Chapter 2
$\mathcal{N}$=2 Multiplets and Lagrangians

2.1 Microscopic Lagrangian

2.1.1 $\mathcal{N}$=1 Superfields

Let us now move on to the construction of the Lagrangian with $\mathcal{N}$=2 supersymmetry. An $\mathcal{N}$=2 supersymmetric theory is in particular an $\mathcal{N}$=1 supersymmetric theory. Therefore it is convenient to use $\mathcal{N}$=1 superfields to describe $\mathcal{N}$=2 systems. For this purpose let us quickly recall the $\mathcal{N}$=1 formalism. In this section only, we distinguish the imaginary unit by writing it as i.

An $\mathcal{N}$=1 vector multiplet consists of a Weyl fermion λ_α and a vector field A_μ, both in the adjoint representation of the gauge group G. We combine them into the superfield W_α with the expansion

$$W_\alpha = \lambda_\alpha + F_{(\alpha\beta)}\theta^\beta + D\theta_\alpha + \cdots \tag{2.1.1}$$

where D is an auxiliary field, again in the adjoint of the gauge group. $F_{\alpha\beta} = \frac{\mathrm{i}}{2}\sigma^{\mu}{}_{\dot\gamma}^{\beta}\bar\sigma^{\nu}{}^{\dot\gamma}_{\alpha}F_{\mu\nu}$ is the anti-self-dual part of the field strength $F_{\mu\nu}$.

The kinetic term for a vector multiplet is given by

$$\int d^2\theta \frac{-\mathrm{i}}{8\pi}\tau \operatorname{tr} W_\alpha W^\alpha + cc. \tag{2.1.2}$$

where

$$\tau = \frac{4\pi\mathrm{i}}{g^2} + \frac{\theta}{2\pi} \tag{2.1.3}$$

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is a complex number combining the inverse of the coupling constant and the theta angle. We call it the complexified coupling of the gauge multiplet. Expanding in components, we have

$$\frac{1}{2g^2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2}\operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{g^2}\operatorname{tr} D^2 - \frac{2\mathrm{i}}{g^2}\operatorname{tr}\bar{\lambda}\not{D}\lambda. \tag{2.1.4}$$

We use the convention that $\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab}$ for the standard generators of gauge algebras, which explain why we have the factors $1/(2g^2)$ in front of the gauge kinetic term. The θ term is a total derivative of a gauge-dependent term. Therefore, it does not affect to perturbative computations. It does affect non-perturbative computations, to which we will come back later.

An $\mathcal{N}$=1 chiral multiplet Q consists of a complex scalar Q and a Weyl fermion ψ_α, both in the same representation of the gauge group. In terms of a superfield we have

$$Q = Q\big|_{\theta=0} + 2\psi_\alpha\theta^\alpha + F\theta_\alpha\theta^\alpha \tag{2.1.5}$$

where F is auxiliary. The coefficient 2 in front of the middle component is unconventional, but this choice allows us to remove various annoying factors of $\sqrt{2}$ appearing in the formulas later. The chiral multiplet $Q_{1,\ldots}$ can be in an arbitrary complex representation R of the gauge group G. The Lagrangian density is then

$$\int d^4\theta Q^{\dagger j} e^{V^a \rho_a{}^i{}_j} Q_i + \int d^2\theta W(Q) + cc. \tag{2.1.6}$$

where V is the vector superfield, $\rho_a{}^i{}_j$ is the matrix representation of the gauge algebra, and $W(Q)$ is a gauge invariant holomorphic function of $Q_{1,\ldots}$.

