Chapter 2
Solution, Stability and Realization of Fractional Order Differential Equation

2.1 Introduction

Classical calculus has provided an efficient tool for modeling and exploring the properties of the dynamical system problems concerning of physics, biology, engineering and applied sciences. However, experiments with a realistic approach teach us that there are a large class of complex systems where microscopic and macroscopic behaviors are not captured or properly explained using classical calculus. Some examples can be stated: relaxation in viscoelastic materials like polymers, the spread of contaminants in underground water, network traffic, charge transport in amorphous semiconductors, cell diffusion process, the transmission of signals through strong magnetic fields such as those found within confined plasma etc. After several years of research and discussion, it has been found that these major classes of complex systems which contains non-local dynamics involving long-memory are captured using a more general class of operators known as fractional operators. The differential equations involving these operators are known as fractional order differential equation.

Stability is the one of the most frequent terms used in literature whenever we deal with the dynamical systems and their behaviors. In mathematical terminology, stability theory addresses the convergence of solutions of differential or difference equations and of trajectories of dynamical systems under small perturbations of initial conditions. Same as classical differential or difference equations a lot of stress has been given to the stability and stabilization of the systems represented by fractional order differential equations.

Up to this point it is quit obvious that the fractional order calculus is more appropriate to capture the real dynamical behavior rather than integer order calculus. However, fractional order systems have an infinite dimension, while the integer order systems is only finite dimensional. Therefore, to realize the fractional order controllers perfectly, all past inputs should be memorized, which is not possible without proper approximation. This issue is also discussed in this chapter.

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The brief outline of this chapter is as follows. Section 2.2 describes the solution of fractional differential equations and Mittag-Leffler function. Section 2.3 discusses the brief summary of the notation of stability and stabilization. A brief review on linear matrix inequality (LMI) stability conditions for LTI fractional order systems are analyzed in Sect. 2.4. A deep discussion on the realization issue of fractional-order controller is presented in Sect. 2.5. A brief review of fractional order PID control is surveyed in Sect. 2.6 followed by the concluding Sect. 2.7.

2.2 Solution of Fractional Differential Equations and Mittag-Leffler Function

As already discussed in Chap. 1 the most popular fractional order derivative has been given by Riemann-Liouville and Caputo. Therefore, in this book we concentrate on fractional order differential equations formed by these two derivatives only. Riemann-Liouville's fractional-order derivative has been most widely used for capturing the physical problems because it places less constraints on the concerned function. However, fractional order differential equations involving Riemann-Liouville's fractional-order derivative has some practical issues, related to initial value problem. This is because the initial problems contain the fractional operator which does not have a straightforward physical meaning. Initial value problem for a non-homogeneous fractional differential equation under non-zero initial conditions, is expressed as [1, 2]

$$ {}_0D_t^{\alpha} x(t) - \lambda x(t) = f(t, x(t)), \quad \left[{}_0D_t^{\alpha-k} x(t) \right]_{t=0} = c_k \quad (k = 1, 2, \ldots, n), $$

where $n - 1 < \alpha < n$. However, in the case of Caputo's fractional differential equation, the initial value problem can be represented as

$$ {}_0D_t^{\alpha} x(t) - \lambda x(t) = f(t, x(t)), \quad \left[{}_0D_t^{k} x(t) \right]_{t=0} = \hat{c}_k \quad (k = 0, 1, 2, \ldots, n-1). $$

When we are solving fractional order differential equations and fractional order integral equations, the *Mittag-Leffler* function comes into picture. Therefore, before presenting the solution of fractional differential equations, we begin with the discussions on definition and various properties of *Mittag-Leffler* functions.

2.2.1 Mittag-Leffler Function

The function $E_\alpha(t)$ was defined by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential series [1, 2]. For $\alpha = 1$ we have the exponential series.

Definition 2.1 The Mittag-Leffler function function $E_\alpha(t)$ and the generalized Mittag-Leffler function $E_{\alpha,\beta}(t)$ are defined as:

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{2.1}$$

For $\alpha = 1$, we have the exponential series. Similarly,

$$E_{\alpha,\ \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{2.2}$$

The other well known function is *Miller-Ross function* which is defined as:

Definition 2.2

$$\xi_{\alpha,\ a}(t) = \sum_{k=0}^{\infty} \frac{a^k t^{k+\alpha}}{\Gamma(\alpha + k + 1)} = t^\alpha E_{1,\alpha+1}(at). \tag{2.3}$$

Some special cases of Mittag-leffler functions are summarized as,

$$\xi_{0,1}(t) = E_1(t) = E_{1,1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t, \tag{2.4}$$

$$\xi_{0,\ a}(t) = E_1(at) = E_{1,1}(at) = e^{at}. \tag{2.5}$$

The following Laplace transform is involved frequently while solving fractional differential equation:

Theorem 2.3 *The Laplace transform of* $t^{\alpha k+\beta-1} \frac{d^k E_{\alpha,\beta}(\pm a t^\alpha)}{d(\pm a t^\alpha)^k}$,

$$\mathscr{L}\left[t^{\alpha k+\beta-1} \frac{d^k E_{\alpha,\beta}(\pm a t^\alpha)}{d(\pm a t^\alpha)^k} \right] = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}. \tag{2.6}$$

Proof Consider the following integral

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(\pm\eta t^\alpha) dt = \int_0^\infty e^{-t} t^{\beta-1} \sum_{k=0}^{\infty} \frac{(\pm\eta)^k t^{\alpha k}}{\Gamma(\alpha k + \beta)} dt$$

$$= \sum_{k=0}^{\infty} \frac{(\pm\eta)^k}{\Gamma(\alpha k + \beta)} \int_0^\infty e^{-t} t^{\alpha k+\beta-1} dt. \tag{2.7}$$

Using the following relation

$$\int_0^\infty e^{-t}t^{\alpha k+\beta-1}dt = \Gamma(\alpha k+\beta),$$

(2.7) can be written as,

$$\begin{aligned}\int_0^\infty e^{-t}t^{\beta-1}E_{\alpha,\beta}(\pm\eta t^\alpha)dt &= \sum_{k=0}^{\infty}\frac{(\pm\eta)^k}{\Gamma(\alpha k+\beta)}\Gamma(\alpha k+\beta)\\ &= \frac{1}{1\mp\eta}.\end{aligned} \tag{2.8}$$

kth differentiation of (2.8) is given as

$$\frac{d^k}{d\eta^k}\int_0^\infty e^{-t}t^{\beta-1}E_{\alpha,\beta}(\pm\eta t^\alpha)dt = \frac{k!(\pm1)^k}{(1\mp\eta)^{k+1}}$$

$$\int_0^\infty e^{-t}t^{\beta-1}(\pm t^\alpha)^k\frac{d^k}{d(\pm\eta t^\alpha)^k}E_{\alpha,\beta}(\pm\eta t^\alpha)dt = \frac{k!(\pm1)^k}{(1\mp\eta)^{k+1}}. \tag{2.9}$$

Now changing the variable t with st

$$\frac{k!(\pm1)^k}{(1\mp\eta)^{k+1}} = \int_0^\infty e^{-st}s^{\beta-1}t^{\beta-1}(\pm1)^k s^{\alpha k}t^{\alpha k}\frac{d^k E_{\alpha,\beta}(\pm\eta s^\alpha t^\alpha)}{d(\pm\eta s^\alpha t^\alpha)^k}sdt. \tag{2.10}$$

Also by replacing ηs^α by a

$$\frac{k!}{s^\beta s^{\alpha k}\left(1\mp\frac{a}{s^\alpha}\right)^{k+1}} = \int_0^\infty e^{-st}t^{\alpha k+\beta-1}\frac{d^k E_{\alpha,\beta}(\pm a t^\alpha)}{d(\pm a t^\alpha)^k}dt. \tag{2.11}$$

This completes the proof.

Some special cases of the Laplace transform (2.6) :

- Substituting $k=0$ in (2.6)

$$\mathscr{L}\left[t^{\beta-1}E_{\alpha,\beta}(\pm a t^\alpha)\right] = \frac{s^{\alpha-\beta}}{s^\alpha\mp a}. \tag{2.12}$$

- Substituting $\alpha = \beta$ in (2.12)

$$\mathscr{L}\left[t^{\alpha-1}E_{\alpha,\alpha}(\pm at^{\alpha})\right] = \frac{1}{s^{\alpha} \mp a}. \tag{2.13}$$

- *Example* Inverse Laplace transform of $\frac{4s^{1/2}-1}{s+s^{1/2}-2}$ using (2.13)

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{4s^{1/2}-1}{s+s^{1/2}-2}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s^{1/2}-1} + \frac{3}{s^{1/2}+2}\right] \\ &= t^{-1/2}E_{1/2,1/2}(t^{1/2}) + 3t^{-1/2}E_{1/2,1/2}(-2t^{1/2}).\end{aligned} \tag{2.14}$$

2.2.2 Solution the Fractional Differential Using Laplace Transform

Consider the following fractional differential equation,

$$\begin{aligned}{}_0D_t^{\alpha}x(t) - \lambda x(t) = f(t, x(t)), \quad &\left[{}_0D_t^{\alpha-k}x(t)\right]_{t=0} \\ &= c_k \quad (k = 1, 2, \ldots, n).\end{aligned} \tag{2.15}$$

Laplace transform of Riemann-Liouville's derivative is given as,

$$\int_0^{\infty} e^{-st}\,{}_0D_t^{\alpha}x(t)dt = s^{\alpha}X(s) - \sum_{k=0}^{n-1} s^k\left[{}_0D_t^{\alpha-k-1}x(t)\right]_{t=0}.$$

Taking Laplace transform of (2.15), one can write

$$s^{\alpha}X(s) - \lambda X(s) = F(s) + \sum_{k=1}^{n} c_k s^{k-1}$$

$$X(s) = \frac{F(s)}{(s^{\alpha}-\lambda)} + \sum_{k=1}^{n} c_k \frac{s^{k-1}}{(s^{\alpha}-\lambda)}. \tag{2.16}$$

Using (2.6) the inverse Laplace transformation of (2.16) can be found as,

$$x(t) = \sum_{k=1}^{n} c_k t^{\alpha-k}E_{\alpha,\alpha-k+1}(\lambda t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})f(\tau)d\tau. \tag{2.17}$$

Example Consider an example of fractional differential equation as,

$$ {}_0D_t^{\frac{1}{2}}x(t) + ax(t) = 0, \quad (t > 0); \quad \left[{}_0D_t^{-1/2}x(t)\right]_{t=0} = C. \tag{2.18}$$

Applying Laplace transform, one can write

$$ X(s) = \frac{C}{s^{1/2} + a}. \tag{2.19}$$

Taking inverse Laplace transform, one can write solution of (2.19), as discussed in (2.17) as

$$ x(t) = Ct^{-1/2}E_{1/2,1/2}(-a\sqrt{t}). \tag{2.20}$$

2.2.3 *More Proper Way to Impose Initial Condition to Fractional Order Differential Equation*

Initialized fractional order Riemann-Liouville derivative is expressed in the following way

$$ \begin{gathered} {}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau, \quad t > a \\ f = \varphi(t), \quad f \in C^{n-1}, \quad t_0 < t \le a \end{gathered} \tag{2.21}$$

where n is a positive integer satisfying $n - 1 < \alpha \le n$, and $f = \varphi(t)$ represents the initial history over $(t_0, \ a]$, and $f = 0$, for $t \le t_0$.

