L}_2^S \rangle_{1,2}$ of the generator $\mathscr{L}_2^S$ verifies:

$$\left\langle \mathscr{L}_2^S \right\rangle_{1,2} = \frac{\partial}{\partial t} + r(\frac{\partial}{\partial S_t} S_t - .) + \frac{1}{2} \frac{\partial^2}{\partial S_t^2} S_t^2 \left\langle \beta^2 f_1^2(y) + f_2^2(z) \right\rangle_{1,2}.$$

It can be deduced that $\langle \mathscr{L}_2^S \rangle_{1,2} = \mathscr{L}_{BS}^S(\bar{\sigma}_S)$ where $\bar{\sigma}_S^2 = \beta^2 \langle f_1^2 \rangle_{1,2} + \langle f_2^2 \rangle_{1,2}$.

Consequently, $P_0^{S,\varepsilon}$ is the solution of the following problem:

$$\begin{aligned} \mathscr{L}_{BS}(\bar{\sigma}_S) P_0^{S,\varepsilon} &= 0, \\ P_0^{S,\varepsilon}(T, S_T) &= h(S_T). \end{aligned}$$

It can be easily seen that the quantity $P_0^{S,\varepsilon}$ represents the Black-Scholes price of the option with implied volatility equal to $\bar{\sigma}_S$:

$$P_0^{S,\varepsilon} = P_{BS}^S(t, S_t, \bar{\sigma}_S).$$

The term of order 1 in $\sqrt{\varepsilon}$ is a Poisson equation for $P_3^{S,\varepsilon}$ with respect to $\mathscr{L}_0^S$, whose solvability condition is:

$$\left\langle \mathscr{L}_2^S \right\rangle_{1,2} P_1^{S,\varepsilon} = -\left\langle \mathscr{L}_1^S P_2^{S,\varepsilon} \right\rangle_{1,2} = \left\langle \mathscr{L}_1^S (\mathscr{L}_0^S)^{-1} (\mathscr{L}_2^S - \left\langle \mathscr{L}_2^S \right\rangle_{1,2}) \right\rangle_{1,2} P_0^{S,\varepsilon}. \quad (18)$$

The quantity $P_1^{S,\varepsilon}$ is the solution of the last equation with terminal condition $P_1^{S,\varepsilon}(T, S_T) = 0$.

The function f_1 is independent of z and the function f_2 is independent of y, it follows that: $\langle f_1^2 \rangle_{1,2} = \langle f_1^2 \rangle_1$ and $\langle f_2^2 \rangle_{1,2} = \langle f_2^2 \rangle_2$. In addition, the function ϕ_I which is the solution of (7), doesn't depend on z. This implies:

$$\mathscr{L}_0^S \phi_I(y) = \mathscr{L}_0^I \phi_I(y) = f_1^2(y) - \left\langle f_1^2 \right\rangle_{1,2}.$$

Let the function ϕ_{Idios} be the solution of the following equation:

$$\mathcal{L}_0^S \phi_{Idios}(z) = f_2^2(z) - \left\langle f_2^2 \right\rangle_{1,2}. \tag{19}$$

The function ϕ_{Idios} doesn't depend on y, thus it can be deduced:

$$\mathcal{L}_0^S(\beta^2\phi_I(y) + \phi_{Idios}(z)) = \beta^2(f_1^2(y) - \left\langle f_1^2 \right\rangle_{1,2}) + (f_2^2(z) - \left\langle f_2^2 \right\rangle_{1,2}).$$

Building on this, it can be obtained:

$$\mathcal{L}_1^S(\mathcal{L}_0^S)^{-1}(\mathcal{L}_2^S - \left\langle \mathcal{L}_2^S \right\rangle_{1,2}) = (\beta^2\mathcal{L}_1^S\phi_I(y) + \mathcal{L}_1^S\phi_{Idios}(z))\frac{1}{2}S_t^2\frac{\partial^2}{\partial S_t^2}.$$

The development of the right-hand side of the previous equation yields:

$$\begin{aligned}
&\frac{1}{2}(\beta^2\left\langle \mathcal{L}_1^S\phi_I(y) \right\rangle_{1,2} + \left\langle \mathcal{L}_1^S\phi_{Idios}(z) \right\rangle_{1,2}) \\
&\quad = (\frac{\beta^3\nu_Y\rho_Y}{\sqrt{2}}\left\langle \phi_I' f_1 \right\rangle_{1,2} + \frac{\rho_Z\nu_Z\sqrt{\alpha}}{\sqrt{2}}\left\langle \phi_{Idios}' f_2 \right\rangle_{1,2})S_t\frac{\partial}{\partial S_t} \\
&\quad\quad - (\frac{\beta^2\nu_Y}{\sqrt{2}}\left\langle \phi_I'\chi_1 \right\rangle_{1,2} + \frac{\nu_Z\sqrt{\alpha}}{\sqrt{2}}\left\langle \phi_{Idios}'\chi_2 \right\rangle_{1,2}).
\end{aligned}$$

Let the quantities $V_2^{S,\varepsilon}$ and $V_3^{S,\varepsilon}$ be defined as follows:

$$\begin{aligned}
V_3^{S,\varepsilon} &= \frac{\sqrt{\varepsilon}}{\sqrt{2}}(\beta^3\nu_Y\rho_Y\left\langle \phi_I' f_1 \right\rangle_{1,2} + \rho_Z\nu_Z\sqrt{\alpha}\left\langle \phi_{Idios}' f_2 \right\rangle_{1,2}), \\
V_2^{S,\varepsilon} &= -\frac{\sqrt{\varepsilon}}{\sqrt{2}}(\beta^2\nu_Y\left\langle \phi_I'\chi_1 \right\rangle_{1,2} + \nu_Z\sqrt{\alpha}\left\langle \phi_{Idios}'\chi_2 \right\rangle_{1,2}).
\end{aligned}$$

The Eq. (18) becomes:

$$\left\langle \mathcal{L}_2^S \right\rangle_{1,2}\sqrt{\varepsilon}P_1^{S,\varepsilon} = V_2^{S,\varepsilon}S_t^2\frac{\partial^2 P_0^{S,\varepsilon}}{\partial S_t^2} + V_3^{S,\varepsilon}S_t\frac{\partial}{\partial S_t}(S_t^2\frac{\partial^2 P_0^{S,\varepsilon}}{\partial S_t^2}). \tag{20}$$

Consequently, $P_1^{S,\varepsilon}$ is the solution of (20) with the final condition $P_1^{S,\varepsilon}(T, S_T) = 0$. Since the differential operator $\left\langle \mathcal{L}_2^S \right\rangle_{1,2} = \mathcal{L}_{BS}^S(\bar{\sigma}_S)$ commits with the operators $\mathcal{D}_{1,S} = S_t\frac{\partial}{\partial S_t}$ and $\mathcal{D}_{2,S} = S_t^2\frac{\partial^2}{\partial S_t^2}$, and that $\left\langle \mathcal{L}_2^S \right\rangle_{1,2} P_0^{S,\varepsilon} = 0$, the solution to the last problem can be given explicitly as:

$$\sqrt{\varepsilon}P_1^{S,\varepsilon} = -(T-t)(V_2^{S,\varepsilon}S_t^2\frac{\partial^2 P_0^{S,\varepsilon}}{\partial S_t^2} + V_3^{S,\varepsilon}S_t\frac{\partial}{\partial S_t}(S_t^2\frac{\partial^2 P_0^{S,\varepsilon}}{\partial S_t^2})). \tag{21}$$

In order to check the validity of the solution, the following verification can be made:

$$\begin{aligned}\left\langle \mathscr{L}_2^S \right\rangle_{1,2} \sqrt{\varepsilon} P_1^{S,\varepsilon} &= (V_2^{S,\varepsilon} \mathscr{D}_{2,S} P_0^{S,\varepsilon} + V_3^{S,\varepsilon} \mathscr{D}_{1,S} \mathscr{D}_{2,S} P_0^{S,\varepsilon}) \left\langle \mathscr{L}_2^S \right\rangle_{1,2} ((t-T)) \\ &\quad - (T-t)(V_2^{S,\varepsilon} \mathscr{D}_{2,S} \left\langle \mathscr{L}_2^S \right\rangle_{1,2} (P_0^{S,\varepsilon}) \\ &\quad + V_3^{S,\varepsilon} \mathscr{D}_{1,S} \mathscr{D}_{2,S} \left\langle \mathscr{L}_2^S \right\rangle_{1,2} (P_0^{S,\varepsilon})) \\ &= V_2^{S,\varepsilon} \mathscr{D}_{2,S} P_0^{S,\varepsilon} + V_3^{S,\varepsilon} \mathscr{D}_{1,S} \mathscr{D}_{2,S} P_0^{S,\varepsilon}.\end{aligned}$$

By neglecting terms of order higher to 1 in $\sqrt{\varepsilon}$, the stock option price can be approximated by $(P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon})$. In order to reduce the number of the parameters in the approximation, a second approximation is derived here. Let $\mathscr{L}_{BS}(\bar{\sigma}_S^*)$ be the Black-Scholes differential operator with volatility $\bar{\sigma}_S^*$:

$$(\bar{\sigma}_S^*)^2 = \bar{\sigma}_S^2 - 2V_2^{S,\varepsilon}.$$

Let the quantity $\tilde{P}_0^{S,\varepsilon}$ be introduced as the solution of the following problem:

$$\begin{aligned}&\mathscr{L}_{BS}(\bar{\sigma}_S^*)\tilde{P}_0 = 0, \\ &\tilde{P}_0^{S,\varepsilon}(T, S_T) = (S_T - K_S)^+.\end{aligned}$$

It follows that $\tilde{P}_0^{S,\varepsilon} = P_{BS}(t, S_t, \bar{\sigma}_S^*)$. Following that, the quantity $\tilde{P}_1^{S,\varepsilon}$ is defined as the solution of the following problem:

$$\begin{aligned}\mathscr{L}_{BS}(\bar{\sigma}_S^*)\sqrt{\varepsilon}\tilde{P}_1^{S,\varepsilon} &= V_3^{S,\varepsilon} S_t \frac{\partial}{\partial S_t}(S_t^2 \frac{\partial^2}{\partial S_t^2} \tilde{P}_0^{S,\varepsilon}), \\ \tilde{P}_1^{S,\varepsilon}(T, S_T) &= 0.\end{aligned}$$

Using the same arguments as before, it can be deduced that $\tilde{P}_1^{S,\varepsilon} = -(T - t)$ $V_3^{S,\varepsilon} S_t \frac{\partial}{\partial S_t}(S_t^2 \frac{\partial^2}{\partial S_t^2} \tilde{P}_0^{S,\varepsilon})$.