The supersymmetric vacua is obtained by demanding that the supersymmetry transformation of various fields are zero. The nontrivial conditions come from

$$\delta\lambda_\alpha = 0, \qquad \delta\psi_\alpha = 0 \tag{2.1.7}$$

which give

$$D_a = 0, \quad F_i = 0. \tag{2.1.8}$$

By solving the algebraic equations of motion of the auxiliary fields, we find

$$Q^\dagger_{\bar{j}}\rho_a^{\bar{j}i}Q_i = 0, \quad \frac{\partial W}{\partial Q_i} = 0. \tag{2.1.9}$$

2.1.2 Vector Multiplets and Hypermultiplets

An $\mathcal{N}=2$ vector multiplet consists of the following $\mathcal{N}=1$ multiplets, both in the adjoint of the gauge group G:

$$\begin{array}{ccccclr} & \swarrow\!\!\!\nearrow & \lambda_\alpha & \leftrightarrow & A_\mu & \qquad \mathcal{N}=1 \text{ vector multiplet,} & \\ \Phi & \leftrightarrow & \tilde{\lambda}_\alpha & \swarrow\!\!\!\nearrow & & \qquad \mathcal{N}=1 \text{ chiral multiplet.} & \end{array} \tag{2.1.10}$$

Here, the horizontal arrows signify the $\mathcal{N}=1$ sub-supersymmetry generator manifest in the $\mathcal{N}=1$ superfield formalism, and the slanted arrows are for the second $\mathcal{N}=1$ sub-supersymmetry.

One easy way to construct the second supersymmetry action is to demand that the theory is symmetric under the SU(2) rotation acting on λ_α and $\tilde{\lambda}_\alpha$. A symmetry which does not commute with the supersymmetry generators is called an R-symmetry in general. Therefore this SU(2) symmetry is often called the $\mathrm{SU}(2)_R$ symmetry. It is by now a standard technique to combine the supersymmetry manifest in a superfield formalism and an R-symmetry to construct a theory with more supersymmetries, see e.g. [1] for an application. It is also to be kept in mind that there can be and indeed are $\mathcal{N}=2$ supersymmetric theories without $\mathrm{SU}(2)_R$ symmetry: there can just be two sets of supersymmetry generators without SU(2) symmetry relating them, see e.g. [2, 3]. That said, for simplicity, we only deal with $\mathcal{N}=2$ supersymmetric systems with $\mathrm{SU}(2)_R$ symmetry in this lecture note.

The Lagrangian is then

$$\frac{\operatorname{Im}\tau}{4\pi}\int d^4\theta \operatorname{tr}\Phi^\dagger e^{[V,\cdot]}\Phi + \int d^2\theta \frac{-\mathrm{i}}{8\pi}\tau \operatorname{tr} W_\alpha W^\alpha + cc. \tag{2.1.11}$$

The ratio between the prefactors of the Kähler potential and of the gauge kinetic term is fixed by demanding $\mathrm{SU}(2)_R$ symmetry.

An $\mathcal{N}=2$ hypermultiplet[1] consists of the following fields:

$$\begin{array}{ccccclr} & \swarrow\!\!\!\nearrow & Q & \leftrightarrow & \psi & \qquad \mathcal{N}=1 \text{ chiral multiplet} & \\ \tilde{\psi}^\dagger & \leftrightarrow & \tilde{Q}^\dagger & \swarrow\!\!\!\nearrow & & \qquad \mathcal{N}=1 \text{ antichiral multiplet} & \end{array} \tag{2.1.12}$$

They are both in the same representation R of the gauge group. Therefore, the $\mathcal{N}=1$ chiral multiplets Q and $\tilde{Q}$ are in the conjugate representations of the gauge group. We demand again that the theory is symmetric under the SU(2) rotation acting on Q and $\tilde{Q}^\dagger$, to have $\mathcal{N}=2$ supersymmetry.

For definiteness, let us consider $G = \mathrm{SU}(N)$ and N_f hypermultiplets Q_i^a, $\tilde{Q}_a^i$ in the fundamental N-dimensional representation, where $a = 1, \ldots, N$ and $i =$

[1]There is a stupid convention that we use a space between 'vector' and 'multiplets' to spell "vector multiplets", but not for "hypermultiplets". Colloquially, hypermultiplets are often just called hypers.

$1, \ldots, N_f$. This set of fields is often called N_f flavors of fundamentals of SU(N). The gauge transformation acts on them as

$$Q_i \to e^{\Lambda} Q_i, \qquad \tilde{Q}^i \to \tilde{Q}^i e^{-\Lambda} \tag{2.1.13}$$

where Λ is a traceless $N \times N$ matrix of chiral superfields; the gauge indices are suppressed.

