Chapter 2
Recursive Filtering with Missing Measurements and Quantized Effects

In this chapter, we consider the recursive filtering problems for discrete time-varying non-linear systems with missing measurements over a finite horizon. Firstly, the EKF problem is investigated for a class of discrete time-varying non-linear stochastic systems with multiple missing measurements. Both deterministic and stochastic non-linearities are considered in the system model, where the stochastic non-linearities are described by statistical means that could reflect the multiplicative stochastic disturbances. The phenomenon of measurement missing occurs in a random way, and the missing probability for each sensor is governed by an individual random variable satisfying a certain probability distribution over the interval [0, 1]. Such a probability distribution is allowed to be any commonly used probability distribution over the interval [0, 1] with known conditional probability. The focus of the addressed filtering problem is to design a recursive time-varying filter such that, in the presence of both the stochastic non-linearities and multiple missing measurements, there exists an upper bound for the filtering error covariance. Subsequently, such an upper bound is minimized by properly designing the filter gain at each sampling instant. It is shown that the desired filter can be obtained in terms of the solutions to two Riccati-like difference equations that are of a recursive form suitable for computation in online applications. Secondly, the proposed recursive filtering scheme is extended to study the filtering problem for time-varying systems with missing measurements, quantization effects, and multiplicative noises. The quantization phenomenon is described by using the logarithmic function and the multiplicative noises are considered to account for the stochastic disturbances on the system states. Accordingly, a set of parallel results is obtained by using the similar techniques. Finally, three illustrative examples are provided to demonstrate the effectiveness and applicability of the proposed filter design schemes.

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2.1 Extended Kalman Filtering with Multiple Missing Measurements

In this section, the EKF is designed for a general class of time-varying non-linear stochastic systems with multiple missing measurements and stochastic non-linearities. By employing the Riccati-like difference equation approach, the filter gains are obtained such that the upper bound of the filtering error covariance is minimized.

2.1.1 Problem Formulation

In this section, we consider the filtering problem for a general class of discrete time-varying non-linear systems with stochastic non-linearities and multiple missing measurements, where the schematic diagram is shown in Fig. 2.1. The plant under consideration is of the following form:

$$x_{k+1} = f(x_k) + g(x_k, \eta_k) + D_k \omega_k, \tag{2.1}$$

$$y_k = \Xi_k h(x_k) + s(x_k, \zeta_k) + \nu_k, \tag{2.2}$$

where k is the sampling instant, $x_k \in \mathbb{R}^n$ is the state vector to be estimated, $y_k \in \mathbb{R}^q$ is the measurement output, η_k and ζ_k are zero-mean Gaussian noise sequences, D_k is a known matrix of appropriate dimension, $\omega_k \in \mathbb{R}^m$ is the process noise, and $\nu_k \in \mathbb{R}^q$ is the measurement noise. $\Xi_k := \mathrm{diag}\{\alpha_k^1, \alpha_k^2, \ldots, \alpha_k^q\}$ where α_k^i $(i = 1, 2, \ldots, q)$ are q independent random variables in k and i and are independent of all noise signals. It is assumed that α_k^i has the probability density function $p_k^i(s)$ on the interval $[0, 1]$ with mathematical expectation μ_k^i and variance $(\sigma_k^i)^2$ $(i = 1, 2, \ldots, q)$. Also, the noise signals $\eta_k, \zeta_k, \omega_k$, and ν_k are uncorrelated with each other.

The deterministic non-linearities $f(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ and $h(x_k) : \mathbb{R}^n \to \mathbb{R}^q$ are known and continuously differentiable with

$$\|h(x_k)\| \leq a_1 \|x_k\| + a_2, \tag{2.3}$$

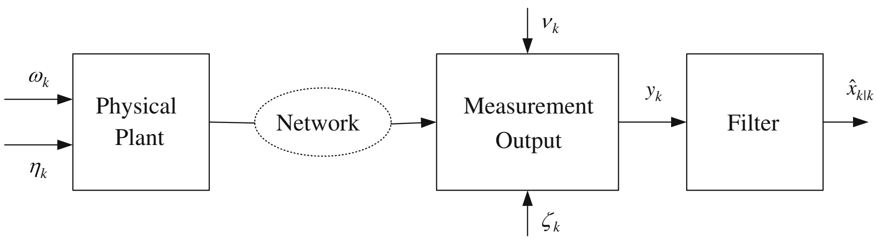

Fig. 2.1 Schematic structure for the plant and filter over network

for some non-negative scalars a_1 and a_2. On the other hand, the stochastic non-linearities $g(x_k, \eta_k) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $s(x_k, \zeta_k) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$ satisfy $g(0, \eta_k) = 0$ and $s(0, \zeta_k) = 0$, respectively, and are assumed to have the following first moment for all x_k:

$$\mathbb{E}\left\{\left[\begin{array}{c} g(x_k, \eta_k) \\ s(x_k, \zeta_k) \end{array}\right] \middle| x_k\right\} = 0, \tag{2.4}$$

and the covariance given by

$$\mathbb{E}\left\{\left[\begin{array}{c} g(x_k, \eta_k) \\ s(x_k, \zeta_k) \end{array}\right]\left[\begin{array}{c} g(x_j, \eta_j) \\ s(x_j, \zeta_j) \end{array}\right]^T \middle| x_k\right\} = 0, \quad k \neq j, \tag{2.5}$$

$$\mathbb{E}\left\{\left[\begin{array}{c} g(x_k, \eta_k) \\ s(x_k, \zeta_k) \end{array}\right]\left[\begin{array}{c} g(x_k, \eta_k) \\ s(x_k, \zeta_k) \end{array}\right]^T \middle| x_k\right\} = \sum_{i=1}^{r} \Pi_k^i x_k^T \Gamma_k^i x_k, \tag{2.6}$$

where r is a known positive integer, $\Pi_k^i = \operatorname{diag}\left\{\Pi_k^{1i}, \Pi_k^{2i}\right\}$ and Γ_k^i $(i = 1, 2, \ldots, r)$ are known matrices of appropriate dimensions.

The initial state x_0, the process noise ω_k, and the measurement noise ν_k are mutually uncorrelated and have the following statistical properties:

$$\begin{aligned} \mathbb{E}\{x_0\} = \bar{x}_0, \quad \mathbb{E}\left\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\right\} &= P_{0|0}, \\ \mathbb{E}\{\omega_k\} = 0, \quad \mathbb{E}\{\nu_k\} &= 0, \\ \mathbb{E}\left\{\omega_k \omega_k^T\right\} = Q_k, \quad \mathbb{E}\left\{\nu_k \nu_k^T\right\} &= R_k, \end{aligned} \tag{2.7}$$

where $P_{0|0} > 0$, $Q_k > 0$, and $R_k > 0$ are known matrices of appropriate dimensions.

We construct the following recursive filter:

$$\hat{x}_{k+1|k} = f\left(\hat{x}_{k|k}\right), \tag{2.8}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left[y_{k+1} - \bar{\Xi}_{k+1} h\left(\hat{x}_{k+1|k}\right)\right], \quad \hat{x}_{0|0} = \bar{x}_0, \tag{2.9}$$

where $\hat{x}_{k|k}$ is the estimate of x_k at time k, $\hat{x}_{k+1|k}$ is the one-step prediction at time k, $\bar{\Xi}_{k+1} := \mathbb{E}\{\Xi_{k+1}\} := \operatorname{diag}\{\mu_{k+1}^1, \mu_{k+1}^2, \ldots, \mu_{k+1}^q\}$, and K_{k+1} is the filter gain to be determined.

The aim is to design a finite-horizon filter of the structure (2.8)–(2.9) such that, for both stochastic non-linearities and multiple missing measurements, an upper bound for the filtering error covariance is guaranteed, that is, there exists a sequence of positive-definite matrices $\Sigma_{k+1|k+1}$ $(0 \leq k \leq N)$ satisfying

$$\mathbb{E}\left\{\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)^T\right\} \leq \Sigma_{k+1|k+1}, \quad \forall\, k. \tag{2.10}$$

Moreover, the designed filter gain K_{k+1} is expected to minimize the upper bound $\Sigma_{k+1|k+1}$ through a recursive scheme.

Remark 2.1 In (2.2), $h(x_k)$ represents the sensor outputs coupled with non-linearities. In engineering practice, the non-linearities in the sensor outputs result primarily from the sensor saturations due to finite register-length of digital hardware, and such kind of non-linearities can be covered by the assumption made in (2.3). To be more specific, the assumption in (2.3) could encompass a number of frequently occurred sensor-related non-linearities such as sector-bounded non-linearities, quantization, overflow non-linearities, etc. Note that, under the same norm-bounded assumption, the control and filtering problems have been extensively studied for non-linear stochastic systems, see, for example [1, 2].

Remark 2.2 In recent years, it is quite common that the measurement signals are transmitted through a large number of sensors in a network. Due to the limited bandwidth of a network, the missing measurement phenomenon may occur intermittently and the data-missing probability may be different for individual sensor. In (2.2), the multiple missing measurements (i.e., data missing with multiple sensors) are taken into account, where the diagonal matrix Ξ_k represents the missing status for all sensors as a whole and the random variable α_k^i corresponds to the ith sensor ($i = 1, 2, \ldots, q$). As discussed in [3], the random variable α_k^i can take any value over the interval [0, 1], and the probability for α_k^i to take different values may vary with the sensors. Moreover, α_k^i can obey any discrete probability distributions over the interval [0, 1] that includes the Bernoulli (binary) distribution as a special case. By considering the phenomenon of the multiple missing measurements, the new measurement model (2.2) is capable of describing the actual arrivals of the measured information from multiple sensors especially when only partial data are missing.

Before proceeding further, we are in a position to introduce the following lemmas, which will be used in subsequent developments.

Lemma 2.1 [4] *Let $A = [a_{ij}]_{p\times p}$ be a real-valued matrix and $B = \mathrm{diag}\{b_1, b_2, \ldots, b_p\}$ be a diagonal stochastic matrix. Then,*

$$\mathbb{E}\{BAB^T\} = \begin{bmatrix} \mathbb{E}\{b_1^2\} & \mathbb{E}\{b_1b_2\} & \cdots & \mathbb{E}\{b_1b_p\} \\ \mathbb{E}\{b_2b_1\} & \mathbb{E}\{b_2^2\} & \cdots & \mathbb{E}\{b_2b_p\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}\{b_pb_1\} & \mathbb{E}\{b_pb_2\} & \cdots & \mathbb{E}\{b_p^2\} \end{bmatrix} \circ A,$$

where $\circ$ is the Hadamard product.

Lemma 2.2 [5] *Given matrices A, H, E, and F of appropriate dimensions such that $FF^T \leq I$. Let X be a symmetric positive-definite matrix and γ be an arbitrary positive constant such that $\gamma^{-1}I - EXE^T > 0$. Then, the following inequality holds*

$$(A + HFE)\, X\, (A + HFE)^T \le A \left(X^{-1} - \gamma E^T E \right)^{-1} A^T + \gamma^{-1} H H^T. \quad (2.11)$$

Lemma 2.3 [6] *For* $0 \le k \le N$*, suppose that* $X = X^T > 0$*,* $\mathscr{S}_k(X) = \mathscr{S}_k^T(X) \in \mathbb{R}^{n \times n}$ *and* $\mathscr{H}_k(X) = \mathscr{H}_k^T(X) \in \mathbb{R}^{n \times n}$*. Suppose that*

$$\mathscr{S}_k(Y) \ge \mathscr{S}_k(X), \quad \forall\ X \le Y = Y^T, \quad (2.12)$$

and

$$\mathscr{H}_k(Y) \ge \mathscr{S}_k(Y). \quad (2.13)$$

Then, the solutions M_k *and* N_k *to the following discrete difference equations*

$$M_{k+1} = \mathscr{S}_k(M_k), \quad N_{k+1} = \mathscr{H}_k(N_k), \quad M_0 = N_0 > 0, \quad (2.14)$$

satisfy

$$M_k \le N_k.$$

2.1.2 Design of EKF

In this section, our aim is to establish a unified framework to deal with the addressed filtering problem in the simultaneous presence of stochastic non-linearities and multiple missing measurements. The linearization is firstly enforced to facilitate the further developments. Subsequently, the one-step prediction error covariance and the filtering error covariance are calculated so as to design the EKF, where special effort is made to compensate the effects of multiple missing measurements. Next, the upper bound of the filtering error covariance is derived and the filter gain is designed to ensure that such an upper bound is minimized.