It can be proved that the option price $P^{S,\varepsilon}$ can be approximated up to order 1 in $\sqrt{\varepsilon}$ by $\tilde{P}^{S,\varepsilon}$ which is defined as:

$$\tilde{P}^{S,\varepsilon} = \tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon}\tilde{P}_1^{S,\varepsilon}.$$

The proof of this result is detailed in (Appendix 2).

3.3 Calibration of Implied Beta Using Options Prices

In the last section, approximations for the prices of European options on the index and the stock are given respectively in (2) and (8). Using these results, an estimator of the parameter β is provided in this model setting.

3.3.1 Approximation Formula of the Implied Volatility Smile

For the purpose of the beta estimation, a Taylor expansion is carried out in (Appendix 3) in order to provide approximations of the implied volatilities of the stock and the index. The following results are obtained:

Proposition 3.2 *The implied volatility of an European call on the index with strike K_I and maturity T can be approximated, at order 1 in $\sqrt{\varepsilon}$, by $\Sigma_I(K_I, T)$:*

$$\Sigma_I(K_I, T) = b_I + a_I \frac{\log(\frac{F_I}{K_I})}{T}, \tag{22}$$

where $b_I = \bar{\sigma}_I^* - \frac{V_3^{I,\varepsilon}}{2\bar{\sigma}_I^*}$, $a_I = \frac{V_3^{I,\varepsilon}}{(\bar{\sigma}_I^*)^3}$ and $F_I = I_t e^{r(T-t)}$.

Likewise, the implied volatility of an European call on the stock with strike K_S and maturity T can be approximated, at order 1 in $\sqrt{\varepsilon}$, by $\Sigma_S(K_S, T)$:

$$\Sigma_S(K, T) = b_S + a_S \frac{\log(\frac{F_S}{K_S})}{T}, \tag{23}$$

where $b_S = \bar{\sigma}_S^* - \frac{V_3^{S,\varepsilon}}{2\bar{\sigma}_S^*}$, $a_S = \frac{V_3^{S,\varepsilon}}{(\bar{\sigma}_S^*)^3}$ and $F_S = S_t e^{r(T-t)}$.

3.3.2 Comparison with the Model with Constant Idiosyncratic Volatility

The approximations of the smiles of the stock and the index, given in (22) and (23), are used here to estimate the parameter β. Indeed, based on the definitions of $V_3^{S,\varepsilon}$ and $V_3^{I,\varepsilon}$, it can be written that:

$$V_3^{S,\varepsilon} = \beta^3 V_3^{I,\varepsilon} + \frac{\sqrt{\varepsilon}}{\sqrt{2}} \rho_Z \nu_Z \sqrt{\alpha} \left\langle \phi'_{Idios} f_2 \right\rangle_{1,2},$$
$$\frac{V_3^{S,\varepsilon}}{V_3^{I,\varepsilon}} = \beta^3 + \frac{\rho_Z \nu_Z \sqrt{\alpha}}{\rho_Y \nu_Y} \frac{\left\langle \phi'_{Idio} f_2 \right\rangle_{1,2}}{\left\langle \phi'_I f_1 \right\rangle_{1,2}}.$$

The estimator $\hat{\beta}$ proposed in [9] and introduced in (1) verifies:

$$\hat{\beta}^3 = \frac{V_3^{S,\varepsilon}}{V_3^{I,\varepsilon}} = \beta^3 + \frac{\rho_Z \nu_Z \sqrt{\alpha}}{\rho_Y \nu_Y} \frac{\langle \phi'_{Idio} f_2 \rangle_{1,2}}{\langle \phi'_I f_1 \rangle_{1,2}}.$$

it can be deduced that, in the case of stochastic idiosyncratic volatility, the quantity $\hat{\beta}$ is a biased estimator of the parameter β. Thus, it would be useful to provide an unbiased estimator of β in the new model setting.

3.3.3 Alternative Method for the Estimation

It can be recalled that:

$$\bar{\sigma}_S^2 = \beta^2 \bar{\sigma}_I^2 + \left\langle f_2^2 \right\rangle_{1,2}.$$

Using the relations between $\bar{\sigma}_I^2$ and $(\bar{\sigma}_I^*)^2$ as well as between $\bar{\sigma}_S^2$ and $(\bar{\sigma}_S^*)^2$, the following result is deduced:

$$\beta^2 = \frac{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}}{(\bar{\sigma}_I^*)^2 + 2V_2^{I,\varepsilon}} + \frac{2V_2^{S,\varepsilon}}{(\bar{\sigma}_I^*)^2 + 2V_2^{I,\varepsilon}}.$$

Based on the smile approximation formula, given in (15), for an asset A denoting either the stock S or the index I, it can be stated:

$$b_A = \bar{\sigma}_A^* - \frac{1}{2} a_A (\bar{\sigma}_A^*)^2.$$

The latter second order equation in $\bar{\sigma}_A^*$ has two admissible solutions:

$$x_1 = \frac{1 - \sqrt{1 - 2a_A b_A}}{a_A},$$
$$x_2 = \frac{1 + \sqrt{1 - 2a_A b_A}}{a_A}.$$

Since $V_2^{A,\varepsilon}$ and $V_3^{A,\varepsilon}$ are of order 1 in $\sqrt{\varepsilon}$, then a_A is of order 1 in $\sqrt{\varepsilon}$ and $b_A = \bar{\sigma}_A^* + o\left(\sqrt{\varepsilon}\right)$. Thus, it can be deduced that the appropriate solution of the second-order equation is x_1, and that the quantities $\bar{\sigma}_S^*$ and $\bar{\sigma}_I^*$ can be written as below:

$$\bar{\sigma}_S^* = \frac{1 - \sqrt{1 - 2a_S b_S}}{a_S}, \quad \bar{\sigma}_I^* = \frac{1 - \sqrt{1 - 2a_I b_I}}{a_I}.$$

Using previous results, the parameter β can be approximated using $\tilde{\beta}$:

$$\tilde{\beta} = \sqrt{\frac{(\frac{1-\sqrt{1-2a_S b_S}}{a_S})^2 - \langle f_2^2 \rangle_{1,2}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2 + 2V_2^{I,\varepsilon}} + \frac{2V_2^{S,\varepsilon}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2 + 2V_2^{I,\varepsilon}}}. \tag{24}$$

Through the use of a Taylor expansion, it can be shown that $\tilde{\beta}$ writes:

$$\tilde{\beta} = \sqrt{\frac{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}}{(\bar{\sigma}_I^*)^2}} \left(1 + \frac{V_2^{S,\varepsilon}}{(\bar{\sigma}_S^*)^2 - \langle f_2^2 \rangle_{1,2}} - \frac{V_2^{I,\varepsilon}}{(\bar{\sigma}_I^*)^2} \right) + o(\varepsilon)\,.$$

In order to compute the value of $\tilde{\beta}$, the quantities $\langle f_2^2 \rangle_{1,2}$ and $V_2^{S,\varepsilon}$ can be estimated statistically using historical data from both option and underlying prices. The estimation procedure will be detailed in the next section.

3.3.4 Numerical Simulations

The accuracy of the estimators $\tilde{\beta}$ and $\hat{\beta}$ are tested in this subsection on simulated data. Indeed, Monte Carlo simulations of the CAPM model with stochastic idiosyncratic volatility are performed using the following parameters:

$$\begin{cases} \varepsilon = 0.1, & \alpha = 1, \\ \rho_Y = -0.8, & \rho_Z = -0.5, \\ \nu_Y = 0.15, & \beta = 1.5, \\ r = 0, & \mu_S = \mu_I = 0, \\ \gamma_1 = 0, & \gamma_2 = 0, \end{cases}$$

The choice of the parameters $r = \mu_S = \mu_I = 0$ and $\gamma_1 = \gamma_2 = 0$ enables to have $V_2^{S,\varepsilon} = V_2^{I,\varepsilon} = 0$. Consequently, the parameter $\tilde{\beta}$ in (24) becomes:

$$\tilde{\beta} = \sqrt{\frac{(\frac{1-\sqrt{1-2a_S b_S}}{a_S})^2 - \langle f_2^2 \rangle_{1,2}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2}}.$$

Several experiments are carried out here according to the value of the parameter ν_Z. For each value of $\nu_Z \in \{0.05, 0.15, 0.25, 0.4\}$, we launch 100 Monte Carlo pricing algorithms whose random number generators have different seeds. In each of the 100 pricing algorithms, we generate 20,000 paths of the Brownian motions $\left(W_t^{*,(1)}, W_t^{*,(2)}, W_t^{*,(3)}, W_t^{*,(4)}\right)_{\{0 \le t \le T\}}$ as well as their antithetic. We deduce afterward the paths of the processes $(I_t, S_t, y_t, z_t)_{\{0 \le t \le T\}}$ and we price, using the Monte Carlo method, options on the stock and the index with different strikes and with

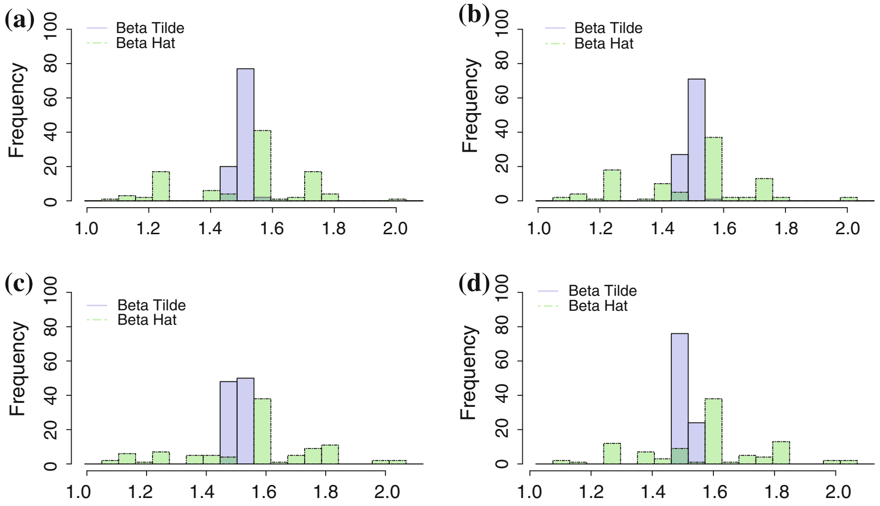

Fig. 2 Histogram of the estimators $\tilde{\beta}$ and $\hat{\beta}$ for $\beta = 1.5$. **a** $v_Z = 0.05$. **b** $v_Z = 0.15$. **c** $v_Z = 0.25$. **d** $v_Z = 0.4$

maturity $T = 0.5$. Finally, we compute the estimators $\tilde{\beta}$ and $\hat{\beta}$ using (1) and (24) respectively, and we obtain the following histograms:

The histograms show that the estimator $\tilde{\beta}$ insures a faster convergence towards the parameter β. Indeed, by using only 40,000 antithetic paths in each Monte Carlo pricing algorithm, the bias of the estimator $\tilde{\beta}$ is lower than the one of $\hat{\beta}$ (Figs. 2, 3).