The Lagrangian for the hypermultiplets is

$$c\int d^4\theta(Q^{\dagger i}e^V Q_i + \tilde{Q}^i e^{-V}\tilde{Q}^{\dagger}{}_i) + c'(\int d^2\theta\,\tilde{Q}^i\Phi Q_i + cc.) + (\int d^2\theta\mu^i_j\,\tilde{Q}^j Q_i + cc.) \tag{2.1.14}$$

where the gauge index a is suppressed again. The existence of SU$(2)_R$ symmetry fixes the ratio of c and c': it can be done e.g. by comparing the coefficients of $Q^i\lambda\psi$ from the first term and of $\tilde{Q}^i\tilde{\lambda}\psi$ from the second term. We find the choice $c = c'$ does the job. In the following we take $c = c' = 1$ unless otherwise mentioned. The SU$(2)_R$ symmetry also demands that the mass term μ^i_j satisfies $[\mu, \mu^\dagger] = 0$. Then μ can be diagonalized, and consequently the mass term is often written as

$$\sum_i \int d^2\theta\mu_i\,\tilde{Q}^i Q_i + cc. \tag{2.1.15}$$

As another example, let us consider the case when we have a hypermultiplet $(Z, \tilde{Z})$ in the adjoint representation, i.e. they are both $N \times N$ traceless matrices. The following discussion can easily be generalized to arbitrary gauge group too. When the hypermultiplet is massless, the total Lagrangian has the form

$$\begin{aligned}&\int d^2\theta\frac{-\mathrm{i}}{8\pi}\tau\,\mathrm{tr}\,W_\alpha W^\alpha + cc. + \frac{\mathrm{Im}\,\tau}{4\pi}\int d^4\theta\,\mathrm{tr}\,\Phi^\dagger e^{[V,\cdot]}\Phi\\ &\quad+\frac{\mathrm{Im}\,\tau}{4\pi}\int d^4\theta(Z^\dagger e^{[V,\cdot]}Z + \tilde{Z}e^{-[V,\cdot]}\tilde{Z}^\dagger) + \frac{\mathrm{Im}\,\tau}{4\pi}\int d^2\theta\tilde{Z}[\Phi, Z] + cc.\end{aligned} \tag{2.1.16}$$

where we made a different choice of $c = c'$ in (2.1.14). This Lagrangian clearly has SU$(3)_F$ flavor symmetry rotating Φ, Z and $\tilde{Z}$. This commutes with the $\mathcal{N}$=1 supersymmetry manifest in the superfield formalism. We also know that this theory has an SU$(2)_R$ symmetry rotating Z and $\tilde{Z}^\dagger$. These two symmetries SU$(3)_F$ and SU$(2)_R$ does not commute: we find that there is an SO$(6)_R$ symmetry, acting on

$$\mathrm{Re}\,\Phi, \mathrm{Im}\,\Phi, \mathrm{Re}\,Z, \mathrm{Im}\,Z, \mathrm{Re}\,\tilde{Z}, \mathrm{Im}\,\tilde{Z}. \tag{2.1.17}$$

Note that $\mathrm{SO}(6)_R$ can also be regarded as $\mathrm{SU}(4)_R$, as SO(6) and SU(4) have the same Lie algebra. Then the $\mathrm{SU}(4)_R$ symmetry acts on the four Weyl fermions

$$\lambda, \tilde{\lambda}, \psi, \tilde{\psi} \tag{2.1.18}$$

in the system, where λ and $\tilde{\lambda}$ are in the $\mathcal{N}=2$ vector multiplet, and ψ, $\tilde{\psi}$ are in the $\mathcal{N}=2$ hypermultiplet. We conclude that this system has in fact $\mathcal{N}=4$ supersymmetry, whose four supersymmetry generators are acted on by $\mathrm{SU}(4)_R \simeq \mathrm{SO}(6)_R$. The argument here is another application of the combination of the manifest and non-manifest symmetries in the superfield formalism.