To begin, denote the one-step prediction error as $\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$ and the filtering error as $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$. Subtracting (2.8) from (2.1), we have

$$\tilde{x}_{k+1|k} = f(x_k) - f\left(\hat{x}_{k|k}\right) + g(x_k, \eta_k) + D_k \omega_k. \quad (2.15)$$

By using the Taylor series expansion around $\hat{x}_{k|k}$, we linearize the non-linear function $f(x_k)$ as follows:

$$f(x_k) = f\left(\hat{x}_{k|k}\right) + A_k \tilde{x}_{k|k} + o(|\tilde{x}_{k|k}|), \quad (2.16)$$

where

$$A_k = \frac{\partial f(x_k)}{\partial x_k}\Big|_{x_k = \hat{x}_{k|k}},$$

and $o(|\tilde{x}_{k|k}|)$ stands for the high-order terms of the Taylor series expansion. For presentation convenience, following [7, 8], the high-order terms are transformed into the following easy-to-handle formulation:

$$o(|\tilde{x}_{k|k}|) = B_k \aleph_{1,k} L_k \tilde{x}_{k|k}, \tag{2.17}$$

where B_k is a problem-dependent scaling matrix, L_k is introduced to provide an extra degree of freedom to tune the filter, and $\aleph_{1,k}$ is an unknown time-varying matrix accounting for the linearization errors of the dynamical model that satisfies

$$\aleph_{1,k} \aleph_{1,k}^T \leq I. \tag{2.18}$$

It follows from (2.15)–(2.17) that

$$\tilde{x}_{k+1|k} = \left(A_k + B_k \aleph_{1,k} L_k\right) \tilde{x}_{k|k} + g\left(x_k, \eta_k\right) + D_k \omega_k. \tag{2.19}$$

Similarly, by applying the Taylor series expansion for $h(x_{k+1})$ around $\hat{x}_{k+1|k}$, the innovation of the filter can be obtained as follows:

$$\begin{aligned}\tilde{y}_{k+1} &= y_{k+1} - \bar{\Xi}_{k+1} h\left(\hat{x}_{k+1|k}\right) \\ &= \left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right) h(x_{k+1}) + \bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1} \aleph_{2,k+1} L_{k+1}\right) \tilde{x}_{k+1|k} \\ &\quad + s\left(x_{k+1}, \zeta_{k+1}\right) + \nu_{k+1},\end{aligned} \tag{2.20}$$

where

$$C_{k+1} = \frac{\partial h\left(x_{k+1}\right)}{\partial x_{k+1}}|_{x_{k+1}=\hat{x}_{k+1|k}},$$

E_{k+1} is a problem-dependent scaling matrix, and $\aleph_{2,k+1}$ is an unknown time-varying matrix representing the linearization errors of the dynamical model that satisfies

$$\aleph_{2,k+1} \aleph_{2,k+1}^T \leq I. \tag{2.21}$$

According to (2.9) and (2.20), the filtering error can be written as:

$$\begin{aligned}\tilde{x}_{k+1|k+1} = &\left[I - K_{k+1} \bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1} \aleph_{2,k+1} L_{k+1}\right)\right] \tilde{x}_{k+1|k} \\ &- K_{k+1}\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right) h(x_{k+1}) - K_{k+1} s\left(x_{k+1}, \zeta_{k+1}\right) \\ &- K_{k+1} \nu_{k+1}.\end{aligned} \tag{2.22}$$

Subsequently, according to (2.19) and (2.22), the covariances for the one-step prediction error and filtering error can be derived, respectively, in the following theorems.

Theorem 2.1 *The one-step prediction error covariance $P_{k+1|k}$ is given by*

$$
\begin{aligned}
P_{k+1|k} = & \left(A_k + B_k \aleph_{1,k} L_k\right) P_{k|k} \left(A_k + B_k \aleph_{1,k} L_k\right)^T \\
& + \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left(\mathbb{E}\left\{x_k x_k^T\right\} \Gamma_k^i\right) + D_k Q_k D_k^T. \quad (2.23)
\end{aligned}
$$

Proof It can be shown that (2.23) follows directly from (2.6) to (2.7) and (2.19), and therefore, the proof is omitted for conciseness.

Theorem 2.2 *The recursion of the filtering error covariance $P_{k+1|k+1}$ satisfies*

$$
\begin{aligned}
P_{k+1|k+1} = & \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right] P_{k+1|k} \\
& \times \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right]^T \\
& + K_{k+1}\Bigg[\breve{\Xi}_{k+1} \circ \mathbb{E}\left\{h(x_{k+1})h^T(x_{k+1})\right\} \\
& + \sum_{i=1}^{r} \Pi_{k+1}^{2i} \mathrm{tr}\left(\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\}\Gamma_{k+1}^i\right) + R_{k+1}\Bigg]K_{k+1}^T, \quad (2.24)
\end{aligned}
$$

where

$$
\breve{\Xi}_{k+1} = \mathrm{diag}\left\{(\sigma_{k+1}^1)^2, (\sigma_{k+1}^2)^2, \ldots, (\sigma_{k+1}^q)^2\right\}. \quad (2.25)
$$

Proof Considering (2.22), we obtain

$$
\begin{aligned}
& P_{k+1|k+1} \\
& = \mathbb{E}\left\{\tilde{x}_{k+1|k+1}\tilde{x}_{k+1|k+1}^T\right\} \\
& = \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right] P_{k+1|k} \\
& \quad \times \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right]^T \\
& \quad + K_{k+1}\mathbb{E}\left\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)h(x_{k+1})h^T(x_{k+1})\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\right\}K_{k+1}^T \\
& \quad + K_{k+1}\mathbb{E}\left\{s\left(x_{k+1}, \zeta_{k+1}\right)s^T\left(x_{k+1}, \zeta_{k+1}\right)\right\}K_{k+1}^T + K_{k+1}\mathbb{E}\left\{\nu_{k+1}\nu_{k+1}^T\right\}K_{k+1}^T \\
& \quad - \mathcal{P}_{k+1} - \mathcal{P}_{k+1}^T - \mathcal{Q}_{k+1} - \mathcal{Q}_{k+1}^T - \mathcal{R}_{k+1} - \mathcal{R}_{k+1}^T \\
& \quad + \mathcal{X}_{k+1} + \mathcal{X}_{k+1}^T + \mathcal{Y}_{k+1} + \mathcal{Y}_{k+1}^T + \mathcal{Z}_{k+1} + \mathcal{Z}_{k+1}^T, \quad (2.26)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{P}_{k+1} = & \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right] \\
& \times \mathbb{E}\left\{\tilde{x}_{k+1|k}h^T(x_{k+1})\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\right\}K_{k+1}^T,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{Q}_{k+1} &= \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right] \\
&\quad \times \mathbb{E}\left\{\tilde{x}_{k+1|k}s^T\left(x_{k+1}, \zeta_{k+1}\right)\right\} K_{k+1}^T, \\
\mathcal{R}_{k+1} &= \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right]\mathbb{E}\left\{\tilde{x}_{k+1|k}\nu_{k+1}^T\right\} K_{k+1}^T, \\
\mathcal{X}_{k+1} &= K_{k+1}\mathbb{E}\left\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)h(x_{k+1})s^T\left(x_{k+1}, \zeta_{k+1}\right)\right\} K_{k+1}^T, \\
\mathcal{Y}_{k+1} &= K_{k+1}\mathbb{E}\left\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)h(x_{k+1})\nu_{k+1}^T\right\} K_{k+1}^T, \\
\mathcal{Z}_{k+1} &= K_{k+1}\mathbb{E}\left\{s\left(x_{k+1}, \zeta_{k+1}\right)\nu_{k+1}^T\right\} K_{k+1}^T.
\end{aligned}$$

It is easy to show that the terms $\mathcal{P}_{k+1}$, $\mathcal{Q}_{k+1}$, $\mathcal{R}_{k+1}$, $\mathcal{X}_{k+1}$, $\mathcal{Y}_{k+1}$, and $\mathcal{Z}_{k+1}$ are all equal to zero. It follows from (2.6)–(2.7) that (2.26) can be rewritten as:

$$\begin{aligned}
&P_{k+1|k+1} \\
&\quad = \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right] P_{k+1|k} \\
&\quad\quad \times \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right]^T \\
&\quad\quad + K_{k+1}\mathbb{E}\left\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)h(x_{k+1})h^T(x_{k+1})\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\right\} K_{k+1}^T \\
&\quad\quad + K_{k+1}\left[\sum_{i=1}^{r}\Pi_{k+1}^{2i}\operatorname{tr}\left(\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\}\Gamma_{k+1}^i\right) + R_{k+1}\right] K_{k+1}^T.
\end{aligned} \tag{2.27}$$

By applying Lemma 2.1 and using the property of conditional expectation, the second term of the right-hand side of (2.27) can be determined as follows:

$$\begin{aligned}
&\mathbb{E}\left\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)h(x_{k+1})h^T(x_{k+1})\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\right\} \\
&\quad = \breve{\Xi}_{k+1} \circ \mathbb{E}\left\{h(x_{k+1})h^T(x_{k+1})\right\},
\end{aligned} \tag{2.28}$$

where $\breve{\Xi}_{k+1}$ is defined in (2.25). Then, it follows from (2.27) and (2.28) that (2.24) holds. The proof is now complete.

Remark 2.3 In Theorem 2.2, the recursive form of the filtering error covariance has been established. Note that the linearization is enforced to deal with the nonlinearities $f(\cdot)$ and $h(\cdot)$. Therefore, (2.23) and (2.24) involve $\aleph_{1,k}$ and $\aleph_{2,k+1}$, which add extra computational difficulties for the design of filter gain. Actually, due to the consideration of the linearization errors, it is literally impossible to obtain the accurate value of the filtering error covariance $P_{k+1|k+1}$, and a seemingly natural way is to design appropriate filter gain K_{k+1} in order to guarantee an upper bound for the filtering error covariance that can then be minimized at each sampling instant.

Motivated by [9], in the following theorem, an upper bound is proposed for the filtering error covariance, and the filter gain is then designed to minimize such an upper bound.

Theorem 2.3 *Consider the covariance matrices of the one-step prediction error and the filtering error in* (2.23) *and* (2.24). *Assume that* (2.18) *and* (2.21) *are true. Let* $\gamma_{1,k}$, $\gamma_{2,k+1}$, *and* ε_j $(j = 1, 2)$ *be positive scalars. Suppose that the following two discrete-time Riccati-like difference equations:*

$$
\begin{aligned}
&\Sigma_{k+1|k} \\
&\quad = A_k \left(\Sigma_{k|k}^{-1} - \gamma_{1,k} L_k^T L_k\right)^{-1} A_k^T + \gamma_{1,k}^{-1} B_k B_k^T + D_k Q_k D_k^T \\
&\qquad + \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left\{\left[(1+\varepsilon_1)\,\Sigma_{k|k} + \left(1+\varepsilon_1^{-1}\right)\hat{x}_{k|k}\hat{x}_{k|k}^T\right]\Gamma_k^i\right\}, \qquad (2.29)\\
&\Sigma_{k+1|k+1} \\
&\quad = \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\left(\Sigma_{k+1|k}^{-1} - \gamma_{2,k+1} L_{k+1}^T L_{k+1}\right)^{-1}\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T \\
&\qquad + \gamma_{2,k+1}^{-1} K_{k+1}\bar{\Xi}_{k+1}E_{k+1}E_{k+1}^T\bar{\Xi}_{k+1}K_{k+1}^T + K_{k+1}\Big\{\breve{\Xi}_{k+1}\circ[2(a_1^2\mathrm{tr}\left(\Omega_{k+1|k}\right) \\
&\qquad + a_2^2)I] + \sum_{i=1}^{r}\Pi_{k+1}^{2i}\mathrm{tr}\left(\Omega_{k+1|k}\Gamma_{k+1}^i\right) + R_{k+1}\Big\}K_{k+1}^T, \qquad (2.30)
\end{aligned}
$$

with initial condition $\Sigma_{0|0} = P_{0|0} > 0$ *have positive- definite solutions* $\Sigma_{k+1|k}$ *and* $\Sigma_{k+1|k+1}$ *such that, for all* $0 \le k \le N$, *the following two constraints*

$$\gamma_{1,k}^{-1} I - L_k \Sigma_{k|k} L_k^T > 0, \qquad (2.31)$$

$$\gamma_{2,k+1}^{-1} I - L_{k+1} \Sigma_{k+1|k} L_{k+1}^T > 0, \qquad (2.32)$$

are satisfied where

$$\Omega_{k+1|k} = (1+\varepsilon_2)\,\Sigma_{k+1|k} + \left(1+\varepsilon_2^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T. \qquad (2.33)$$

Then, with the filter gain K_{k+1} *given by*

$$
\begin{aligned}
K_{k+1} = &\left(\Sigma_{k+1|k}^{-1} - \gamma_{2,k+1}L_{k+1}^T L_{k+1}\right)^{-1} C_{k+1}^T\bar{\Xi}_{k+1}\Big\{\bar{\Xi}_{k+1}C_{k+1} \\
&\times\left(\Sigma_{k+1|k}^{-1} - \gamma_{2,k+1}L_{k+1}^T L_{k+1}\right)^{-1} C_{k+1}^T\bar{\Xi}_{k+1} \\
&+ \gamma_{2,k+1}^{-1}\bar{\Xi}_{k+1}E_{k+1}E_{k+1}^T\bar{\Xi}_{k+1} + \breve{\Xi}_{k+1}\circ\left[2\left(a_1^2\mathrm{tr}\left(\Omega_{k+1|k}\right) + a_2^2\right)I\right] \\
&+ \sum_{i=1}^{r}\Pi_{k+1}^{2i}\mathrm{tr}\left(\Omega_{k+1|k}\Gamma_{k+1}^i\right) + R_{k+1}\Big\}^{-1}, \qquad (2.34)
\end{aligned}
$$

the matrix $\Sigma_{k+1|k+1}$ *is an upper bound for* $P_{k+1|k+1}$, *that is,*

$$P_{k+1|k+1} \le \Sigma_{k+1|k+1}. \tag{2.35}$$

Moreover, the filter gain K_{k+1} given by (2.34) minimizes the upper bound $\Sigma_{k+1|k+1}$.

Proof To begin with, based on (2.23) and (2.24), rewrite the covariance matrices $P_{k+1|k}$ and $P_{k+1|k+1}$ as the functions of $P_{k|k}$ and $P_{k+1|k}$ as follows:

$$\begin{aligned}
P_{k+1|k}\left(P_{k|k}\right) = & \left(A_k + B_k \aleph_{1,k} L_k\right) P_{k|k} \left(A_k + B_k \aleph_{1,k} L_k\right)^T \\
& + \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left(\mathbb{E}\left\{x_k x_k^T\right\} \Gamma_k^i\right) + D_k Q_k D_k^T, \\
P_{k+1|k+1}\left(P_{k+1|k}\right) = & \left[I - K_{k+1} \bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1} \aleph_{2,k+1} L_{k+1}\right)\right] P_{k+1|k} \\
& \times \left[I - K_{k+1} \bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1} \aleph_{2,k+1} L_{k+1}\right)\right]^T \\
& + K_{k+1}[\breve{\Xi}_{k+1} \circ \mathbb{E}\left\{h(x_{k+1}) h^T(x_{k+1})\right\} \\
& + \sum_{i=1}^{r} \Pi_{k+1}^{2i} \mathrm{tr}\left(\mathbb{E}\left\{x_{k+1} x_{k+1}^T\right\} \Gamma_{k+1}^i\right) + R_{k+1}] K_{k+1}^T.
\end{aligned}$$

Then, it is not difficult to verify that the condition (2.12) in Lemma 2.3 is satisfied.