4 Applications for the Estimation of the Parameter β

In this section, we highlight the importance of the estimation of the parameter β under the risk-neutral measure $\mathbb{P}^*$. We present two main applications which emphasize the utility of this estimation for econometric studies or also for derivatives hedging. In the first subsection, we investigate whether the implied beta $\tilde{\beta}$ can predict the future value of the parameter β under the real-world probability measure $\mathbb{P}$. This question is legitimate to be posed since option-implied information under the risk-neutral pricing measure $\mathbb{P}^*$ may help to predict information under the physical measure $\mathbb{P}$. In the second subsection, we show that the estimator $\tilde{\beta}$ may be crucial for hedging purposes. Indeed, we point out that the quantity $\tilde{\beta}$ can be used in order to hedge the volatility or the delta risk of stock options using instruments on the index. Thus, we conclude that having a good estimation of the value of the parameter β under the risk-neutral measure $\mathbb{P}^*$ can be very useful from a risk-management perspective (Figs. 4, 5).

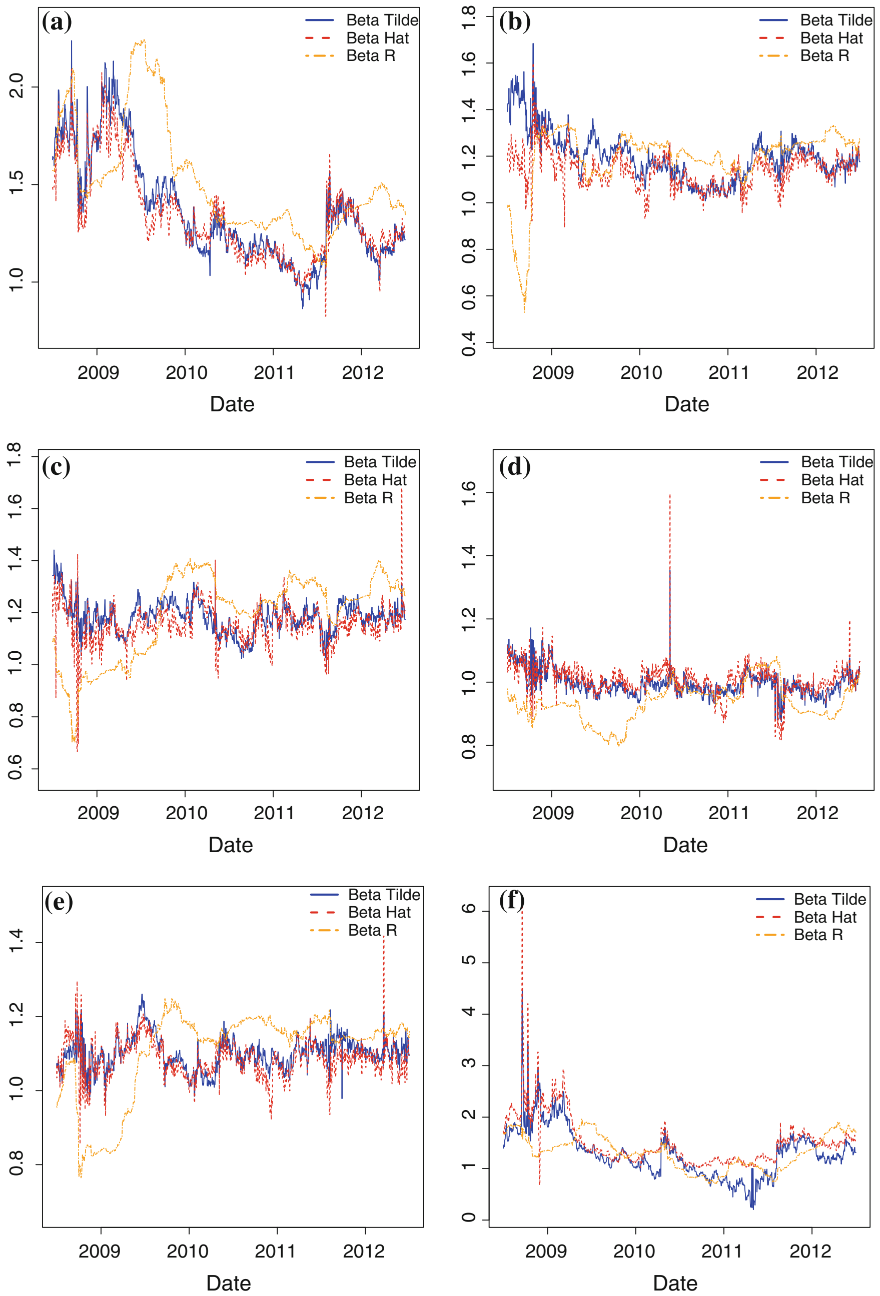

Fig. 3 Comparison between $\tilde{\beta}$ (*blue* line) and $\hat{\beta}$ (*red* line). **a** XLF vs SPX. **b** XLE vs SPX. **c** XLB vs SPX. **d** XLK vs SPX. **e** XLI vs SPX. **f** GS vs SPX

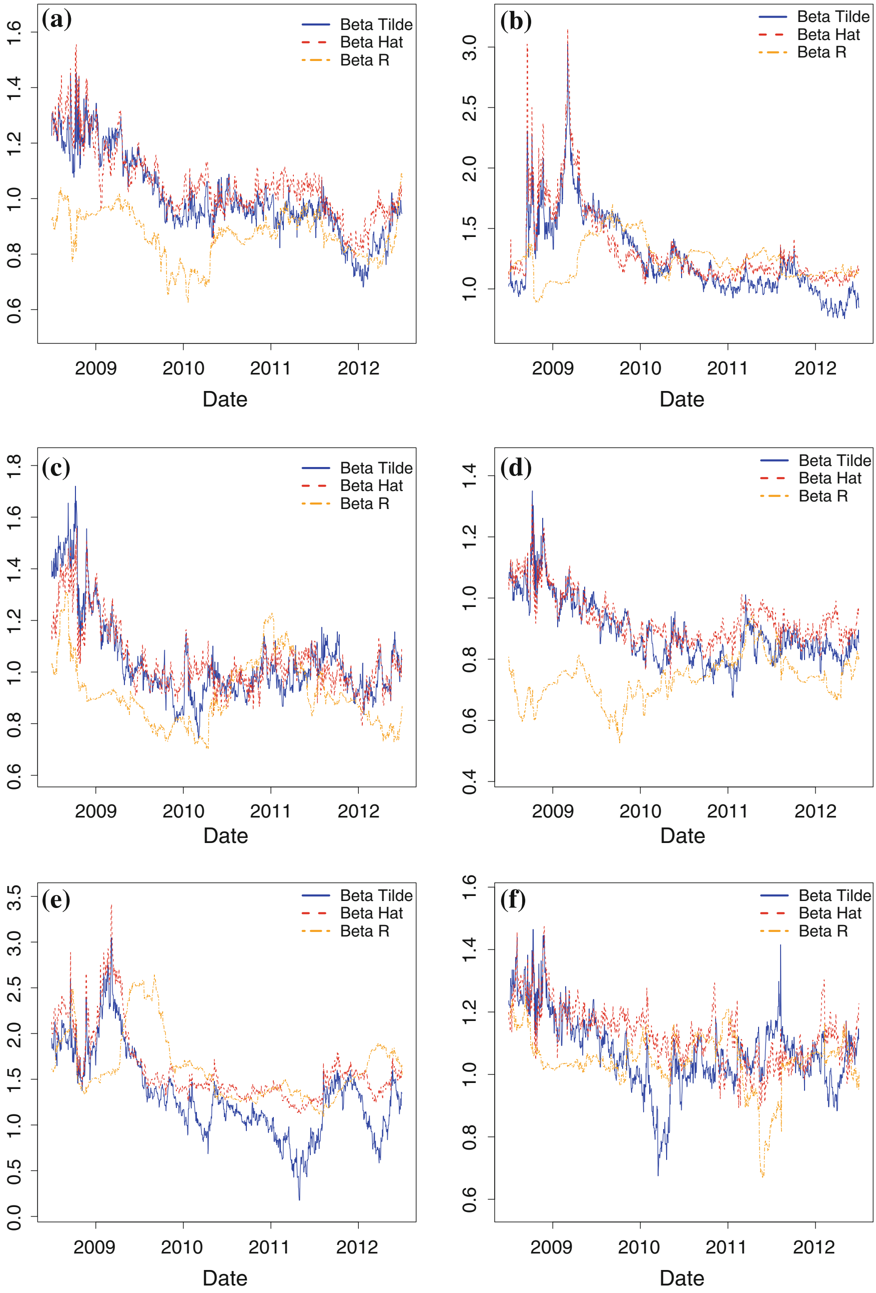

Fig. 4 Comparison between $\tilde{\beta}$ (*blue line*) and $\hat{\beta}$ (*red line*). **a** MSFT versus SPX. **b** GE versus SPX. **c** GOOG versus SPX. **d** IBM versus SPX. **e** JPM versus SPX. **f** CSCO versus SPX