We can add the mass term $\int d^2\theta \mu Z\tilde{Z} + cc.$ to (2.1.16). This preserves the $\mathcal{N}=2$ supersymmetry but it breaks $\mathcal{N}=4$ supersymmetry. The resulting theory is sometimes called the $\mathcal{N}=2^*$ theory.

Before closing this section, we should mention the concept of half-hypermultiplet. Let us start from a full hypermultiplet $(Q_a, \tilde{Q}^a)$ so that Q_a and $\tilde{Q}^a$ are in the representations R, $\bar{R}$, respectively. When R is pseudo-real, or equivalently when there is an antisymmetric invariant tensor ϵ_{ab}, we can impose the constraint

$$Q_a = \epsilon_{ab}\tilde{Q}^b \tag{2.1.19}$$

compatible with $\mathcal{N}=2$ supersymmetry, which halves the number of degrees of freedom in the multiplet. The resulting multiplet is called a half-hypermultiplet in the representation R. We will come back to this in Sect. 7.2.

2.2 Vacua

The combined system of the vector multiplet and the hypermultiplets has the Lagrangian which is the sum of (2.1.11) and (2.1.14). The supersymmetric vacua are given by the following conditions.

First, the variation of the D auxiliary fields gives

$$\frac{1}{g^2}[\Phi^\dagger, \Phi] + (Q_i Q^{\dagger i} - \tilde{Q}^\dagger{}_i \tilde{Q}^i)\big|_{\text{traceless}} = 0, \tag{2.2.1}$$

where $X|_{\text{traceless}}$ for an $N \times N$ matrix is defined by

$$X|_{\text{traceless}} = X - \frac{1}{N}\operatorname{tr} X. \tag{2.2.2}$$

We use the convention that a scalar is multiplied by a unit matrix when necessary.

Second, the variation of the F auxiliary field of Φ gives

$$Q_i \tilde{Q}^i\big|_{\text{traceless}} = 0 \tag{2.2.3}$$

and the F auxiliary fields of Q_i, $\tilde{Q}^i$ give

$$\Phi Q_i + \mu_i^j Q_j = 0, \qquad \tilde{Q}^i \Phi + \mu^i_j \tilde{Q}^j = 0 \tag{2.2.4}$$

for all i. The total scalar potential is a weighted sum of absolute values squared of (2.2.1), (2.2.3) and (2.2.4).

So far we only used the supersymmetry condition with respect to the $\mathcal{N}$=1 supersymmetry manifest in the superfield notation. By massaging the cross terms between the first term and the second term of (2.2.1) and combining them with the squares of (2.2.4), we can re-write the total scalar potential as a weighted sum of the following objects. First, we have one term purely of Φ:

$$[\Phi^\dagger, \Phi] = 0. \tag{2.2.5}$$

Second, we have terms purely of Q and $\tilde{Q}$: one is

$$(Q_i Q^{\dagger i} - \tilde{Q}^\dagger{}_i \tilde{Q}^i)\big|_{\text{traceless}} = 0 \tag{2.2.6}$$

and another is (2.2.3). Finally, we have terms mixing Φ and Q, which are (2.2.4) together with

$$\Phi^\dagger Q_i + \mu^\dagger{}_i^j Q_j = 0, \qquad \tilde{Q}^i \Phi^\dagger + \mu^{\dagger i}{}_j \tilde{Q}^j = 0. \tag{2.2.7}$$

Note that (2.2.5) and (2.2.6) are the $\mathrm{SU}(2)_R$ singlet and triplet parts of Eq. (2.2.1), respectively. Furthermore, Eq. (2.2.6) together with the real and the imaginary parts of Eq. (2.2.3) form the triplet of $\mathrm{SU}(2)_R$. Finally, Eqs. (2.2.4) and (2.2.7) transform as a doublet of $\mathrm{SU}(2)_R$.

