Chapter 2
Oscillators and Time References

The word oscillator *originates from the verb* to oscillate *which in its turn originates from the Latin word* oscillātus *(past participle of* oscillāre*, to swing). In English the word can be understood as 'any instrument for producing oscillations, a person or thing that oscillates'. Although the use of electrical oscillators has only started in the twentieth century, the word* oscillator *nowadays mostly refers to an electrical oscillator* [74].

2.1 Introduction

The application field of oscillators is wide and contains many applications such as clock for digital logic, watches, motor controllers, audio applications and even cruise controls or autopilots. Moreover, oscillators are an essential part of both wired and wireless communication systems. Obviously, the requirements regarding oscillator specifications such as power, frequency accuracy and phase noise, are different in each application. In order to develop an efficient strategy in designing an oscillator with the right specifications, the trade-offs and the relation between the design parameters and these specifications need to be examined. A good insight in these principles is essential to understand the observed behavior and the design choices made in the following chapters.

This chapter handles about oscillators and time references in general and is organized as follows. In Sect. 2.2 it will be shown that some basic requirements need to be fulfilled to obtain an oscillation. Next, in Sect. 2.3, oscillators will be divided in two main categories; harmonic and relaxation oscillators. This categorization will have an impact on the generated output waveforms as well as the circuit topology. Although the circuit components used are equal in both cases, a difference is seen in temperature- and supply-voltage-dependency (Chap. 4) but also in the phase noise behavior (Chap. 3). Afterwards, in Sect. 2.4, different representations of an oscillator signal will be introduced. In Sect. 2.5, the main properties of an oscillator will be discussed. Section 2.6 concludes.

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2.2 The Phase Space Description of an Oscillator

Oscillators are observed everywhere in nature: the leaves of a tree waving in the wind, the waves on the ocean, the noise of a waterfall, etc. All of these oscillators can be isolated from their (endless) environment and described as an autonomous system. No influence from outside the system is allowed, i.e. one has to make sure that, after simplification, the environmental parameters with a considerable influence are still present inside the system. This results in a system, which can be described by a finite set of equations and variables. If the system is simplified in the right way, it is still oscillating. The question is how complex a system must be to obtain an oscillation. To answer this, a closer look at the phase space description of a system is required [242].

2.2.1 The Phase Space Description

The phase space description of a system is the mathematical space in which all possible states of a system are represented. Each possible state corresponds to one unique point in the phase space. To obtain the phase space, one must identify the degrees of freedom or free (independent) parameters x_i of the system. A complete set of parameter values gives a complete description of a system at a certain moment. When putting all these variables on a separate axis in an n-dimensional space, the phase space is obtained. To describe the behavior of the system over time, i.e. to be able to predict the state of the system in (near) future, the relation between these parameters and a time variable is needed. This relationship looks as follows:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n) \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n) \end{cases} \tag{2.1}$$

where $x_i(t)$ is a real-valued function of time t, and $f_i(x_1, \ldots, x_n)$ is a smooth, real-valued function of $x_1, \ldots, x_n$ and not a function of time. Any n-th order system (described by an n-th order differential equation) can be transformed to a system similar to (2.1), a set of n first-order differential equations. Even when this set of equations can be solved in a closed form, it is often difficult to truly understand the behavior of the system. One of the most basic techniques in dynamic system analysis is to interpret a differential equation as a vector field in the n-dimensional phase space. This graphical interpretation will be used often throughout this work.

2.2.2 One-Dimensional Systems

One-dimensional systems only have one parameter x_1 and are described using one differential equation:

$$\begin{cases} \dot{x}_1 = f_1(x_1) \end{cases} \tag{2.2}$$

when $f_1(x_1)$ is a function of time, the system must be described as a higher-order system. The phase space of this first-order system (2.2) is one-dimensional, which means that all possible states of the system are located on a single line.

Example 2.1 Suppose the following system:

$$\begin{cases} \dot{x}_1 = x_1^2 - 1 \end{cases} \tag{2.3}$$

This equation has a closed form solution equal to:

$$t = -\text{arctanh}(x_1) + C \tag{2.4}$$

In which C is a constant depending on the start conditions. Suppose $x_1 = x_{1,0}$ at $t = 0$, then (2.4) this simplifies to:

$$t = -\text{arctanh}(x_1) + \text{arctanh}(x_{1,0}) \tag{2.5}$$

The interpretation of this solution is much easier when using the graphical representation of Fig. 2.1: the horizontal axis represents the 1-dimensional phase space, on the vertical axis $\dot{x}_1$ is drawn. The state of the system is completely defined by its point on the horizontal axis. The arrows on the horizontal axis show for each value of x_1 the direction in which the system will evolve. The velocity of this evolution is proportional to the y-value of the parabolic curve. At points where $\dot{x}_1 = 0$ the system is static. Such points are therefore called *fixed points*. The solid black point is a *stable* fixed point or an *attractor* or *sink*; the open circle represents an *unstable* fixed point or *repeller* or *source*. For a starting point $x_{1,0} < 1$, the system will evolve to $x_1 = -1$; for $x_{1,0} > 1$, x_1 will go to infinity. A picture showing the different trajectories of the system is called a *phase portrait* (see Fig. 2.1).

It is clear from the phase portrait in Fig. 2.1 that a 1-dimensional system can only evolve linearly, in one direction. The evolution of the system is dominated by fixed

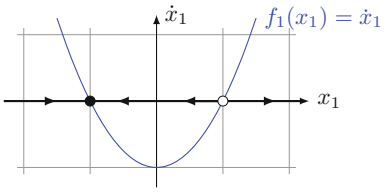

Fig. 2.1 The graphical representation of the 1-dimensional phase space gives a good insight in the first-order system's behavior. All the possible states of the system are represented by unique points on the *horizontal axis*. The *vertical axis* is only added for better understanding

points. All the possible trajectories start and end in a fixed point or diverge to $\pm\infty$. Overshoot or (damped) oscillations are therefore not possible in first-order systems. This corresponds to the fact that a first-order equation has no periodic solutions.

★ **1-Dimensional Oscillations** When x_1 is interpreted as an angle, a first kind of oscillation is possible because the output variable 'wraps around'. This, however, is a mathematical artifact which does not *define* a time reference as such: since there is only one parameter allowed, the resulting oscillation does not have an amplitude. Furthermore, in every physical system which can be described by a 1-dimensional flow on a circle, the oscillation frequency (in this case it's preferred to use *repetition* frequency instead) depends on other time constants, starting conditions or are defined by a higher-order system. This discrepancy between theory and practice comes forth from the wrapping of the state variable, which is a purely mathematical operation.

2.2.3 Two-Dimensional Systems

A second-order or 2-dimensional system is described by two first-order time-invariant equations:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \tag{2.6}$$

In the best case these equations have a closed-form solution; most often, however, this is not the case. The use of a phase portrait also helps here to explain the different properties of the system, for instance an oscillator. A distinction is made between two cases: linear and nonlinear systems. For a linear system, Eq. (2.6) simplifies to:

$$\begin{cases} \dot{x}_1 = a \cdot x_1 + b \cdot x_2 \\ \dot{x}_2 = c \cdot x_1 + d \cdot x_2 \end{cases} \tag{2.7}$$

Note that every linear combination of the solutions of this system, will also be a solution. This system can be written as:

$$\dot{\mathbf{x}} = A \cdot \mathbf{x} \tag{2.8}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. The general solution of this set of equations is equal to:

$$\mathbf{x(t)} = c_1 \cdot e^{\lambda_1 \cdot t} \cdot \mathbf{u_1} + c_2 \cdot e^{\lambda_2 \cdot t} \cdot \mathbf{u_2} \tag{2.9}$$

where $\mathbf{u_i}$ are the eigenvectors of matrix A, λ_i are the corresponding eigenvalues, and c_i are coefficients depending on the starting conditions. If the two corresponding eigenvectors are linearly independent, a unique combination of c_1 and c_2 can be found for any start condition $\mathbf{x_0} = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$. When the eigenvectors $\mathbf{u}$ are linearly dependent, however, the eigenspace corresponding to λ is only 1-dimensional. The general solution (2.9) in this case only represents the solution for start conditions on this line. The complete solution makes use of a generalized eigenvector $\boldsymbol{\rho}$ for which:

$$(A - \lambda \cdot I) \cdot \boldsymbol{\rho} = \mathbf{u} \text{ and } (A - \lambda \cdot I)^2 \cdot \boldsymbol{\rho} = 0 \tag{2.10}$$