For visualizing the effect of initial history Du and Wang [3] considered the example of axially loaded viscoelastic bar, of which elongation $x(t)$ and longitudinal force F satisfy the following equation

$$ F = {}_0D_t^{0.5}x(t). \tag{2.22}$$

Suppose that the elongation $x(t)$ is initialized as $x(t) = \varphi(t) = t$ for $t \in (0, 1]$ and it is kept constant ($x(t) = 1$) when $t > 1$; then the force can be calculated as

$$ \begin{aligned} F &= {}_0D_1^{0.5}x(t) + {}_1D_t^{0.5}x(t) \\ &= \frac{1}{\Gamma(0.5)} \frac{d}{dt} \left[\int_0^1 \frac{\tau}{(t-\tau)^{0.5}} d\tau + \int_1^t \frac{1}{(t-\tau)^{0.5}} d\tau \right] \\ &= \frac{2(\sqrt{t} - \sqrt{t-1})}{\Gamma(0.5)}. \end{aligned} \tag{2.23}$$

Using (2.23) it can be concluded that viscoelastic force is still dependent on t, although the elongation $x(t)$ is constant after $t > 1$. It can be also seen that the force is sensitive to the initial history. If the initial history is given as $\varphi(t) = \sqrt{t}$, then the force is given as

$$F = \frac{1}{\Gamma(0.5)} \sin^{-1}\left(\frac{1}{\sqrt{t}}\right). \tag{2.24}$$

In both the cases one can see that the net viscoelastic force depends on the initial history.

Very often the initial value problem of a fractional differential equation is converted to an equivalent integral equation. Consider the following fractional order differential equation

$$\begin{aligned} {}_{t_0}D_t^{\alpha} x(t) &= f(t, x(t)), \quad (t > a) \\ x(t) &= \varphi(t_0, a], \ x_0 \in C^n. \end{aligned} \tag{2.25}$$

Theorem 2.4 *The initial value problem* (2.25) *is equivalent to the following integral equation*

$$x(t) = \begin{cases} F(\varphi(t); t, x(t)) & \text{if } t > a \\ \varphi(t) & \text{if } t_0 < t \le a \end{cases} \tag{2.26}$$

where

$$\begin{aligned} F(\varphi(t); t, x(t)) = {}_aD_t^{-\alpha} f(t, x(t)) - {}_aD_t^{n-\alpha}\left[{}_{t_0}D_a^{-(n-\alpha)}\varphi(t)\right] \\ + \sum_{j=1}^{n} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha - j + 1)}\left[{}_{t_0}D_t^{(\alpha-j)}\varphi(t)\right]_{t=a}. \end{aligned} \tag{2.27}$$

Proof Following Lemma is important before proving the main result which is given by Du and Wang [3].

Lemma 2.5 *For a given* $\alpha > 0$, *one has*

$$\begin{aligned} {}_aD_t^{-\alpha}\left[{}_{t_0}D_t^{\alpha} x(t)\right] = x(t) + {}_aD_t^{n-\alpha}\left[{}_{t_0}D_a^{-(n-\alpha)} x(t)\right] \\ - \sum_{j=1}^{m} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha - j + 1)}\left[{}_{t_0}D_t^{(\alpha-j)} x(t)\right]_{t=a}. \end{aligned} \tag{2.28}$$

Proof Let

$$L(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{x(\tau)}{(t-\tau)^{1+\alpha-n}}d\tau={}_{t_0}D_t^{-(n-\alpha)}x(t), \tag{2.29}$$

then

$$L^{(n)}(t)={}_{t_0}D_t^{\alpha}x(t), \tag{2.30}$$

$$L^{(n-j)}(a)=\left[L^{(n-j)}(t)\right]_{t=a}=\left[{}_{t_0}D_t^{\alpha-j}\varphi(t)\right]_{t=a}. \tag{2.31}$$

Now

$$\begin{aligned}{}_aD_t^{-\alpha}{}_{t_0}D_t^{\alpha}x(t)&=\frac{d}{dt}\,{}_aD_t^{-\alpha-1}{}_{t_0}D_t^{\alpha}x(t)\\&=\frac{d}{dt}\left[\frac{1}{\Gamma(\alpha+1)}\int_a^t(t-\tau)^{\alpha}L^{(n)}(\tau)d\tau\right]\\&=\frac{d}{dt}\left[\frac{1}{\Gamma(\alpha+1-n)}\int_a^t\frac{L(\tau)}{(t-\tau)^{n-\alpha}}d\tau-\sum_{j=1}^{n}\frac{(t-a)^{\alpha-j+1}}{\Gamma(\alpha-j+2)}L^{(n-j)}(a)\right]\\&=\frac{d}{dt}\,{}_aD_t^{-(\alpha-n+1)}L(t)-\sum_{j=1}^{n}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}L^{(n-j)}(a).\end{aligned} \tag{2.32}$$

As $L(t)=\left({}_{t_0}D_a^{-(n-\alpha)}+{}_aD_t^{-(n-\alpha)}\right)x(t)$, by substituting $L(t)$ in (2.32) one can write

$$\begin{aligned}{}_aD_t^{-\alpha}{}_{t_0}D_t^{\alpha}x(t)&=\frac{d}{dt}\left[{}_aD_t^{-1}x(t)+{}_aD_t^{-(\alpha-n+1)}{}_{t_0}D_{\varphi}^{-(n-\alpha)}(a)\right]\\&\quad-\sum_{j=1}^{n}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}L^{(n-j)}(a)\\&=x(t)+\frac{d}{dt}\left[{}_aD_t^{-(\alpha-n+1)}{}_{t_0}D_a^{-(n-\alpha)}\varphi(t)\right]\\&\quad-\sum_{j=1}^{n}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}\left[{}_{t_0}D_t^{(\alpha-j)}\varphi(t)\right]_{t=a}\\&=x(t)+\left[{}_aD_t^{(n-\alpha)}{}_{t_0}D_a^{-(n-\alpha)}\varphi(t)\right]-\sum_{j=1}^{n}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}\left[{}_{t_0}D_t^{(\alpha-j)}\varphi(t)\right]_{t=a}.\end{aligned} \tag{2.33}$$

This proves the Lemma.
If $x(t)$ is the solution of (2.25), then

$$ {}_aD_t^{-\alpha}\,{}_{t_0}D_t^{\alpha}x(t) = {}_aD_t^{-\alpha}f(t, x(t)). \tag{2.34}$$

It can be verified that $x(t)$ satisfies (2.26), due to above Lemma.

Conversely, if $x(t)$ is a solution of (2.26), then

$$\begin{aligned} {}_aD_t^{\alpha}x(t) &= f(t, x(t)) - {}_aD_t^{\alpha}\left({}_aD_t^{n-\alpha}\left[{}_{t_0}D_a^{-(n-\alpha)}\varphi(t)\right]\right) \\ &\quad + {}_aD_t^{\alpha}\left(\sum_{j=1}^{n}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}\left[{}_{t_0}D_t^{(\alpha-j)}\varphi(t)\right]_{t=a}\right). \end{aligned} \tag{2.35}$$

Adding ${}_{t_0}D_a^{\alpha}x(t)$ to both sides and, since ${}_aD_t^{\alpha}(t-a)^{\alpha-j} = 0$ for $j = 1, 2, \ldots, n$. Equation (2.35) is simplified as,

$$\begin{aligned} {}_{t_0}D_t^{\alpha}x(t) &= f(t, x(t)) + {}_{t_0}D_a^{\alpha}x(t) - {}_aD_t^{\alpha}\left({}_aD_t^{n-\alpha}\left[{}_{t_0}D_a^{-(n-\alpha)}\varphi(t)\right]\right) \\ &\quad + {}_aD_t^{\alpha}\left(\sum_{j=1}^{n}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}\left[{}_{t_0}D_t^{(\alpha-j)}\varphi(t)\right]_{t=a}\right) \\ &= f(t, x(t)) + {}_{t_0}D_a^{\alpha}x(t) - {}_aD_t^{\alpha}\left({}_aD_t^{n-\alpha}\left[{}_{t_0}D_a^{-(n-\alpha)}\varphi(t)\right]\right). \end{aligned} \tag{2.36}$$

Also, one can write

$$ {}_{t_0}D_a^{\alpha}x(t) = \frac{d^n}{dt^n}\,{}_{t_0}D_a^{-(n-\alpha)}\varphi(t) = {}_aD_t^{\alpha}\left({}_aD_t^{n-\alpha}\left[{}_{t_0}D_a^{-(n-\alpha)}\varphi(t)\right]\right). \tag{2.37}$$

Thus, ${}_{t_0}D_t^{\alpha}x(t) = f(t, x(t))$, $t > a$. It implies that $x(t)$ is a solution of (2.25). This completes the proof.

Remark 2.6 The initial value problem of fractional differential equation can be converted to an equivalent integral equation and it is easy for both theoretical and numerical analysis.

2.3 Stability and Stabilization

Control system problems generally cater to two categories, first is the stabilization or regulation and second is the tracking or servo. Stabilization problems, aim to design a control system, known as stabilizer or a regulator, so that the state of the closed-loop system will be stabilized around the desired point also known as an equilibrium point. In tracking problems, the design objective is to construct a suitable controller, called a tracker, so that the system output tracks a given time-varying reference trajectory. When we see the tracking problem in the frame of difference between reference trajectory and trajectories generated by system, which is called error; then tracking problem is converted into a stabilization problem of the error variable. Therefore, in this book we restrict our analysis on stabilization problem.