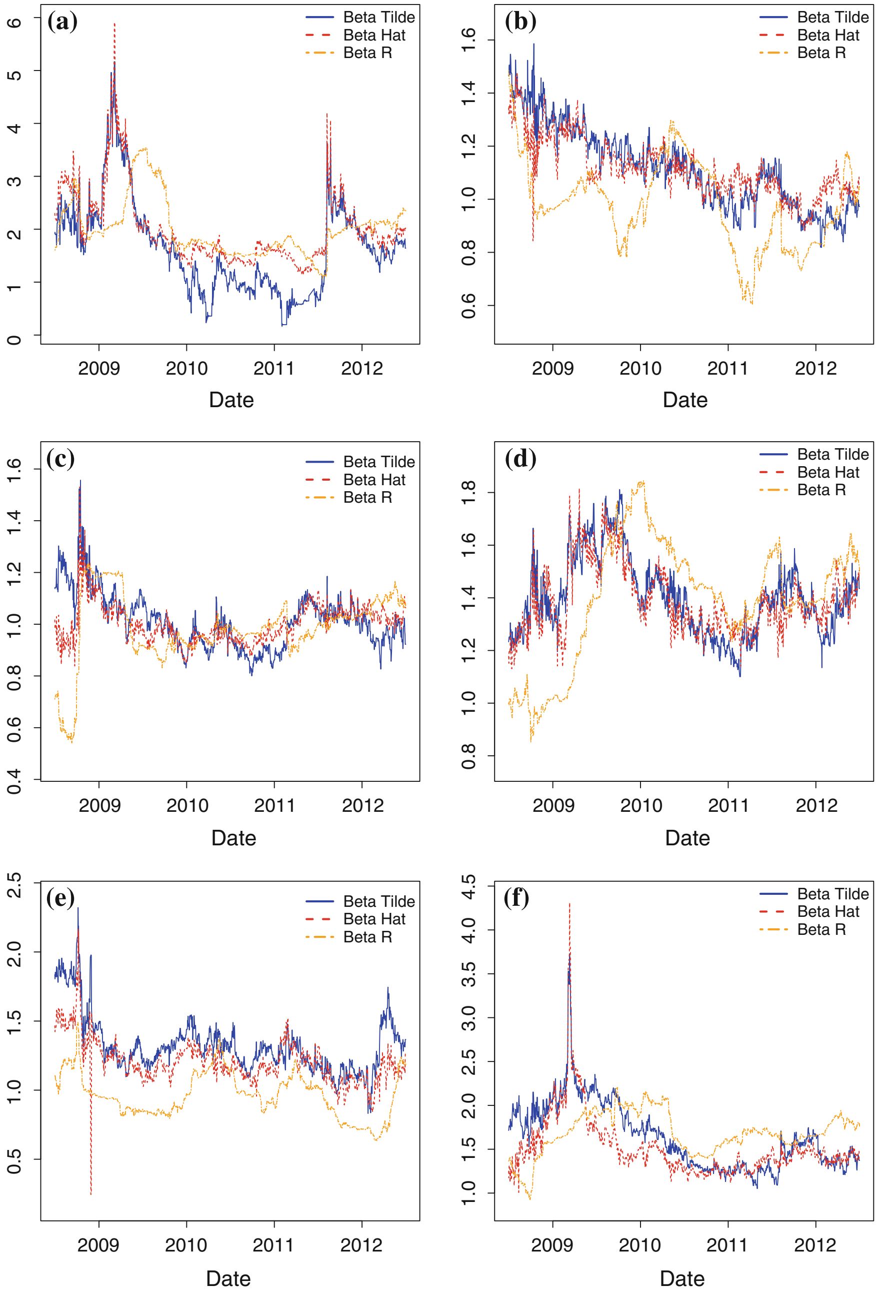

Fig. 5 Comparison between $\tilde{\beta}$ (*blue line*) and $\hat{\beta}$ (*red line*). **a** BAC versus SPX. **b** INTC versus SPX. **c** CVX versus SPX. **d** CAT versus SPX. **e** AAPL versus SPX. **f** AA versus SPX

4.1 Prediction of Forward Beta

We use spot and option data of several US stocks and ETF between 01/01/2008 and 31/12/2012. The option prices have a maturity equal to $T = 0.5$ and a moneyness ranging between 80 to 120 %. The data sample used in this empirical study includes the following instruments: Financial Select Sector (XLF), Energy Select Sector (XLE), Materials Select Sector (XLB), Technology Select Sector (XLK), Industrial Select Sector (XLI), Goldman Sachs (GS), Microsoft (MSFT), General Electric (GE), Google (GOOG), IBM (IBM), JP Morgan (JPM), Cisco Systems (CSCO), Bank Of America (BAC), Intel Corporation (INTC), Chevron Corporation (CVX), Caterpillar Inc(CAT), Apple Inc (AAPL), Alcoa Inc (AA).

For every date t of the sample, the estimators $\hat{\beta}(t)$ and $\tilde{\beta}(t)$ are computed according to (1) and (24) respectively. In order to compute $\tilde{\beta}$, the methodology given below is followed:

1. The quantities $\bar{\sigma}_I$ and $\bar{\sigma}_S$ are approximated by the historical volatilities of the market index and the stock respectively using underlying log-returns.
2. The implied volatilities $\Sigma_I(K_I, T)$ and $\Sigma_S(K_S, T)$ are regressed on the variables $\frac{\log(\frac{F_I(T)}{K_I})}{T-t}$ and $\frac{\log(\frac{F_S(T)}{K_S})}{T-t}$ respectively, thus the slopes a_S, a_I and the intercepts b_S, b_I are deduced.
3. From the estimated slope a_S, the intercept b_S and the effective volatility $\bar{\sigma}_S$, the following quantities can be calculated:

   - $\bar{\sigma}_S^* = \frac{1-\sqrt{1-2a_S b_S}}{a_S}$
   - $V_2^{S,\varepsilon} = \frac{\bar{\sigma}_S^2-(\bar{\sigma}_S^*)^2}{2}$

4. Likewise, from the estimated slope a_I, the intercept b_I and the effective volatility $\bar{\sigma}_I$, the following quantities are computed:

   - $\bar{\sigma}_I^* = \frac{1-\sqrt{1-2a_I b_I}}{a_I}$
   - $V_2^{I,\varepsilon} = \frac{\bar{\sigma}_I^2-(\bar{\sigma}_I^*)^2}{2}$

5. The time series of the idiosyncratic volatility $(f(Z_t))_t$ is obtained through the regression of the log-returns of the stock (S) on those of the index (I). Using these data, the parameters m_Z, ν_Z and ρ_Z are calibrated using the maximum likelihood method as suggested in [16]. The quantity $\langle f_2^2 \rangle_{1,2}$ is then evaluated. For example, if f_2 denotes the exponential function, then:

$$\left\langle f_2^2 \right\rangle_{1,2} = e^{2m_Z+2\nu_Z^2}$$

6. The estimator $\tilde{\beta}$ can finally be computed using (24).

Let the quantity β_H denote the historical measure of the parameter beta. Thus, at a given date t, $\beta_H(t)$ is defined as the slope of the linear regression of the stock log-returns on the index log-returns between $t - T$ and t. The estimator β_H is computed

Table 1 Bias of the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H

Stock	$E(\hat{\beta} - \beta_F)$	$E(\tilde{\beta} - \beta_F)$	$E(\beta_H - \beta_F)$
XLF.US	–0.148	–0.121	0.054
XLE.US	–0.05	0.0094	–0.0313
XLB.US	–0.0231	0.0002	–0.0345
XLK.US	0.0679	0.0542	–0.0158
XLI.US	–0.0258	–0.0183	–0.0215
AAPL.US	0.2649	0.4073	0.0005
AA.US	–0.1814	–0.0738	–0.0333
BAC.US	–0.1414	–0.3557	0.0222
CAT.US	–0.0453	–0.0574	–0.0512
CSCO.US	0.0854	0.0216	–0.0073
CVX.US	0.0252	0.0366	–0.0187
GE.US	0.0375	–0.0534	0.0016
GOOG.US	0.1693	0.1742	0.0193
GS.US	0.1306	–0.0962	0.0113
IBM.US	0.2063	0.1575	–0.0233
INTC.US	0.1462	0.1451	–0.0069
JPM.US	–0.0415	–0.347	0.0383
MSFT.US	0.1717	0.1102	–0.0234

on a backward window of length T, and so can be compared to the estimators $\hat{\beta}(t)$ and $\tilde{\beta}(t)$ which are estimated using option prices with maturity T.

The graphs below represent the time series $(\hat{\beta}(t))_{t_0 \leq t \leq t_N}$, $(\tilde{\beta}(t))_{t_0 \leq t \leq t_N}$ and $(\beta_H(t))_{t_0 \leq t \leq t_N}$ for $t_0 = 01/01/2008$ and $t_N = 31/12/2012$.

Let the parameter β_F denote the forward realized beta, which is obtained as the slope of the linear regression of stock returns on index returns performed on a forward window with length T. In other words, $\beta_F(t) = \beta_H(t + T)$. It is then interesting to see how well the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H predict the quantity β_F. The predictive power of these estimators is tested using several statistical measures, the bias and the root-mean-square error are retained here (Tables 1, 2):

It can be seen through the tables above that the historical beta estimator β_H is unbiased for all the considered stocks and has in average the lowest RMSE compared to the estimators $\hat{\beta}$ and $\tilde{\beta}$. It can also be noticed that the estimator $\tilde{\beta}$ has a significant positive bias for the stocks AAPL, GOOG, IBM, INTC, MSFT and a negative bias for XLF and BAC. Thus, the RMSE of $\tilde{\beta}$ for these stocks is higher than the one of β_H. It can be deduced that option market participants may have an expectation of the parameter β which is temporarily different from its value under the physical probability measure $\mathbb{P}$, this finding proves the existence of discrepancies between the risk-neutral measure and the physical measure.

Table 2 RMSE of the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H

Stock	$\sqrt{E((\frac{\hat{\beta}-\beta_F}{\beta_F})^2)}$	$\sqrt{E((\frac{\tilde{\beta}-\beta_F}{\beta_F})^2)}$	$\sqrt{E((\frac{\beta_H-\beta_F}{\beta_F})^2)}$
XLF.US	0.143	0.138	0.195
XLE.US	0.183	0.207	0.128
XLB.US	0.196	0.209	0.116
XLK.US	0.132	0.124	0.095
XLI.US	0.128	0.112	0.108
AAPL.US	0.41	0.547	0.252
AA.US	0.187	0.188	0.181
BAC.US	0.234	0.407	0.337
CAT.US	0.118	0.123	0.149
CSCO.US	0.159	0.162	0.147
CVX.US	0.198	0.238	0.165
GE.US	0.154	0.189	0.158
GOOG.US	0.277	0.31	0.165
GS.US	0.341	0.292	0.303
IBM.US	0.369	0.348	0.134
INTC.US	0.279	0.294	0.244
JPM.US	0.158	0.292	0.265
MSFT.US	0.297	0.27	0.163

4.2 Hedging of Options on the Stock by Instruments on the Index

The capital asset pricing model offers a practical framework where the stock dynamics are linked to those of the market index. This section aims to show that the parameter beta may be very useful to hedge the delta risk or volatility risk of a stock option using instruments on the index.

4.2.1 Hedging Volatility Risk

The natural way to hedge the volatility risk of a stock option is to use other stock options with a longer maturity. Nevertheless, if options on this stock are not sufficiently liquid, it won't be possible to carry out this hedging strategy. In this case, it may be judicious to use options on the index in order to perform the hedging.