The general solution to the differential equation is then written as:

$$\mathbf{x(t)} = c_1 \cdot e^{\lambda \cdot t} \cdot \mathbf{u} + c_2 \cdot (t \cdot e^{\lambda \cdot t} \cdot \mathbf{u} + e^{\lambda \cdot t} \cdot \boldsymbol{\rho}) \tag{2.11}$$

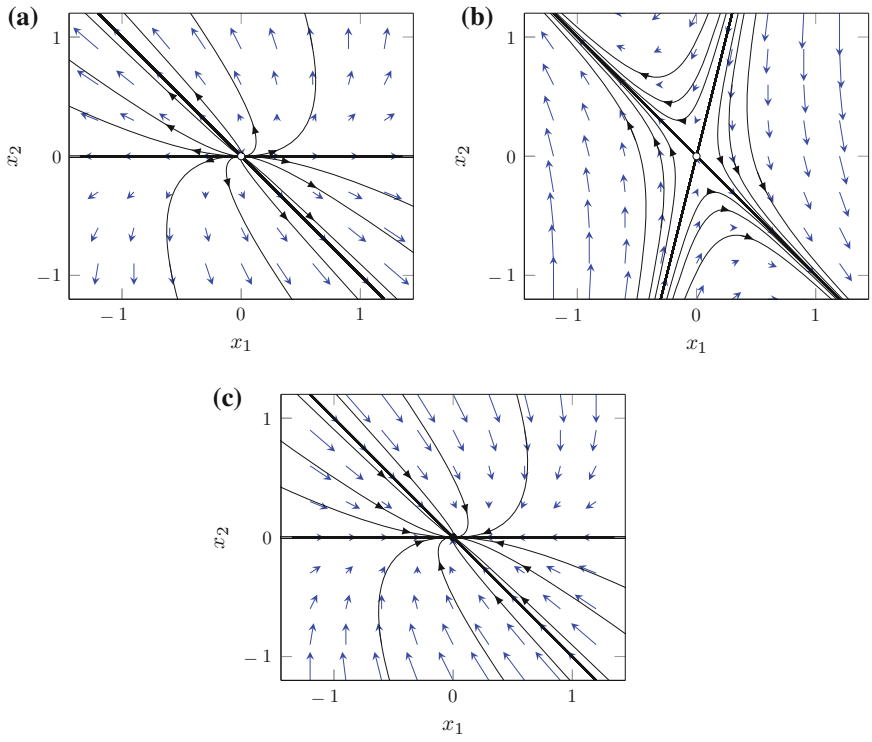

Fig. 2.2 Phase portraits of different linear second-order systems with **a** two positive eigenvalues, **b** a positive and a negative eigenvalue, **c** two negative eigenvalues. **a** and **b** are both called unstable systems, **c** is a stable system

which spans the entire plane. The fixed point at (0, 0) is called a degenerate node. Interesting is to take a closer look at the resulting phase portrait for different eigenvalues λ. Figure 2.2 shows the phase portrait for 3 different systems. The vector field as well as some trajectories are plotted to show the behavior of the system. The matrix, eigenvalues and eigenvectors of the corresponding systems are respectively:

$$\begin{aligned}
&(a)\ A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad \lambda_1 = 1,\ \lambda_2 = 2 \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&(b)\ A = \begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix} \quad \lambda_1 = -3,\ \lambda_2 = 2 \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&(c)\ A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \quad \lambda_1 = -1,\ \lambda_2 = -2 \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{aligned} \tag{2.12}$$

It is easy to see in these graphs that all trajectories are starting and ending in a fixed point or at $\pm\infty$. This means that there is no return path available to accommodate any oscillations. The situation, however, is more interesting when taking a look at systems with two complex conjugate eigenvalues (for a real matrix this means the eigenvectors are also complex conjugate). Again three systems are observed: the main difference between them is the sign of the real part of the eigenvalue. The properties of the systems are:

$$\begin{aligned}
&(a)\ A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \lambda_1 = 1 - j,\ \lambda_2 = 1 + j \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix} \\
&(b)\ A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \quad \lambda_1 = -j,\ \lambda_2 = j \quad \boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 1 + j \end{bmatrix} \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 - j \end{bmatrix} \\
&(c)\ A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad \lambda_1 = -1 - j,\ \lambda_2 = -1 + j \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ j \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix}
\end{aligned} \tag{2.13}$$

As can be seen in Fig. 2.3, this results in an oscillator. In (a) the real part of the eigenvalues is positive, which means the amplitude of the oscillation is increasing. In (b) the real part is equal to zero and a stable oscillation will exist (at any amplitude). In (c) the amplitude is decreasing due to the negative real part of the eigenvalue. An important observation is that the amplitude of a linear, oscillating system is undefined. When it is increasing (decreasing), it will be increasing (decreasing) forever. Therefore, a nonlinear component is essential to control the oscillation amplitude and to force the system to a so-called limit cycle. Small amplitudes will increase until this limit cycle is reached; larger amplitudes will decrease to finally reach the same limit cycle. Nonlinear second-order systems can contain zero, one or more limit cycles, depending on the topology. Every real oscillator will contain at least one stable (which means the limit cycle attracts other trajectories) limit cycle. The shift from a linear oscillator to a nonlinear oscillator can be demonstrated using the *van der Pol* oscillator.

2.2.4 The van der Pol Oscillator

The van der Pol oscillator is a second-order system, which is described by the following equation:

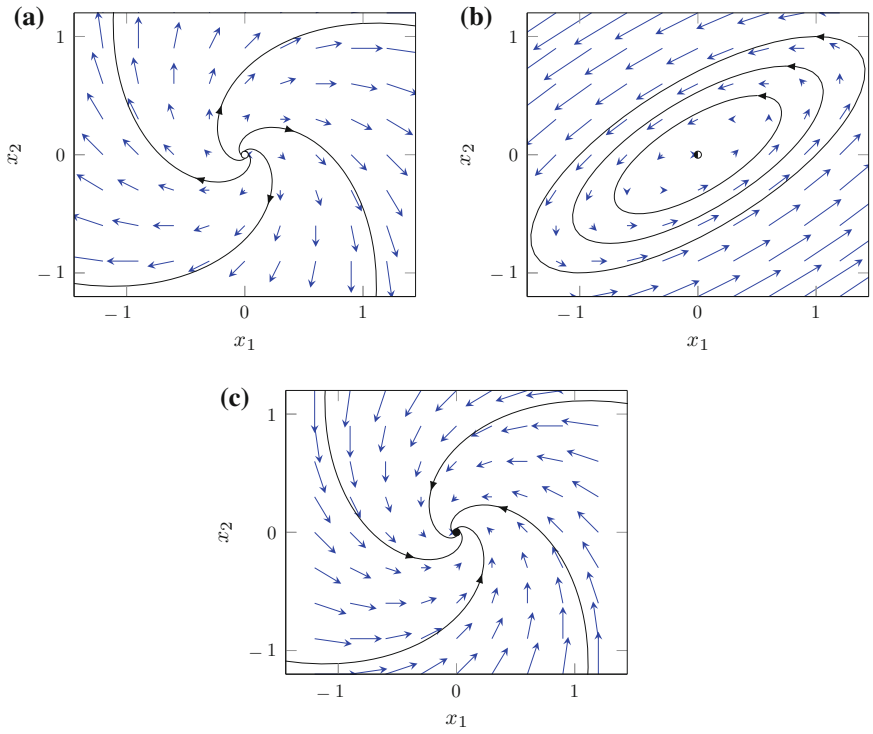

Fig. 2.3 Phase portraits of different linear second-order systems with two complex conjugate eigenvalues. The real part has **a** a positive sign, **b** a value equal to zero, **c** a negative real part. **a** is called an unstable system, **b** is marginally stable and **c** is a stable system