Let us summarize. We first demanded that one $\mathcal{N}$=1 sub-supersymmetry is unbroken in (2.2.1), (2.2.3) and (2.2.4). We found the equations satisfied are automatically $\mathrm{SU}(2)_R$ invariant, and therefore we see that all the $\mathcal{N}$=2 supersymmetry is automatically unbroken.

One easy way to have supersymmetry is to demand (2.2.5) and set $Q = \tilde{Q} = 0$. This subspace of the supersymmetric vacuum moduli is called the Coulomb branch, since there usually remain a number of Abelian gauge fields in the infrared.

When the mass terms μ^i_j are nonzero, it is not straightforward to discuss other vacuum configurations in general. When $\mu^i_j = 0$, there is another class of vacuum configurations, given by just demanding (2.2.6) and (2.2.3), and setting $\Phi = 0$. This is called the Higgs branch. Some people in the field reserve the word the Higgs branch for the branch where the gauge group is completely broken, but theoretically the Higgs branch as defined here behaves more uniformly under various operations.

The branches with when both the hypermultiplet scalars Q, $\tilde{Q}$ and the vector multiplet scalars Φ are nonzero are called the mixed branches.

From (2.2.5) we see that Φ can be diagonalized in the supersymmetric vacua. For definiteness let $G = \mathrm{SU}(2)$. Then $\Phi = \mathrm{diag}(a, -a)$. When $a \neq 0$ this breaks the gauge group to U(1). As there is a Coulomb field remaining in the infrared, these vacua are called the Coulomb branch. Let us compute the mass of the resulting W-bosons. From

$$\frac{1}{g^2}\,\mathrm{tr}\,|D_\mu\Phi|^2 = \frac{1}{g^2}\,\mathrm{tr}(\partial_\mu\Phi + [A_\mu,\Phi])^2 \tag{2.2.8}$$

we have a term

$$\frac{1}{g^2}\,\mathrm{tr}[A_\mu,\langle\Phi\rangle]^2 \tag{2.2.9}$$

in the Lagrangian, which gives a mass to the vector field. Writing

$$A_\mu = \begin{pmatrix} A^0 & W^+ \\ W^- & -A^0 \end{pmatrix}_\mu, \tag{2.2.10}$$

we find

$$\left[\begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\right] = -2a\begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix}. \tag{2.2.11}$$

The kinetic term in our convention is $\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu}/(2g^2)$, and therefore this gives the mass

$$M_W = |2a|. \tag{2.2.12}$$

The mass terms of the fields Q_i, $\tilde{Q}^i$ for fixed i are

$$\tilde{Q}^i\Phi Q_i + \mu_i\tilde{Q}^i Q_i = (\tilde{Q}^i_1, \tilde{Q}^i_2)\begin{pmatrix} a+\mu_i & 0 \\ 0 & -a+\mu_i \end{pmatrix}\begin{pmatrix} Q^i_1 \\ Q^i_2 \end{pmatrix}. \tag{2.2.13}$$

Therefore we have

$$M_{Q_{i,1}} = |a+\mu|, \qquad M_{Q_{i,2}} = |-a+\mu|. \tag{2.2.14}$$

We studied the classical mass of the monopole in this model in (1.3.12) when $\theta = 0$. In general, this is given by

$$M_{\mathrm{monopole}} = |2\tau a|. \tag{2.2.15}$$

Classically, there is a general inequality for the mass of a particle

$$M \geq |na + m(2\tau a) + \sum_i f_i\mu_i| \tag{2.2.16}$$

where n, m, f_i are the electric, magnetic and flavor charges of the particle. Here the i-th flavor charges are associated to the symmetry

$$Q_i \to e^{\mathrm{i}\varphi_i} Q_i, \quad \tilde{Q}^i \to e^{-\mathrm{i}\varphi_i} \tilde{Q}^i. \tag{2.2.17}$$

This inequality, called the Bogomolnyi-Prasad-Sommerfield (BPS) bound, persists in the quantum system, once quantum corrections are taken into account to a and $2\tau a$. Let us study this point next.