2.3.1 Concept of Equilibrium Point

Equilibrium point of fractional order system is defined same as in integer order [4]. Consider the Riemann-Liouville fractional order autonomous system

$$ {}_{t_0}^{RL}D_t^{\alpha}x(t) = f(t, x), \tag{2.38} $$

with initial condition $x(t_0)$, where $\alpha \in (0 \;\; 1)$, $f : [t_o, \infty] \times \boldsymbol{\Omega} \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_o, \infty] \times \boldsymbol{\Omega}$, $\boldsymbol{\Omega} \in \mathbb{R}^n$ is a domain that contains the equilibrium point $x = 0$.

Definition 2.7 The constant x_0 is an equilibrium point of the Riemann-Liouville fractional dynamic system (2.38), if and only if

$$ {}_{t_0}^{RL}D_t^{\alpha}x_0 = f(t, x_0). \tag{2.39} $$

Just like integer order, shifting of equilibrium point is valid for fractional order. Without loss of generality any equilibrium point can be shifted to origin via a change of variables. Suppose the equilibrium point for (2.38) is $\bar{x} \neq 0$ and consider the change of variable $y = x - \bar{x}$. The αth order derivative of y is given by Riemann-Liouville fractional order autonomous system

$$ \begin{aligned} {}_{t_0}^{RL}D_t^{\alpha}y = {}_{t_0}^{RL}D_t^{\alpha}(x - \bar{x}) &= f(t, x) - \frac{\bar{x}t^{-\alpha}}{\Gamma(1-\alpha)} \\ &= f(t, y + \bar{x}) - \frac{\bar{x}t^{-\alpha}}{\Gamma(1-\alpha)} = \bar{g}(t, y), \end{aligned} \tag{2.40} $$

$\bar{g}(t, 0) = 0$ and in terms of the new variable y, the system has equilibrium at the origin.

Another popular definition used to represent dynamical system governed by fractional order is the one by Caputo. For defining the equilibrium point consider the following Caputo fractional order autonomous system

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = f(t, x), \tag{2.41} $$

with initial condition $x(t_0)$, where $\alpha \in (0 \;\; 1)$, $f : [t_o, \infty] \times \boldsymbol{\Omega} \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_o, \infty] \times \boldsymbol{\Omega}$, $\boldsymbol{\Omega} \in \mathbb{R}^n$ is a domain that contains the equilibrium point $x = 0$.

Definition 2.8 The constant x_0 is an equilibrium point of the Caputo fractional dynamic system (2.41), if and only if $f(t, x_0) = 0$.

Remark 2.9 When $\alpha \in (0 \;\; 1)$, the Caputo fractional order system (2.41) has the same equilibrium points as the integer-order system $\dot{x}(t) = f(t, x)$.

After defining the equilibrium point, the most fundamental aspect about any dynamical system is the stability of system with respect to the equilibrium point. So in the next subsection we review some of the fundamental definitions of stability. It can be seen that these concepts are similar to those of integer order.

2.3.2 Fundamental of Stability

Definition 2.10 [5] The zero solution of ${}_0D_t^{\alpha}x(t) = f(t, x)$ is said to be stable if, for any initial conditions $x(0) \in \mathbb{R}^n$, there exists $\delta > 0$ such that any solution $x(t)$ of ${}_0D_t^{\alpha}x(t) = f(t, x)$ satisfies $\|x(t)\| < \delta$ for all $t > t_0$. Further, the zero solution of fractional differential system is said to be asymptotically stable if, in addition to being stable, $\|x(t)\| \to 0$ as $t \to \infty$.

Similarly $L^p(\boldsymbol{\Omega})$ stability of fractional order system with Riemann-Liouville derivative is defined as follows:

Definition 2.11 Suppose that $1 \leq p \leq \infty$ and $\boldsymbol{\Omega} \subset [t_0, \infty]$, then the solution $x(t)$ of the fractional order differential system ${}_0D_t^{\alpha}x(t) = f(t, x)$ (where $0 < \alpha < 1$, $x(0) \in \mathbb{R}^n$ is the initial condition , $f \in \mathbb{C}([t_0, \infty)$ is a continuous positive function) is called $L^p(\boldsymbol{\Omega})$ stability if

$$x(t) = \frac{x(0)}{\Gamma(\alpha)}(t - t_0)^{\alpha - 1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha - 1} f(\tau, x(\tau))d\tau,$$

belongs to $L^p(\boldsymbol{\Omega})$.

Other more generalized stability concept similar to asymptotic stability will be discussed in the next subsection. It is found that it is more convenient to characterize stability according to linear and non linear fractional order systems.

2.3.3 $t^{-\alpha}$ Stability

Decay rate of a simple fractional order autonomous system is not the same as of integer order. The components of the state variables in fractional order system have anomalous decay, due to the fact that fractional order systems have memory features. Asymptotic stability is also called $t^{-\alpha}$ stability. Following definition has been suggested by Sabatier et al. (2010) [6] regarding this proposition.

Definition 2.12 The trajectories $x(t) = 0$ of the system $d^\alpha x(t)/dt^\alpha = f(t, x(t))$ is $t^{-\alpha}$ asymptotic stable if the system is uniformly asymptotically stable and if there is a positive real α such that:

$$\begin{aligned}&\forall\ \|x(t)\|,\ t \le t_0 \quad \exists\ N\ (x(t),\ t \le t_0),\ t_1(x(t),\ t \le t_0)\\ &\text{such that } \forall\ t > t_0 \quad \|x(t)\| \le N\ (t - t_1)^{-\alpha}.\end{aligned} \tag{2.42}$$

2.3.4 Mittag-Leffler Stability

Before defining the Mittag-Leffler stability, the following concepts are primarily required.

Lemma 2.13 [4] *Fractional integral of real valued function $f(t, x)$ satisfies the following inequality*

$$\|{}_{t_0}D_t^{-\alpha} f(t, x(t))\| \le {}_{t_0}D_t^{-\alpha} \|f(t, x(t))\|, \tag{2.43}$$

where $\alpha \ge 0$ and $\|.\|$ denotes the arbitrary norm.

Proof Taking the arbitrary norm on the fractional order integral, one can write

$$\begin{aligned}\|{}_{t_0}D_t^{-\alpha} f(t, x(t))\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right\|\\ &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\|f(\tau, x(\tau))\|}{(t-\tau)^{1-\alpha}} d\tau\\ &\le {}_{t_0}D_t^{-\alpha} \|f(t, x(t))\|.\end{aligned} \tag{2.44}$$

This ends the proof.

Theorem 2.14 [4] *If $x = 0$ is the equilibrium point of the system ${}_{t_0}^{C}D_t^{\alpha} x(t) = f(t, x)$, f is Lipschitz by a constant L and is piecewise continuous with respect to t, then the solution of the system satisfies $\|x(t)\| \le \|x(t_0)\|\ E_\alpha(L(t - t_0)^\alpha)$, where $\alpha \in (0, 1)$.*

Proof Applying ${}_{t_0}D_t^{-\alpha} f(t, x(t))$ to both side of ${}_{t_0}^{C}D_t^{\alpha} x(t) = f(t, x)$, one can write

$$x(t) = x(t_0) + {}_{t_0}D_t^{-\alpha} f(t, x(t)). \tag{2.45}$$

Using norm-inequality, the above equation can be rewritten as,

$$\|x(t)\| - \|x(t_0)\| \le \|x(t) - x(t_0)\| \le \left\|{}_{t_0}D_t^{-\alpha} f(t, x(t))\right\|. \tag{2.46}$$

Further, using Lipschitz condition and Lemma 2.13, one can write

$$\left\| {}_{t_0}D_t^{-\alpha} f(t, x(t)) \right\| \leq {}_{t_0}D_t^{-\alpha} \left\| f(t, x(t)) \right\| \leq L \; {}_{t_0}D_t^{-\alpha} \left\| x(t) \right\|. \tag{2.47}$$

There exists a nonnegative function $\eta(t)$ satisfying

$$\|x(t)\| - \|x(t_0)\| = L \; {}_{t_0}D_t^{-\alpha} \|x(t)\| - \eta(t). \tag{2.48}$$

By applying Laplace transform to (2.48), one can further write

$$\|x(t)\| = \frac{\|x(t_0)\| s^{\alpha-1} - s^{\alpha} \eta(s)}{s^{\alpha} - L}. \tag{2.49}$$

Now applying Inverse Laplace transform to (2.49), which gives

$$\|x(t)\| = \|x(t_0)\| E_{\alpha}(L(t-t_0)^{\alpha}) - \eta(t) * \left[t^{-1} E_{\alpha,0}(L(t-t_0)^{\alpha}) \right], \tag{2.50}$$

where $*$ denotes the convolution operator and

$$t^{-1} E_{\alpha,0}(L(t-t_0)^{\alpha}) = \frac{dE_{\alpha}(L(t-t_0)^{\alpha})}{dt} \geq 0. \tag{2.51}$$

Using (2.50) and (2.51), one can write

$$\|x(t)\| \leq \|x(t_0)\| E_{\alpha}(L(t-t_0)^{\alpha}). \tag{2.52}$$

This ends the proof.

The following Lemma is important for establishing the relation between Riemann-Lioville and Caputo fractional derivative

Lemma 2.15 *Suppose that* $\alpha \in (0, 1)$ *and* $f(0) \geq 0$ *then*

$${}_{t_0}^{C}D_t^{\alpha} f(t) \leq {}_{t_0}^{RL}D_t^{\alpha} f(t). \tag{2.53}$$

Proof One can write

$${}_{t_0}^{C}D_t^{\alpha} f(t) = {}_{t_0}^{RL}D_t^{\alpha-1} \frac{d}{dt} f(t) = {}_{t_0}^{RL}D_t^{\alpha} f(t) - \frac{f(0) t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2.54}$$

Now substituting the condition $\alpha \in (0, 1)$ and $f(0) \geq 0$, then ${}_{t_0}^{C}D_t^{\alpha} f(t) \leq {}_{t_0}^{RL}D_t^{\alpha} f(t)$. This ends the proof.

Based on Theorem 2.14, the following stability condition has been proposed by Li et al. (2009) [4], which has been termed as Mittag-Leffler stability.

Definition 2.16 The solution of ${}_{t_0}D_t^{\alpha}x(t) = f(t, x)$ is said to be Mittag-Leffler stable if

$$||x(t)|| \leq \{m[x(t_0)]E_{\alpha}(-\lambda(t-t_0)^{\alpha})\}^b \tag{2.55}$$

where $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, t_0 is the initial time $\alpha \in (0, 1)$, $\lambda > 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \in \mathbb{R}^n$ with Lipschitz constant m_0.