$$\ddot{x}_1 - \mu \cdot (1 - x_1^2) \cdot \dot{x}_1 + x_1 = 0 \tag{2.14}$$

where $\mu > 0$ is a scalar parameter indicating the nonlinearity and the strength of the damping. Using the *Liénard* transformation $x_2 = \mu \cdot (x_1 - x_1^3/3) - \dot{x_1}$, this equation can be written in its familiar second-order form [242]:

$$\begin{cases} \dot{x}_1 = \mu \cdot (x_1 - x_1^3/3) - x_2 \\ \dot{x}_2 = x_1 \end{cases} \tag{2.15}$$

when $\mu = 0$, no damping is present in the equation and the system is a linear oscillator similar to Fig. 2.3b. When μ is slightly increased, after some transient effects, the system will enter a limit cycle. The higher the nonlinearity factor, the more the oscillator will move from a linear and 'soft' behavior towards a nonlinear 'switched' oscillator, as shown in Fig. 2.4.

For low μ, the two state variables are continuously changing or interacting. For higher values of μ, however, this is not the case: on the left hand side the system

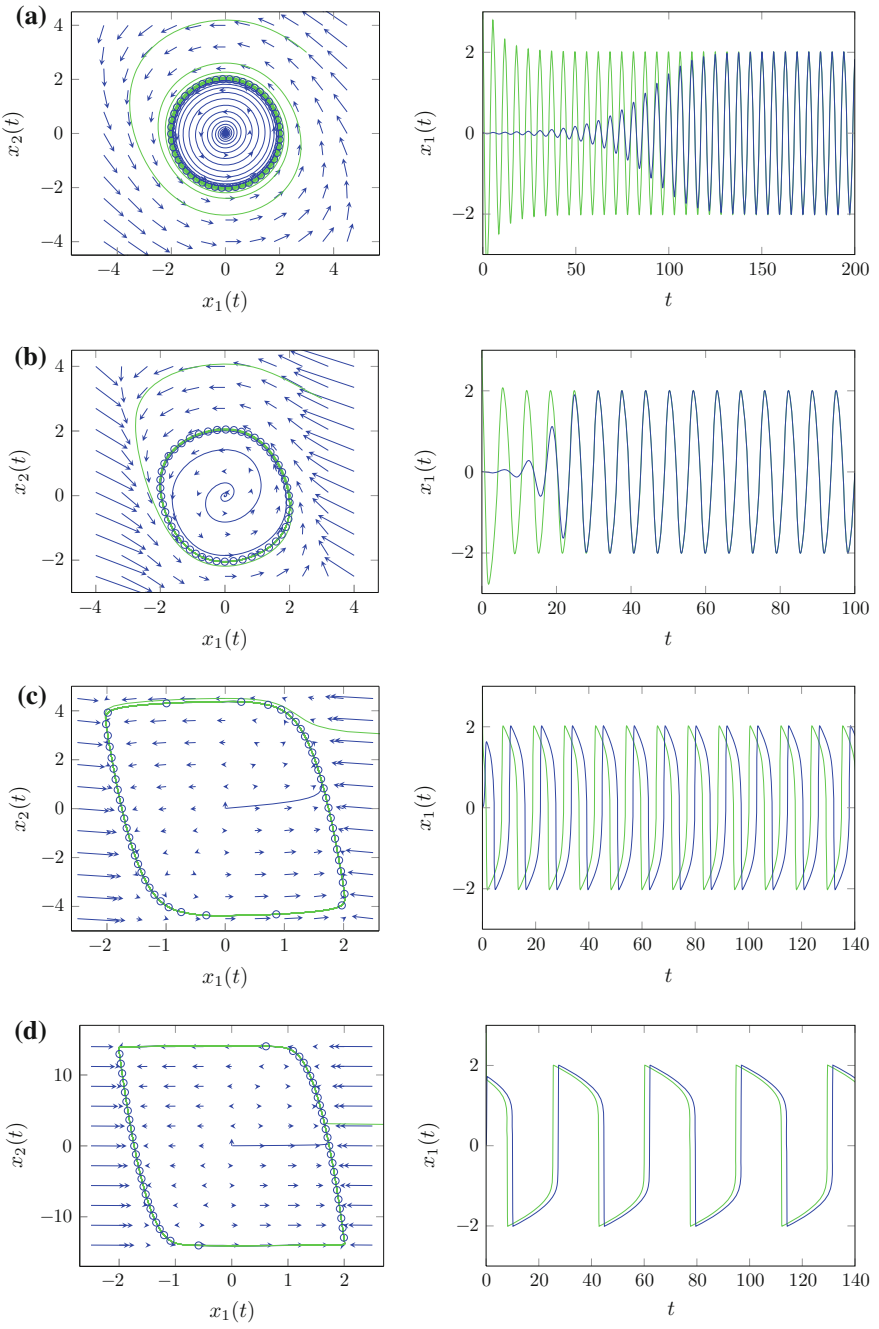

Fig. 2.4 Phase portraits and output signal of the van der Pol oscillator for different values of μ. The initial value for the *blue curve* is (0.01, 0.01), for the *green curve* this is (3, 3). On each phase portrait the 50 *blue circles* are equidistant in time to show the behavior of the oscillator during the limit cycle. **a** $\mu = 0.1$, **b** $\mu = 0.5$, **c** $\mu = 5$, **d** $\mu = 20$

is moving downward; at a certain level it will suddenly switch to the right, where it will move slowly upward, suddenly switch to the left and so on. The period of the oscillator is then mainly determined by the phases where the oscillator is moving upward or downward and not by the switching. For high values of μ the first state of the oscillator can even be considered to be discrete (*left* or *right*) and determines the evolution in the second state in a discrete way (*move downward* or *move upward*). The switch between these states happens when x_2 reaches a certain threshold.

Since a phase portrait shows every possible state of a system, the amplitude of a second-order system with an attracting limit cycle cannot overshoot, neither oscillate. This is because the trajectories in the phase portrait can never cross, they always (exponentially) reach the limit cycle. Once the state of the system is on its limit cycle, it will stay there. As a result, to study the stability of an amplitude regulator, the order of the mathematical model (which is often a reduction of the real system) must always be higher or equal to 3. More in general, for a second-order system, this is described by the *Poincaré-Bendixson Theorem* [242]:

Theorem 2.1 *Suppose that:*

- *R is a closed bounded subset of the plane;*
- $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ *is a continuously differentiable vector field on an open set containing R;*
- *R does not contain any fixed points;*
- *There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time.*

Then either C is a closed orbit or spirals towards a closed orbit when $t \to \infty$. *In either case R contains a closed orbit.*

To prove that the van der Pol oscillator has a limit cycle, it is sufficient to construct a ring-shaped trapping region around the fixed point at the origin for which, at every border of the region, the vector field is pointing inward. Furthermore, this theorem implies that there is no chaos in a second-order system; a system evolving within a limited sub plane will have a predictable behavior.

2.2.5 n-Dimensional Systems

Most practical systems have an order higher than two. An example is the instability of an amplitude regulation [270], which can only occur in higher-order systems. Without going into detail, two examples are briefly discussed. The first example is a van der Pol oscillator with an unstable amplitude control; the second example shows chaotic behavior in the three-dimensional phase space.