Now, we are ready to deal with the term of the right-hand side of (2.23). Note that the following elementary inequality

$$\left(\varepsilon_1^{\frac{1}{2}} \tilde{x}_{k|k} - \varepsilon_1^{-\frac{1}{2}} \hat{x}_{k|k}\right)\left(\varepsilon_1^{\frac{1}{2}} \tilde{x}_{k|k} - \varepsilon_1^{-\frac{1}{2}} \hat{x}_{k|k}\right)^T \ge 0,$$

yields

$$\tilde{x}_{k|k} \hat{x}_{k|k}^T + \hat{x}_{k|k} \tilde{x}_{k|k}^T \le \varepsilon_1 \tilde{x}_{k|k} \tilde{x}_{k|k}^T + \varepsilon_1^{-1} \hat{x}_{k|k} \hat{x}_{k|k}^T, \tag{2.36}$$

where $\varepsilon_1 > 0$ is a scalar. Based on (2.36), the second term of the right-hand side of (2.23) can be rearranged as

$$\begin{aligned}
& \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left(\mathbb{E}\left\{x_k x_k^T\right\} \Gamma_k^i\right) \\
& = \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left(\mathbb{E}\left\{\left(\hat{x}_{k|k} + \tilde{x}_{k|k}\right)\left(\hat{x}_{k|k} + \tilde{x}_{k|k}\right)^T\right\} \Gamma_k^i\right) \\
& \le \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left(\mathbb{E}\left\{(1 + \varepsilon_1) \tilde{x}_{k|k} \tilde{x}_{k|k}^T + \left(1 + \varepsilon_1^{-1}\right) \hat{x}_{k|k} \hat{x}_{k|k}^T\right\} \Gamma_k^i\right) \\
& = \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left\{\left[(1 + \varepsilon_1) P_{k|k} + \left(1 + \varepsilon_1^{-1}\right) \hat{x}_{k|k} \hat{x}_{k|k}^T\right] \Gamma_k^i\right\}.
\end{aligned} \tag{2.37}$$

Together with (2.23) and (2.37), we obtain

$$\begin{aligned} P_{k+1|k} \leq & \left(A_k + B_k \aleph_{1,k} L_k\right) P_{k|k} \left(A_k + B_k \aleph_{1,k} L_k\right)^T + D_k Q_k D_k^T \\ & + \sum_{i=1}^{r} \Pi_k^{1i} \mathrm{tr}\left\{\left[(1+\varepsilon_1) P_{k|k} + \left(1+\varepsilon_1^{-1}\right) \hat{x}_{k|k}\hat{x}_{k|k}^T\right] \Gamma_k^i\right\}. \end{aligned} \tag{2.38}$$

On the other hand, let us handle the terms of the right-hand side of (2.24). It follows from (2.3) that

$$\begin{aligned} & \mathbb{E}\left\{h\left(x_{k+1}\right) h^T\left(x_{k+1}\right)\right\} \\ & \leq \mathbb{E}\left\{\|h\left(x_{k+1}\right)\|^2\right\} I \\ & \leq \mathbb{E}\left\{\left(a_1\|x_{k+1}\| + a_2\right)^2\right\} I \\ & \leq \left(2a_1^2 \mathbb{E}\left\{\|x_{k+1}\|^2\right\} + 2a_2^2\right) I \\ & = 2\left[a_1^2 \mathrm{tr}\left(\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\}\right) + a_2^2\right] I. \end{aligned} \tag{2.39}$$

Notice that, when deriving (2.39), we have used the elementary inequality $2ab \leq a^2 + b^2$. Taking the following inequality into consideration

$$\tilde{x}_{k+1|k}\hat{x}_{k+1|k}^T + \hat{x}_{k+1|k}\tilde{x}_{k+1|k}^T \leq \varepsilon_2 \tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T + \varepsilon_2^{-1}\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T, \tag{2.40}$$

with $\varepsilon_2 > 0$ being a scalar, we have

$$\begin{aligned} & \mathbb{E}\left\{h\left(x_{k+1}\right) h^T\left(x_{k+1}\right)\right\} \\ & \leq 2\left[a_1^2 \mathrm{tr}\left(\mathbb{E}\left\{(1+\varepsilon_2)\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T + \left(1+\varepsilon_2^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T\right\}\right) + a_2^2\right] I \\ & = 2\left[a_1^2 \mathrm{tr}\left((1+\varepsilon_2) P_{k+1|k} + \left(1+\varepsilon_2^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T\right) + a_2^2\right] I. \end{aligned} \tag{2.41}$$

Subsequently, by considering (2.24), (2.40), and (2.41), we obtain

$$\begin{aligned} P_{k+1|k+1} \leq & \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right] P_{k+1|k} \\ & \times \left[I - K_{k+1}\bar{\Xi}_{k+1}\left(C_{k+1} + E_{k+1}\aleph_{2,k+1}L_{k+1}\right)\right]^T \\ & + K_{k+1}\Bigg[\breve{\Xi}_{k+1} \circ \left(2\left[a_1^2 \mathrm{tr}\left(\Psi_{k+1|k}\right) + a_2^2\right] I\right) \\ & + \sum_{i=1}^{r} \Pi_{k+1}^{2i} \mathrm{tr}\left(\Psi_{k+1|k}\Gamma_{k+1}^i\right) + R_{k+1}\Bigg] K_{k+1}^T, \end{aligned} \tag{2.42}$$

where

$$\Psi_{k+1|k} = (1+\varepsilon_2)\,P_{k+1|k} + \left(1+\varepsilon_2^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T.$$

Next, according to (2.29) and (2.30), we continue to rewrite $\Sigma_{k+1|k}$ and $\Sigma_{k+1|k+1}$ as the function of $\Sigma_{k|k}$ and $\Sigma_{k+1|k}$ as follows:

$$\begin{aligned}
&\Sigma_{k+1|k}\left(\Sigma_{k|k}\right)\\
&= A_k\left(\Sigma_{k|k}^{-1} - \gamma_{1,k}L_k^T L_k\right)^{-1} A_k^T + \gamma_{1,k}^{-1}B_k B_k^T + D_k Q_k D_k^T\\
&\quad + \sum_{i=1}^{r}\Pi_k^{1i}\mathrm{tr}\left\{\left[(1+\varepsilon_1)\,\Sigma_{k|k} + \left(1+\varepsilon_1^{-1}\right)\hat{x}_{k|k}\hat{x}_{k|k}^T\right]\Gamma_k^i\right\},
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&\Sigma_{k+1|k+1}\left(\Sigma_{k+1|k}\right)\\
&= \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\left(\Sigma_{k+1|k}^{-1} - \gamma_{2,k+1}L_{k+1}^T L_{k+1}\right)^{-1}\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T\\
&\quad + \gamma_{2,k+1}^{-1}K_{k+1}\bar{\Xi}_{k+1}E_{k+1}E_{k+1}^T\bar{\Xi}_{k+1}K_{k+1}^T + K_{k+1}\Bigg\{\breve{\Xi}_{k+1}\circ[2(a_1^2\mathrm{tr}\left(\Omega_{k+1|k}\right)\\
&\quad + a_2^2)I] + \sum_{i=1}^{r}\Pi_{k+1}^{2i}\mathrm{tr}\left(\Omega_{k+1|k}\Gamma_{k+1}^i\right) + R_{k+1}\Bigg\}K_{k+1}^T,
\end{aligned} \tag{2.44}$$

where $\breve{\Xi}_{k+1}$ and $\Omega_{k+1|k}$ are defined in (2.25) and (2.33), respectively. Combining (2.38), (2.42), (2.43), and (2.44), we can show that the condition (2.13) in Lemma 2.3 is satisfied. Therefore, it follows from Lemmas 2.2–2.3 that

$$P_{k+1|k+1} \le \Sigma_{k+1|k+1}.$$

Next, we are in a position to show that the filter gain given by (2.34) is optimal in the sense that it minimizes the upper bound $\Sigma_{k+1|k+1}$. Taking the partial derivative of $\Sigma_{k+1|k+1}$ with respect to K_{k+1} and letting the derivative be zero, we have

$$\begin{aligned}
&\frac{\partial\mathrm{tr}\left(\Sigma_{k+1|k+1}\right)}{\partial K_{k+1}}\\
&\quad = -2\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\left(\Sigma_{k+1|k}^{-1} - \gamma_{2,k+1}L_{k+1}^T L_{k+1}\right)^{-1} C_{k+1}^T\bar{\Xi}_{k+1}\\
&\qquad + 2K_{k+1}\Bigg\{\gamma_{2,k+1}^{-1}\bar{\Xi}_{k+1}E_{k+1}E_{k+1}^T\bar{\Xi}_{k+1} + \breve{\Xi}_{k+1}\circ\left[2\left(a_1^2\mathrm{tr}\left(\Omega_{k+1|k}\right) + a_2^2\right)I\right]\\
&\qquad + \sum_{i=1}^{r}\Pi_{k+1}^{2i}\mathrm{tr}\left(\Omega_{k+1|k}\Gamma_{k+1}^i\right) + R_{k+1}\Bigg\}\\
&\quad = 0.
\end{aligned}$$

Based on the above equation, the optimal filter gain K_{k+1} can be determined as

$$\begin{aligned}
&K_{k+1}\\
&= \left(\Sigma_{k+1|k}^{-1} - \gamma_{2,k+1} L_{k+1}^T L_{k+1}\right)^{-1} C_{k+1}^T \bar{\Xi}_{k+1} \Bigg\{ \bar{\Xi}_{k+1} C_{k+1}\\
&\quad \times \left(\Sigma_{k+1|k}^{-1} - \gamma_{2,k+1} L_{k+1}^T L_{k+1}\right)^{-1} C_{k+1}^T \bar{\Xi}_{k+1} + \gamma_{2,k+1}^{-1} \bar{\Xi}_{k+1} E_{k+1} E_{k+1}^T \bar{\Xi}_{k+1}\\
&\quad + \breve{\Xi}_{k+1} \circ \left[2\left(a_1^2 \mathrm{tr}\left(\Omega_{k+1|k}\right) + a_2^2\right) I\right] + \sum_{i=1}^{r} \Pi_{k+1}^{2i} \mathrm{tr}\left(\Omega_{k+1|k} \Gamma_{k+1}^i\right) + R_{k+1} \Bigg\}^{-1},
\end{aligned}$$

which is identical to (2.34). It is clear that the filter gain given by (2.34) is optimal that minimizes the upper bound $\Sigma_{k+1|k+1}$ for the filtering error covariance. The proof of this theorem is complete.

Remark 2.4 The recursive EKF problem is investigated in Theorems 2.1–2.3 for a general class of discrete time-varying non-linear systems with stochastic non-linearities and multiple missing measurements. Unlike most existing literature, the EKF scheme presented in this chapter has an advantage to cope with the multiple missing measurements where each sensor is allowed to have individual data-missing probability especially when only partial information is missing. Note that such a missing measurement phenomenon is typically encountered in practical engineering systems including networked control systems. To handle the emergence of multiple missing measurements, we have made specific efforts to design a recursive filter and derive the upper bound for the filtering error covariance that are dependent on the individual missing probability. In particular, the Hadamard product has been applied to facilitate the algorithm developments. It is worth pointing out that the related (first to third) terms in (2.30) caused by multiple missing measurements and the fourth term in (2.30) due to the consideration of stochastic non-linearities constitute the main difference between our work and the work of [9].

Remark 2.5 In this chapter, our aim is to study the recursive filter design problem for *time-varying* non-linear systems with stochastic non-linearities and multiple missing measurements. Due to such a complicated time-varying nature, we carry out the research for the *finite horizon* case, that is, we wish the filtering criteria to be satisfied over a finite horizon. Instead of the asymptotic behavior (over an infinite horizon), in this chapter, we are only interested in the *transient property over the finite horizon* $k \in [0, N]$, that is, the upper bound for the filtering error covariance is obtained at every sampling instant $k \in [0, N]$, and such an upper bound is minimized by properly designing the filter gain at each sampling instant. Nevertheless, in case that the convergence analysis of the proposed filter approach becomes a concern, as discussed in [10], some additional assumptions can be made on the system parameters in order to ensure the global boundedness of the estimation errors, which constitutes one of our future research topics.

Remark 2.6 At each sampling instant, the filter gain K_{k+1} is designed in Theorem 2.3 to guarantee that the upper bound for the filtering error covariance is minimized. The system (2.1)–(2.2) under consideration is comprehensive that includes two phenomena of the stochastic non-linearities and the multiple missing measurements, hence reflects the reality more closely especially in a networked environment. In our main results, these two phenomena are dealt with in a unified yet effective framework and are explicitly reflected in the design procedure. In particular, the matrices Π_k^{ij} and Γ_k^j $(i = 1, 2;\ j = 1, 2, \ldots, r)$ quantify the effects of the stochastic non-linearities, and the constants μ_k^i and $(\sigma_k^i)^2$ $(i = 1, 2, \ldots, q)$ are there to account for the multiple missing measurements. Furthermore, the proposed filter is derived in terms of two discrete Riccati-like difference equations, which are suitable for recursive computation in online applications.

2.2 Quantized Filtering with Missing Measurements and Multiplicative Noises

In this section, the focus is on the design of the recursive filtering for a class of time-varying non-linear systems with missing measurements, quantization effects, and multiplicative noises. In particular, a new time-varying filter is constructed based on the available information of the missing probability and the quantized measurements.

2.2.1 Problem Formulation

The recursive filter design problem with missing measurements, quantization effects, and multiplicative noises is illustrated in Fig. 2.2. In this figure, the signals are measured by multiple sensors where the measurement missing phenomenon might occur intermittently. Furthermore, due to the fact that communication cables are of limited capacity, the measurement signals are quantized before being transmitted to the filter. In the following, let us model the physical plant, quantization process, and missing measurements in a mathematical way.