Remark 2.17 Mittag-Leffler stability implies asymptotic stability.

2.3.5 Stability Using Ω Plane Analysis

For the simplicity let us assume that the following fractional order differential equation [7]

$${}_0D_t^{\alpha}x(t) = -ax(t) + bu(t), \tag{2.56}$$

where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$ is the control input. It is assumed that all initial conditions, or initialization functions, are zero. Then the Laplace transform of (2.56) is given as

$$s^{\alpha}X(s) = -aX(s) + bU(s). \tag{2.57}$$

System transfer of (2.56) is given as

$$G(s) = \frac{X(s)}{U(s)} = \frac{b}{s^{\alpha} + a}. \tag{2.58}$$

Impulse response of (2.58)

As b is the constant, it can be assumed to be unity without any loss of generality. Now expanding the right hand side of (2.58) about $s = \infty$, one can write

$$G(s) = \frac{1}{s^{\alpha} + a} = \frac{1}{s^{\alpha}} - \frac{a}{s^{2\alpha}} + \frac{a^2}{s^{3\alpha}} - \cdots = \frac{1}{s^{\alpha}} \sum_{j=0}^{\infty} \frac{(-a)^j}{s^{j\alpha}}. \tag{2.59}$$

The inverse Laplace transform of (2.59), using the fact $\frac{1}{s^{\alpha}} = \mathscr{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}$

$$\begin{aligned} g(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^{\alpha}} - \frac{a}{s^{2\alpha}} + \frac{a^2}{s^{3\alpha}} - \cdots\right\} \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{at^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{a^2t^{3\alpha-1}}{\Gamma(3\alpha)} + \cdots . \end{aligned} \tag{2.60}$$

Hence generalized impulse response of function is given as

$$g(t) = t^{\alpha-1} \sum_{j=0}^{\infty} \frac{(-a)^j t^{j\alpha}}{\Gamma(j\alpha + \alpha)}. \tag{2.61}$$

Unit step response of (2.58)

If the input function $u(t)$ is a unit step function, (2.58) can be written as

$$X(s) = \frac{1}{s}\left[\frac{1}{s^\alpha + a}\right]. \tag{2.62}$$

Further, after rearranging (2.62)

$$X(s) = \frac{1/a}{s}\left[\frac{a}{s^\alpha + a}\right] = \frac{1/a}{s}\left[1 - \frac{s^\alpha}{s^\alpha + a}\right] = \frac{1/a}{s} - \frac{s^\alpha/a}{s(s^\alpha + a)}. \tag{2.63}$$

For obtaining Inverse Laplace transform of the above equation, the following definition of Mittage Leffler function in summation form is needed

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0. \tag{2.64}$$

Assuming $x = -at^\alpha$, (2.64) becomes

$$E_\alpha(-at^\alpha) = \sum_{k=0}^{\infty} \frac{(-a)^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad \alpha > 0. \tag{2.65}$$

Taking Laplace transform of (2.65)

$$\begin{aligned} \mathscr{L}\left\{E_\alpha(-at^\alpha)\right\} &= \mathscr{L}\left\{\frac{1}{\Gamma(1)} - \frac{at^\alpha}{\Gamma(1+\alpha)} + \frac{a^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots\right\} \\ &= \frac{1}{s} - \frac{a}{s^{\alpha+1}} + \frac{a^2}{s^{2\alpha+1}} + \cdots, \end{aligned} \tag{2.66}$$

or, equivalently

$$\begin{aligned} \mathscr{L}\left\{E_\alpha(-at^\alpha)\right\} &= \frac{1}{s}\left[1 - \frac{a}{s^\alpha} + \frac{a^2}{s^{2\alpha}} + \cdots\right] = \frac{1}{s}\sum_{j=0}^{\infty}\left(\frac{-a}{s^\alpha}\right)^j \\ &= \frac{1}{s}\left[\frac{s^\alpha}{s^\alpha + a}\right]. \end{aligned} \tag{2.67}$$

Using (2.67), the step response of the system can be obtained by taking the inverse Laplace transform of (2.63), which is given as

$$x(t) = \frac{1}{a}\left[H(t) - E_{\alpha}(-at^{\alpha})\right], \tag{2.68}$$

where $H(t)$ is the Heaviside unit step function.

Stability using (2.58)

Perform the following conformal transformation of s

$$\boldsymbol{\Omega} = s^{\alpha}. \tag{2.69}$$

Then (2.58) is transformed as

$$G(s) = \frac{b}{s^{\alpha} + a} \Leftrightarrow \frac{1}{\boldsymbol{\Omega} + a}. \tag{2.70}$$

Using the above transformation, we will study the $\boldsymbol{\Omega}$-plane poles. Once the time domain responses are obtained corresponding to the $\boldsymbol{\Omega}$-plane pole locations, their behavior in new complex plane can be characterized.

For this, it is necessary to map the s-plane, along with the time-domain function properties associated with each point, into the new complex $\boldsymbol{\Omega}$-plane. For simplicity assume that $0 < \alpha \leq 1$. Then (2.69) can be written as

$$\boldsymbol{\Omega} = s^{\alpha} = \left(re^{j\theta}\right)^{\alpha} = r^{\alpha}e^{j\alpha\theta}. \tag{2.71}$$

Using (2.71), it is possible to map s-plane into the $\boldsymbol{\Omega}$-plane. For the stability, mapping of imaginary axis $s = re^{\pm j\pi/2}$ is important. The image of this axis in the $\boldsymbol{\Omega}$-plane is

$$\boldsymbol{\Omega} = r^{\alpha}e^{\pm j\dfrac{\alpha\pi}{2}}, \tag{2.72}$$

which is the pair of lines at $\phi = \frac{\pm\alpha\pi}{2}$, where ϕ is the angle in $\boldsymbol{\Omega}$ plane and $\boldsymbol{\Omega} = \rho e^{j\phi}$. Thus, the right half of the s-plane maps into a wedge in the $\boldsymbol{\Omega}$-plane of angle less than $\pm\frac{\pi}{2}\alpha$ degrees, that is the right half s-plane maps into $|\phi| < \frac{\alpha\pi}{2}$, which is shown in Fig. 2.1. Similar, kind of situation in the case of $1 < \alpha \leq 2$ is which shown in Fig. 2.2.

Example Inductor terminated semi-infinite lossy line

Consider the system shown in Fig. 2.3, where the inductor is terminated on the lossy line. The input to the system is voltage $v_i(t)$ and output $v_0(t)$ will be selected as the terminal of the lossy line. Assume that the $L = 1$, then transfer function is expressed as [7]

$$G(s) = \frac{V_0(s)}{V_I(s)} = \frac{\frac{1}{\sqrt{s}}}{s + \frac{1}{\sqrt{s}}} = \frac{1}{s^{3/2} + 1}. \tag{2.73}$$

Taking the inverse Laplace transformation, the above problem can be expressed in time domain as

$${}_0D_t^{\frac{3}{2}} v_0(t) + v_0(t) = v_i(t), \tag{2.74}$$

where all the initial conditions are assumed to be zero. One can also note the following:

- Impulse response of the system is given as (2.61) by substituting $\alpha = 3/2$

$$v_0(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^{3/2} + 1}\right\} = t^{1/2} \sum_{j=0}^{\infty} \frac{(-1)^j t^{3/2j}}{\Gamma(j3/2 + 3/2)}. \tag{2.75}$$

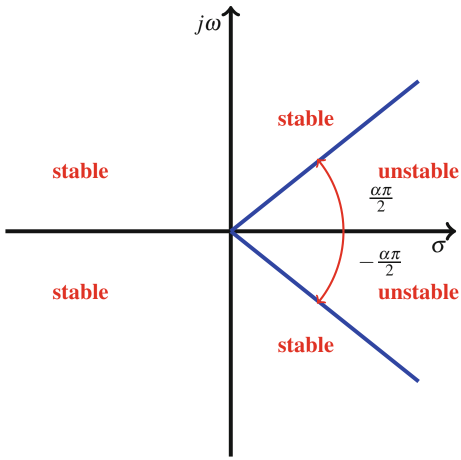
- The step response of the system is given as (2.68) by substituting $\alpha = 3/2$

$$v_0(t) = \mathscr{L}^{-1}\left\{\frac{1}{s(s^{3/2} + 1)}\right\} = H(t) - E_{3/2}[-t^{3/2}]. \tag{2.76}$$

- For the stability of (2.73), let the transformation is taken as $s^{1/2} = \boldsymbol{\Omega}$. Then the transfer function is given as

$$G(s) = \frac{V_0(s)}{V_I(s)} = \frac{1}{\boldsymbol{\Omega}^3 + 1}. \tag{2.77}$$

Fig. 2.1 $0 < \alpha < 1$



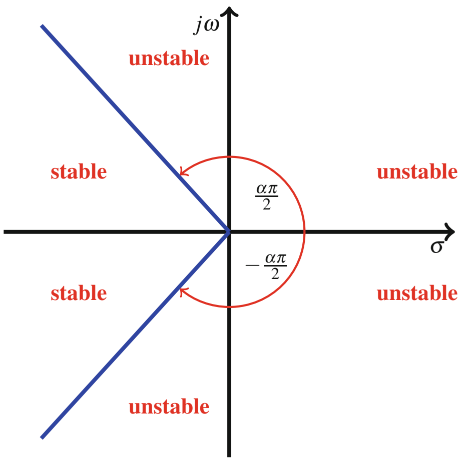


Fig. 2.2 $1 < \alpha < 2$

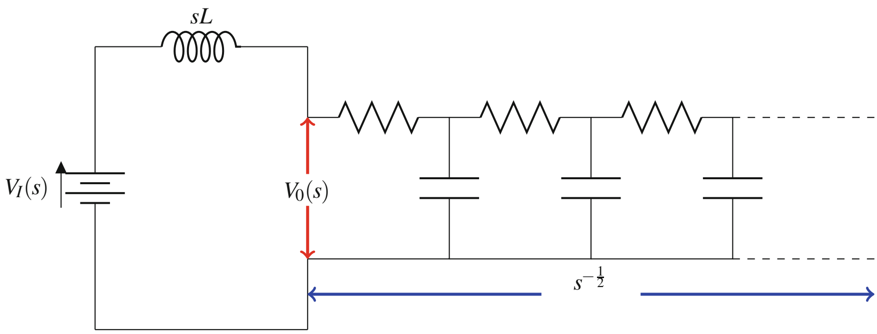


Fig. 2.3 Semi-infinite lossy line

The poles in the $\boldsymbol{\Omega}$-plane is $\boldsymbol{\Omega}_1 = -1$, $\boldsymbol{\Omega}_2 = e^{+\frac{j\pi}{3}}$ and $\boldsymbol{\Omega}_3 = e^{-\frac{j\pi}{3}}$. Hence, all the poles lie on the left of the instability wedge $\phi = \pm\frac{\pi}{4}$ and the system is stable. One can further note that instability wedge is calculated based on the mapping $s^{\frac{1}{2}} = \boldsymbol{\Omega}$, therefore it is $\pm\frac{\pi}{4}$.