Let X be the value of the portfolio containing an option on the stock, a quantity $-\vartheta$ of an option on the index, a quantity $-\Delta_S$ of the stock and a quantity Δ_I of the index. The value of this portfolio at time t is $X_t = P_t^{S,\varepsilon} - \vartheta_t P_t^{I,\varepsilon} + \Delta_{I,t} I_t - \Delta_{S,t} S_t$ and X_t satisfies the following SDE:

$$dX_t = \left(\mathscr{L}^S P_t^{S,\varepsilon} + r P_t^{S,\varepsilon}\right) dt + \frac{\partial P_t^{S,\varepsilon}}{\partial S} (dS_t - rS_t dt) + \frac{\partial P_t^{S,\varepsilon}}{\partial y} \frac{\nu_Y \sqrt{2}}{\sqrt{\varepsilon}} dW_t^{*,(3)}$$
$$+ \frac{\partial P_t^{S,\varepsilon}}{\partial z} \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\varepsilon}} dW_t^{*,(4)} - \vartheta_t \left(\mathscr{L}^I P_t^{I,\varepsilon} + r P_t^{I,\varepsilon}\right) dt - \vartheta_t \frac{\partial P_t^{I,\varepsilon}}{\partial I} (dI_t - rI_t dt)$$
$$- \vartheta_t \frac{\partial P_t^{I,\varepsilon}}{\partial y} \frac{\nu_Y \sqrt{2}}{\sqrt{\varepsilon}} dW_t^{*,(3)} - \Delta_{S,t} dS_t + \Delta_{I,t} dI_t.$$

By rearranging the terms, the last equation becomes:

$$dX_t = r\left(P_t^{S,\varepsilon} - \vartheta_t P_t^{I,\varepsilon} + \vartheta_t \frac{\partial P_t^{I,\varepsilon}}{\partial I} I_t - \frac{\partial P_t^{S,\varepsilon}}{\partial S} S_t\right) dt + \left(\frac{\partial P_t^{S,\varepsilon}}{\partial S} - \Delta_{t,S}\right) dS_t$$
$$+ \left(\Delta_{I,t} - \vartheta_t \frac{\partial P_t^{I,\varepsilon}}{\partial I}\right) dI_t + \left(\frac{\partial P_t^{S,\varepsilon}}{\partial y} - \vartheta_t \frac{\partial P_t^{I,\varepsilon}}{\partial y}\right) \frac{\nu_Y \sqrt{2}}{\sqrt{\varepsilon}} dW_t^{*,(3)}$$
$$+ \frac{\partial P_t^{S,\varepsilon}}{\partial z} \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\varepsilon}} dW_t^{*,(4)},$$

If the parameters $(\vartheta_t, \Delta_{S,t}, \Delta_{I,t})$ are chosen in the following way:

$$\vartheta_t = \frac{\frac{\partial P^{S,\varepsilon}}{\partial y}}{\frac{\partial P^{I,\varepsilon}}{\partial y}}, \quad \Delta_{S,t} = \frac{\partial P^{S,\varepsilon}}{\partial S_t}, \quad \Delta_{I,t} = \vartheta_t \frac{\partial P^{I,\varepsilon}}{\partial I_t},$$

then the portfolio X is delta-hedged continuously in the stock and the index, and is made insensitive to the variations of the process y. It follows that:

$$dX_t = r\left(P_t^{S,\varepsilon} - \vartheta_t P_t^{I,\varepsilon} + \Delta_{I,t} I_t - \Delta_{S,t} S_t\right) dt + \frac{\partial P_t^{S,\varepsilon}}{\partial z} \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\varepsilon}} dW_t^{*,(4)},$$
$$= rX_t dt + \frac{\partial P_t^{S,\varepsilon}}{\partial z} \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\varepsilon}} dW_t^{*,(4)},$$

which means that the risk of the volatility of the index is canceled, and the only remaining risk comes from the idiosyncratic volatility.

The hedging parameter ϑ_t can be obtained through the computation of the terms $\frac{\partial P^{S,\varepsilon}}{\partial y}$ and $\frac{\partial P^{I,\varepsilon}}{\partial y}$ using Monte-Carlo simulations. This computation method is time-consuming, so it may be interesting to have a closed-form approximation for this hedging ratio. For this purpose, the quantities $\frac{\partial P^{S,\varepsilon}}{\partial y}$ and $\frac{\partial P^{I,\varepsilon}}{\partial y}$ should be approximated at order 1 in ε as explained below:

$$P(t, I_t, Y_t) = P_0^{I,\varepsilon}(t, I_t) + \sqrt{\varepsilon} P_1^{I,\varepsilon}(t, I_t) + \varepsilon P_2^{I,\varepsilon}(t, I_t, Y_t) + o(\varepsilon)$$
$$P(t, S_t, Y_t, Z_t) = P_0^{S,\varepsilon}(t, S_t) + \sqrt{\varepsilon} P_1^{S,\varepsilon}(t, S_t) + \varepsilon P_2^{S,\varepsilon}(t, S_t, Y_t, Z_t) + o(\varepsilon)$$

where:

$$P_2^{I,\varepsilon}(t, I_t, Y_t) = -\frac{1}{2} I_t^2 \frac{\partial^2 P_0^{I,\varepsilon}}{\partial I_t^2} \phi_I(Y_t),$$
$$P_2^{S,\varepsilon}(t, S_t, Y_t, Z_t) = -\frac{1}{2} S_t^2 \frac{\partial^2 P_0^{S,\varepsilon}}{\partial S_t^2} \left(\beta^2 \phi_I(Y_t) + \phi_{Idios}(Z_t)\right).$$

Consequently, the quantity ϑ_t writes:

$$\vartheta_t = \beta^2 \frac{S_t^2 \frac{\partial^2 P_0^{S,\varepsilon}}{\partial S_t^2}}{I_t^2 \frac{\partial^2 P_0^{I,\varepsilon}}{\partial I_t^2}} + o(\varepsilon).$$

Since $P_0^{S,\varepsilon}$ and $P_0^{I,\varepsilon}$ have known closed-form expressions, the quantity ϑ_t can be approximated analytically by $\hat{\vartheta}_t = \beta^2 \frac{S_t^2 \frac{\partial^2 P_0^{S,\varepsilon}}{\partial S_t^2}}{I_t^2 \frac{\partial^2 P_0^{I,\varepsilon}}{\partial I_t^2}}$, the approximation error is at order 1 in ε.

4.2.2 Delta and Vega Hedging

The lack of liquidity or the presence of transaction costs on the stock can make the delta-hedging of the stock option costly. Therefore, it may be useful to have an alternative hedging strategy in circumstances where there are trading constraints on the stock. The setting of the CAPM model enables the decomposition of the stock risk into two parts: the index risk and the idiosyncratic risk. It is then reasonable in this framework to hedge the delta of a stock option using the index.

Let L be the portfolio containing an option on the stock, a quantity $-\vartheta$ of an option on the index and a quantity $-\varphi_I$ of the index. We have then $L_t = P_t^{S,\varepsilon} - \vartheta_t P^{I,\varepsilon} - \varphi_{I,t} I_t$. It follows that:

$$\begin{aligned}
dL_t = {} & r\left(P_t^{S,\varepsilon} - \vartheta_t P_t^{I,\varepsilon} - \varphi_{I,t} I_t\right) dt \\
& + \left(\frac{\partial P_t^{S,\varepsilon}}{\partial S} \beta S_t f_1(y_t) - \vartheta_t \frac{\partial P_t^{I,\varepsilon}}{\partial I} I_t f_1(y_t) - \varphi_{I,t} I_t f_1(y_t)\right) dW_t^{*,(1)} \\
& + \left(\frac{\partial P_t^{S,\varepsilon}}{\partial y} - \vartheta_I \frac{\partial P_t^{I,\varepsilon}}{\partial y}\right) \frac{\nu_Y \sqrt{2}}{\sqrt{\varepsilon}} dW_t^{*,(3)} + \frac{\partial P_t^{S,\varepsilon}}{\partial S} S_t f_2(z_t) dW_t^{*,(2)} \\
& + \frac{\partial P_t^{S,\varepsilon}}{\partial z} \frac{\nu_Z \sqrt{2}}{\sqrt{\varepsilon}} dW_t^{*,(4)}.
\end{aligned}$$

By taking $(\vartheta_t, \varphi_{I,t})$ such that:

$$\vartheta_t = \frac{\frac{\partial P^{S,\varepsilon}}{\partial y}}{\frac{\partial P^{I,\varepsilon}}{\partial y}}, \quad \varphi_{I,t} = \frac{\beta S_t}{I_t}\frac{\partial P^{S,\varepsilon}}{\partial S} - \vartheta_t \frac{\partial P^{I,\varepsilon}}{\partial I},$$

the risks related to the index and its volatility are hedged. It follows that:

$$dL_t = rL_t dt + \frac{\partial P^{S,\varepsilon}}{\partial S} S_t f_2(z_t) dW_t^{*,(2)} + \frac{\partial P^{S,\varepsilon}}{\partial z}\frac{\nu_Z\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{*,(4)}.$$

Here again, the computation of the quantities ϑ_t and $\varphi_{I,t}$ can be made easier through the estimation of the parameter β. Indeed, $\varphi_{I,t}$ and $\varphi_{I,t}$ can be approximated respectively by $\hat{\vartheta}_t$ and $\hat{\varphi}_{I,t} = \frac{\beta S_t}{I_t}\frac{\partial P^{S,\varepsilon}}{\partial S} - \hat{\vartheta}_t \frac{\partial P^{I,\varepsilon}}{\partial I}$.

4.2.3 Numerical Simulations

We perform in this subsection numerical simulations in order to test the hedging strategies proposed so far. First, we generate 200 paths of the processes $(I_t), (S_t), (y_t), (z_t)$ for $t \in [0, T]$ with a time step equal to $\delta = \frac{1}{256}$. On each of these simulated paths indexed by $i \in \{1, \dots, 200\}$, and for $k \in [0, \frac{T}{\delta}]$, we generate 50,000 paths of $(I_t^{(i)}, S_t^{(i)}, y_t^{(i)}, z_t^{(i)})_{\{t\in[k\delta,T]\}}$ with a time step equal to δ and we compute the quantities $\Delta_{S,k\delta}^{(i)}, \Delta_{I,k\delta}^{(i)}, \left(P_{k\delta}^{S,\varepsilon}\right)^{(i)}, \left(P_{k\delta}^{I,\varepsilon}\right)^{(i)}$ using a Monte Carlo method. The quantities $\vartheta_{k\delta}$ and $\varphi_{I,k\delta}$ are approximated using $\hat{\vartheta}_{k\delta}$ and $\hat{\varphi}_{I,k\delta}$ respectively. We can then obtain $X_{k\delta}$ denoting the portfolio value at time $k\delta$, and we deduce the final value X_T at the maturity date T. It should be precised here that $\tilde{\beta}$ is used instead of β for the computation of $\hat{\vartheta}$ and $\hat{\varphi}_I$, so that the error of estimation is accounted in the hedging error (Figs. 6, 7 and Tables 3, 4).