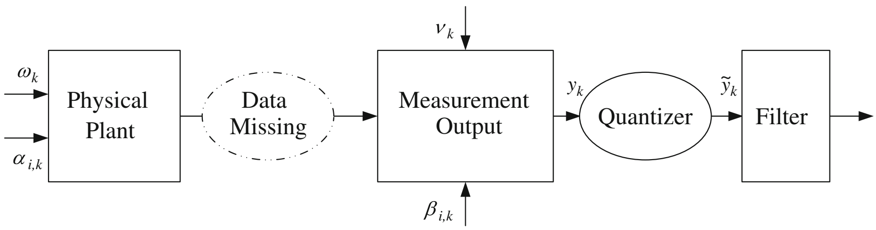

Fig. 2.2 Filtering problem with missing measurements and quantization effects

We consider the following class of time-varying non-linear stochastic systems:

$$x_{k+1} = f(x_k) + \sum_{i=1}^{n_1} \alpha_{i,k} A_{i,k} x_k + D_k \omega_k, \tag{2.45}$$

$$y_k = \Xi_k C_k x_k + \sum_{i=1}^{m_1} \beta_{i,k} C_{i,k} x_k + \nu_k, \tag{2.46}$$

where $x_k \in \mathbb{R}^n$ is the system state to be estimated, the initial value x_0 has mean $\bar{x}_0$ and covariance $P_{0|0}$, $y_k \in \mathbb{R}^m$ is the output vector, $\alpha_{i,k} \in \mathbb{R}$ and $\beta_{i,k} \in \mathbb{R}$ are multiplicative noises with zero-mean and unity variances, and are mutually uncorrelated in k and i, $\omega_k \in \mathbb{R}^r$ is the process noise with zero-mean and covariance $Q_k > 0$, and $\nu_k \in \mathbb{R}^m$ is the zero-mean measurement noise with covariance $R_k > 0$. The non-linear function $f(x_k)$ is continuously differentiable with known form, $A_{i,k}$, $C_{i,k}$, D_k, and C_k are known matrices of appropriate dimensions. $\Xi_k = \text{diag}\{\xi_k^1, \xi_k^2, \ldots, \xi_k^m\}$ is to account for the missing measurements where the mutually uncorrelated (in k and i) random variables $\xi_k^i \in \mathbb{R}$ ($i = 1, 2, \ldots, m$) take values of 1 and 0 with

$$\text{Prob}\left\{\xi_k^i = 1\right\} = \mathbb{E}\left\{\xi_k^i\right\} := \vartheta_k^i, \tag{2.47}$$

$$\text{Prob}\left\{\xi_k^i = 0\right\} = 1 - \mathbb{E}\left\{\xi_k^i\right\} := 1 - \vartheta_k^i. \tag{2.48}$$

Here, $\vartheta_k^i \in [0, 1]$ is a known constant, ξ_k^i is assumed to be independent with $\alpha_{i,k}$, $\beta_{i,k}$, ω_k, ν_k, and x_0. Also, the noise signals mentioned above are uncorrelated with each other.

In a networked environment, it is quite common that the data are quantized before being transmitted to another node (as illustrated in Fig. 2.2). The map of the quantization process is given by

$$\tilde{y}_k = q(y_k) = \left[\, q_1(y_k^1)\ q_2(y_k^2)\ \ldots\ q_m(y_k^m) \,\right]^T.$$

In this chapter, the quantizer is assumed to be of the logarithmic type. For each $q_j(\cdot)$ ($j = 1, 2, \ldots, m$), the set of quantization levels is described by

$$\mathcal{U}_j = \left\{ \pm u_i^{(j)},\ u_i^{(j)} = \left(\chi^{(j)}\right)^i u_0^{(j)},\ i = 0, \pm 1, \pm 2, \ldots \right\} \cup \{0\},\ 0 < \chi^{(j)} < 1,\ u_0^{(j)} > 0,$$

where $\chi^{(j)}$ ($j = 1, 2, \ldots, m$) is called the quantization density. Each of the quantization level corresponds to a segment such that the quantizer maps the whole segment to this quantization level.

Following [11, 12], the logarithmic quantizer is given by

$$q_j(y_k^j) = \begin{cases} u_i^{(j)}, & \frac{1}{1+\delta_j}u_i^{(j)} < y_k^j \leq \frac{1}{1-\delta_j}u_i^{(j)}, \\ 0, & y_k^j = 0, \\ -q_j(-y_k^j), & y_k^j < 0, \end{cases}$$

where

$$\delta_j = \frac{1-\chi^{(j)}}{1+\chi^{(j)}}.$$

From the above definition, it is not difficult to see that $q_j(y_k^j) = \left(1 + \Delta_k^{(j)}\right) y_k^j$ with $|\Delta_k^{(j)}| \leq \delta_j$, and the quantization effects can then be transformed into the sector-bounded uncertainties [12].

Defining $\Delta_k = \operatorname{diag}\left\{\Delta_k^{(1)}, \Delta_k^{(2)}, \ldots, \Delta_k^{(m)}\right\}$ and considering (2.46), the measurements with quantization effects can be expressed as

$$\tilde{y}_k = (I + \Delta_k)\,\Xi_k C_k x_k + (I + \Delta_k)\sum_{i=1}^{m_1} C_{i,k}\beta_{i,k}x_k + (I + \Delta_k)\,\nu_k. \quad (2.49)$$

In fact, setting $\Lambda = \operatorname{diag}\{\delta_1, \delta_2, \ldots, \delta_m\}$ and letting $\mathscr{F}_k = \Delta_k\Lambda^{-1}$, we can obtain an unknown real-valued time-varying matrix $\mathscr{F}_k$ satisfying $\mathscr{F}_k\mathscr{F}_k^T = \mathscr{F}_k^T\mathscr{F}_k \leq I$.

In the sequel, we construct the following filter:

$$\hat{x}_{k+1|k} = f\left(\hat{x}_{k|k}\right), \quad (2.50)$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(\tilde{y}_{k+1} - \bar{\Xi}_{k+1}C_{k+1}\hat{x}_{k+1|k}\right), \quad (2.51)$$

where $\hat{x}_{k|k}$ is the estimate of x_k at time k with $\hat{x}_{0|0} = \bar{x}_0$, $\hat{x}_{k+1|k}$ is the one-step prediction at time k, K_{k+1} is the filter parameter to be determined, and $\bar{\Xi}_{k+1} := \mathbb{E}\{\Xi_{k+1}\} := \operatorname{diag}\{\vartheta_{k+1}^1, \vartheta_{k+1}^2, \ldots, \vartheta_{k+1}^m\}$.

The aim of this section is twofold. First, we aim to design a finite-horizon filter of form (2.50) to (2.51) such that, for all missing measurements, quantization effects, and multiplicative noises, an upper bound for the filtering error covariance is guaranteed, that is, there exists a sequence of positive-definite matrices $\Sigma_{k+1|k+1}$ $(0 \leq k \leq N)$ satisfying

$$\mathbb{E}\left\{\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)^T\right\} \leq \Sigma_{k+1|k+1}, \quad \forall\, k. \quad (2.52)$$

Second, we shall minimize such an upper bound $\Sigma_{k+1|k+1}$ by appropriately designing the filter gain at each sampling instant.

Remark 2.7 In the model (2.46), $C_k x_k$ represents the measurement output subject to probabilistic information loss characterized by the matrix Ξ_k, $\beta_{i,k}$ $(i = 1, 2, \ldots, m_1)$ describes the inherent state-dependent noises that are unrelated with the sensor failures or network congestions, and ν_k is the random exogenous noise acting on the measurement output. In other words, the model (2.46) is quite comprehensive to include the practical cases of probabilistic missing measurements, internal multiplicative noises, and external additive disturbances, thereby reflecting the engineering practice in a more realistic way.

Remark 2.8 In this section, the phenomena of measurements missing and signal quantization are considered simultaneously. In (2.46), Ξ_k is introduced to characterize the missing measurements where the random variable ξ_k^i $(i = 1, 2, \ldots, m)$ corresponds to the i sensor operating at the kth sampling time point. For different sensors, it would be more reasonable to allow multiple sensors to have different missing probabilities (or failure rates [13]). On the other hand, due to limited transmission capacity of the communication channel, the signals are commonly quantized before transmitted to other nodes in a networked system and, as such, the logarithmic-type quantization is brought to discussion here with hope to better reflect such a reality.

2.2.2 *Design of Quantized Filter*

In this subsection, the recursive filter design problem over a finite-horizon is studied for a class of time-varying non-linear stochastic systems in the simultaneous presence of missing measurements, quantization effects, and multiplicative noises. A sufficient condition for the design of filter gain is given by solving two Riccati-like difference equations in order to guarantee an upper bound of the filtering error covariance. Moreover, such an upper bound can be minimized based on the designed filter. The proposed algorithm is of a form suitable for recursive computation in online applications.

To proceed, set the one-step prediction error as $\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$ and the filtering error as $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$. Subtracting (2.50) from (2.45), we have

$$\tilde{x}_{k+1|k} = f(x_k) - f\left(\hat{x}_{k|k}\right) + \sum_{i=1}^{n_1} \alpha_{i,k} A_{i,k} x_k + D_k \omega_k. \tag{2.53}$$

By using the Taylor series expansion around $\hat{x}_{k|k}$, we linearize $f(x_k)$ as follows:

$$f(x_k) = f\left(\hat{x}_{k|k}\right) + A_k \tilde{x}_{k|k} + o(|\tilde{x}_{k|k}|), \tag{2.54}$$

where

$$A_k = \frac{\partial f(x_k)}{\partial x_k}|_{x_k=\hat{x}_{k|k}},$$

and $o(|\tilde{x}_{k|k}|)$ represents the high-order terms of the Taylor series expansion. For presentation convenience, along the similar line of [7, 8], the high-order terms are transformed into the following easy-to-handle formulation:

$$o(|\tilde{x}_{k|k}|) = B_k \aleph_k L_k \tilde{x}_{k|k}, \tag{2.55}$$

where B_k is a bounded problem-dependent scaling matrix, L_k is a bounded matrix for providing an extra degree of freedom to tune the filter, and $\aleph_k$ is an unknown time-varying matrix accounting for the linearization errors of the dynamical model and satisfies

$$\aleph_k \aleph_k^T \leq I. \tag{2.56}$$

Remark 2.9 In traditional EKF algorithm, the Taylor series expansion is used to linearize the non-linearity $f(x_k)$, and the linearization errors are simply neglected, which would inevitably lead to conservatism in certain cases. Recently, a more accurate approach has been proposed in [7] to describe the higher-order terms in the Taylor series in terms of parameter uncertainties. In this chapter, as in [7, 9], we use the deterministic matrix $\aleph_k$ and the scaling matrix B_k in (2.55)–(2.56) to account for the linearization errors in obtaining the matrix A_k. For more details, we refer the reader to Appendix C of [7] where a nice interpretation has been given. It is worthwhile to further mention that, in practice, the high-order terms in the Taylor series expansion are commonly bounded and it is reasonable to regard them as deterministic uncertainties affecting the system matrix A_k.

It follows from (2.53) to (2.55) that the one-step prediction error is given by

$$\tilde{x}_{k+1|k} = (A_k + B_k \aleph_k L_k)\, \tilde{x}_{k|k} + \sum_{i=1}^{n_1} \alpha_{i,k} A_{i,k} x_k + D_k \omega_k. \tag{2.57}$$

On the other hand, it follows from (2.49) and (2.51) that the filtering error $\tilde{x}_{k+1|k+1}$ can be described by

$$\begin{aligned} &\tilde{x}_{k+1|k+1} \\ &\quad = \left(I - K_{k+1} \bar{\Xi}_{k+1} C_{k+1}\right) \tilde{x}_{k+1|k} - K_{k+1}\, (I + \Delta_{k+1}) \left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right) C_{k+1} x_{k+1} \\ &\qquad - K_{k+1} \Delta_{k+1} \bar{\Xi}_{k+1} C_{k+1} x_{k+1} - K_{k+1}\, (I + \Delta_{k+1}) \sum_{i=1}^{m_1} \beta_{i,k+1} C_{i,k+1} x_{k+1} \\ &\qquad - K_{k+1}\, (I + \Delta_{k+1})\, \nu_{k+1}. \end{aligned} \tag{2.58}$$

Based on (2.57) and (2.58), we are in a position to introduce the following lemmas that give the recursion of the one-step prediction error covariance and filtering error covariance, respectively.

Lemma 2.4 *The one-step prediction error covariance $P_{k+1|k}$ obeys the following recursion:*

$$\begin{aligned} P_{k+1|k} &= (A_k + B_k \aleph_k L_k)\, P_{k|k}\, (A_k + B_k \aleph_k L_k)^T \\ &\quad + \sum_{i=1}^{n_1} A_{i,k} \mathbb{E}\left\{ x_k x_k^T \right\} A_{i,k}^T + D_k Q_k D_k^T, \end{aligned} \tag{2.59}$$

where $P_{k|k} = \mathbb{E}\{\tilde{x}_{k|k}\tilde{x}_{k|k}^T\}$ is the filtering error covariance.

Proof Since (2.59) follows from (2.57) directly, the proof is omitted for brevity.