Linear matrix inequality plays a very important role in control theory for both stability and stabilization of dynamical systems. In the next section, a brief review of LMI formulation of the fractional order linear systems is given.

2.4 A Brief Review on Linear Matrix Inequality (LMI) Stability Conditions for LTI Fractional Order Systems

State space representation [8] of a fractional order linear time-invariant system is given as

$$\begin{aligned} {}_0D_t^{\alpha}x(t) &= Ax(t) + Bu(t),\ x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{2.78}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$ and $y(t) \in \mathbb{R}^p$ are states, input and output vectors of the system and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ and $0 < \alpha < 2$ is the fractional commensurate order and pair (A, B) is controllable.

It has been well established in literature that the controllability and observability conditions of the continuous-time commensurate fractional order systems are same as that of the integer order case [9]. Thus, the system (5.1) is controllable if the rank of the controllability matrix

$$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}, \tag{2.79}$$

is equal to n. Similarly, the system (5.1) is observable if the rank of the observability matrix

$$\mathbb{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{2.80}$$

Theorem 2.18 [6, 10] *The system* ${}_0D_t^{\alpha}x(t) = Ax(t)$ *is asymptotically stable if the following condition is satisfied*

$$|arg(eig(A))| > \frac{\alpha\pi}{2}, \tag{2.81}$$

where $0 < \alpha < 2$ *and* $eig(A)$ *are eigenvalues of matrix* A.

Based on the mapping in $\boldsymbol{\Omega}$ plane the following theorem has been proved for the stability of linear time invariant fractional order system

Theorem 2.19 [11, 12] *The fractional order linear time invariant system* ${}_0D_t^{\alpha}x(t) = Ax(t)$ *where* $1 < \alpha < 2$, *is* $t^{-\alpha}$ *stable if and only if there exist a positive definite Hermitian matrix* $P = P^* > 0$ *such that*

$$\beta PA + \beta^* A^T P < 0, \tag{2.82}$$

where $\beta = \eta + j\zeta$ *and* η, ζ *are defined from* $tan(\frac{\pi}{2} - \theta) = \frac{\eta}{\zeta}$ *with* $\theta = (\alpha - 1)\frac{\pi}{2}$.

Based on the above Theorem, the following lemma is recently derived.

Lemma 2.20 [13] *The fractional order linear time invariant system* ${}_0D_t^{\alpha}x(t) = Ax(t)$ *where* $1 < \alpha < 2$, *is stable (regarding input and output) if and only if the following integer order system is stable*

$$\dot{x}(t) = (\beta A)x(t). \tag{2.83}$$

where $\beta = \eta + j\zeta$ *and* η, ζ *are defined from* $tan(\frac{\pi}{2} - \theta) = \frac{\eta}{\zeta}$ *and* $\theta = (\alpha - 1)\frac{\pi}{2}$.

Remark 2.21 The above lemma establishes the relationship of the LMI inequality (2.82) with an integer order linear system (2.83) which ensures the stability of the linear time invariant fractional order system ${}_0D_t^{\alpha}x(t) = Ax(t)$. Therefore (2.83) is nothing but a shadow (equivalent) system of ${}_0D_t^{\alpha}x(t) = Ax(t)$ from the stability point of view.

The above mentioned stability condition can be used to illustrate the state feedback control design for the fractional system (5.1). Let us consider a stabilizing control of the form $u = Kx$. The closed loop system becomes

$${}_0D_t^{\alpha}x(t) = (A + BK)x(t) \tag{2.84}$$

The necessary and sufficient condition for stability of system (2.84), according to Lemma (2.20) and Theorem (2.19),

$$\beta P(A + BK) + \beta^*(A + BK)^T P \leq -R \ \ \forall x \in \mathbb{R}^n, \tag{2.85}$$

where P and R are the symmetric positive definite matrices.

Following remark is necessary for checking the negative definiteness of complex Hermitian matrix H.

Remark 2.22 [11] A complex Hermitian matrix H is negative definite($H < 0$), if and only if

$$\begin{bmatrix} \mathrm{Re}(H) & \mathrm{Im}(H) \\ -\mathrm{Im}(H) & \mathrm{Re}(H) \end{bmatrix} < 0, \tag{2.86}$$

where $\mathrm{Re}(H)$ and $\mathrm{Im}(H)$ are the real and imaginary part of Hermitian matrix H respectively.

Theorem 2.23 [11, 12] *The fractional order linear time invariant system* ${}_0D_t^{\alpha}x(t) = Ax(t)$ *where* $1 < \alpha < 2$, *is* $t^{-\alpha}$ *asymptotically stable if and only if there exist a positive symmetric definite matrix* $P = P^{\top} > 0$, $P \in \mathbb{R}^{n\times n}$, *such that*

$$\begin{bmatrix} (A^{\top}P + PA)\sin(\alpha\frac{\pi}{2}) & (A^{\top}P - PA)\cos(\alpha\frac{\pi}{2}) \\ (PA - A^{\top}P)\cos(\alpha\frac{\pi}{2}) & (A^{\top}P + PA)\sin(\alpha\frac{\pi}{2}) \end{bmatrix} < 0. \tag{2.87}$$

Proof The system ${}_0D_t^{\alpha} = Ax(t)$ is $t^{-\alpha}$ asymptotically stable if the following condition is satisfied

$$|\arg(\lambda)| > \frac{\alpha\pi}{2},$$

where $0 < \alpha < 2$ and λ are eigenvalues of the matrix A. Now define following regions

- rotate λ by an angles $(\alpha - 1)\frac{\pi}{2}$

$$R1 = \left\{\lambda \in \mathbb{C} : \mathrm{Re}\left(\lambda e^{j(\alpha-1)\frac{\pi}{2}}\right) < 0\right\}, \tag{2.88}$$

  and
- rotate λ by an angles $(1 - \alpha)\frac{\pi}{2}$

$$R2 = \left\{\lambda \in \mathbb{C} : \mathrm{Re}\left(\lambda e^{j(1-\alpha)\frac{\pi}{2}}\right) < 0\right\}, \tag{2.89}$$

where λ belongs to $R = R1 + R2$, $\mathbb{C}$: represents complex number and Re: represents real part. As for some $\lambda \in \mathrm{spec}(A)$, conjugate $\lambda^* \in \mathrm{spec}(A)$, and as $R1$ and $R2$ are symmetric with respect to the real axis of the complex plane.

$$\exists\lambda_1 \in \mathrm{spec}(A),\ \lambda_1 \in R1 \Leftrightarrow \exists\lambda_2 \in \mathrm{spec}(A),\ \lambda_2 \in R2, \tag{2.90}$$

hence only relation (2.89) is necessary to derive the stability. Also, (2.89) can be written as

$$\begin{aligned}
&\lambda e^{j(1-\alpha)\frac{\pi}{2}} + \lambda^* e^{-j(1-\alpha)\frac{\pi}{2}} < 0 \\
&\quad \Rightarrow \lambda\left(\cos\left((1-\alpha)\frac{\pi}{2}\right) + j\sin\left((1-\alpha)\frac{\pi}{2}\right)\right) \\
&\qquad + \lambda^*\left(\cos\left((\alpha-1)\frac{\pi}{2}\right) + j\sin\left((\alpha-1)\frac{\pi}{2}\right)\right) < 0,
\end{aligned} \tag{2.91}$$

because for any complex number z, $\mathrm{Re}(z) = \frac{z+z^*}{2}$. From, above relation is true (see Boyd 1994 [14]) if and only if $\exists P > 0,\ P \in \mathbb{R}^{n\times n}$ the following LMI is feasible

$$(A^\top P + PA)\sin\left(\alpha\frac{\pi}{2}\right) + j(A^\top P - PA)\cos\left(\alpha\frac{\pi}{2}\right) \le 0. \tag{2.92}$$

As an LMI involving real term can be derived from a complex one, the problem becomes:

$$\begin{bmatrix} (A^\top P + PA)\sin(\alpha\frac{\pi}{2}) & (A^\top P - PA)\cos(\alpha\frac{\pi}{2}) \\ (PA - A^\top P)\cos(\alpha\frac{\pi}{2}) & (A^\top P + PA)\sin(\alpha\frac{\pi}{2}) \end{bmatrix} < 0.$$

When $0 < \alpha < 1$, the stability domain is not convex. Due to the absence of the convexity property, the LMI conditions can not be derived directly as in the case of integer order or fractional order with $1 < \alpha < 2$. However, in literature different approaches are suggested to by pass this problem and LMI condition are derived indirectly. Some of the well recognized results which exist in literature are discussed here.

Theorem 2.24 *The fractional order linear time invariant system ${}_0D_t^{\alpha}x(t) = Ax(t)$ where $0 < \alpha < 1$, is $t^{-\alpha}$ asymptotically stable if and only if there exist positive definite Hermitian matrices $H_1 = H_1^* \in \mathbb{C}^{n\times n}$ and $H_2 = H_2^* \in \mathbb{C}^{n\times n}$ such that*

$$\bar{r}H_1A^\top + rAH_1 + rH_2A^\top + \bar{r}AH_2 < 0, \tag{2.93}$$

where $r = e^{j(1-\alpha)\frac{\pi}{2}}$.