Example 2.2 The van der Pol equation (2.14) has a damping which is proportional to the squared value of the output waveform. However, it is often not feasible to implement an amplitude control which exactly follows this

behavior. Furthermore, the amplitude will often be limited or clipped due to the limited output swing of the oscillator circuit, which introduces hard distortion. A better solution is to use an average or rms value of the output signal in the feedback circuit. The adapted van der Pol equation looks like this:

$$\ddot{x}_1 - \mu \cdot (1 - x_1^2) \cdot \dot{x}_1 - \eta \int (A - x_1^2) dt + x_1 = 0 \tag{2.16}$$

where η determines the integration speed of the amplitude control and A is the desired rms output of the circuit. To transform this system to its autonomous equations, three state variables are needed. A possible set of equations is:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_2 \cdot x_3 - x_1 \\ \dot{x}_3 = -2 \cdot \mu \cdot x_1 \cdot x_2 + \eta \cdot (A - x_1^2) \end{cases} \tag{2.17}$$

The behavior of this system can be compared to the working principle of a PI regulator on the squared value of the output signal. The proportional path is determined by μ; the integrating path is determined by the value of η. When η is equal to zero, the equation simplifies to (2.14) and the amplitude regulation is perfectly stable for $\mu > 0$. However, with a proportional regulator, the error on the amplitude can only be decreased by increasing μ which increases the nonlinearity of the circuit. By slightly increasing η, the resulting rms amplitude error is integrated and the damping of the oscillator is adapted. An example for $A = 1/\sqrt{2}$, $\mu = 0.05$ and $\eta = 0.01$ is shown in Fig. 2.5. Although the oscillator converges to a limit cycle, the amplitude regulator is under-damped and overshoots several times before converging to its final value. When $\mu = 0$, the amplitude overshoot is not damped at all.

Example 2.3 The second example is to show the more unpredictable behavior in a third-order system compared to a second-order system, called chaos. This odd behavior can be observed in the forced van der Pol equation and was already noticed by Balthasar van der Pol himself. However, other researchers [256] classified these phenomena as quasi-periodic but non-chaotic. At the same moment they proposed a modified van der Pol equation [139] which *has* chaotic behavior:

$$\ddot{x}_1 - \mu \cdot (1 - x_1^2) \cdot \dot{x}_1 + x_1^3 = B \cdot \cos(t) \tag{2.18}$$

Again, μ is a nonlinearity parameter, B determines the amplitude of the applied signal which forces the van der Pol equation. An autonomous set of equations is equal to:

$$\begin{cases} \dot{x}_1 = \mu \cdot (x_1 - x_1^3/3) - x_2 \\ \dot{x}_2 = x_1^3 - B \cdot \sin(x_3) \\ \dot{x}_3 = 1 \end{cases} \tag{2.19}$$

Chaotic behavior means that, although the behavior of the system is deterministic, it is unpredictable and never repeats itself. Chaotic systems are therefore very sensitive to the initial conditions and to minor numerical errors during calculation. Since chaotic systems can perfectly occur within a bounded volume within phase space, the periodic (or quasi-periodic) trajectories are very dense. Figure 2.6 pictures the evolution of 32 start conditions over time. The initial distance between these points is equal to 0.01. To show the overall behavior and the density of the trajectories, one trajectory is plotted too. For this simulation $\mu = 0.1$ and $B = 1$. A more quantitative way to characterize the sensitivity to initial conditions is the use of Lyapunov exponents, which falls beyond the scope of this work [139, 242]. An example application is estimating the reliability of a weather forecast (which is an extremely complex dynamical system) [84].

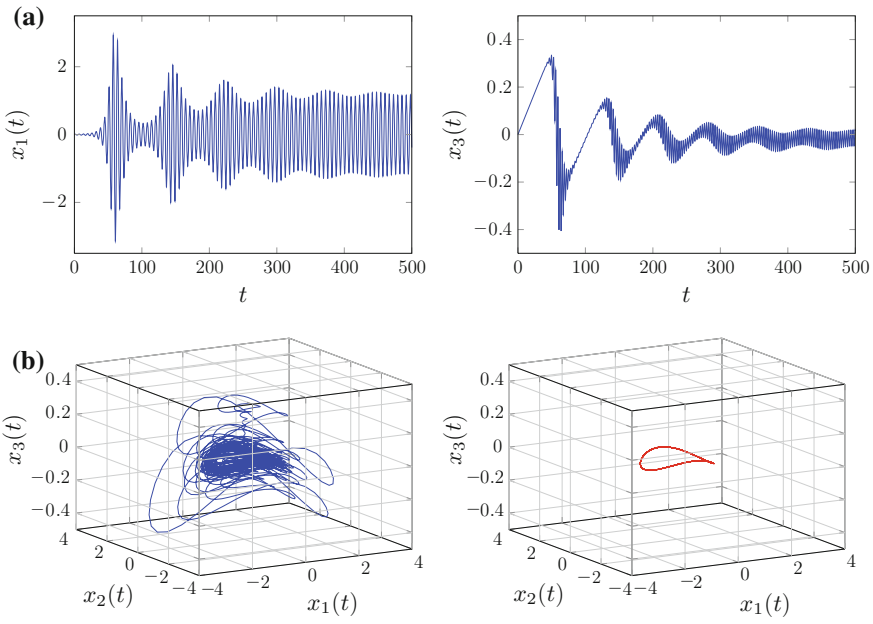

Fig. 2.5 Behavior of the van der Pol oscillator with a PI amplitude regulator, $A = 1/\sqrt{2}$, $\mu = 0.05$ and $\eta = 0.01$. **a** Shows the behavior in the time domain. The output signal $x_1(t)$ is shown *on the left*. *On the right* $x_3(t)$ is plotted, which corresponds to the resulting output signal of the amplitude regulator. **b** Shows the behavior in the phase plane/space; the overshoot of the amplitude is clearly visible. It takes a lot of periods before the limit cycle (*on the right*) is reached

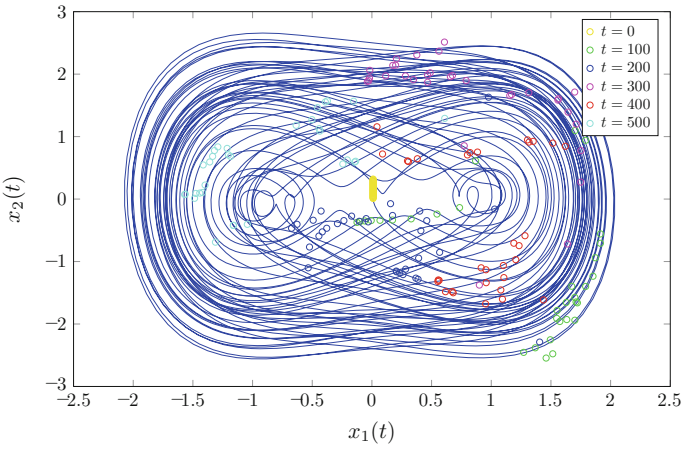

Fig. 2.6 Chaotic behavior of the forced modified van der Pol equation. The *circles* show the evolution of 32 near start conditions over time. The *blue line* is an $x_1 - x_2$ plot of one of the trajectories

2.3 Minimum Requirements for a Time Reference

What are the minimum requirements to define a time reference? This question is different from the discussion in the previous section: in this case it is acceptable that the system starts at its starting condition and evolves to another state *after a certain time*. No repetition of this process is needed.

A difference can be made between state variables which are connected to an energy level (the voltage over a capacitor, the speed of a moving object, the water level in a tank, etc.) and state variables which are not related to an energy level (position of an object on an equipotential surface, angular position of a carousel, etc.). The first category of state variables all have a connection to time, or are dependent on the definition of one second. This corresponds to the physical principle that the evolution of the energy level of a system will always be downwards. Otherwise, there is no driving force and the state variable or energy level will be static. Hence, to define a time reference, at least one energy tank is needed. Of course, since the unit of energy is not equal to *one second*, also another component is needed. The evolution of the system is the exchange of energy between these components, which can happen in two different manners.