Lemma 2.5 *The filtering error covariance $P_{k+1|k+1}$ is given by:*

$$\begin{aligned} &P_{k+1|k+1} \\ &\quad = \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right) P_{k+1|k} \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T + \mathbb{E}\left\{ \mathcal{H}_{k+1} + \mathcal{H}_{k+1}^T \right\} \\ &\quad + K_{k+1}\Delta_{k+1}\bar{\Xi}_{k+1}C_{k+1}\mathbb{E}\left\{ x_{k+1}x_{k+1}^T \right\} C_{k+1}^T \bar{\Xi}_{k+1}\Delta_{k+1}^T K_{k+1}^T \\ &\quad + K_{k+1}\left(I + \Delta_{k+1}\right)\left[\mathcal{P}_{k+1} + \mathcal{Q}_{k+1} + R_{k+1} \right] \left(I + \Delta_{k+1}\right)^T K_{k+1}^T, \end{aligned} \tag{2.60}$$

where

$$\begin{aligned} \mathcal{P}_{k+1} &:= \check{\Xi}_{k+1} \circ \left(C_{k+1}\mathbb{E}\left\{ x_{k+1}x_{k+1}^T \right\} C_{k+1}^T \right), \\ \mathcal{Q}_{k+1} &:= \sum_{i=1}^{m_1} C_{i,k+1}\mathbb{E}\left\{ x_{k+1}x_{k+1}^T \right\} C_{i,k+1}^T, \\ \mathcal{H}_{k+1} &:= -\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\tilde{x}_{k+1|k}x_{k+1}^T C_{k+1}^T \bar{\Xi}_{k+1}\Delta_{k+1}^T K_{k+1}^T, \\ \check{\Xi}_{k+1} &:= \mathrm{diag}\left\{ \vartheta_{k+1}^1\left(1 - \vartheta_{k+1}^1\right), \vartheta_{k+1}^2\left(1 - \vartheta_{k+1}^2\right), \ldots, \vartheta_{k+1}^m\left(1 - \vartheta_{k+1}^m\right) \right\} \end{aligned} \tag{2.61}$$

Proof According to (2.58), we have

$$\begin{aligned} &P_{k+1|k+1} \\ &\quad = \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right) P_{k+1|k} \left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T + \mathbb{E}\left\{ \mathcal{H}_{k+1} + \mathcal{H}_{k+1}^T \right\} \\ &\quad + K_{k+1}\Delta_{k+1}\bar{\Xi}_{k+1}C_{k+1}\mathbb{E}\left\{ x_{k+1}x_{k+1}^T \right\} C_{k+1}^T \bar{\Xi}_{k+1}\Delta_{k+1}^T K_{k+1}^T \\ &\quad + K_{k+1}\left(I + \Delta_{k+1}\right)\mathbb{E}\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right) C_{k+1}x_{k+1} \end{aligned}$$

$$\times x_{k+1}^T C_{k+1}^T \left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\} \left(I + \Delta_{k+1}\right)^T K_{k+1}^T$$
$$+ K_{k+1}\left(I + \Delta_{k+1}\right)\left(\mathcal{Q}_{k+1} + R_{k+1}\right)\left(I + \Delta_{k+1}\right)^T K_{k+1}^T, \tag{2.62}$$

where $\mathcal{H}_{k+1}$ and $\mathcal{Q}_{k+1}$ are defined in (2.61).

Next, together with the property of conditional expectation and applying Lemma 2.1, we obtain

$$\begin{aligned}&\mathbb{E}\left\{\left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right) C_{k+1} x_{k+1} x_{k+1}^T C_{k+1}^T \left(\Xi_{k+1} - \bar{\Xi}_{k+1}\right)\right\}\\&\quad = \check{\Xi}_{k+1} \circ \left(C_{k+1}\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\} C_{k+1}^T\right),\end{aligned} \tag{2.63}$$

where $\check{\Xi}_{k+1}$ is defined in (2.61). Therefore, (2.60) follows directly from (2.62) and (2.63), and the proof of this lemma is complete.

Remark 2.10 It can be seen that the linearization has been enforced to facilitate the recursive filtering algorithm developments. From Lemmas 2.4–2.5, the filtering error covariance can be obtained for all missing measurements, quantization effects, and multiplicative noises, provided that the matrix Eqs. (2.59) and (2.60) are solvable. Unfortunately, due to the simultaneous consideration of the non-linearity and the signal quantization, (2.59) and (2.60) are contaminated by some uncertain terms $\aleph_k$, $\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\}$, and Δ_{k+1}, which lead to essential difficulty in determining the accurate value of the filtering error covariance $P_{k+1|k+1}$. In the following, an alternatively way is employed to design an appropriate filter gain K_{k+1} such that there exists an upper bound for the filtering error covariance. It will be shown that the designed filter is optimal in the sense of minimizing such an upper bound at each sampling instant. Moreover, the developed algorithm is of an easy-to-implement form suitable for online applications.

Now, we are ready to present the main results of this section. According to Lemmas 2.4–2.5, the filter gain is designed such that an optimized upper bound for the filtering error covariance is achieved at each sampling instant.

Theorem 2.4 *Consider the one-step prediction error covariance and the filtering error covariance in* (2.59)–(2.60)*, respectively. Assume that* (2.56) *holds. Let* $\gamma_{1,k}$*,* $\gamma_{2,k+1}$*, and* ε_j *(*$j = 1, 2, 3$*) be positive scalars. Suppose that the following two Riccati-like difference equations*

$$\begin{aligned}\Sigma_{k+1|k} &= A_k\left(\Sigma_{k|k}^{-1} - \gamma_{1,k}L_k^T L_k\right)^{-1} A_k^T + \gamma_{1,k}^{-1}B_k B_k^T + D_k Q_k D_k^T\\&\quad + \sum_{i=1}^{n_1} A_{i,k}\left[\left(1+\varepsilon_1\right)\Sigma_{k|k} + \left(1+\varepsilon_1^{-1}\right)\hat{x}_{k|k}\hat{x}_{k|k}^T\right] A_{i,k}^T,\end{aligned} \tag{2.64}$$

$$\begin{aligned}\Sigma_{k+1|k+1} &= \left(1+\varepsilon_2\right)\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\Sigma_{k+1|k}\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T\\&\quad + K_{k+1}\left\{\left(1+\varepsilon_2^{-1}\right)\mathrm{tr}\left(\Lambda\bar{\Xi}_{k+1}C_{k+1}\Phi_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Lambda\right) I\right.\end{aligned}$$

$$+\operatorname{tr}\left(\Psi_{k+1|k}\right)\left[\left(I-\gamma_{2,k+1}\Lambda\Lambda\right)^{-1}+\gamma_{2,k+1}^{-1}I\right]\Bigg\}K_{k+1}^{T}, \tag{2.65}$$

with initial condition $\Sigma_{0|0} = P_{0|0} > 0$ *have positive-definite solutions* $\Sigma_{k+1|k}$ *and* $\Sigma_{k+1|k+1}$ *such that, for all* $0 \leq k \leq N$, *the following two constraints*

$$\gamma_{1,k}^{-1}I - L_k\Sigma_{k|k}L_k^T > 0, \tag{2.66}$$

$$\gamma_{2,k+1}^{-1}I - \Lambda\Lambda > 0, \tag{2.67}$$

are satisfied where

$$\begin{aligned}\Phi_{k+1|k} &:= (1+\varepsilon_3)\,\Sigma_{k+1|k} + \left(1+\varepsilon_3^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T,\\ \Psi_{k+1|k} &:= \check{\Xi}_{k+1}\circ\left(C_{k+1}\Phi_{k+1|k}C_{k+1}^T\right) + \sum_{i=1}^{m_1} C_{i,k+1}\Phi_{k+1|k}C_{i,k+1}^T + R_{k+1}.\end{aligned} \tag{2.68}$$

Then, with the filter parameter K_{k+1} *given by*

$$\begin{aligned}K_{k+1} = {} & (1+\varepsilon_2)\,\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Bigg\{(1+\varepsilon_2)\,\bar{\Xi}_{k+1}C_{k+1}\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\\ & + \left(1+\varepsilon_2^{-1}\right)\operatorname{tr}\left(\Lambda\bar{\Xi}_{k+1}C_{k+1}\Phi_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Lambda\right)I\\ & + \operatorname{tr}\left(\Psi_{k+1|k}\right)\left[\left(I-\gamma_{2,k+1}\Lambda\Lambda\right)^{-1}+\gamma_{2,k+1}^{-1}I\right]\Bigg\}^{-1},\end{aligned} \tag{2.69}$$

the matrix $\Sigma_{k+1|k+1}$ *is an upper bound for* $P_{k+1|k+1}$, *that is,*

$$P_{k+1|k+1} \leq \Sigma_{k+1|k+1}. \tag{2.70}$$

Moreover, the filter gain K_{k+1} *given by* (2.69) *minimizes the upper bound* $\Sigma_{k+1|k+1}$.

Proof Note that the covariance matrices $P_{k+1|k}$ and $P_{k+1|k+1}$ can be rewritten as the functions of $P_{k|k}$ and $P_{k+1|k}$, respectively. Then, it is not difficult to verify that the condition (2.12) in Lemma 2.3 is satisfied.

Now, we are in a position to deal with the terms of the right-hand side of (2.59). Considering the following elementary inequality

$$\left(\varepsilon_1^{\frac{1}{2}}\tilde{x}_{k|k} - \varepsilon_1^{-\frac{1}{2}}\hat{x}_{k|k}\right)\left(\varepsilon_1^{\frac{1}{2}}\tilde{x}_{k|k} - \varepsilon_1^{-\frac{1}{2}}\hat{x}_{k|k}\right)^T \geq 0,$$

we have

$$\tilde{x}_{k|k}\hat{x}_{k|k}^T + \hat{x}_{k|k}\tilde{x}_{k|k}^T \leq \varepsilon_1\tilde{x}_{k|k}\tilde{x}_{k|k}^T + \varepsilon_1^{-1}\hat{x}_{k|k}\hat{x}_{k|k}^T, \tag{2.71}$$

with $\varepsilon_1 > 0$ being a scalar. Taking (2.71) into consideration, the second term of the right-hand side of (2.59) can be rearranged as follows:

$$\begin{aligned}
&\sum_{i=1}^{n_1} A_{i,k}\mathbb{E}\left\{x_k x_k^T\right\} A_{i,k}^T \\
&\quad \le \sum_{i=1}^{n_1} A_{i,k}\mathbb{E}\left\{(1+\varepsilon_1)\,\tilde{x}_{k|k}\tilde{x}_{k|k}^T + \left(1+\varepsilon_1^{-1}\right)\hat{x}_{k|k}\hat{x}_{k|k}^T\right\} A_{i,k}^T \\
&\quad = \sum_{i=1}^{n_1} A_{i,k}\left[(1+\varepsilon_1)\,P_{k|k} + \left(1+\varepsilon_1^{-1}\right)\hat{x}_{k|k}\hat{x}_{k|k}^T\right] A_{i,k}^T. \qquad (2.72)
\end{aligned}$$

According to (2.59) and (2.72), we have

$$\begin{aligned}
P_{k+1|k} \le & (A_k + B_k\aleph_k L_k)\,P_{k|k}\,(A_k + B_k\aleph_k L_k)^T + D_k Q_k D_k^T \\
& + \sum_{i=1}^{n_1} A_{i,k}\left[(1+\varepsilon_1)\,P_{k|k} + \left(1+\varepsilon_1^{-1}\right)\hat{x}_{k|k}\hat{x}_{k|k}^T\right] A_{i,k}^T. \qquad (2.73)
\end{aligned}$$

Subsequently, let us now tackle the terms of the right-hand side of (2.60). Noting the following inequality

$$\begin{aligned}
\mathcal{H}_{k+1} + \mathcal{H}_{k+1}^T \le & \varepsilon_2\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T \\
& + \varepsilon_2^{-1}K_{k+1}\Delta_{k+1}\bar{\Xi}_{k+1}C_{k+1}x_{k+1}x_{k+1}^T C_{k+1}^T\bar{\Xi}_{k+1}\Delta_{k+1}^T K_{k+1}^T,
\end{aligned}$$

where $\varepsilon_2 > 0$ is a scalar, we have

$$\begin{aligned}
&\mathbb{E}\left\{\mathcal{H}_{k+1} + \mathcal{H}_{k+1}^T\right\} \\
&\quad \le \varepsilon_2\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)P_{k+1|k}\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T \\
&\quad\quad + \varepsilon_2^{-1}K_{k+1}\Delta_{k+1}\bar{\Xi}_{k+1}C_{k+1}\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\}C_{k+1}^T\bar{\Xi}_{k+1}\Delta_{k+1}^T K_{k+1}^T. \qquad (2.74)
\end{aligned}$$

Following the same line of the derivation of (2.71), we can obtain the following inequality

$$\tilde{x}_{k+1|k}\hat{x}_{k+1|k}^T + \hat{x}_{k+1|k}\tilde{x}_{k+1|k}^T \le \varepsilon_3\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T + \varepsilon_3^{-1}\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T,$$

with $\varepsilon_3 > 0$ being a scalar, which yields

$$\begin{aligned}
\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\} &\le \mathbb{E}\left\{(1+\varepsilon_3)\,\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T + \left(1+\varepsilon_3^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T\right\} \\
&= (1+\varepsilon_3)\,P_{k+1|k} + \left(1+\varepsilon_3^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T. \qquad (2.75)
\end{aligned}$$

Then, together with (2.60) and (2.75), and noticing $\Delta_{k+1} = \mathscr{F}_{k+1}\Lambda$, the third term of the right-hand side of (2.60) can be tackled as follows:

$$\begin{aligned}
& K_{k+1}\Delta_{k+1}\bar{\Xi}_{k+1}C_{k+1}\mathbb{E}\left\{x_{k+1}x_{k+1}^T\right\}C_{k+1}^T\bar{\Xi}_{k+1}\Delta_{k+1}^T K_{k+1}^T \\
&\quad \le K_{k+1}\mathscr{F}_{k+1}\Lambda\bar{\Xi}_{k+1}C_{k+1}\mathfrak{M}_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Lambda\mathscr{F}_{k+1}^T K_{k+1}^T \\
&\quad \le K_{k+1}\mathrm{tr}\left(\Lambda\bar{\Xi}_{k+1}C_{k+1}\mathfrak{M}_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Lambda\right)K_{k+1}^T,
\end{aligned} \tag{2.76}$$

where

$$\mathfrak{M}_{k+1|k} := (1+\varepsilon_3)\,P_{k+1|k} + \left(1+\varepsilon_3^{-1}\right)\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T.$$