Theorem 2.25 [15] *The fractional order linear time invariant system ${}_0D_t^{\alpha}x(t) = Ax(t)$ where $0 < \alpha < 1$, is asymptotically stable if and only if there exist two real positive symmetric definite matrices $P_{k1} \in \mathbb{R}^{n\times n}$, $k = 1, 2$, and two skew-symmetric matrices $P_{k2} \in \mathbb{R}^{n\times n}$, $k = 1, 2$ such that*

$$\sum_{i=1}^{2}\sum_{j=1}^{2} Sym\left\{\Theta_{ij} \otimes (AP_{ij})\right\} < 0, \tag{2.94}$$

$$\begin{bmatrix} P_{11} & P_{12} \\ -P_{12} & P_{11} \end{bmatrix} > 0, \quad \begin{bmatrix} P_{21} & P_{22} \\ -P_{22} & P_{21} \end{bmatrix} > 0, \tag{2.95}$$

where

$$\begin{aligned} \Theta_{11} &= \begin{bmatrix} \sin\left(\alpha\frac{\pi}{2}\right) & -\cos\left(\alpha\frac{\pi}{2}\right) \\ \cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \end{bmatrix}, \quad \Theta_{12} = \begin{bmatrix} \cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \\ -\sin\left(\alpha\frac{\pi}{2}\right) & \cos\left(\alpha\frac{\pi}{2}\right) \end{bmatrix} \\ \Theta_{11} &= \begin{bmatrix} \sin\left(\alpha\frac{\pi}{2}\right) & \cos\left(\alpha\frac{\pi}{2}\right) \\ -\cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \end{bmatrix}, \quad \Theta_{22} = \begin{bmatrix} -\cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \\ -\sin\left(\alpha\frac{\pi}{2}\right) & -\cos\left(\alpha\frac{\pi}{2}\right) \end{bmatrix}, \end{aligned} \tag{2.96}$$

where $Sym\{X\}$ denotes the expression $X^\top + X$ and $\otimes$ is the Kronecker product of two matrices.

Proof Let us define $P_{k1} = \mathrm{Re}(H_k)$, $P_{k2} = \mathrm{Re}(H_k)$, $k = 1, 2$. Since $P_{k1} - jP_{k2}^\top = P_{k1} + jP_{k2}$, $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$, using Theorem 2.24 one can write

$$P_{11} + jP_{12} > 0, \quad P_{21} + jP_{22} > 0, \tag{2.97}$$

which is equivalent to (2.95) and

$$\begin{aligned}
&(\cos\theta + j\sin\theta)(P_{11} - jP_{12}^{\top})A^{\top} + (\cos\theta - j\sin\theta)A(P_{11} + jP_{12}) \\
&+ (\cos\theta - j\sin\theta)(P_{21} - jP_{22}^{\top})A^{\top} + (\cos\theta + j\sin\theta)A(P_{21} + jP_{22}) \\
&< 0
\end{aligned} \tag{2.98}$$

where $\theta = (1-\alpha)\frac{\pi}{2}$. Further, (2.98) can be written as

$$\begin{aligned}
&\mathrm{Sym}\{AP_{11}\cos\theta + AP_{12}\sin\theta + AP_{12}\cos\theta - AP_{22}\sin\theta\} + j(P_{11}A^{\top} - AP_{11})\sin\theta \\
&+ j(-P_{12}^{\top}A^{T} + AP_{12})\cos\theta + j(-P_{12}A^{-\top} + AP_{21})\sin\theta + j(-P_{22}^{\top}A^{\top} \\
&+ AP_{22})\cos\theta < 0,
\end{aligned} \tag{2.99}$$

which can be further written as

$$\begin{aligned}
&\mathrm{Sym}\left\{AP_{11}\sin\left(\alpha\frac{\pi}{2}\right) + AP_{12}\cos\left(\alpha\frac{\pi}{2}\right)\right\} + \mathrm{Sym}\left\{AP_{21}\sin\left(\alpha\frac{\pi}{2}\right) - AP_{22}\cos\left(\alpha\frac{\pi}{2}\right)\right\} \\
&+ j(P_{11}A^{\top} - AP_{11})\cos\left(\alpha\frac{\pi}{2}\right) + j(AP_{12} - P_{12}^{\top}A^{\top})\sin\left(\alpha\frac{\pi}{2}\right) \\
&+ j(P_{21}A^{\top} - P_{21}A^{T})\cos\left(\alpha\frac{\pi}{2}\right) + j(AP_{22} - P_{22}^{\top}A^{\top})\sin\left(\alpha\frac{\pi}{2}\right) < 0,
\end{aligned} \tag{2.100}$$

this is equivalent to (2.94). This ends the proof.

One more LMI based stability theorem for $0 < \alpha < 1$ is proposed by Sabatier et al.(2010) [6] based on the equivalence of fractional order system with integer order and by analyzing the geometric property of stability domain, which is stated as:

Theorem 2.26 *The fractional order linear time invariant system* ${}_0D_t^{\alpha}x(t) = Ax(t)$ *where* $0 < \alpha < 1$*, is asymptotically stable if and only if there exists a positive definite matrix* $P \in \mathbb{S}$*, where* $\mathbb{S}$ *denotes the set of symmetric matrices, such that*

$$\left(-(-A)^{\frac{1}{2-\alpha}}\right)^{\top} P + P\left(-(-A)^{\frac{1}{2-\alpha}}\right) < 0. \tag{2.101}$$

A Large class of the dynamical systems used for practical applications, are nonlinear in nature. So stability and stabilization of this class of the systems are also very important. In the next subsection a brief review of the stability of fractional order nonlinear system based on the second method of Lyapunov is discussed.

2.4.1 A Brief Review of the Stability of Nonlinear Fractional Order Systems Based on Lyapunov Function

It is well known fact that Lyapunov's second method provides a platform to analyze the stability of the system without solving explicitly the differential equations. This analysis is recently extended by the Li et al. [16] for the fractional order system also. While obtaining the solution of fractional order differential equation, it is seen that it contains more generalized function that is called Mittag-Leffler function, rather than the exponential function (in case of integer order differential equation). Therefore, more generalized stability concept, which is called Mittag-Leffler stability is defined for the fractional order. Following Theorems are reported in Li et al. [16].

Theorem 2.27 *Let $x = 0$ be an equilibrium point for the autonomous fractional-order system*

$$ {}_0D_t^{\alpha} x(t) = f(t, x), \tag{2.102}$$

and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exist a Lyapunov candidate $V(t, x(t)) : [0, \infty) \times \mathbb{D} \rightarrow \mathbb{R}$, which is a continuously differentiable function and locally Lipschitz with respect to x such that

$$\begin{aligned} \alpha_1 ||x||^a \leq V(t, x) \leq \alpha_2 ||x||^{ab} \\ {}_0^C D_t^{\alpha} V(t, x) \leq -\alpha_3 ||x||^{ab}, \end{aligned} \tag{2.103}$$

where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0, 1)$, $\alpha_1, \alpha_2, \alpha_3$, α, a and b are arbitrary positive constants. Then $x = 0$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag-Leffler stable.

Proof Using (2.103), one can write

$$ {}_0^C D_t^{\alpha} V(t, x) \leq -\frac{\alpha_3}{\alpha_2} V(t, x). \tag{2.104}$$

It is always possible to find a function $\eta(t)$, such that

$$ {}_0^C D_t^{\alpha} V(t, x) + \eta(t) = -\frac{\alpha_3}{\alpha_2} V(t, x). \tag{2.105}$$

After taking Laplace transform,

$$\begin{aligned} s^{\alpha} V(s) - V(0) s^{\alpha-1} + \eta(s) = -\frac{\alpha_3}{\alpha_2} V(s) \\ V(s) = \frac{V(0) s^{\alpha-1} - \eta(s)}{s^{\alpha} + \frac{\alpha_3}{\alpha_2}}, \end{aligned} \tag{2.106}$$

where $V(0) = V(0, x(0))$ and $V(s) = \mathscr{L}\{V(t, x)\}$. If $x(0) = 0 \Rightarrow V(0) = 0$, the solution of (6.2) is $x = 0$. For the case when $x(0) \neq 0 \Rightarrow V(0) > 0$, the inverse

Laplace of (2.106) is given as

$$V(t) = V(0)E_\alpha\left(-\frac{\alpha_3}{\alpha_2}t^\alpha\right) - \eta(t) * \left[t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\alpha_3}{\alpha_2}t^\alpha\right)\right]. \tag{2.107}$$

Since $t^{\alpha-1} > 0$ and $E_{\alpha,\alpha}\left(-\frac{\alpha_3}{\alpha_2}t^\alpha\right) > 0$, it follows from (2.107) that

$$V(t) \le V(0)E_\alpha\left(-\frac{\alpha_3}{\alpha_2}t^\alpha\right). \tag{2.108}$$

After substitution of (2.108) into (2.103)

$$\|x(t)\| \le \left[\frac{V(0)}{\alpha_1}E_\alpha\left(-\frac{\alpha_3}{\alpha_2}t^\alpha\right)\right]^{\frac{1}{a}}, \tag{2.109}$$

where $\frac{V(0)}{\alpha_1} > 0$ for $x(0) \neq 0$. Also, $x(0) = 0$ only when $\frac{V(0)}{\alpha_1} = 0$, because $V(t, x)$ is locally Lipschitz with respect to x. Further, using (2.109) it is concluded that (6.2) is Mittag-Leffler stable.

When system (6.2) is represented using Riemann-Liouville definition, then following inequality is required which is already discussed in (2.53)

$${}_{t_0}^{C}D_t^\alpha V(t) \le {}_{t_0}^{RL}D_t^\alpha V(t) \le -\alpha_3\|x\|^{ab}, \tag{2.110}$$

which further implies

$$\|x(t)\| \le \left[\frac{V(0)}{\alpha_1}E_\alpha\left(-\frac{\alpha_3}{\alpha_2}t^\alpha\right)\right]^{\frac{1}{a}}.$$

This ends the proof.

Similar kind of proof can be extended for these two theorems also.

Theorem 2.28 [4] *Let $x = 0$ be an equilibrium point for (either Caputo or Riemann-Liouville) autonomous fractional-order system (6.2), where $f(t, x)$ satisfies the Lipschitz condition with Lipschitz constant $L > 0$ and $\alpha \in (0, 1)$. Assume that there exist a Lyapunov candidate $V(t, x(t))$ satisfying*

$$\begin{aligned} \alpha_1||x||^a \le V(t, x) &\le \alpha_2||x|| \\ \dot{V}(t, x) &\le -\alpha_3||x||, \end{aligned} \tag{2.111}$$

where $\alpha_1, \alpha_2, \alpha_3$ and a are positive constants and $||.||$ denotes an arbitrary norm. Then the equilibrium point of the system (6.2) is Mittag-Leffer stable.