★ Definition of time

The unit of time, one second, is one of the 7 SI base units. Moreover, it is next to Kelvin the only SI base unit of which the definition does not depend on any other SI base units. Originally, one second was defined as $1/(24 \cdot 60 \cdot 60)$ of the

day, but since the rotation speed of the earth is slowly decreasing this definition had to be adapted. Since 1967, one second is 'autonomously' defined as *"the duration of 9.192.631.770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom."* Recent research shows that even more accurate definitions of time are to be discovered [11]. Except from using an atomic clock, there are of course many other (and more affordable) ways to implement a time reference, be it less accurate. This can be done using components or devices which have a connection with time: their unit is dependent on the definition of one second [310].

2.3.1 An Energy Reservoir and a Resistor

The first possibility is that the energy is dissipated in a resistor. This resistor can be a resistor of any kind, electrical, mechanical, aerodynamical, hydromechanical, etc. It has to be a component or device that dissipates the energy of the energy reservoir. When bringing the focus on electrical components, the derived unit of resistance is ohm or Ω:

$$1\,\Omega = 1\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^3\cdot\mathrm{A}^2} = 1\frac{\mathrm{J}}{\mathrm{A}\cdot\mathrm{s}} \tag{2.20}$$

It is the amount of energy consumed per second when a current of 1 A is flowing through the resistor; which dissipates the energy. On the other hand, there are 2 possible energy reservoirs in electronics: capacitors (to accumulate a charge or voltage) and inductors (to accumulate a magnetic charge or current). By definition their values are equal to:

$$L = U/\frac{dI}{dt} \quad \text{and} \quad C = I/\frac{dU}{dt} \tag{2.21}$$

which means that their units can be written as:

$$1\mathrm{H} = \frac{\mathrm{V}\cdot\mathrm{s}}{\mathrm{A}} = \frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2\cdot\mathrm{A}^2} = \frac{\mathrm{J}}{\mathrm{A}^2} \tag{2.22}$$

$$1\mathrm{F} = \frac{\mathrm{A}\cdot\mathrm{s}}{\mathrm{V}} = \frac{\mathrm{s}^4\cdot\mathrm{A}^2}{\mathrm{kg}\cdot\mathrm{m}^2} = \frac{\mathrm{J}}{\mathrm{V}^2} \tag{2.23}$$

These units indicate the accumulation of energy. The energy is not dissipated; it simply depends on the current through the inductor or the voltage over the capacitor. When connecting one of these devices to a resistor, the energy will be consumed. The speed at which this happens depends on the value of the resistor as well as the

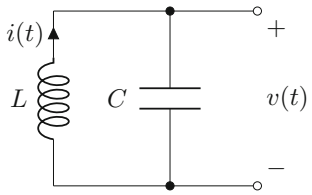

Fig. 2.7 Two coupled energy tanks, an inductor L and a capacitor C, forming an oscillator

energy reservoir. The time to empty the reservoir for 63 % is equal to:

$$\tau_L = L/R \tag{2.24}$$

$$\tau_C = R \cdot C \tag{2.25}$$

and indeed, the unit of the result is in both cases equal to seconds. The link between resistance, voltage and current is given by Ohm's law. Instead of a resistor, also a combination of a voltage and a current can be used, as is often the case in so-called relaxation oscillators.

2.3.2 Two Different Energy Reservoirs

The second possibility to create a time reference is by combining two energy reservoirs. Two identical energy reservoirs can only exchange energy in an infinitesimal short time span, which obviously does not result in a time value. By taking two different energy reservoirs, the exchange of energy will happen in a controlled manner. This means that *it takes some time* to exchange energy. In the case of an inductor and a capacitor, the current through the inductor slowly charges the capacitor. Afterwards, when the current is equal to zero, the capacitor is discharged while generating a current in the inductor. From (2.21) and the schematic in Fig. 2.7, it appears that the equation to describe the energy exchange between an inductor and a capacitor is equal to (using Kirchhoff's law):

$$-L \cdot \frac{di(t)}{dt} - \frac{1}{C} \int_{-\infty}^{t} i(t)dt = 0 \quad \text{or} \quad L \cdot \frac{d^2 i(t)}{dt^2} + \frac{i(t)}{C} = 0 \tag{2.26}$$

The solution to this differential equation is equal to:

$$i(t) = A \cdot e^{j \cdot \omega_n \cdot t + j \cdot \theta_0} \tag{2.27}$$

$$\Re(i(t)) = c_1 \cdot \cos(\omega_n \cdot t) + c_2 \cdot \sin(\omega_n \cdot t) \tag{2.28}$$

where Euler's formula and $\omega_n = 1/\sqrt{L \cdot C}$ are used to obtain (2.28). A and θ_0 or c_1 and c_2 are two constants depending on the start conditions. Again, the resulting unit of $\sqrt{L \cdot C}$ is seconds. The possibility to derive a time reference from the

energy exchange between two energy reservoirs does not only apply to electrical circuits but is also applicable in other branches of science. An important condition is the possibility to exchange energy between two energy tanks. With the necessary amount of engineering it is possible to build a system exchanging electrical and for instance mechanical energy. A simple example is a lossless DC motor connected to an inductor.

Example 2.4 A commonly used DC motor model is an inductor L_m in series with a resistor R_m and a so-called back electromotive force E_M. In this example the series resistor is equal to zero since we assume an ideal motor without any losses. Furthermore, the series inductance is lumped into the external inductor L_e, resulting in one inductor L. The following formulas complete the motor model [17]:

$$\begin{cases} T_M = c \cdot \Phi \cdot i(t) \\ E_M = c \cdot \Phi \cdot \omega_m \end{cases} \tag{2.29}$$

where T_M is the motor torque, ω_m is the angular frequency of the motor and c is a motor constant depending on the construction. Φ is the magnetic flux due to the magnetic poles, expressed in Weber (Wb). Applying Kirchhoff's law on the circuit of Fig. 2.8 leads to the following equation:

$$-\frac{di(t)}{dt} \cdot L = c \cdot \Phi \cdot \omega_m \tag{2.30}$$

$$= \frac{c^2 \cdot \Phi^2}{I} \cdot \int_{-\infty}^{t} i(t)dt \tag{2.31}$$

where I is the moment of inertia of the rotor (rotating part of the motor). The solution to this equation is equal to (2.27). However, in this case, ω_n (the angular frequency of the oscillation, not that of the rotor) is equal to:

$$\omega_n = \sqrt{\frac{c^2 \cdot \Phi^2}{I \cdot L}} \tag{2.32}$$

The unit of ω is, as expected, equal to $1/s$. The motor in this example acts as an energy converter between electrical energy and kinetic energy, whereas the inductor transfers electrical energy to magnetic energy.

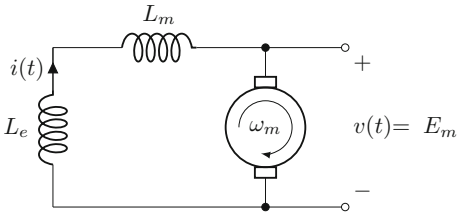

Fig. 2.8 An inductor can exchange its energy with a kinetic energy tank. A DC motor is needed to do the energy conversion

2.3.3 Harmonic Versus Relaxation Oscillators

The difference between a linear and a nonlinear oscillator was demonstrated in Sect. 2.2.4. Based on the linearity or nonlinearity and the necessary oscillator components, oscillators can be divided into two classes: harmonic and relaxation oscillators. The boundary between both categories is rather fuzzy, as previously shown.