Similarly, by Lemma 2.2, the last term of the right-hand side of (2.60) can be determined as

$$\begin{aligned}
& K_{k+1}\left(I+\Delta_{k+1}\right)\left(\mathcal{P}_{k+1}+\mathcal{Q}_{k+1}+R_{k+1}\right)\left(I+\Delta_{k+1}\right)K_{k+1}^T \\
&\quad \le K_{k+1}\left(I+\mathscr{F}_{k+1}\Lambda\right)\Bigg[\check{\Xi}_{k+1}\circ\left(C_{k+1}\mathfrak{M}_{k+1|k}C_{k+1}^T\right)+\sum_{i=1}^{m_1}C_{i,k+1}\mathfrak{M}_{k+1|k}C_{i,k+1}^T \\
&\qquad + R_{k+1}\Bigg]\left(I+\mathscr{F}_{k+1}\Lambda\right)K_{k+1}^T \\
&\quad \le \mathrm{tr}\left(\mathfrak{N}_{k+1|k}\right)K_{k+1}\left(I+\mathscr{F}_{k+1}\Lambda\right)\left(I+\mathscr{F}_{k+1}\Lambda\right)K_{k+1}^T \\
&\quad \le \mathrm{tr}\left(\mathfrak{N}_{k+1|k}\right)K_{k+1}\left[\left(I-\gamma_{2,k+1}\Lambda\Lambda\right)^{-1}+\gamma_{2,k+1}^{-1}I\right]K_{k+1}^T,
\end{aligned} \tag{2.77}$$

where

$$\mathfrak{N}_{k+1|k} := \check{\Xi}_{k+1}\circ\left(C_{k+1}\mathfrak{M}_{k+1|k}C_{k+1}^T\right)+\sum_{i=1}^{m_1}C_{i,k+1}\mathfrak{M}_{k+1|k}C_{i,k+1}^T + R_{k+1}.$$

It then follows from (2.60), (2.74), (2.76), and (2.77) that

$$\begin{aligned}
P_{k+1|k+1} \le{}& (1+\varepsilon_2)\left(I-K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)P_{k+1|k}\left(I-K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)^T \\
& + K_{k+1}\Bigg\{\left(1+\varepsilon_2^{-1}\right)\mathrm{tr}\left(\Lambda\bar{\Xi}_{k+1}C_{k+1}\mathfrak{M}_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Lambda\right)I \\
& + \mathrm{tr}\left(\mathfrak{N}_{k+1|k}\right)\left[\left(I-\gamma_{2,k+1}\Lambda\Lambda\right)^{-1}+\gamma_{2,k+1}^{-1}I\right]\Bigg\}K_{k+1}^T.
\end{aligned} \tag{2.78}$$

Combining (2.64), (2.65), (2.73), and (2.78), we can show that the condition (2.13) in Lemma 2.3 is satisfied. Therefore, it follows directly from Lemmas 2.2–2.3 that

$$P_{k+1|k+1} \le \Sigma_{k+1|k+1}.$$

Having determined the upper bound $\Sigma_{k+1|k+1}$, we are now in a position to show that the filter gain given by (2.69) is optimal in the sense that it minimizes the upper bound $\Sigma_{k+1|k+1}$. Taking the partial derivative of (2.65) with respect to K_{k+1} and letting the derivative be zero, we have

$$\begin{aligned}\frac{\partial \mathrm{tr}(\Sigma_{k+1|k+1})}{\partial K_{k+1}} &= -2\,(1+\varepsilon_2)\left(I - K_{k+1}\bar{\Xi}_{k+1}C_{k+1}\right)\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1} \\ &\quad + 2K_{k+1}\Big\{\left(1+\varepsilon_2^{-1}\right)\mathrm{tr}\left(\Lambda\bar{\Xi}_{k+1}C_{k+1}\Phi_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Lambda\right)I \\ &\quad + \mathrm{tr}\left(\Psi_{k+1|k}\right)\left[\left(I-\gamma_{2,k+1}\Lambda\Lambda\right)^{-1}+\gamma_{2,k+1}^{-1}I\right]\Big\} \\ &= 0. \end{aligned} \tag{2.79}$$

From (2.79), and through straightforward the algebraic manipulations, the optimal filter gain K_{k+1} can be determined as follows:

$$\begin{aligned}K_{k+1} = (1+\varepsilon_2)\,\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Big\{&(1+\varepsilon_2)\,\bar{\Xi}_{k+1}C_{k+1}\Sigma_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1} \\ &+ \left(1+\varepsilon_2^{-1}\right)\mathrm{tr}\left(\Lambda\bar{\Xi}_{k+1}C_{k+1}\Phi_{k+1|k}C_{k+1}^T\bar{\Xi}_{k+1}\Lambda\right)I \\ &+ \mathrm{tr}\left(\Psi_{k+1|k}\right)\left[\left(I-\gamma_{2,k+1}\Lambda\Lambda\right)^{-1}+\gamma_{2,k+1}^{-1}I\right]\Big\}^{-1}. \end{aligned} \tag{2.80}$$

Obviously, the filter gain K_{k+1} in (2.80) is identical to (2.69). To this end, the optimal filter gain K_{k+1} is designed in the sense of minimizing the upper bound $\Sigma_{k+1|k+1}$ for the filtering error covariance and, therefore, the proof of this theorem is complete.

Remark 2.11 At each sampling instant, the filter gain K_{k+1} is designed in Theorem 2.4 to minimize the upper bound of filtering error covariance. The system (2.45)–(2.46) under consideration is comprehensive that includes the aspects of the missing measurements, the quantization effects, and the multiplicative noises, hence reflects the reality more closely especially in a networked environment. In our main results, all these important aspects are dealt with in a unified yet effective framework and are explicitly reflected in the design procedure. In particular, the constants ϑ_k^i ($i = 1, 2, \ldots, m$) are there for the missing measurements where all sensors are allowed to have different missing probabilities, matrix Λ quantifies the effects of signal quantization, and the multiplicative noises covariances account for the effects of the stochastic disturbances on the system states. Furthermore, the proposed filter is derived in terms of the solutions to two Riccati-like difference equations, which is recursive and therefore suitable for online applications.

Remark 2.12 Up to now, the finite-horizon filtering algorithm has been proposed for the addressed time-varying non-linear stochastic systems with network-induced phenomena. With respect to the newly developed filter, it is possible to insert it in a feedback control scheme, which would have significant applications especially in a networked control system. In case the boundedness of the upper bound on the estimation errors becomes a concern, as discussed in [10], it is possible to introduce some additional assumptions/constraints on the system parameters such that the global boundedness of the filtering errors is guaranteed, which constitutes one of our future research topics. Moreover, note that the scalars $\gamma_{1,k}$ and $\gamma_{2,k+1}$ are involved in the discrete Riccati-like difference Eqs. (2.64) and (2.65). In the implementation, the values of $\gamma_{1,k}$ and $\gamma_{2,k+1}$ could be given a prior and adjusted to guarantee the inequalities (2.66) and (2.67) in Theorem 2.4 so as to help enhance the solvability of the proposed filtering algorithm.

2.3 Illustrative Examples

In this section, three simulation examples are presented to demonstrate the effectiveness and applicability of the theory presented in this chapter.

Example 1: EKF with stochastic non-linearities and multiple missing measurements.

As analyzed in [14], consider a maneuvering target that is accelerating with random bursts of gas from its reaction control system thrusters. The state vector could consist of the position and velocity of the target. When tracking a maneuvering target through a radar system equipped with an array of sensors communicating through a (possibly wireless) network, the multiple missing phenomenon might occur due to the bandwidth limit of the signal transmission channel, the sensors aging, and/or sensor temporal failure. Furthermore, the system may contaminate with the stochastic non-linearities owing to a variety of reasons such as random failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. For real-time tracking, the system parameters would have to be time-varying. Our aim is, therefore, to design a filter such that, in the simultaneous presence of stochastic non-linearities and multiple missing measurements, an optimized upper bound for the filtering error covariance is guaranteed.

Motivated by this background, we consider the following discretized maneuvering target-tracking system with stochastic non-linearities and multiple missing measurements:

$$\begin{cases} x_{k+1} = f(x_k) + g(x_k, \eta_k) + D_k\omega_k, \\ y_k \quad = \Xi_k h(x_k) + s(x_k, \zeta_k) + \nu_k, \end{cases}$$

where

$$f(x_k) = \begin{bmatrix} 0.8x_k^1 + x_k^1 x_k^2 \\ 1.5x_k^2 - x_k^1 x_k^2 \end{bmatrix}, \quad D_k = \begin{bmatrix} 0.01 \\ 0.03 \end{bmatrix}, \quad h(x_k) = 7.5\sin(x_k^2),$$

and $x_k = \left[x_k^1 \; x_k^2\right]^T$ is composed of the position and velocity of the target, $\omega_k \in \mathbb{R}$ and $\nu_k \in \mathbb{R}$ are zero-mean Gaussian white noises with covariances 0.05. Consider the following two case of the probability density function for Ξ_k:

$$p_k^1(s) = \begin{cases} 0.05, & s = 0, \\ 0.10, & s = 0.5, \\ 0.85, & s = 1. \end{cases}$$

For this case, the expectation and variance can be easily calculated as $\mu_k^1 = 0.9$ and $\left(\sigma_k^1\right)^2 = 0.065$.

The stochastic non-linearities $g(x_k, \eta_k)$ and $s(x_k, \zeta_k)$ are chosen as follows:

$$\begin{aligned} g(x_k, \eta_k) &= \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \left[0.3\text{sign}\left(x_k^1\right) x_k^1 \eta_k^1 + 0.4\text{sign}\left(x_k^2\right) x_k^2 \eta_k^2\right], \\ s(x_k, \zeta_k) &= 0.5\left[0.3\text{sign}\left(x_k^1\right) x_k^1 \zeta_k^1 + 0.4\text{sign}\left(x_k^2\right) x_k^2 \zeta_k^2\right], \end{aligned}$$

where η_k^i and ζ_k^i $(i = 1, 2)$ stand for zero-mean uncorrelated Gaussian white noises with unity covariances. It is not difficult to verify that the above stochastic non-linearities satisfy

$$\begin{aligned} &\mathbb{E}\left\{\begin{bmatrix} g(x_k, \eta_k) \\ s(x_k, \zeta_k) \end{bmatrix} \middle| x_k \right\} = 0, \\ &\mathbb{E}\left\{\begin{bmatrix} g(x_k, \eta_k) \\ s(x_k, \zeta_k) \end{bmatrix} \begin{bmatrix} g(x_k, \eta_k) \\ s(x_k, \zeta_k) \end{bmatrix}^T \middle| x_k \right\} \\ &\quad = \begin{bmatrix} 0.04 & 0.06 & 0 \\ 0.06 & 0.09 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} x_k^T \begin{bmatrix} 0.09 & 0 \\ 0 & 0.16 \end{bmatrix} x_k. \end{aligned}$$

In the simulation, set the initial value of estimation as $\hat{x}_{0|0} = \bar{x}_0 = \left[1.8 \; 0.2\right]^T$ and $\Sigma_{0|0} = 20 I_2$. The other parameters are chosen as $B_k = \text{diag}\{0.1, 0.2\}$, $E_{k+1} = \left[0.1 \; 0.15\right]^T$, $L_k = L_{k+1} = 0.01 I_2$, $\gamma_{1,k} = 0.002$, $\gamma_{2,k+1} = 0.002$, $\varepsilon_1 = 0.4$, $\varepsilon_2 = 0.35$, $a_1 = 7.5$, and $a_2 = 0.05$. Let MSEi $(i = 1, 2)$ denote the mean square error (MSE) for the estimation of the ith state.

According to (2.29), (2.30), and (2.34) in Theorem 2.3, the upper bound of the filtering error covariance and filter gains at every time step can be recursively calculated. Therefore, the addressed filter design problem can be solved by means of the proposed filter structure (2.8)–(2.9). The filter gains (over certain horizon) are listed in Table 2.1, and the simulation results are shown in Figs. 2.3, 2.4, 2.5 and 2.6. Among them, Figs. 2.3 and 2.4 show the upper bounds $\Sigma_{k|k}^{11}$, $\Sigma_{k|k}^{22}$, and the MSE for the states x_k^1 and x_k^2, which confirm that the MSE stay below their upper bounds. Moreover, the trajectories of the actual states x_k^i and their estimates $\hat{x}_k^i$ $(i = 1, 2)$

Table 2.1 Filter gains

k	1	2	3	$\cdots$	39	40
K_k	$\begin{bmatrix} -0.0618 \\ 0.1007 \end{bmatrix}$	$\begin{bmatrix} -0.1125 \\ 0.0595 \end{bmatrix}$	$\begin{bmatrix} 0.0534 \\ 0.0218 \end{bmatrix}$	$\cdots$	$\begin{bmatrix} 0.0170 \\ 0.1228 \end{bmatrix}$	$\begin{bmatrix} 0.0006 \\ 0.1157 \end{bmatrix}$

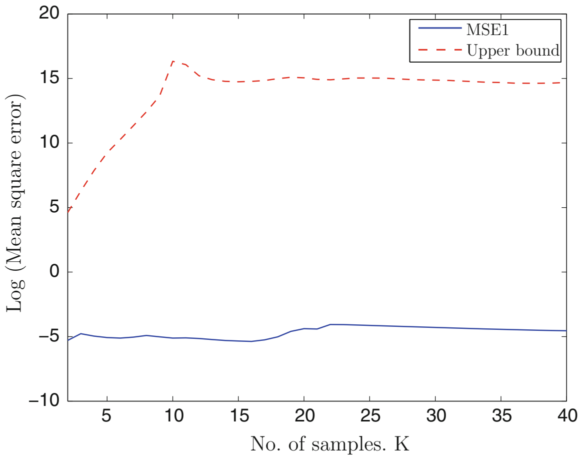

Fig. 2.3 log(MSE1) and its upper bound

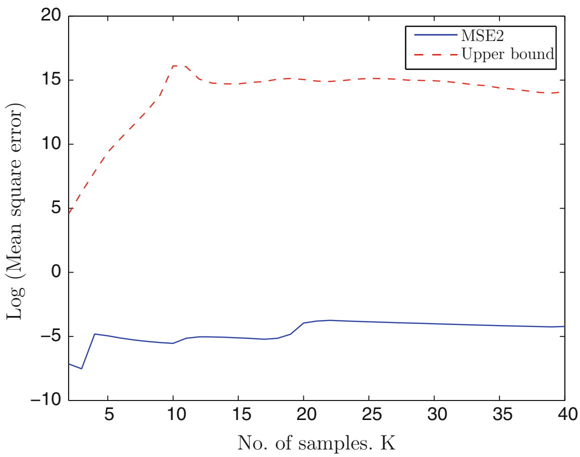

Fig. 2.4 log(MSE2) and its upper bound

are plotted in Figs. 2.5 and 2.6, which illustrate that the presented filter scheme can perform well to estimate the system states. This is due to the fact that we have made specific efforts to compensate the effects of the stochastic non-linearities and multiple missing measurements.