2.3 BPS Bound

The general $\mathcal{N}$=2 supersymmetry algebra has the following form

$$\{Q^I_\alpha, Q^{\dagger \bar{J}}_{\dot\beta}\} = \delta^{I\bar{J}} P_\mu \sigma^\mu_{\alpha\dot\beta}, \tag{2.3.1}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon^{IJ}\epsilon_{\alpha\beta} Z. \tag{2.3.2}$$

Here $I = 1, 2$ are the index distinguishing two supersymmetry generators, and Z is a complex quantity which commutes with everything. Let us take the coordinate system where

$$P_\mu = (M, 0, 0, 0). \tag{2.3.3}$$

This choice breaks the Lorentz symmetry SO(3, 1) to the spatial rotation SO(3), which allows us to identify the undotted and the dotted spinor indices. Let us then define

$${}^{(\varphi)}Q_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha + e^{-\mathrm{i}\varphi}\sigma^0{}_\alpha{}^{\dot\beta} Q^{\dagger 2}_{\dot\beta}) \tag{2.3.4}$$

for which we have

$$\{{}^{(\varphi)}Q_\alpha, {}^{(\varphi)}Q^\dagger_\beta\} = \delta_{\alpha\beta}(M - \mathrm{Re}(e^{-\mathrm{i}\varphi} Z)). \tag{2.3.5}$$

In general, if there is an operator a satisfying $\{a, a^\dagger\} = c$ with a constant c, c is necessarily non-negative. Indeed, take a ket vector $|\psi\rangle$ then

$$\left|a^\dagger|\psi\rangle\right|^2 + |a|\psi\rangle|^2 = \langle\psi|aa^\dagger|\psi\rangle + \langle\psi|a^\dagger a|\psi\rangle = c\langle\psi|\psi\rangle, \tag{2.3.6}$$

meaning that $c \geq 0$. From (2.3.5), then, we see

$$M \geq \mathrm{Re}(e^{-\mathrm{i}\varphi} Z) \tag{2.3.7}$$

for all φ. Choosing $\varphi = \operatorname{Arg} Z$, we find the inequality

$$M \geq |Z|. \tag{2.3.8}$$

In general, the multiplet of the supertranslations Q^I_α and $Q^{J\dagger}_\alpha$ generates $2^4 = 16$ states in the supermultiplet. When the inequality (2.3.8) is saturated, c in Eq. (2.3.6) for $a_\alpha = {}^{(\operatorname{Arg} Z)}Q_\alpha$ is zero, forcing the operators ${}^{(\operatorname{Arg} Z)}Q_\alpha$ themselves to vanish. Then the supertranslations only generate $2^2 = 4$ states. Such multiplets are called BPS, and those multiplets with 16 states under the action of supertranslations are called non-BPS. A BPS state is rather robust: under a generic perturbation, the number of states in a multiplet can not jump. Therefore the BPS state will generically stay BPS.

What is this quantity Z, which commutes with everything? A quantity commuting with everything is by definition a conserved charge. When the low-energy theory is a weakly-coupled U(1) gauge theory, Z is a linear combination of the electric charge n, the magnetic charge m, and the flavor charges f_i. We define the coefficients appearing in the linear combination to be a, a_D and μ_i in the quantum theory:

$$Z = na + ma_D + \sum_i \mu_i f_i. \tag{2.3.9}$$

When the theory is weakly-coupled, we can identify a to be the diagonal entry of the field Φ, a_D to be $2\tau a$, and μ_i to be the coefficients of the mass terms in the Lagrangian, by comparing the quantum BPS mass formula (2.3.8) and its classical counterpart (2.2.16). In the strongly-coupled regime, there is no meaning in saying that a is the diagonal entry of a gauge-dependent field Φ. Rather, we should think of (2.3.9) as the definition of the quantity a.