Theorem 2.29 [4] *Let $x = 0$ be an equilibrium point for the autonomous fractional-order system (6.2). Assume that there exists a Lyapunov function $V(t, x(t))$ and* *class-κ functions* $\alpha_1 (i = 1, 2, 3)$ *satisfying*

$$\begin{aligned}\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||)\\ {}_0^C D_t^\alpha V(t, x) \leq -\alpha_3(||x||).\end{aligned} \tag{2.112}$$

Then the system (6.2) is asymptotically stable.

In general fractional calculus offers the following advantages to control engineering.

- adequate modeling of control plant's dynamic features
- effective robust control design
- reasonable realization by approximation.

A brief survey has been already presented in the initial part of the book about the first feature. Second feature is actually the main concern of this monograph and the last point is discussed in the next section.

2.5 Realization Issue of Fractional-Order Controller

It is obvious from the definition of fractional order operators that, to realize fractional order controllers perfectly, all the past input should be memorized. However, this is not possible in real scenario. Therefore, proper approximation by finite differential or difference equation must be introduced. There are many approximation methods exist in the literature. But, the most commonly used discretization method of a fractional-order controller is termed as *short memory principle* [2]. This discretization is based on the philosophy that, for the Grünwald-Letnikov definition, the values of the binomial coefficients near "*starting point*" $t = 0$ are small enough to be neglected or "forgotten" for large t. Therefore short memory principle takes into account the behavior of $f(t)$ only in "*recent past*", i.e., in the interval $[t - L, t]$, where L is the length of "*memory*"

$${}_{t_0}D_t^\alpha f(t) \approx {}_{t-L}D_t^\alpha f(t), \; t > t_0 + L, \; \alpha > 0. \tag{2.113}$$

Based on approximation of the time increment h through the sampling time T, the discrete equivalent of the fractional-order α derivative is given by

$$Z\left\{D^\alpha f(t)\right\} \approx \left(\frac{1}{T^\alpha}\sum_{j=0}^{m} c_j z^{-j}\right) F(z), \tag{2.114}$$

where $F(z) = Z\{f(t)\}$ $m = [L/T]$ and the coefficients c_j are

$$c_0 = 1$$

$$c_j = (-1)^j \binom{\alpha}{j} = \frac{j-\alpha-1}{j} c_{j-1}, \quad j \geq 1. \tag{2.115}$$

The following theorem is useful for showing the reasonability of the above approximation.

Theorem 2.30 *When the Riemann-Liouville (or the Grünwald-Letnikoff) definition is used, if* $|f(t)| < M$, $\forall t > t_0$, *then the error* ε *committed by approximation* (2.113) *is bounded by*

$$|\varepsilon| < \frac{M}{L^\alpha |\Gamma(1-\alpha)|}. \tag{2.116}$$

Proof When $\alpha < 0$,

$$|\varepsilon| < \left| {}_{t_0}D_t^\alpha f(t) - {}_{t-L}D_t^\alpha f(t) \right| = \left| {}_{t_0}D_{t-L}^\alpha f(t) \right|. \tag{2.117}$$

$$\begin{aligned} |\varepsilon| &\leq \left| \int_{t_0}^{t-L} \frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)} f(\tau) d\tau \right| \leq \left| \int_{t_0}^{t-L} \frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)} M d\tau \right| \\ &= \left| \frac{M}{\Gamma(-\alpha)} \left[\frac{-1}{\alpha} (t-\tau)^{-\alpha} \right]_{t_0}^{t-L} \right| = \left| \frac{M[(t-t_0)^{-\alpha} - L^{-\alpha}]}{\Gamma(1-\alpha)} \right|. \end{aligned} \tag{2.118}$$

Note This is not a very useful bound, because it grows with $t - t_0$ and can become very large. But if $\alpha > 0$

$$\begin{aligned} |\varepsilon| &\leq \left| \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} {}_{t_0}D_t^{\alpha-\lceil\alpha\rceil} f(t) - \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} {}_{t-L}D_t^{\alpha-\lceil\alpha\rceil} f(t) \right| \\ &= \left| \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} {}_{t_0}D_{t-L}^{\alpha-\lceil\alpha\rceil} f(t) \right|. \end{aligned} \tag{2.119}$$

Using (2.118)

$$|\varepsilon| \leq \left| \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \frac{M[(t-t_0)^{-\alpha+\lceil\alpha\rceil} - L^{-\alpha+\lceil\alpha\rceil}]}{\Gamma(1-\alpha+\lceil\alpha\rceil)} \right|. \tag{2.120}$$

Note that

$${}_0D_t^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}, \ t \in \mathbb{R}^+, \ \lambda \notin \mathbb{Z}^-. \tag{2.121}$$

Using (2.121), one can write

$$|\varepsilon| \le \left| \frac{M(t-t_0)^{-\alpha}\Gamma(1-\alpha+\lceil\alpha\rceil)}{\Gamma(1-\alpha+\lceil\alpha\rceil-\lceil\alpha\rceil)\Gamma(1-\alpha+\lceil\alpha\rceil)} \right| = \frac{M}{(t-t_0)^{\alpha}|\Gamma(1-\alpha)|}. \quad (2.122)$$

Since $t - t_0 > 0$ which implies $(t-t_0)^{\alpha} > L^{\alpha} \Rightarrow \frac{1}{(t-t_0)^{\alpha}} < \frac{1}{L^{\alpha}}$

$$|\varepsilon| \le \frac{M}{(t-t_0)^{\alpha}|\Gamma(1-\alpha)|} < \frac{M}{L^{\alpha}\Gamma(1-\alpha)}. \quad (2.123)$$

This ends the proof.

Remark 2.31 The above theorem is known as short memory principle, because (2.113) corresponds to a shortening of the memory of operator D, which remembers nothing older than L (thereby called memory length).

Corollary 2.32 *Thus to ensure the absolute value of error ε should not be larger than a certain value, the memory length in approximation* (2.113) *must satisfy*

$$L \ge \left(\frac{M}{|\varepsilon\Gamma(1-\alpha)|} \right)^{\frac{1}{\alpha}}, \quad \alpha > 0. \quad (2.124)$$

2.6 A Brief Review of Fractional Order PID Control

History of fractional order control started from the work of Bode [17, 18]. He formulated a problem to design a feedback amplifier to devise a feedback loop, so that the performance of the close-loop is invariant to changes in the amplifier gain. He gave a simple and elegant solution for this specified problem, which is termed as Bode's ideal loop transfer function, whose Nyquist plot is a straight line through the origin giving a phase margin invariant to gain changes. The Bode's ideal transfer function is represented as $G(s) = (\omega_0/s)^{\alpha}$, where $0 < \alpha < 1$, ω_0 is the gain crossover frequency and the constant phase margin is $\varphi_m = \pi - \frac{\alpha\pi}{2}$.

Above frequency characteristic is very interesting in term of robustness of the system to parameter changes or uncertainties. In fact, the fractional order integrator can be used as an alternative for more robust reference system for control. The frequency characteristics and the transient response of the non-integer order integral and its application to the control system was introduced by Manabe [19] and more recently by Barbosa et al. [20].

Analysis and design of controller for linear and nonlinear fractional order dynamical systems are easy and more efficient in time domain. Because, frequency domain approach is not easily extendable for the nonlinear or linear fractional order systems with disturbances which occurs in most of the cases. Several time design based controller design approaches are existing in the literature for the fractional order system as that for integer order system. One of the most successful controllers which is popular and useful in the practical industries is fractional order PID. The fractional order PID controller, namely the $PI^{\lambda}D^{\mu}$, which is the generalization of the classical PID controller is proposed by Podlubny [21] and Oustaloup [22–24]. In their series of papers and books Podlubny [2] and Oustaloup [22–24] successfully used the fractional order controller to develop the CRONE-controller (Commande Robuste d'Ordre Non Entrier controller), which is an interesting example of application of fractional calculus in control. He also demonstrated the superiority of fractional order $PI^{\lambda}D^{\mu}$ controller in comparison to the classical PID controller both for the fractional order and integer order dynamical systems.

Fractional order dynamical system can be represented in the time domain by the following differential equation

$$\left[\sum_{j=0}^{n} a_{n-j} D^{\alpha_{n-j}}\right] y(t) = f(t), \tag{2.125}$$

where $\alpha_{n-j} > \alpha_{n-j-1} (j = 0, 1, 2, \cdots, n) \in \mathbb{R}^{+}$, α_{n-j} are arbitrary constants, and $D^{\alpha} = {}_{0}^{C}D_{t}^{\alpha}$ denotes Caputo's fractional-order derivative of order α. The fractional-order transfer function for the system represented as (2.125) is given by

$$G_n(s) = \left[\sum_{j=0}^{n} a_{n-j} s^{\alpha_{n-j}}\right]^{-1}. \tag{2.126}$$

The unit-impulse response $y_i(t)$ of the system is given as follows

$$y_i(t) = \mathscr{L}^{-1}\{G_n(s)\} = g_n(t), \tag{2.127}$$

and the unit-step response function is given by the integral of the $g_n(t)$ so that

$$y_s(t) = \int_{0}^{t} g_n(\tau) d\tau. \tag{2.128}$$

2.6.1 Brief Overview of Fractional Order Integral Action

Following main effects are observed in case of integral actions

- it makes the system response slower
- decreases the system relative stability
- eliminates the steady state error for those inputs, for which the system had a finite error.

The effects of PID actions of a controller are analyzed using complex plane, time domain and frequncy domain methods. Similarly the fractional PID controller can be analyzed using these same techniques. For example consider the close loop system as shown in Fig. 2.4.

2.6.1.1 Complex Plane Analysis

In the complex plane, root locus of the system is displaced towards the right half plane after applying the integral action. Mathematically, the root locus of the system with control action is governed by

$$1 + Ks^{\alpha}G(s) = 0 \tag{2.129}$$

Its magnitude and phase is given as

$$\begin{aligned} |K| &= \frac{1}{|s^{\alpha}||G(s)|} \\ \arg\left[s^{\alpha}G(s)\right] &= (2n+1)\pi, \quad n = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{2.130}$$

$s = |s|e^{j\theta}$ can be written as

$$s^{\alpha} = |s|^{\alpha}e^{j\alpha\theta}. \tag{2.131}$$

The conditions of phase can be further expressed as

$$\arg\left[s^{\alpha}G(s)\right] = \arg\left[G(s)\right] + \alpha\theta = (2n+1)\pi, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{2.132}$$

Therefore, it is obvious that, by choosing $\alpha \in (-1, 0)$, the root locus is displaced towards the right half plane.