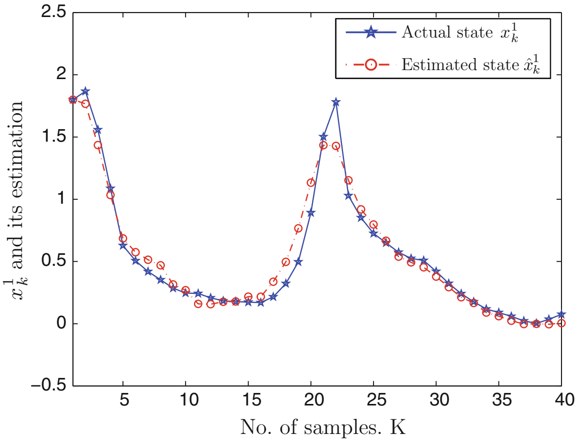

Fig. 2.5 The actual state x_k^1 and its estimation $\hat{x}_k^1$

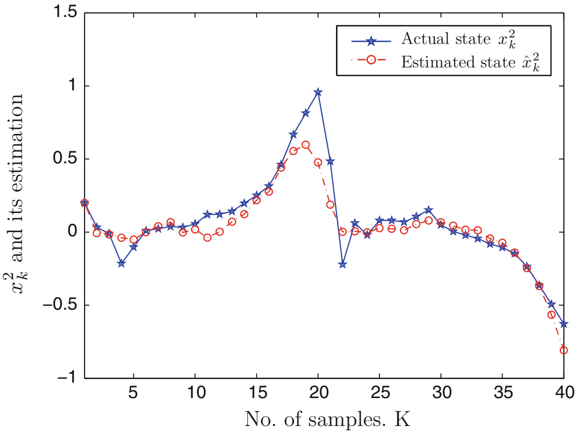

Fig. 2.6 The actual state x_k^2 and its estimation $\hat{x}_k^2$

Remark 2.13 As discussed in [8], the matrices B_k, E_{k+1}, and L_k are used to quantitatively characterize the upper bound of the linearization errors obtained from the Taylor series expansion for the non-linearities. Accordingly, by taking the inequalities (2.18) and (2.21) into consideration, the high-order terms in the Taylor series expansions can be approximated. In the simulation, we set the matrix L_k as $\delta_k I$ (δ_k is a positive constant) in order to enhance the feasibility of (2.31) and (2.32), and then, we can always adjust the values of scaling matrices B_k and E_{k+1} to guarantee the inequalities (2.18) and (2.21). In particular, it is worth mentioning that we can simply set $B_k = 0$ and $E_{k+1} = 0$ suppose that the effects of the linearization errors are negligible for some problems.

Example 2: Quantized recursive filter design with missing measurements and multiplicative noises.

Consider the following non-linear system in the simultaneous presence of missing measurements, quantization effects, and multiplicative noises:

$$\begin{cases} x_{k+1} = f(x_k) + \alpha_{1,k} A_{1,k} x_k + D_k \omega_k, \\ y_k \quad = \Xi_k C_k x_k + \beta_{1,k} C_{1,k} x_k + \nu_k, \end{cases}$$

where

$$f(x_k) = \begin{bmatrix} 0.8x_k^1 + x_k^1 x_k^2 \\ 1.5x_k^2 - x_k^1 x_k^2 \end{bmatrix}, \quad A_{1,k} = \begin{bmatrix} 0.15\sin(2k) & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$D_k = \begin{bmatrix} 0.06 \\ 0.03 + 0.5e^{-5k} \end{bmatrix}, \quad C_k = \begin{bmatrix} 0.85 & 0 \\ 0 & -1.5 \end{bmatrix},$$
$$C_{1,k} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03\sin(k+2) \end{bmatrix},$$

and $x_k = \begin{bmatrix} x_k^1 & x_k^2 \end{bmatrix}^T$ is the state vector with x_k^i $(i = 1, 2)$ being the i-th element of the system state, $\alpha_{1,k} \in \mathbb{R}$, $\beta_{1,k} \in \mathbb{R}$, $\omega_k \in \mathbb{R}$, and $\nu_k \in \mathbb{R}^2$ are zero-mean Gaussian white noises with covariances 1, 1, 0.5, and $0.1I_2$ with $I_2 \in \mathbb{R}^{2\times 2}$ being the identity matrix, respectively.

In the simulation, let the initial value of estimation as $\hat{x}_{0|0} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}^T$ and $\Sigma_{0|0} = 20I_2$ (Initial condition 1) or $\hat{x}_{0|0} = \begin{bmatrix} 0.65 & 0.25 \end{bmatrix}^T$ and $\Sigma_{0|0} = 15I_2$ (Initial condition 2). Set the parameters of the logarithmic quantizer be $u_0^1 = 0.16$, $u_0^2 = 0.3$, $\chi^{(1)} = 0.6$, and $\chi^{(2)} = 0.35$. For comparison, consider two cases of Ξ_k, i.e., $\bar{\Xi}_k = \mathrm{diag}\{0.48, 0.54\}$ for **Case I**, and $\bar{\Xi}_k = \mathrm{diag}\{0.98, 0.78\}$ for **Case II**. The other parameters are chosen as $B_k = \mathrm{diag}\{0.1, 0.2\}$, $L_k = 0.1I_2$, $\gamma_{1,k} = 0.05$, $\gamma_{2,k+1} = 0.02$, $\varepsilon_1 = 0.4$, $\varepsilon_2 = 0.35$, and $\varepsilon_3 = 0.55$. By solving (2.64) and (2.65), the filter gain can be obtained recursively and the simulation results are shown in Figs. 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17 and 2.18. Here, MSEi $(i = 1, 2)$ denotes the mean square error (MSE) for the estimation of the ith state.

In the figures, Figs. 2.7, 2.8, 2.9 and 2.10 plot the measurement signals without and with quantization. Figures 2.11, 2.12, 2.13 and 2.14 (Case II for comparison) show the log(MSE) for the states x_k^1 and x_k^2, and the upper bounds, which confirm that the MSE stays below their upper bounds. Moreover, the trajectories of the actual states x_k^i and their estimations $\hat{x}_k^i$ $(i = 1, 2)$ are plotted in Figs. 2.15, 2.16, 2.17 and 2.18, which illustrate that the presented scheme can perform well to estimate the system states. This is very well expected since specific efforts have been made to compensate the effects of the missing measurements, the signals quantization, and the multiplicative noises in the system model and measurement model. With respect to Case II, that is, the missing measurement phenomenon is less severe, and it can be seen that the filter performance shown in Figs. 2.17 and 2.18 is better than that

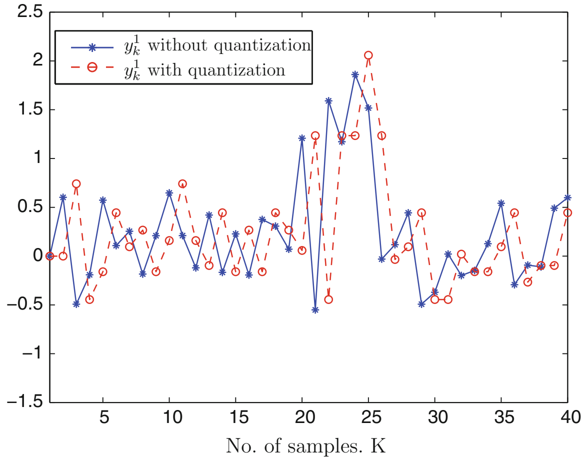

Fig. 2.7 y_k^1 without and with quantization (Case I)

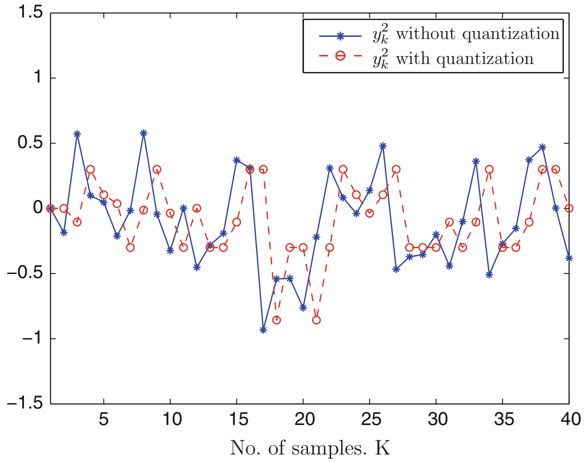

Fig. 2.8 y_k^2 without and with quantization (Case I)

shown in Figs. 2.15 and 2.16, which is a natural result because more information is used in the measurement update for Case II.

Example 3: Quantized recursive filter design for a ballistic object tracking system.

Following [15], we consider the recursive filter design problem for a ballistic object tracking system. When tracking a ballistic object, the measurements are collected sequentially by a radar system equipped with an array of sensors communicating through a (possibly wireless) network. The phenomena of missing measurements and quantization effects might occur due to the finite word length of the packets, the bandwidth limit of the signal transmission channel. Moreover, the system may suffer

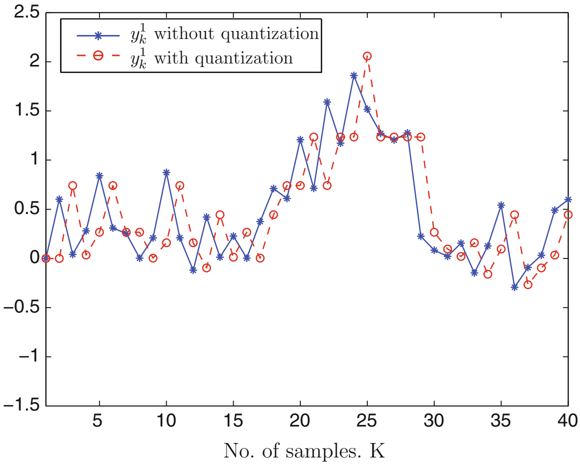

Fig. 2.9 y_k^1 without and with quantization (Case II)

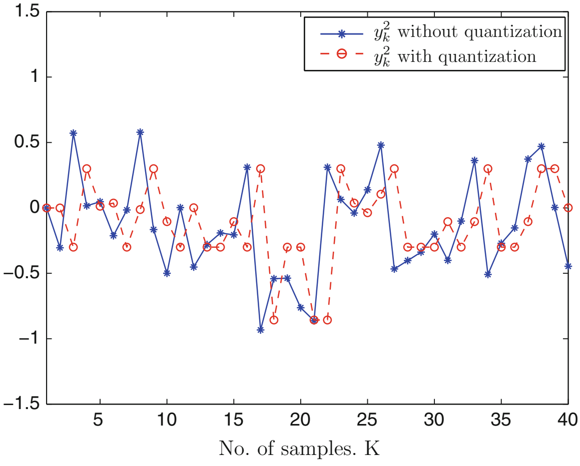

Fig. 2.10 y_k^2 without and with quantization (Case II)

from the multiplicative noises owing to a variety of reasons such as random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, and modification of the operating point of the model. To this end, we aim to design a filter such that, for all missing measurements, quantization effects, and multiplicative noises, the filter gains can be obtained by minimizing the upper bound of the filtering error covariance. The dynamic equations are given as follows:

$$\begin{cases} x_{k+1} = f(x_k) + \alpha_{1,k} A_{1,k} x_k + \omega_k, \\ y_k \quad\;\; = \Xi_k C_k x_k + \beta_{1,k} C_{1,k} x_k + \nu_k, \end{cases}$$

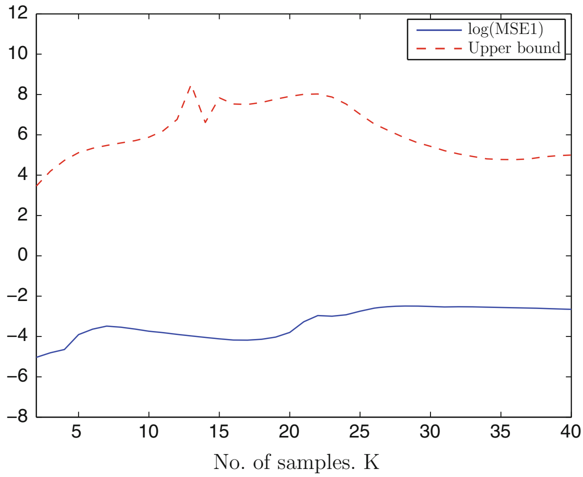

Fig. 2.11 log(MSE1) and its upper bound (Case I)

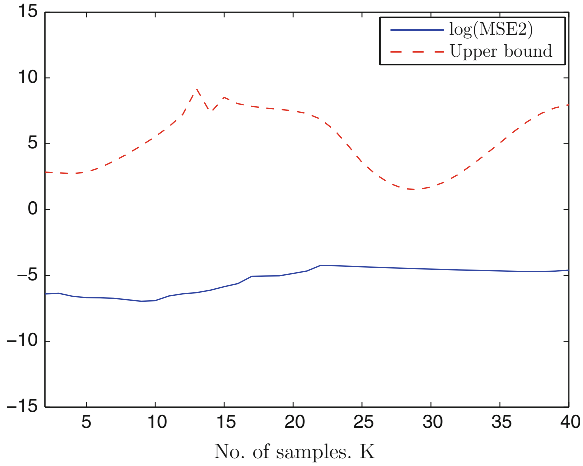

Fig. 2.12 log(MSE2) and its upper bound (Case I)

with

$$
\begin{aligned}
f(x_k) &= \Phi_k x_k + G(h(x_k) + H),\\
h(x_k) &= -\frac{g\rho(x_{2,k})}{2\beta}\sqrt{\dot{x}_{1,k}^2 + \dot{x}_{2,k}^2}\begin{bmatrix}\dot{x}_{1,k}\\ \dot{x}_{2,k}\end{bmatrix},\\
\rho(x_{2,k}) &= \theta_1 \cdot \exp(-\theta_2 x_{2,k}),
\end{aligned}
$$