2.4 Low Energy Lagrangian

Let us consider a general effective Lagrangian which describes $\mathrm{U}(1)^n$ gauge fields in the infrared. Let us denote n U(1) vector multiplets by

$$\begin{array}{ccccc@{\qquad}l} & \swarrow\!\!\!\!\nearrow & \lambda_\alpha & \leftrightarrow & A_\mu & \mathcal{N}{=}1 \text{ vector multiplet} \\ a & \leftrightarrow & \tilde{\lambda}_\alpha & \swarrow\!\!\!\!\nearrow & & \mathcal{N}{=}1 \text{ chiral multiplet} \end{array} \tag{2.4.1}$$

with additional scripts $i = 1, \ldots, n$. A general $\mathcal{N}{=}1$ supersymmetric Lagrangian is given by

$$\frac{1}{8\pi}\int d^4\theta K(\bar{a}_i, a_j) + \int d^2\theta \frac{-\mathrm{i}}{8\pi}\tau^{ij}(a) W_{\alpha,i} W^{\alpha}{}_j + cc. \tag{2.4.2}$$

Note that we allowed the Kähler potential and the gauge coupling matrix to have nontrivial dependence on a_i.

We demand the existence of the $\mathrm{SU}(2)_R$ symmetry rotating λ_α and $\tilde{\lambda}_\alpha$ to guarantee the existence of $\mathcal{N}$=2 supersymmetry. The kinetic matrix of $\tilde{\lambda}_\alpha$ is

$$\frac{1}{4\pi}\frac{\partial^2 K}{\partial a_i \partial \bar{a}_j} \tag{2.4.3}$$

and that of λ is

$$\frac{\mathrm{Im}\,\tau^{ij}}{2\pi} = \frac{\tau^{ij} - \bar{\tau}^{ij}}{4\pi\mathrm{i}}. \tag{2.4.4}$$

Equating them, we have

$$\frac{\tau^{ij} - \bar{\tau}^{ij}}{\mathrm{i}} = \frac{\partial^2 K}{\partial a_i \partial \bar{a}_j}. \tag{2.4.5}$$

Taking the derivative of both sides by a_k, we have

$$\frac{\partial}{\partial a_k}\frac{\tau^{ij}}{\mathrm{i}} = \frac{\partial^3 K}{\partial a_k \partial a_i \partial \bar{a}_j}. \tag{2.4.6}$$

The left hand side is symmetric under $i \leftrightarrow j$, and the right hand side is symmetric under $k \leftrightarrow i$. Therefore, at least locally, τ^{ij} can be integrated twice:

$$\tau^{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} \tag{2.4.7}$$

for a locally holomorphic function $F(a)$. We define

$$a_D^i = \frac{\partial F}{\partial a_i}, \tag{2.4.8}$$

then we have

$$K = \mathrm{i}(\bar{a}_D^i a_i - \bar{a}_i a_D^i). \tag{2.4.9}$$

A Kähler manifold with this additional structure is often called a special Kähler manifold. With supergravity, a slightly different structure appears. To distinguish from it, it is also called a rigid special Kähler manifold. The same geometry is also called a Seiberg–Witten integrable system, or a Donagi–Witten integrable system. See e.g. [4–6] for a review. In this context, the fields a_i and a_D^i are called the special coordinates.

The notations a_i and a^i_D can be justified as follows. Suppose we have a hypermultiplet Q, $\tilde{Q}$ charged under the i-th vector multiplet only. It has the superpotential

$$W = Qa_i\tilde{Q}, \tag{2.4.10}$$

which gives the mass

$$M_Q = |a_i|. \tag{2.4.11}$$

Therefore, a_i is indeed the coefficient appearing in (2.3.9). To justify the notation a^i_D, write down the Lagrangian for the bosons in components:

$$\frac{\operatorname{Im}\tau^{ij}}{4\pi}\partial_\mu \bar{a}_i \partial^\mu a_j + \frac{\operatorname{Im}\tau^{ij}}{8\pi}F_{\mu\nu\ i}F^{\mu\nu}_j + \frac{\operatorname{Re}\tau^{ij}}{8\pi}F_{\mu\nu\ i}\tilde{F}^{\mu\nu}_j. \tag{2.4.12}$$