2.3.3.1 Harmonic Oscillators

When building an oscillator using only linear components, two continuous state variables or energy reservoirs are needed. The frequency of the oscillator is then defined by the properties of the two reservoirs. But, what about the amplitude or startup behavior? A completely linear oscillator has an ever increasing, decreasing or perfectly constant amplitude. A nonlinear component is needed to control the amplitude. *Oscillators consisting of (mainly) linear components are therefore called harmonic oscillators.* Due to their linearity, the behavior of harmonic oscillators can be described in terms of transfer functions, loop gain and phase margin. An important criterion to have a working harmonic oscillator is the *Barkhausen Criterion*:

Theorem 2.2 *A feedback system will only generate a stable oscillation when the loop gain is equal to one and the complete phase shift is equal to* $z \cdot 2 \cdot \pi$ *where* $z \in \mathbb{Z}$.

$$\begin{cases} |H(s)| = 1 \\ \angle H(s) = z \cdot 2 \cdot \pi \end{cases} \quad \text{and } z \in \mathbb{Z} \tag{2.33}$$

where $H(s)$ *is the loop transfer function.*

The output waveform is a sine wave and only contains a limited amount of harmonics. In Sect. 4.4.1, some examples from literature will be discussed.

2.3.3.2 Relaxation Oscillators

When looking back to van der Pol's equation, it is observed that, when increasing the nonlinearity parameter μ, one of the states slowly converts from a continuous state towards a discrete state. The system slowly evolves over the continuous state and

then switches its discrete state, after which this process is repeated. Such oscillators can therefore be implemented using only one energy reservoir and a discrete memory element (a Schmitt trigger or a latch) to implement the discrete state. *An oscillator containing strongly nonlinear elements is called a relaxation oscillator.* The term relaxation oscillator was used by Balthasar van der Pol to express the period of 'building up a tension' in the continuous state and then suddenly relax by switching the discrete state. As a result, the output is non-sinusoidal and the first derivative of the output waveform is often not continuous.

Relaxation oscillators cannot be described using transfer functions. It is therefore difficult to mathematically predict their behavior. Often, a piecewise combination of (differential) equations is needed to describe the behavior. Typically, relaxation oscillators are oscillators where an energy tank is combined with a resistive element. However, in Chap. 6, a relaxation oscillator will be built using a harmonic LC tank. In Sect. 4.4.2, different examples found in literature will be discussed.

2.4 Representation of an Oscillator Signal

The output signal of an oscillator has a representation in the time domain as well as in the frequency domain. Different representations are discussed in this section. The introduced symbols will be used in the remainder of this work while discussing the frequency stability as well as the noise performance. The relationship between the measures in the frequency and the time domain will be discussed into more detail in Chap. 3, handling about phase noise and jitter.

2.4.1 Oscillator Signals in the Time Domain

The most basic representation of (the first harmonic of) an oscillator output signal can be written as [92]:

$$v(t) = A(t) \cdot \sin(\Phi(t)) \tag{2.34}$$

where $A(t)$ is the instantaneous amplitude and $\Phi(t)$ is the instantaneous phase of the oscillator signal. More in general, however, an oscillator signal can have a completely different, but periodic waveform:

$$v(t) = A(t) \cdot f(\Phi(t)) \tag{2.35}$$

where f is a periodic function with period $2 \cdot \pi$. Since this is an oscillator, an *almost cyclostationary* signal is expected,[1] which means that the instantaneous phase is

[1] As will be seen in Chap. 3, a real cyclostationary signal can only be expected from PLL output signals, which are corrected every cycle.

expected to increase linearly. The instantaneous angular frequency $\omega(t)$ is written as:

$$\omega(t) = \frac{d\Phi(t)}{dt} \tag{2.36}$$

and is expressed in *radians per second*. Hence, in an ideal oscillator $\omega(t)$ is constant. For a Voltage-Controlled Oscillator (VCO), $\omega(t)$ depends on the input control signal. When the relationship between the angular frequency and the control signal is linear:

$$\omega(t) = \omega_0 + K_{VCO} \cdot v_c(t) \tag{2.37}$$

which means that the VCO acts as an integrator for the control voltage $v_c(t)$, with integration constant K_{VCO} and the phase deviation (compared to an identical non-modulated VCO) as an output signal. Since the phase deviation is unbounded, a VCO can be considered to be an ideal integrator [92].

2.4.1.1 The Oscillator Phase

In a real oscillator, random as well as deterministic variations are observed in the phase of the oscillator output signal. In most cases the deterministic and the systematic variations are treated separately from the random fluctuations. This means:

$$\Phi(t) = \omega_0 \cdot t + \phi(t) + \Psi(t) \tag{2.38}$$

where ω_0 is the constant mean angular frequency, $\phi(t)$ represents the random phase variations or noise, and $\Psi(t)$ implements both the systematic and deterministic variations in the phase function. The function $T(t)$, often called the *phase-time*, is equal to the instantaneous phase divided by ω_0 and gives the time of a clock that is run by the oscillator:

$$T(t) = t + \frac{\phi(t)}{\omega_0} + \frac{\Psi(t)}{\omega_0} \tag{2.39}$$

For an ideal oscillator this is equal to t. When neglecting the systematic and deterministic phase variations, the random instantaneous phase-time fluctuation is defined as:

$$x(t) = \frac{\phi(t)}{\omega_0} \tag{2.40}$$

which can be understood as the time difference between corresponding zero crossings of a phase-noise-contaminated oscillator and its noise-free replica. The instantaneous fractional frequency deviation can be defined as:

$$y(t) = \frac{dx(t)}{dt} = \frac{\frac{d\phi(t)}{dt}}{\omega_0} = \frac{\Delta\omega(t)}{\omega_0} = \frac{\Delta f(t)}{f_0} \tag{2.41}$$

which allows to compare frequency fluctuations between oscillators at different frequencies. Furthermore, it is insensitive to frequency multiplication or division.

2.4.1.2 The Oscillator Amplitude

For the amplitude, similar equations can be used. Although it is not commonly used in literature, apart from the random amplitude variations (noise), also some systematic or deterministic amplitude variations can be identified:

$$A(t) = A_0 + \varepsilon(t) + \Upsilon(t) \tag{2.42}$$

where A_0 is the ideal amplitude, $\varepsilon(t)$ are the random amplitude variations and $\Upsilon(t)$ represents the deterministic or systematic amplitude errors. When talking about or when measuring amplitude noise, $\Upsilon(t)$ is mostly neglected. The first reason for this is that when measuring a signal, this mostly happens only over a short time which makes a measurement of the (slow) systematic amplitude variations impossible. The second reason is that in most applications, the amplitude of the clock signal is not important, certainly when it is used as a digital clock. The amplitude function of the oscillator does not affect the times of the zero crossings (or the noise on the zero crossings, called jitter) which is often used as a benchmark to evaluate the quality of an oscillator. However, when measuring the spectrum of an oscillator, both the phase noise $\phi(t)$ and the amplitude noise $\varepsilon(t)$ are measured. For a sinusoidal signal it is mostly assumed that the amplitude and phase noise each add one half of the noise spectrum. The phase noise spectrum is then 3 dB lower than the measured spectrum [38].

2.4.2 Oscillator Signals in the Frequency Domain

Oscillator signals also have a representation in the frequency domain. For an ideal sinusoidal oscillator (2.34) where $\Phi(t)$ increases linearly, the output spectrum results in two Dirac impulses $\pm\omega_0$. When the waveform is more irregular, harmonics are present in the output signal, represented in the spectrum by Dirac impulses at $\pm n \cdot \omega_0$. As previously shown, the frequency and the phase are closely connected:

$$\omega = \frac{d\Phi(t)}{dt} = \omega_0 + \frac{d\phi(t)}{dt} + \frac{d\Psi(t)}{dt} \tag{2.43}$$

of which the last term is supposed to be close to zero. The consequence of the time dependency of A and ϕ is a spectrum with sidebands around every harmonic. To characterize these frequency or phase deviations, the Power Spectral Density (PSD)

Table 2.1 Commonly used PSD functions to characterize the frequency stability of an oscillator signal