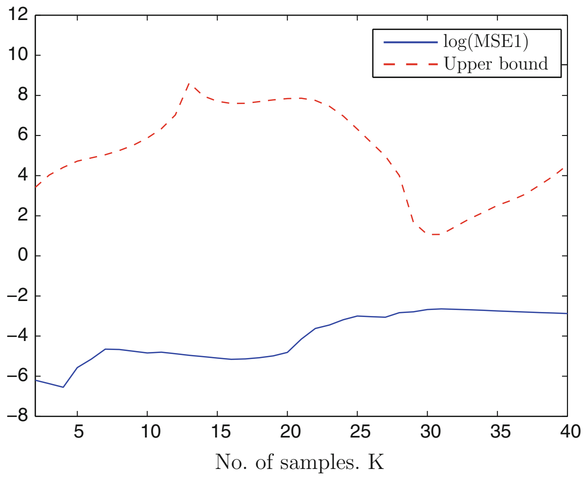

Fig. 2.13 log(MSE1) and its upper bound (Case II)

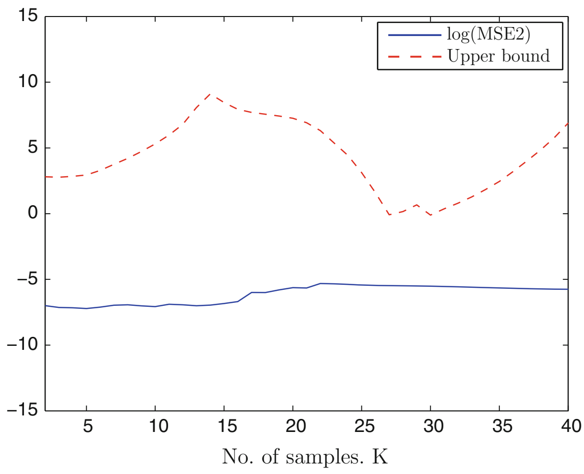

Fig. 2.14 log(MSE2) and its upper bound (Case II)

$$\Phi_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix},$$

$$A_{1,k} = \begin{bmatrix} 0.12\sin(k) & 0 & 0 & 0 \\ 0 & -0.02 & 0 & 0 \\ 0 & 0 & 0.1\sin(2k) & 0 \\ 0 & 0 & 0 & 0.15 \end{bmatrix},$$

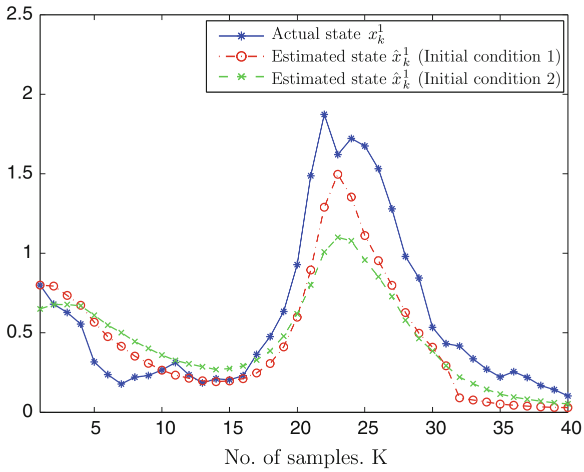

Fig. 2.15 The actual state x_k^1 and its estimation $\hat{x}_k^1$ (Case I)

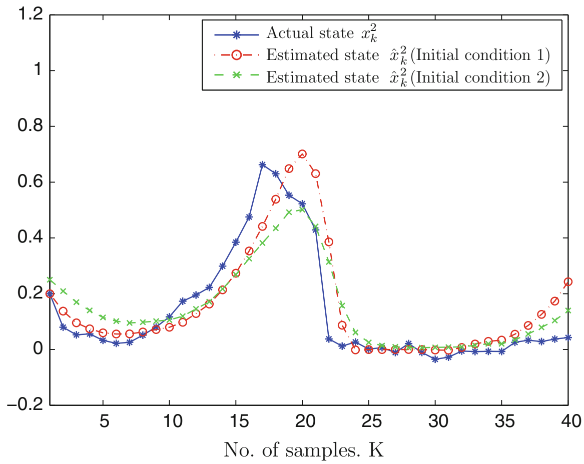

Fig. 2.16 The actual state x_k^2 and its estimation $\hat{x}_k^2$ (Case I)

$$C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_{1,k} = \begin{bmatrix} 0.15 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ -g \end{bmatrix},$$

where $x_k = \begin{bmatrix} x_{1,k} & \dot{x}_{1,k} & x_{2,k} & \dot{x}_{2,k} \end{bmatrix}^T$ is the state vector, $x_{1,k}$ is the target abscissa, $x_{2,k}$ is the target ordinate, T is the sampling period, g is the gravity acceleration, β is the ballistic coefficient (depending on the object mass, shape, and cross-sectional area), $\rho(\cdot)$ is the air density, typically an exponentially decaying function of object height ($\theta_1 = 1.227$, $\theta_2 = 1.093 \times 10^{-4}$ for the object height $x_{2,k} < 9{,}144$ m, and

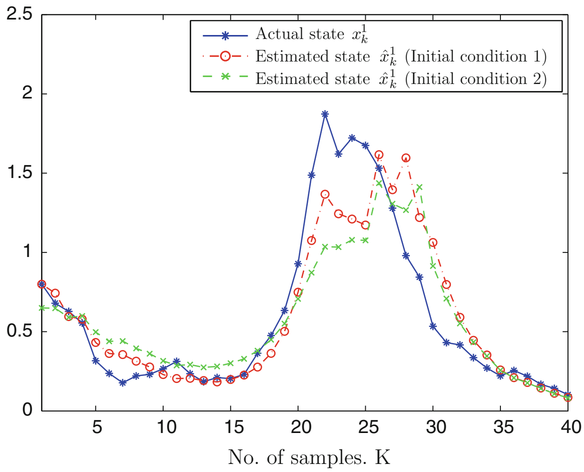

Fig. 2.17 The actual state x_k^1 and its estimation $\hat{x}_k^1$ (Case II)

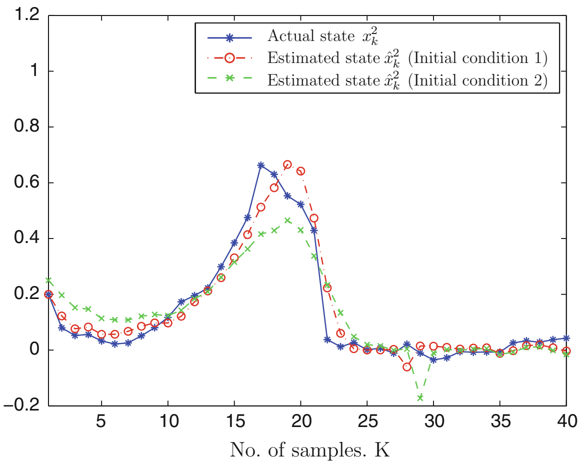

Fig. 2.18 The actual state x_k^2 and its estimation $\hat{x}_k^2$ (Case II)

$\theta_1 = 1.754$, $\theta_2 = 1.49 \times 10^{-4}$ for the object height $x_{2,k} \geq 9{,}144$ m), $\alpha_{1,k} \in \mathbb{R}$, $\beta_{1,k} \in \mathbb{R}$, $\omega_k \in \mathbb{R}^4$, and $\nu_k \in \mathbb{R}^2$ are zero-mean Gaussian white noises with covariances 1, 1, Q_k, and $R_k = 100\, I_2$. Here,

$$Q_k = c \cdot \mathrm{diag}\{q, q\}, \quad q = \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{bmatrix}.$$

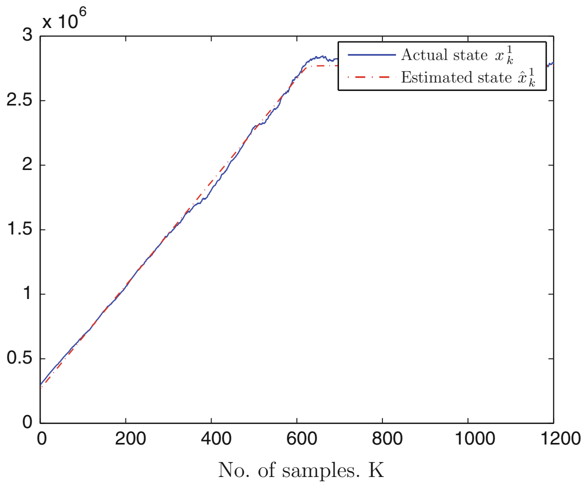

Fig. 2.19 The actual state x_k^1 and the estimated state $\hat{x}_k^1$

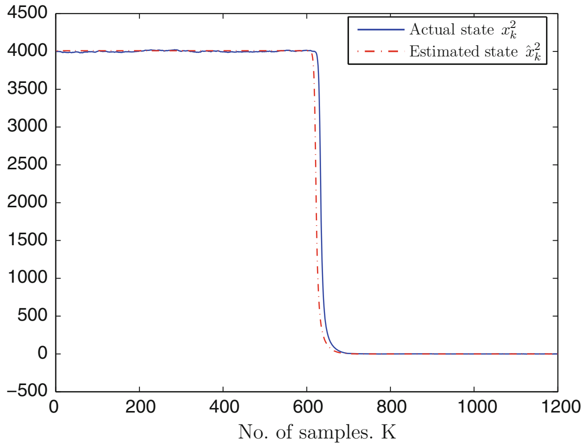

Fig. 2.20 The actual state x_k^2 and the estimated state $\hat{x}_k^2$

In the simulation, the parameters are chosen as $g = 9.81\,\mathrm{m/s}^2$, $\beta = 4 \times 10^4\,\mathrm{kg/ms}^2$, $c = 0.1\,\mathrm{m}^2/\mathrm{s}^3$, $T = 1\,\mathrm{s}$, $u_0^1 = 9 \times 10^5$, $u_0^2 = 8 \times 10^4$, $\chi^{(1)} = 0.9$, $\chi^{(2)} = 0.9$, $\bar{\Xi}_k = \mathrm{diag}\{0.85, 0.85\}$, $x_0 = 10^3 \times \begin{bmatrix} 300 & 4 & 90 & 3 \end{bmatrix}^T$, $\bar{x}_0 = 10^3 \times \begin{bmatrix} 270 & 4.01 & 95 & 2.9 \end{bmatrix}^T$, $B_k = \mathrm{diag}\{15, 1.2, 4, 0.1\}$, $L_k = 0.01 I_4$, $\gamma_{1,k} = 0.005$, $\gamma_{2,k+1} = 0.002$, $\varepsilon_1 = 0.4$, $\varepsilon_2 = 0.3$, and $\varepsilon_3 = 0.5$. Similarly, according to (2.64), (2.65), and (2.69) in Theorem 2.4, the upper bound of the filtering error covariance and filter gain can be recursively calculated at each sampling instant. Therefore, the addressed filter design problem can be solved by using the proposed filter structure

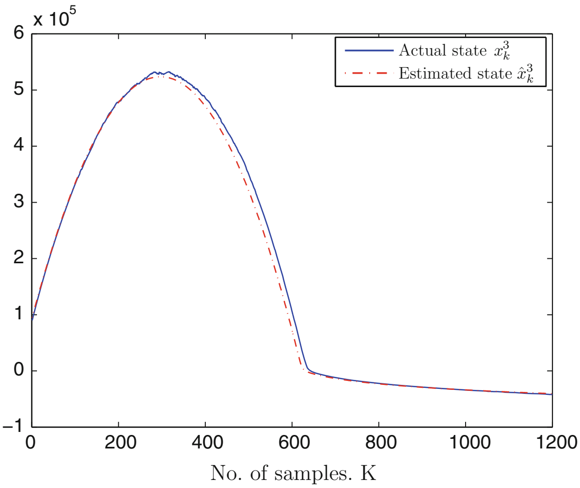

Fig. 2.21 The actual state x_k^3 and the estimated state $\hat{x}_k^3$

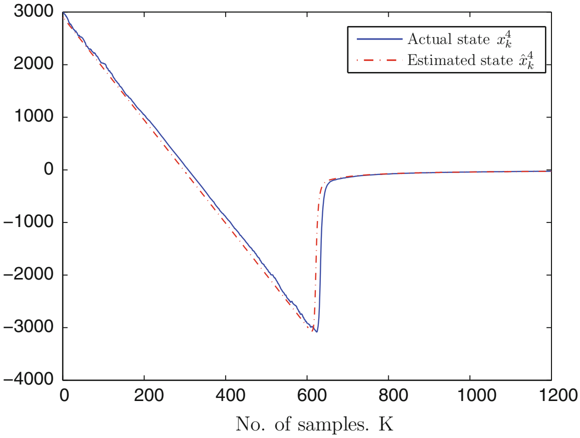

Fig. 2.22 The actual state x_k^4 and the estimated state $\hat{x}_k^4$

(2.50)–(2.51). The trajectories of the actual states and their estimations are plotted in Figs. 2.19, 2.20, 2.21, and 2.22. In summary, all the simulation results have further confirmed our theoretical analysis for the recursive filtering problem for a class of time-varying non-linear systems with missing measurements, quantization effects, and multiplicative noises.

2.4 Summary

In this chapter, we have made one of the first few attempts to design the finite-horizon recursive filters for time-varying non-linear systems with missing measurements. Firstly, the stochastic non-linearities described by statistical means have been taken into account. The phenomenon of multiple missing measurements has been described by any discrete-time distributions with known probability density function. A series of mutually independent random variables has been introduced to characterize the operation behavior of each sensor. By means of Riccati-like difference equation approach, we have designed the EKF such that, for both the stochastic non-linearities and multiple missing measurements, the upper bound of the filtering error covariance exits and is then minimized by properly designing the filter gain at each sampling instant. Moreover, the logarithmic quantization has been considered to characterize the signal quantization. Accordingly, the recursive filter has been designed for a class of non-linear systems with missing measurements, quantization effects, and multiplicative noises. It has been shown that the proposed filter schemes are of a recursive form that are suitable for recursive computation in online applications. Finally, the effectiveness and applicability of the developed algorithms have been demonstrated by three simulation examples.

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