Generalizing the argument in Sect. 1.2, the dual electromagnetic field F_D is given by

$$F_D{}^i_{\mu\nu} = \operatorname{Im}\tau^{ij}F_{\mu\nu\ j} + \operatorname{Re}\tau^{ij}\tilde{F}_{\mu\nu\ j}, \tag{2.4.13}$$

in terms of which the kinetic term of the gauge fields is

$$\frac{1}{8\pi}\left(\operatorname{Im}\tau_{D\,ij}F_D{}^i_{\mu\nu}F_D{}^{\mu\nu\ j} + \operatorname{Re}\tau_{D\,ij}F_{D\,\mu\nu\ i}\tilde{F}_D{}^{\mu\nu\ j}\right) \tag{2.4.14}$$

where

$$\tau_{D\,ij} = (-\tau^{-1})_{ij}. \tag{2.4.15}$$

Then we find

$$\frac{1}{4\pi}\operatorname{Im}\tau^{ij}\partial_\mu\bar{a}_i\partial^\mu a_j = \frac{1}{4\pi}\operatorname{Im}\tau_{D\,ij}\partial_\mu\bar{a}^i_D\partial^\mu a^j_D \tag{2.4.16}$$

where a_D is as defined in (2.4.8). This means that we have the dual $\mathcal{N}$=2 multiplets

$$\begin{array}{lll} & \swarrow\!\!\nearrow\ \lambda_{D\alpha} \leftrightarrow A_{D\mu} & \mathcal{N}{=}1 \text{ vector multiplet} \\ a_D \leftrightarrow \tilde{\lambda}_{D\alpha}\ \swarrow\!\!\nearrow & & \mathcal{N}{=}1 \text{ chiral multiplet} \end{array} \tag{2.4.17}$$

where $A_{D\mu}$ is the gauge potential of $F_{D\mu\nu}$ introduced above, with additional superscripts i.

We introduced the prepotential F in a rather indirect manner in this section, by saying that the kinetic term of the U(1) vector multiplets (2.4.2) should be given by (2.4.7) and (2.4.9). This can be better understood using $\mathcal{N}$=2 superspace, since it is known that the prepotential is the Lagrangian density in the $\mathcal{N}$=2 superspace. This is similar to the situation where the Kähler potential gives the Lagrangian density in the $\mathcal{N}$=1 superspace.

Recall that the multiplets (2.4.1) can be summarized in $\mathcal{N}$=1 superfields

$$\Phi_i = a_i + 2\tilde{\lambda}_i{}^\alpha \theta_\alpha + \cdots , \qquad W_i = \lambda_{\alpha\, i} + F_{\alpha\beta}\theta^\beta + \cdots . \tag{2.4.18}$$

We can introduce another set of supercoordinates $\tilde{\theta}_\alpha$ to combine them:

$$\boldsymbol{\Phi}_i = \Phi_i + 2W_{\alpha\, i}\tilde{\theta}^\alpha = a_i + 2\tilde{\lambda}_{\alpha\, i}\theta^\alpha + 2\lambda_{\alpha\, i}\tilde{\theta}^\alpha + 2F_{\alpha\beta\, i}\theta^{(\alpha}\tilde{\theta}^{\beta)} + \cdots . \tag{2.4.19}$$

Then the SU(2) R-symmetry rotating λ and $\tilde{\lambda}$ acts on the two sets of supercoordinates θ_α and $\tilde{\theta}_\alpha$.

Now, take an arbitrary holomorphic function of n variables $F(a_1, \ldots, a_n)$, and consider its integral over the chiral $\mathcal{N}$=2 superspace:

$$\int d^2\theta d^2\tilde{\theta} F(\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_n) + cc. \tag{2.4.20}$$

It is clear that this gives rise to the structure (2.4.7) for the gauge kinetic matrix. To obtain the Kähler potential (2.4.9) one needs to study the structure of the constraints and the auxiliary fields of the $\mathcal{N}$=2 superfields, see e.g. Sect. 2.10 of [7]. The non-Abelian microscopic action (2.1.11) has the prepotential $F(\boldsymbol{\Phi}) = \frac{1}{2}\tau \operatorname{tr} \boldsymbol{\Phi}^2$.

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