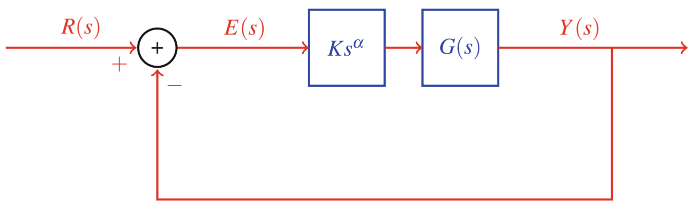


Fig. 2.4 Fractional integral action $\alpha \in (-1, 0)$

2.6.2 Frequency Domain Analysis

In frequency domain, a pole at zero adds $-20\,\text{dB/dec}$ in the magnitude curve and decreases the phase plot by $\pi/2$ rad. The effect of fractional order integral is explained as follows.

The magnitude curve in the frequency domain is given as

$$20\log\left[s^{\alpha}G(s)\right]_{s=jw} = 20\log|G(j\omega)| + 20\alpha\log\omega \tag{2.133}$$

and the phase plot is given by

$$\arg\left[s^{\alpha}G(s)\right]_{s=jw} = \arg|G(s)| + \alpha\frac{\pi}{2}. \tag{2.134}$$

Therefore, by varying the value of α between -1 and 0, it is possible to introduce a constant increment in the slope of the magnitude curve by introducing a fractional order integrator, which varies between -20 and 0 dB/dec. Similarly, a constant delay in phase plot, which varies between $-\frac{\pi}{2}$ and 0 rad.

2.6.3 Time Domain Analysis

By introducing a fractional order integrator, there are clear cut effects over the transient response, which consists of the decrease in the rise time, increase of the settling time and the overshoot. Mathematically, this effects can be studied considering the error signal of the following form

$$e(t) = \sum_{j=0}^{n}(-1)^{j}u_0(t - jT), \quad j = 0, 1, 2, \ldots, n \tag{2.135}$$

where $u_0(t)$ represents the unit step input. Its Laplace equivalent is given as

$$E(s) = \sum_{j=0}^{n}(-1)^{j}\frac{e^{-jTs}}{s}. \tag{2.136}$$

Therefore, the control action can be expressed as

$$\begin{aligned} u(t) &= \mathscr{L}^{-1}\{U(s)\} \\ &= \mathscr{L}^{-1}\left\{K\sum_{j=0}^{n}(-1)^{j}\frac{e^{-jTs}}{s^{1-\alpha}}\right\} \end{aligned}$$

$$= K \sum_{j=0}^{n} \frac{(-1)^j}{\Gamma(1-\alpha)} (t - jT)^{-\alpha} u_0 (t - jT) \tag{2.137}$$

It is clear that the control action over the error signal, vary between the effects of a proportional action $\alpha = 0$ (square signal) and an integral action $\alpha = -1$ (straight line curve). For the intermediate value of α, the control action increases for a constant error, which results in the elimination of the steady state error and decrease when error is zero, resulting a more stable system.

2.6.4 Brief Overview of Fractional Order Derivative Action

Same as fractional integral action, the derivative action of fractional order controller can be analyze in complex domain, frequency domain and time domain. For example consider the close loop system as shown in Fig. 2.5.

2.6.5 Complex Plane Analysis

In the complex plane, root locus of the system is displaced towards the left half plane after applying the derivative action.

2.6.6 Frequency Domain Analysis

In frequency domain, a derivative action of a controller adds a slope of +20 db/dec in the magnitude plot and adds $\pi/2$ radian in phase plot. Similarly in its fractional counter part, fractional derivative can add a slope of 0–20 db/dec, when β is varied. Similarly, a constant delay in phase plot, which varies between 0 and $\frac{\pi}{2}$ radian.

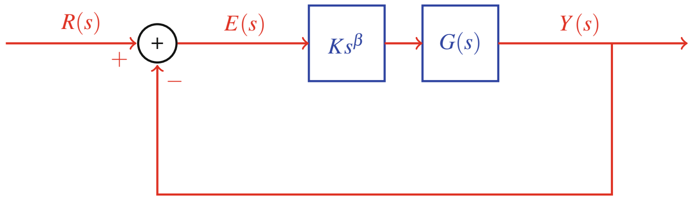


Fig. 2.5 Fractional derivative action $\beta \in (0, 1)$

2.6.7 Time Domain Analysis

In the time domain, a decrease in the overshoot and the settling time is observed. This can be studied using the trapezoidal error signal given as

$$e(t) = tu_0(t) - t(t-T)u_0(t-T) - t(t-2T)u_0(t-2T) + t(t-3T)u_0(t-3T), \tag{2.138}$$

where u_0 represents the unit step input. The laplace transform of (2.138) can be written as

$$E(s) = \frac{1}{s^2} - \frac{e^{-Ts}}{s^2} - \frac{e^{-2Ts}}{s^2} + \frac{e^{-3Ts}}{s^2}. \tag{2.139}$$

Therefore, the control action can be expressed as

$$\begin{aligned} u(t) &= \mathscr{L}^{-1}\{U(s)\} \\ &= \mathscr{L}^{-1}\left\{K\left(\frac{1}{s^{2-\beta}} - \frac{e^{-Ts}}{s^{2-\beta}} - \frac{e^{-2Ts}}{s^{2-\beta}} + -\frac{e^{-3Ts}}{s^{2-\beta}}\right)\right\} \\ &= \frac{K}{\Gamma(2-\beta)}\left\{t^{1-\beta}u_0(t) - (t-T)^{1-\beta}u_0(t-T) - (t-2T)^{1-\beta}u_0(t-2T)\right\} \\ &\quad + \frac{K}{\Gamma(2-\beta)}\left\{(t-3T)^{1-\beta}u_0(t-3T)\right\}. \end{aligned} \tag{2.140}$$

The effects of the fractional order control over the error signal vary between the effects of a proportional action $\beta = 0$ (trapezoidal signal) and a derivative action $\beta = 1$ (square signal).

2.6.8 The Fractional Order $PI^{\alpha}D^{\beta}$ Controller

The fractional order $PI^{\alpha}D^{\beta}$ controller is the generalization of the integer order PID controller. The transfer function of $PI^{\alpha}D^{\beta}$ controller is defined as the ratio of the controller output $U(s)$ and error $E(s)$ as

$$G(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\alpha} + K_D s^{\beta}, \quad \alpha, \beta > 0. \tag{2.141}$$

Output $u(t) = \mathscr{L}^{-1}\{U(s)\}$, in the time domain as

$$u(t) = K_P e(t) + K_I D^{-\alpha} e(t) + K_D D^{\beta} e(t), \tag{2.142}$$

It is quite natural to conclude that by introducing more general control actions of the form $PI^{\alpha}D^{\beta}$, one could achieve more satisfactory performances between positive and negative effects of classical PID, and combining the fractional order actions one could develop more powerful and flexible design methods to satisfy the controlled system specifications.

2.6.9 Unit-Impulse and Unit-Step Response of the some Simple Transfer Function

Example Consider the following transfer function

$$G(s) = \left(a_0 s^{\alpha} + b_0\right)^{-1}, \quad \alpha > 0. \tag{2.143}$$

The unit-impulse and unit-step response of (2.143) are given as follows

$$y_i(t) = g(t) = \mathscr{L}^{-1}\left\{a_0 s^{\alpha} + b_0\right\}^{-1} = \frac{1}{a_0}\xi_0\left(t, -\frac{b_0}{a_0}; \alpha, \alpha\right). \tag{2.144}$$

$$y_s(t) = \int_0^t g(t) = \frac{1}{a_0}\xi_0\left(t, -\frac{b_0}{a_0}; \alpha, \alpha + 1\right). \tag{2.145}$$

where, $\xi_0(t, z; \alpha, \beta)$ is defined in terms of Mittage-Leffler's function $E_{\alpha,\beta}(x)$ as

$$\begin{aligned} \xi_0(t, z; \alpha, \beta) &= t^{\alpha j+\beta-1}E^{(j)}_{\alpha,\beta}(zt^{\alpha}), \quad m = 0, 1, \ldots \\ E^{(j)}_{\alpha,\beta}(x) &= \frac{d^j}{dx^j}E_{\alpha,\beta}(x), \\ E_{\alpha,\beta}(x) &= \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0,\ \beta > 0. \end{aligned} \tag{2.146}$$

and the Laplace transfer is

$$\mathscr{L}\left\{t^{\alpha j+\beta-1}E^{(j)}_{\alpha,\beta}(\pm a_0 t^{\alpha})\right\} = \frac{j!s^{\alpha-\beta}}{(s^{\alpha} \mp a_0)^{m+1}}. \tag{2.147}$$

The function ξ_j satisfies the property

$${}_0D^n_t\xi_m(t, z; \alpha, \beta) = \xi_j(t, z; \alpha, \beta - n), \quad \beta > n. \tag{2.148}$$

Example Consider the following transfer function

$$G(s) = \frac{1}{\left(a_0 s^{\alpha} + b_0 s^{\beta} + c_0\right)}, \quad \alpha > \beta > 0 \tag{2.149}$$

The fractional differential equation in the time domain is expressed as

$$a_0 y^{(\alpha)}(t) + b_0 y^{(\beta)}(t) + cy(t) = f(t) \tag{2.150}$$

with the following initial conditions

$$y(0) = y^{(1)}(0) = y^{(2)}(0) = 0. \tag{2.151}$$

The unit-impulse and unit-step response of (2.149) are given as follows

$$y_i(t) = g(t) = \mathscr{L}^{-1}\left\{G(s)\right\}, \tag{2.152}$$

$$y_s(t) = \int_0^t g(t) = \frac{1}{a_0} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left(\frac{c_0}{a_0}\right)^j \xi_j \left(t, \frac{b_0}{a_0}; \alpha - \beta, \alpha + \beta j + 1\right). \tag{2.153}$$

2.7 Summary

This chapter discussed the solution issues of fractional differential equations and Mittag-Leffler function. Along this a brief summary of the notation of stability for the fractional order system are also presented. Linear matrix inequality (LMI) is the one of the most important tool to analyzed the stability and stabilization problem in control theory, this chapter present a brief review on that also. Ideal realization of the fractional order controller requires infinite memory, which is not practically feasible. Therefore, concept of short memory principle comes into picture. This issue is also discussed in this Chapter. Finally, fractional order PID and its benefits over classical PID control is presented.

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