Symbol	Unit	Description
$S_{\Delta f}(f)$	Hz	PSD of frequency fluctuations
$S_{\Delta\omega}(f)$	$(\text{rad/s})^2/\text{Hz}$	PSD of angular frequency fluctuations
$S_y(f)$	1/Hz	PSD of fractional frequency fluctuations
$S_\phi(f)$	rad^2/Hz	PSD of phase fluctuations
$S_x(f)$	s^2/Hz	PSD of phase time fluctuations

is used, as defined in Appendix A.2.2. The PSD can be calculated for almost every time signal. Commonly used PSDs for oscillators are summarized in Table 2.1. From the previous sections it is clear that there are several relations between these PSD functions. From (2.41) it follows that:

$$S_\phi(f) = \frac{S_{\Delta\omega}(f)}{\omega^2} \tag{2.44}$$

From the linearity property of the Fourier transform, it follows that:

$$S_y(f) = \frac{S_{\Delta\omega}(f)}{\omega_0^2} = \frac{S_{\Delta f}(f)}{f_0^2} \tag{2.45}$$

Combining both relationships, it can be concluded that:

$$S_y(f) = \frac{f^2}{f_0^2} \cdot S_\phi(f) = \omega^2 \cdot S_x(f) \tag{2.46}$$

To obtain these relations, the properties from Appendix A.1 are used. From these relations, it is clear that the PSD of the phase and the frequency differ with a factor of f^2. As will be seen in Sect. 3.3, the oscillator spectrum is mostly divided into different sections, depending on the slope of the frequency spectrum. Using the power-law noise model, which describes the phase noise of an oscillator as a sum of different power law curves [92, 164], the (single-sided) PSD of $y(t)$ and $\phi(t)$ is typically written as:

$$S_y(f) = \sum_{\alpha=-2}^{+2} h_\alpha \cdot f^\alpha \tag{2.47}$$

$$S_\phi(f) = f_0^2 \cdot \sum_{\alpha=-2}^{+2} h_\alpha \cdot f^{\alpha-2} \tag{2.48}$$

The different power law curves all represent a type of noise, coming from different noise sources and being the result of different noise mechanisms. The interpretation of these power law terms will elaborately be discussed in Chap. 3. Although this is assumed to be rather uncommon, depending on the noise sources, also higher numbers for α are possible.

2.4.2.1 The Use of $\mathscr{L}(f)$ to Characterize the Phase Noise

A commonly used measure of phase noise is $\mathscr{L}(f)$, especially when measuring the phase noise. Several definitions of this measurement are possible. However, a generally accepted definition as used in [37, 101] is:

$$\mathscr{L}_{total}(\Delta\omega) = \frac{\mathscr{P}_{side}(f_0 + \Delta f, 1\text{ Hz})}{\mathscr{P}_{carrier}} \tag{2.49}$$

which is sometimes expressed in dB and in which $\mathscr{P}_{side}(f_0 + \Delta f, 1\text{ Hz})$ is the single-sideband noise power in a 1 Hz interval at a frequency offset Δf from the carrier frequency f_0. As mentioned earlier, the effect of both amplitude and phase noise is present in the noise spectrum. In Sect. 3.6, it will be shown that the sideband noise in many applications is dominated by phase noise since this part of the noise cannot be eliminated. When considering only the phase noise:

$$\mathscr{L}(\Delta\omega) = \frac{\mathscr{P}_{\phi-side}(f_0 + \Delta f, 1\text{ Hz})}{\mathscr{P}_{carrier}}, \tag{2.50}$$

note that, when using a spectrum analyzer, this portion of the noise cannot be measured separately. As shown in [82], in fact the power of the total signal must be in the denominator of (2.49) and (2.50). However, because the replacement by the power of the carrier only introduces a small error and the entire signal power is much more difficult to measure, this error is in most cases neglected.

2.4.2.2 Relationship Between $\mathscr{L}(f)$ and $S_\phi(f)$

Up till now only the PSD of the phase fluctuations was discussed without taking the carrier signal into account. According to [217], the (single-sided) spectrum of the entire signal is approximated by:

$$S_{RF}(f) \approx \frac{A_0^2}{2} \cdot \left[\delta(f - f_0) + S_\phi(f - f_0)\right] \tag{2.51}$$

where $\delta(f)$ is the delta function or Dirac impulse, $S_\phi(f)$ is the (two-sided) power spectral density, $A_0^2/2$ equals the carrier power and f_0 is the carrier frequency. The

PSD of the phase fluctuations is multiplied by the carrier power to obtain the signal's sideband noise. Furthermore it is clear that this equation only holds for signals where the amplitude noise is negligible and the phase fluctuations are small. The assumption must hold that the spectrum of a phase-modulated signal is (linearly) approximated by the phase spectrum itself. This also means that the baseband PSD of the phase fluctuations is up-converted to the carrier frequency and therefore has the same frequency dependency:

$$\mathscr{P}_{\phi-side}(f_0 + \Delta f, 1\text{ Hz}) = \frac{A_0^2}{2} \cdot S_\phi(\Delta f) \tag{2.52}$$

As a result, the relationship between $\mathscr{L}(f)$ and $S_\phi(f)$ can be written as:

$$\mathscr{L}(\Delta f) = \frac{\mathscr{P}_{\phi-side}(f_0 + \Delta f, 1\text{ Hz})}{\mathscr{P}_{carrier}} \tag{2.53}$$

$$= \frac{\frac{A_0^2}{2} \cdot S_\phi(\Delta f)}{\frac{A_0^2}{2}} = S_\phi(\Delta f) \tag{2.54}$$

which is often used as an alternative definition of $\mathscr{L}(\Delta f)$ [82, 92]. In [92] a lower limit of the frequency offset is calculated down to which this approximation is valid; Δf must be large enough such that:

$$\int_{\Delta f}^{\infty} S_\phi(\Delta f) df \ll 1\ rad^2 \tag{2.55}$$

At small frequency offsets, the phase noise spectrum typically increases drastically, which makes the first-order Taylor approximation of the phase-modulated signal uncertain indeed.

2.5 Properties of an Oscillator

The properties of an oscillator can be described using several parameters, going from the quality of an energy tank to the short- and long-term frequency stability. In Chap. 6, where the focus is on pulsed resonant tanks, a more detailed discussion will be held on the properties of an nth-order energy tank. Often, a tight connection exists between all of these parameters. Theoretical parameters and properties can be interesting to use at design time, but only detailed circuit simulations or even better measurements show the real behavior of a circuit.

2.5.1 The Quality Factor

The quality factor or Q factor is a measure to characterize the quality of an energy tank or, more in general, a two-pole system. As will be shown, the Q factor almost completely determines the properties of the system, going from the filtering characteristics to the losses and the noise generation. Depending on the topology of the tank, a different method can be used to calculate it. Furthermore, for some tank topologies also the meaning and understanding of this factor is non-trivial and against every intuition. Even then, however, it is a useful figure to quantify for instance the noise generation in an oscillator. Four calculation methods are demonstrated, all of them are based on the tank schematics in Fig. 2.9.

Definition 2.1 The most basic definition of the Quality Factor or Q factor is based on the energy losses in the tank.

$$Q = 2 \cdot \pi \cdot \frac{E_{stored}}{E_{Loss-per-Cycle}} \tag{2.56}$$

The Q factor therefore quantifies the quality of a resonator.