For these numerical simulations, the following parameters are used:

$$\begin{cases} \varepsilon = 0.1, & \alpha = 1, \\ \rho_Y = -0.8, & \rho_Z = -0.5, \\ \nu_Y = 0.15, & \nu_Z = 0.05, \\ r = 0, & \mu_I = 0, \\ \beta = 1.5, & \mu_S = 0, \\ \gamma_1 = 0, & \gamma_2 = 0, \\ y_0 = \log(0.4), & z_0 = \log(0.2), \\ I_0 = 1000, & S_0 = 50, \end{cases}$$

It can be precised here, that since the hedging strategies are tested on 200 independent paths, the simulations are done in parallel using the computing cluster of Ecole

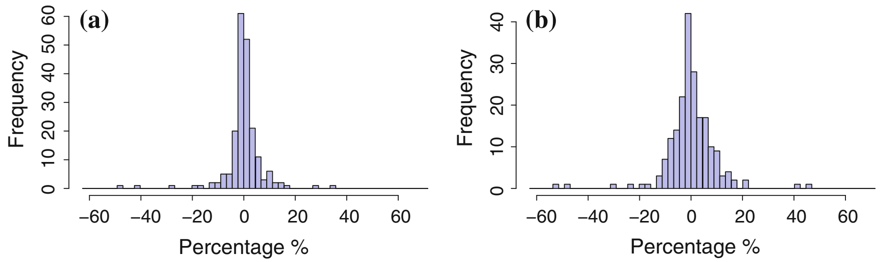

Fig. 6 Histograms of hedging errors for $T = 1$ Month. **a** Histogram of $\frac{X_T - X_0}{X_0} \times 100$. **b** Histogram of $\frac{L_T - L_0}{L_0} \times 100$

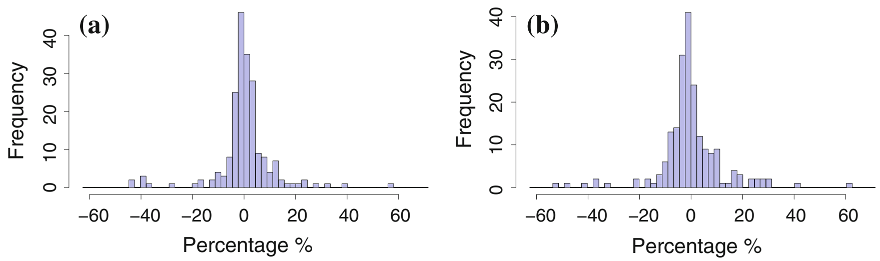

Fig. 7 Histograms of hedging errors for $T = 3$ Months. **a** Histogram of $\frac{X_T - X_0}{X_0} \times 100$. **b** Histogram of $\frac{L_T - L_0}{L_0} \times 100$

Table 3 Statistics of hedging errors for $T = 1$ Month

Statistics of hedging 1M stock options	Median	Mean	Std.	Skewness	Kurtosis
$100 \times \frac{X_T - X_0}{X_0}$	–0.04	0.00	7.51	–1.61	15.87
$100 \times \frac{L_T - L_0}{L_0}$	–0.43	0.07	9.64	–0.63	10.26

Table 4 Statistics of hedging errors for $T = 3$ Months

Statistics of hedging 3M stock options	Median	Mean	Std.	Skewness	Kurtosis
$100 \times \frac{X_T - X_0}{X_0}$	0.07	0.29	10.11	–0.30	8.08
$100 \times \frac{L_T - L_0}{L_0}$	–1.34	–0.15	12.36	0.07	6.33

Centrale Paris. The histograms of the hedging errors $\left(\frac{X_T - X_0}{X_0} \times 100\right)$ in the case of volatility risk hedging and $\left(\frac{L_T - L_0}{L_0} \times 100\right)$ in the case of Delta and volatility risk hedging, are given below:

The same procedure is repeated here in order to perform the numerical simulations for $T = 3$ Months.

5 Conclusion

We presented in this paper a continuous time capital asset pricing model where the index and the stock have both stochastic volatilities. Through the use of singular perturbation technique, we provided approximations of the prices of European options on the index and the stock and we proposed an unbiased estimator $\tilde{\beta}$ for the parameter beta under the risk-neutral pricing measure $\mathbb{P}^*$. We conducted an empirical study and showed that the estimator $\tilde{\beta}$, when compared to the classical estimator β_H, doesn't insure a better prediction of the future realized β under the physical measure $\mathbb{P}$. This proves again that there are some discrepancies between the pricing measure $\mathbb{P}^*$ and the real-world measure $\mathbb{P}$. Besides, we showed that the estimator $\tilde{\beta}$ can be very useful when it comes to hedging stock options using instruments on the index. This approach, considered here as a relative-hedging method, can be very useful when the instruments needed for perfect replication are not liquid.

It would also be quite interesting to extend our study to the use of the implied beta for applications on portfolio construction on option and underlying assets. We keep this subject for a future work.

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Appendices

Appendix 1: Pricing Options on the Index

Let $P_t^{I,\varepsilon} = E^P(h(I_T)|I_t = x, Y_t = y)$ be the price of a european option on the index with payoff $h(I_T)$ and maturity T.

$$P_t^{I,\varepsilon} = P^{I,\varepsilon}(t, I_t, Y_t).$$

Since the process (I, Y) is markovian, applying the Feynman-Kac theorem yields:

$$\mathscr{L}^I P_t^{I,\varepsilon} = 0,$$

where $\mathscr{L}^I$ is a differential operator whose elements can be classified by powers of $\sqrt{\varepsilon}$:

$$\mathscr{L}^I = \mathscr{L}_2^I + \frac{1}{\sqrt{\varepsilon}}\mathscr{L}_1^I + \frac{1}{\varepsilon}\mathscr{L}_0^I,$$

$$\mathscr{L}_0^I = \frac{\partial}{\partial y}(m_Y - y) + \nu_Y^2 \frac{\partial^2}{\partial y^2},$$

$$\mathcal{L}_1^I = -\nu_Y \sqrt{2}\chi_1(y)\frac{\partial}{\partial y} + \sqrt{2}\rho_Y \nu_Y I_t f_1(y)\frac{\partial^2}{\partial I \partial y},$$
$$\mathcal{L}_2^I = \frac{\partial}{\partial t} + r(\frac{\partial}{\partial I_t} I_t - .) + \frac{1}{2}\frac{\partial^2}{\partial I_t^2} I_t^2 f_1(y)^2.$$

The differential operator $\mathcal{L}_0^I$ represents the infinitesimal generator of the Ornstein-Uhlenbeck process $(Y_{1,t})_t$ which has the following dynamics:

$$dY_{1,t} = (m_Y - Y_{1,t})dt + \nu_Y \sqrt{2} dW_t.$$

The price $P^{I,\varepsilon}$ can be expanded in powers of $\sqrt{\varepsilon}$:

$$P^{I,\varepsilon} = \sum_{i=0}^{\infty} (\sqrt{\varepsilon})^i P_i^{I,\varepsilon}.$$

Next to that, the term $\mathcal{L}^I P_t^{I,\varepsilon} = 0$ can be expanded and its elements can be classified by powers of $\sqrt{\varepsilon}$. The terms of orders -2, -1, 0 and 1 in $\sqrt{\varepsilon}$ are written below:

$$\begin{aligned}
(-2) &: \mathcal{L}_0^I P_0^{I,\varepsilon} = 0,\\
(-1) &: \mathcal{L}_1^I P_0^{I,\varepsilon} + \mathcal{L}_0^I P_1^{I,\varepsilon} = 0,\\
(0) &: \mathcal{L}_2^I P_0^{I,\varepsilon} + \mathcal{L}_1^I P_1^{I,\varepsilon} + \mathcal{L}_0^I P_2^{I,\varepsilon} = 0,\\
(1) &: \mathcal{L}_2^I P_1^{I,\varepsilon} + \mathcal{L}_1^I P_2^{I,\varepsilon} + \mathcal{L}_0^I P_3^{I,\varepsilon} = 0,
\end{aligned}$$

The term $P_0^{I,\varepsilon}$ is a solution of $\mathcal{L}_0^I P_0^{I,\varepsilon} = 0$ with final condition $P_0^{I,\varepsilon}(T, I_T, y_T) = h(I_T)$ (h is independent of y_T). By solving this equation, it can be found that:

$$P_0(t, I_t, y_t) = C_1(t, I_t) \int_0^y e^{\frac{u^2}{2\mu_Y^2} - \frac{u m_Y}{\mu^2}} du + C_2(t, I_t).$$

If C_1 is not the null function then the solution diverges when $y \to +\infty$. Nevertheless, in the case of a call option, the option price is bounded ($0 \leq P(t, I_t) \leq I_t$). Then the quantity C_1 has to be null, and then P_0 is independent of y.

The term of order (-1) in $\sqrt{\varepsilon}$ yields $\mathcal{L}_1^I P_0^{I,\varepsilon} + \mathcal{L}_0^I P_1^{I,\varepsilon} = 0$ which reduces to $\mathcal{L}_0^I P_1^{I,\varepsilon} = 0$ (since P_0 doesn't depend on y). Using the same reasoning as before, it can proved that $P_1^{I,\varepsilon} = P_1^{I,\varepsilon}(t, I_t)$ which is independent of y. Consequently:

$$\mathcal{L}_0^I P_0^{I,\varepsilon} = \mathcal{L}_1^I P_0^{I,\varepsilon} = 0,$$
$$\mathcal{L}_0^I P_1^{I,\varepsilon} = \mathcal{L}_1^I P_1^{I,\varepsilon} = 0.$$

Using $\mathscr{L}_1^I P_1^{I,\varepsilon} = 0$, the term of order 0 in $\sqrt{\varepsilon}$ becomes:

$$\mathscr{L}_0^I P_2^{I,\varepsilon} + \mathscr{L}_2^I P_0^{I,\varepsilon} = 0.$$

This is a Poisson equation for $P_2^{I,\varepsilon}$. Its solvability condition is:

$$\left\langle \mathscr{L}_0^I P_2^{I,\varepsilon} \right\rangle_1 + \left\langle \mathscr{L}_2^I P_0^{I,\varepsilon} \right\rangle_1 = 0,$$

where the operator $\langle . \rangle_1$ is the average with respect to the invariant distribution $N(m_Y, \nu_Y^2)$ of the Ornstein-Uhlenbeck process $(Y_{1,t})_t$.

Since $\mathscr{L}_0^I$ is the infinitesimal generator of the process (Y_1), $\left\langle \mathscr{L}_0^I P_2^{I,\varepsilon} \right\rangle_1 = 0$. Then, the solvability condition reduces to:

$$\left\langle \mathscr{L}_2^I \right\rangle_1 P_0^{I,\varepsilon} = 0. \tag{25}$$

The operator $\langle \mathscr{L}_2^I \rangle_1$ has the following form:

$$\left\langle \mathscr{L}_2^I \right\rangle_1 = \frac{\partial}{\partial t} + r(\frac{\partial}{\partial I} I - .) + \frac{1}{2} \frac{\partial^2}{\partial I^2} I^2 \left\langle f_1^2 \right\rangle_1 .$$

Consequently $\langle \mathscr{L}_2^I \rangle_1 = \mathscr{L}_{BS}(\bar{\sigma}_I)$ where $\bar{\sigma}_I^2 = \langle f_1^2 \rangle_1$. The term $P_0^{I,\varepsilon}$ is the solution of the following problem:

$$\begin{aligned} \mathscr{L}_{BS}(\bar{\sigma}_I) P_0^{I,\varepsilon} &= 0, \\ P_0^{\varepsilon}(T, I_T) &= h(I_T). \end{aligned}$$

Therefore, $P_0^{I,\varepsilon} = P_{BS}(t, I_t, \bar{\sigma}_I)$ meaning that $P_0^{I,\varepsilon}$ is the Black-Scholes price of the index option with implied volatility equal to $\bar{\sigma}_I$.

As a result, the term $P_2^{I,\varepsilon}$ can be written as $P_2^{I,\varepsilon} = -(\mathscr{L}_0^I)^{-1}(\mathscr{L}_2^I - \langle \mathscr{L}_2^I \rangle_1) P_0^{I,\varepsilon}$. The term of order 1 in $\sqrt{\varepsilon}$ is a poisson equation for $P_3^{I,\varepsilon}$. Its solvability condition is :

$$\left\langle \mathscr{L}_2^I P_1^{I,\varepsilon} \right\rangle_1 = -\left\langle \mathscr{L}_1^I P_2^{I,\varepsilon} \right\rangle_1 , \tag{26}$$

$$\left\langle \mathscr{L}_2^I \right\rangle_1 P_1^{I,\varepsilon} = \left\langle \mathscr{L}_1^I (\mathscr{L}_0^I)^{-1} (\mathscr{L}_2^I - \left\langle \mathscr{L}_2^I \right\rangle_1) \right\rangle_1 P_0^{I,\varepsilon}. \tag{27}$$

Let ϕ_I the solution of the following Poisson equation:

$$\mathscr{L}_0 \phi_I(y) = f_1^2(y) - \left\langle f_1^2 \right\rangle_1 . \tag{28}$$

Since the difference term between the differential operator $\mathscr{L}_2^I$ and its average is:

$$\mathscr{L}_2^I - \left\langle \mathscr{L}_2^I \right\rangle_1 = \frac{1}{2}(f_1^2(y) - \left\langle f_1^2 \right\rangle_1) I_t^2 \frac{\partial^2}{\partial I_t^2},$$

it can be written that:

$$(\mathscr{L}_0^I)^{-1}(\mathscr{L}_2 - \langle \mathscr{L}_2 \rangle_1) = \frac{1}{2}\phi_I(y) I_t^2 \frac{\partial^2}{\partial I_t^2}.$$

By applying the operator $\mathscr{L}_1$ to the last equation, it follows that:

$$\mathscr{L}_1^I (\mathscr{L}_0^I)^{-1}(\mathscr{L}_2 - \langle \mathscr{L}_2 \rangle_1) = (-\nu_Y \sqrt{2} \left\langle \chi_1 \phi_I' \right\rangle_1 + \sqrt{2}\rho_Y \nu_Y \left\langle f_1 \phi_I' \right\rangle_1 I_t \frac{\partial}{\partial I_t}) \frac{1}{2} I_t^2 \frac{\partial^2}{\partial I_t^2}.$$

Let the quantities $V_2^{I,\varepsilon}$ and $V_3^{I,\varepsilon}$ be defined as below:

$$\begin{aligned} V_2^{I,\varepsilon} &= -\frac{\sqrt{\varepsilon}}{\sqrt{2}} \nu_Y \left\langle \phi_I' \chi_1 \right\rangle_1, \\ V_3^{I,\varepsilon} &= \frac{\sqrt{\varepsilon}}{\sqrt{2}} \rho_Y \nu_Y \left\langle \phi_I' f \right\rangle_1. \end{aligned}$$

Using $V_2^{I,\varepsilon}$ and $V_3^{I,\varepsilon}$, the Eq. (27) becomes:

$$\left\langle \mathscr{L}_2^I \right\rangle_1 \sqrt{\varepsilon} P_1^{I,\varepsilon} = V_2^{I,\varepsilon} I_t^2 \frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\varepsilon} I_t \frac{\partial}{\partial I_t}(I_t^2 \frac{\partial^2 P_0}{\partial I_t^2}).$$

Therefore, $P_1^{I,\varepsilon}$ is the solution of the following problem :

$$\left\langle \mathscr{L}_2^I \right\rangle_1 \sqrt{\varepsilon} P_1^{I,\varepsilon} = V_2^{I,\varepsilon} I_t^2 \frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\varepsilon} I_t \frac{\partial}{\partial I_t}(I_t^2 \frac{\partial^2 P_0}{\partial I_t^2}), \tag{29}$$

$$P_1^{I,\varepsilon}(T, I_T) = 0. \tag{30}$$

In order to simplify the notations, the following differential operators are defined:

$$\begin{aligned} \mathscr{D}_{1,I} &= I_t \frac{\partial}{\partial I_t}, \\ \mathscr{D}_{2,I} &= I_t^2 \frac{\partial^2}{\partial I_t^2}. \end{aligned}$$

Using the fact that $\left\langle \mathscr{L}_2^I \right\rangle_1 = \mathscr{L}_{BS}(\bar{\sigma}_I)$ commits with $\mathscr{D}_{1,I}$ and $\mathscr{D}_{2,I}$, and that $\left\langle \mathscr{L}_2^I \right\rangle_1 P_0 = 0$, the solution to the last problem can be given explicitly by:

$$\sqrt{\varepsilon}P_1^{I,\varepsilon} = -(T-t)(V_2^{I,\varepsilon} I_t^2 \frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\varepsilon} I_t \frac{\partial}{\partial I_t}(I_t^2 \frac{\partial^2 P_0}{\partial I_t^2})).$$

By neglecting terms of order higher or equal to 2 in $\sqrt{\varepsilon}$, the option's price can be approximated by $(P_0^{I,\varepsilon} + \sqrt{\varepsilon}P_1^{I,\varepsilon})$. As it was proven by the authors in [9], it is possible to carry out a parameter reduction method and approximate $P^{I,\varepsilon}$ by the following formula:

$$P^{I,\varepsilon} \sim \tilde{P_0}^{I,\varepsilon} + \sqrt{\varepsilon}\tilde{P_1}^{I,\varepsilon},$$

such as:

$$\tilde{P_0}^{I,\varepsilon} = P_{BS}(\bar{\sigma}_I^*), \tag{31}$$

$$(\bar{\sigma}_I^*)^2 = \bar{\sigma}_I^2 - 2V_2^{I,\varepsilon}, \tag{32}$$

$$\sqrt{\varepsilon}\tilde{P_1}^{I,\varepsilon} = -(T-t)V_3^{I,\varepsilon} I_t \frac{\partial}{\partial I_t}(I_t^2 \frac{\partial^2 \tilde{P}_0}{\partial I_t^2}). \tag{33}$$

Appendix 2: Accuracy of the Approximation

It can be seen here that:

$$\mathscr{L}_{BS}(\bar{\sigma}_S^*) = \mathscr{L}_{BS}(\bar{\sigma}_S) - V_2^{S,\varepsilon} S_t^2 \frac{\partial^2}{\partial S_t^2}. \tag{34}$$

Using (34), it can be proved that:

$$\mathscr{L}_{BS}(\bar{\sigma}_S)(P_0^{S,\varepsilon} - \tilde{P}_0^{S,\varepsilon}) = -V_2^{S,\varepsilon} S_t^2 \frac{\partial^2 \tilde{P}_0^{S,\varepsilon}}{\partial S_t^2},$$
$$(P_0^{S,\varepsilon} - \tilde{P}_0^{S,\varepsilon})(T, S_T) = 0.$$

The source term is $O\left(\sqrt{\varepsilon}\right)$ because of $V_2^{S,\varepsilon}$, then the difference term $(P_0^{S,\varepsilon} - \tilde{P}_0^{S,\varepsilon})$ is also $O\left(\sqrt{\varepsilon}\right)$. Consequently, it follows that:

$$|P^{S,\varepsilon} - (\tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon}\tilde{P}_1^{S,\varepsilon})| \leq |P^{S,\varepsilon} - (P_0^{S,\varepsilon} + \sqrt{\varepsilon}P_1^{S,\varepsilon})| + |(P_0^{S,\varepsilon} + \sqrt{\varepsilon}P_1^{S,\varepsilon}) - (\tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon}\tilde{P}_1^{S,\varepsilon})|,$$

The first term $|P^{S,\varepsilon,\delta} - (P_0^{S,\varepsilon} + \sqrt{\varepsilon}P_1^{S,\varepsilon})|$ is already $o(\varepsilon)$. Therefore, the second term should be studied. To simplify the notations, the error term $\mathscr{R}$ is introduced as following:

$$\mathscr{R} = (P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon}) - (\tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon}).$$

Besides the differential operators $\mathscr{H}_\varepsilon$ and $\mathscr{H}_\varepsilon^*$ are defined:

$$\begin{aligned} \mathscr{H}_\varepsilon &= V_2^{S,\varepsilon} \mathscr{D}_{2,S} + V_3^{S,\varepsilon} \mathscr{D}_{1,S} \mathscr{D}_{2,S}, \\ \mathscr{H}_\varepsilon^* &= V_3^{S,\varepsilon} \mathscr{D}_{1,S} \mathscr{D}_{2,S}. \end{aligned}$$

Using the previous notations, the quantity $\mathscr{L}_{BS}(\bar{\sigma}_S)\mathscr{R}$ can be computed:

$$\begin{aligned} \mathscr{L}_{BS}(\bar{\sigma}_S)\mathscr{R} &= \mathscr{L}_{BS}(\bar{\sigma}_S)((P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon}) - (\tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon})), \\ &= \mathscr{H}_\varepsilon P_0^{S,\varepsilon} - (\mathscr{L}_{BS}(\bar{\sigma}_S^*) + V_2^{S,\varepsilon} \mathscr{D}_{2,S})(\tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon}), \\ &= \mathscr{H}_\varepsilon P_0^{S,\varepsilon} - \mathscr{H}_\varepsilon^* \tilde{P}_0^{S,\varepsilon} - V_2^{S,\varepsilon} \mathscr{D}_{2,S}(\tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon}), \\ &= \mathscr{H}_\varepsilon^* (P_0^{S,\varepsilon} - \tilde{P}_0^{S,\varepsilon}) - V_2^{S,\varepsilon} \mathscr{D}_{2,S}(\tilde{P}_0^{S,\varepsilon} - P_0^{S,\varepsilon} + \sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon}). \end{aligned}$$

Knowing that:

- $(P_0^{S,\varepsilon} - \tilde{P}_0^{S,\varepsilon})$ is $O\big(\sqrt{\varepsilon}\big)$.
- $\mathscr{H}_\varepsilon^*$ is $O\big(\sqrt{\varepsilon}\big)$.
- $V_2^{S,\varepsilon} \mathscr{D}_{2,S}$ is $O\big(\sqrt{\varepsilon}\big)$.
- $\sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon}$ is $O\big(\sqrt{\varepsilon}\big)$.

and additionally $\mathscr{R}(T) = 0$, then it follows that $\mathscr{R} = O(\varepsilon)$. This concludes the derivation of the following result:

$$P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon} = \tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon} + O(\varepsilon)\,.$$

So up to order 1 in $\sqrt{\varepsilon}$, the option price $P^{S,\varepsilon}$ can be approximated by $\tilde{P}^{S,\varepsilon}$ which is defined as:

$$\tilde{P}^{S,\varepsilon} = \tilde{P}_0^{S,\varepsilon} + \sqrt{\varepsilon} \tilde{P}_1^{S,\varepsilon}.$$

The estimation error obtained, when approximating $P^{S,\varepsilon}$ by $\tilde{P}^{S,\varepsilon}$, is at order 1 in ε. Indeed, by neglecting terms of order higher to 1 in $\sqrt{\varepsilon}$, the term $(P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon})$ is obtained as an approximation of the price $P(t, S_t, Y_t)$. It is then important to show that this approximation is of order 1 in ε meaning that:

$$|P(t, S_t, Y_t) - (P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon})| \leq C\varepsilon.$$

The proof of this property is given in [17] in the case where the payoff h is smooth. A summary of this proof is given here in order to make this paper self contained.

Let us introduce the quantity $Z^{S,\varepsilon}$ which verifies that:

$$P(t, S_t, Y_t) = P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon} + \varepsilon P_2^{S,\varepsilon} + \varepsilon^{\frac{3}{2}} P_3^{S,\varepsilon} - Z^{S,\varepsilon}.$$

Since $\mathscr{L}^S P(t, S_t, Y_t) = 0$, it follows that:

$$\mathscr{L}^S Z^{S,\varepsilon} = \mathscr{L}_S(P_0^{S,\varepsilon} + \sqrt{\varepsilon} P_1^{S,\varepsilon} + \varepsilon P_2^{S,\varepsilon} + \varepsilon^{\frac{3}{2}} P_3^{S,\varepsilon}).$$

The differential operator $\mathscr{L}^S$ can be written as $\mathscr{L}^S = \mathscr{L}_2^S + \frac{1}{\sqrt{\varepsilon}}\mathscr{L}_1^S + \frac{1}{\varepsilon}\mathscr{L}_0^S$. By developing $\mathscr{L}^S Z^{S,\varepsilon}$ and regrouping the terms by orders of $\sqrt{\varepsilon}$, it follows that:

$$\begin{aligned}\mathscr{L}^S Z^{S,\varepsilon} &= \frac{1}{\varepsilon}\mathscr{L}_0^S P_0^{S,\varepsilon} + \frac{1}{\sqrt{\varepsilon}}(\mathscr{L}_0^S P_1^{S,\varepsilon} + \mathscr{L}_1^S P_0^{S,\varepsilon}) + (\mathscr{L}_0^S P_2^{S,\varepsilon} + \mathscr{L}_1^S P_1^{S,\varepsilon} + \mathscr{L}_2^S P_0^{S,\varepsilon}) \\ &\quad + \sqrt{\varepsilon}(\mathscr{L}_0^S P_3^{S,\varepsilon} + \mathscr{L}_1^S P_2^{S,\varepsilon} + \mathscr{L}_2^S P_1^{S,\varepsilon}) + \varepsilon(\mathscr{L}_1^S P_3^{S,\varepsilon} + \mathscr{L}_2^S P_2^{S,\varepsilon}) \\ &\quad + \varepsilon^{\frac{3}{2}}\mathscr{L}_2^S P_3^{S,\varepsilon}.\end{aligned}$$

The terms $P_0^{S,\varepsilon}$, $P_1^{S,\varepsilon}$ and $P_2^{S,\varepsilon}$ are chosen to nullify the first four terms in the previous equation, therefore:

$$\mathscr{L}^S Z^{S,\varepsilon} = \varepsilon(\mathscr{L}_1^S P_3^{S,\varepsilon} + \mathscr{L}_2^S P_2^{S,\varepsilon}) + \varepsilon^{\frac{3}{2}}\mathscr{L}_2^S P_3^{S,\varepsilon},$$

and $Z^{S,\varepsilon}$ satisfies the final condition:

$$Z^{S,\varepsilon}(T, S_T, Y_T, Z_T) = \varepsilon P_2(T, S_T, Y_T, Z_T) + \varepsilon^{\frac{3}{2}} P_3(T, S_T, Y_T, Z_T).$$

Using the Feynman-Kac theorem, it follows that:

$$\begin{aligned}Z^{S,\varepsilon}(t,x,y,z) &= \varepsilon E(e^{-r(T-t)}(P_2(T, S_T, Y_T, Z_T) + \varepsilon^{\frac{1}{2}} P_3(T, S_T, Y_T, Z_T)) \\ &\qquad - \int_t^T e^{-r(u-t)}((\mathscr{L}_1^S P_3^{S,\varepsilon} + \mathscr{L}_2^S P_2^{S,\varepsilon}) + \varepsilon^{\frac{1}{2}}\mathscr{L}_2^S P_3^{S,\varepsilon}) \\ &\qquad\qquad \times (u, S_u, Y_u, Z_u)du | S_t = x, Y_t = y, Z_t = z).\end{aligned}$$

Under assumptions on the smoothness of the payoff function h and boundedness of the functions χ_1 and χ_2, the term $Z^{S,\varepsilon}$ is at most linearly growing in $|y|$ and $|z|$ and then $Z^{S,\varepsilon}(t, x, y, z) = O(\varepsilon)$

The demonstration of the accuracy of the approximation for a non smooth payoff h (as in the case of a call option) is derived in [18].

Appendix 3: Approximation of the Implied Volatility

The method developed here was suggested in [9] for the model with constant idiosyncratic volatility.

The symbol A is used to denote either the stock S or the index I. The price of an European option $P^{A,\varepsilon}$ on the asset A can be written as:

$$P^{A,\varepsilon} = \tilde{P}_0^{A,\varepsilon} - (T-t)V_3^{A,\varepsilon} A_t \frac{\partial}{\partial A_t}(A_t^2 \frac{\partial^2 \tilde{P}_0^{A,\varepsilon}}{\partial A_t^2}) + o(\varepsilon)\,, \tag{35}$$

where $\tilde{P}_0^{A,\varepsilon}$ is defined as:

$$\tilde{P}_0^{A,\varepsilon} = P_{BS}(t, A_t, \bar{\sigma}_A^*).$$

The term $P^{A,\varepsilon}$ could represent the price of the option on the index $P^{I,\varepsilon}$ if $A = I$ or the price of the option on the stock $P^{S,\varepsilon}$ in the case where $A = S$.

Let I_A be the implied volatility associated to the asset's option price $P^{A,\varepsilon}$ meaning that $P^{A,\varepsilon} = P_{BS}(t, A_t, I_A(K_A, T))$. An expansion of $I_A(K, T)$ could be made around $\bar{\sigma}_A^*$ in powers of $\sqrt{\varepsilon}$:

$$I_A(K_A, T) = \bar{\sigma}_A^* + \sqrt{\varepsilon} I_1(K_A, T) + O(\varepsilon)\,.$$

Using Taylor's formula, it follows that:

$$P^{A,\varepsilon} = P_{BS}(t, A_t, \bar{\sigma}_A^*) + \frac{\partial P_{BS}}{\partial \sigma}_{|\sigma=\bar{\sigma}_A^*} \sqrt{\varepsilon} I_1 + o(\varepsilon)\,. \tag{36}$$

Combining the Eqs. (35) and (36) gives:

$$\frac{\partial P_{BS}(t, A_t, \bar{\sigma}_A^*)}{\partial \sigma}_{|\sigma=\bar{\sigma}_A^*} \sqrt{\varepsilon} I_1(K, T) = -(T-t)V_3^{A,\varepsilon} A_t \frac{\partial}{\partial A_t}(A_t^2 \frac{\partial^2 \tilde{P}_0}{\partial A_t^2}). \tag{37}$$

Performing simple computations on the derivatives of the Black-Scholes price yields that:

$$\mathscr{D}_{2,A} \tilde{P}_0 = \frac{1}{\bar{\sigma}_A^*(T-t)} \frac{\partial P_{BS}}{\partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

By applying then the operator $\mathscr{D}_{1,A}$ to the last equation, it can be obtained that:

$$\mathscr{D}_{1,A}\mathscr{D}_{2,A} \tilde{P}_0 = \frac{A_t}{\bar{\sigma}_A^*(T-t)} \frac{\partial^2 P_{BS}}{\partial A_t \partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

Using closed-form formulas of Black-Scholes greeks, it can be written that:

$$A_t \frac{\partial^2 P_{BS}}{\partial A_t \partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*) = -\frac{d_2}{\bar{\sigma}_A^* \sqrt{T-t}} \frac{\partial P_{BS}}{\partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

The Eq. (37) can be written as:

$$\sqrt{\varepsilon} I_1(K_A, T) = -\frac{V_3^{A,\varepsilon}}{\bar{\sigma}_A^*} \frac{A_t \frac{\partial^2 \tilde{P}_0}{\partial A_t \partial \bar{\sigma}_A^*}}{\frac{\partial \tilde{P}_0}{\partial \bar{\sigma}_A^*}}.$$

Then, it is straightforward that:

$$\sqrt{\varepsilon} I_1(K_A, T) = \frac{V_3^{A,\varepsilon} d_2(K_A, T)}{(\bar{\sigma}_A^*)^2 \sqrt{T-t}},$$

where:

$$d_2(K_A, T) = \frac{\log(\frac{A_t e^{r(T-t)}}{K_A}) - \frac{(\bar{\sigma}_A^*)^2}{2}(T-t)}{\bar{\sigma}_A^* \sqrt{T-t}}.$$

The implied volatility can then be approximated using the following formula:

$$I_A(K_A, T) = \bar{\sigma}_A^* - \frac{V_3^{A,\varepsilon}}{2\bar{\sigma}_A^*} + \frac{V_3^{A,\varepsilon}}{(\bar{\sigma}_A^*)^3} \frac{\log(\frac{F_A(T)}{K_A})}{T-t}. \tag{38}$$

Then, the following smile approximation can be obtained:

$$I_A(K_A, T) = b_A + a_A \frac{\log(\frac{F_A(T)}{K_A})}{T-t},$$

with:

$$b_A = \bar{\sigma}_A^* - \frac{V_3^{A,\varepsilon}}{2\bar{\sigma}_A^*},$$

$$a_A = \frac{V_3^{A,\varepsilon}}{(\bar{\sigma}_A^*)^3}.$$

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