For an inductor L with series resistance R_s (Fig. 2.9c), driven by a sine wave with amplitude I_A and angular frequency ω, this results in:

$$E_{stored} = \int_0^{T/4} P dt = \frac{L I_A^2}{2} \tag{2.57}$$

$$E_{Loss-per-Cycle} = 4 \int_0^{T/4} I^2 R_s dt = \frac{\pi I_A^2 R_s}{\omega} \tag{2.58}$$

$$\Rightarrow Q_L = \frac{L\omega}{R_s} \tag{2.59}$$

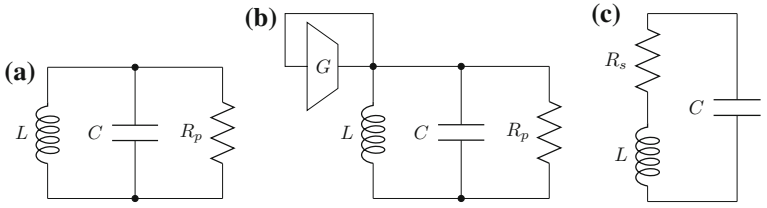

Fig. 2.9 Different RLC tanks. The *left* tank (**a**) is not driven and its losses are represented by the parallel resistor. In (**b**) the same tank is driven by a transconductance amplifier. In (**c**) the losses are represented by a series resistor in the inductor, which is the closest to the real situation

Since, for fully integrated LC tanks, the losses in the tank are dominated by the losses in the inductor, the capacitor is supposed to be ideal compared to the inductor. As a result, the Q factor of the inductor, Q_L, is equal to that of the complete tank. In (2.59), ω can be substituted by $\omega_n = 1/\sqrt{LC}$, the natural angular frequency of the LC tank.

$$Q = Q_L = \sqrt{\frac{L}{C}}\frac{1}{R_s} \tag{2.60}$$

A similar calculation can be made for the parallel RLC network in Fig. 2.9c. This time, the calculation is started from the charging of the capacitor using a sine wave:

$$E_{stored} = \int_0^{T/4} Pdt = \frac{CV_A^2}{2} \tag{2.61}$$

$$E_{Loss-per-Cycle} = 4\int_0^{T/4} V^2/R_p dt = \frac{\pi V_A^2}{R_p \omega} \tag{2.62}$$

$$\Rightarrow Q_C = C\omega/R_p \tag{2.63}$$

Also this time it is assumed that all the losses in the tank are lumped into the parallel resistor R_p. Using the same substitution for ω as in the previous example, the Q factor of the parallel network is equal to:

$$Q = Q_C = \sqrt{\frac{C}{L}}R_p \tag{2.64}$$

Although this parallel network is often not the real situation, it is commonly used for better understanding. In Fig. 2.9b, this parallel resistor is exactly compensated by the transconductance amplifier G and a stable oscillation is obtained. In the neighborhood of the natural angular frequency ω_n, the relation between R_p and R_s is given by:

$$R_p = \frac{1}{R_s}\frac{L}{C} \tag{2.65}$$

This equality and the reason why it is only valid in the neighborhood of ω_n, can be better understood when looking at the transfer functions of both networks. The current-voltage transfer functions of the parallel and series networks in Fig. 2.9a, c are respectively:

$$H_P(s) = \frac{\frac{s}{C}}{s^2 + \frac{1}{C\cdot R_p}\cdot s + \frac{1}{L\cdot C}} = \frac{\omega_n \cdot \sqrt{\frac{L}{C}}\cdot s}{s^2 + \frac{\omega_n}{Q}\cdot s + \omega_n^2} \tag{2.66}$$

$$H_S(s) = \frac{\frac{s}{C} + \frac{R_s}{L\cdot C}}{s^2 + \frac{R_s}{L}\cdot s + \frac{1}{L\cdot C}} = \frac{\omega_n \cdot \sqrt{\frac{L}{C}}\cdot s + \omega_n^2 \cdot R_s}{s^2 + \frac{\omega_n}{Q}\cdot s + \omega_n^2} \tag{2.67}$$

using (2.60), (2.64) and $\omega_n = 1/\sqrt{LC}$. Taking (2.60) into account, in the neighborhood of ω_n both the network with the parallel and series resistor are equivalent as long as $Q \gg 1$. For lower frequencies, however, the network with the series resistance will have a higher equivalent impedance compared to the parallel network. Writing the transfer function (more specifically the denominator) in its standard form (2.66) and (2.67), can be considered as an analytical method to calculate the Q factor. For higher-order systems each pair of (complex conjugate) poles has its own Q.

To explain the other two calculation methods, only the network with the parallel resistor is considered since this will simplify the calculations significantly. The third method to calculate Q is based on the sharpness of the peak in the transfer function. Consider the graph in Fig. 2.10.

Definition 2.2 For a 2-pole system showing a peak in its transfer function, the Q factor is equal to the center frequency ω_n divided by the -3 dB width $\Delta\omega_{3dB}$ of the resonant peak:

$$Q = \frac{\omega_n}{\Delta\omega_{3dB}} \tag{2.68}$$

The fourth and last calculation method is based on the steepness of the output phase of the resonant network. This is shown in Fig. 2.10, the Q factor is then defined as:

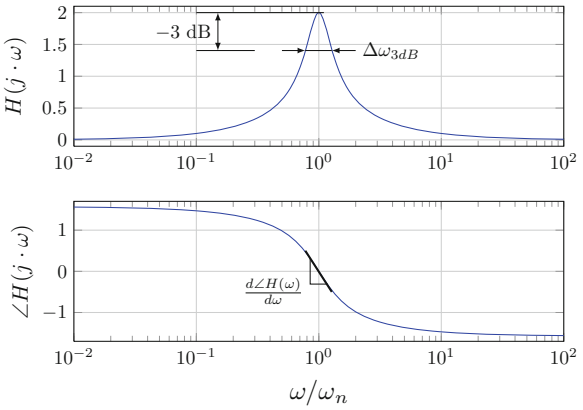

Fig. 2.10 Different definitions of the Q factor of a two-pole system, based on the width (bandwidth) of the resonant peak and based on the steepness of the phase shift of the feedback network

Definition 2.3 The Q factor of a resonant network with natural frequency ω_n is proportional to the steepness of the phase of the network gain at its zero-crossing:

$$Q = \frac{\omega_n}{2} \cdot \frac{d\angle H(j\omega)}{d\omega} \tag{2.69}$$

Depending on the topology, one or more definitions can be used to calculate the Q factor. It must, however, be clear that in the case of a nonlinear (relaxation) oscillator, the use of linear transfer functions does not make sense and only the basic definition (2.56) can be used.

2.5.1.1 Meaning of the Q factor

The meaning of the Q factor can be understood when translating the transfer function to the time domain. Since $H_P(s)$ and $H_S(s)$ are current-voltage transfer functions (or the impedance of the RLC network), $H(s) \cdot I(s) = V(s)$ in which $I(s)$ and $V(s)$ are the Laplace transform of the applied current and the resulting voltage output. If no external current is applied, (2.66) and (2.67) both result in the following differential equation:

$$\frac{d^2v(t)}{dt^2} + \omega_n \left(\frac{1}{Q}\right) \frac{dv(t)}{dt} + \omega_n^2 \cdot v(t) = 0 \tag{2.70}$$

where $v(t)$ is the voltage over the capacitor. The solution to this equation is equal to:

$$\begin{aligned} v(t) = A \cdot e^{\frac{-\omega_n \cdot t}{2Q}} \cdot e^{\frac{\sqrt{1-4Q^2}}{2Q}\omega_n t} \\ + B \cdot e^{\frac{-\omega_n \cdot t}{2Q}} \cdot e^{-\frac{\sqrt{1-4Q^2}}{2Q}\omega_n t} \end{aligned} \tag{2.71}$$

where A and B are constants depending on the initial conditions of the network. This proves that the network is completely characterized by the natural frequency ω_n and the Q factor. The behavior is not determined by the position of the parallel or series resistor. It is now easy to see what the influence is of the Q factor:

- $-1/2 \leqslant Q \leqslant 1/2$: In this case, the result is an ever increasing ($Q < 0$) or decreasing ($Q > 0$) function. Since all exponents are real, there will be no oscillations.
- $Q > 1/2$: Due to the complex exponent, this will result in an oscillation. Since the real part of the exponent is negative, the oscillation will decay.
- $Q < -1/2$: Again, this is an oscillator; the amplitude, however, increases and goes to infinity.

- $Q = \pm\infty$: Since this results in a complex exponential function, there will be an oscillation. The amplitude, however, will be constant and equal to $\sqrt{A^2 + B^2}$. The angular frequency in this case is equal to ω_n; in the other cases the angular frequency is somewhat lower.

Obviously, a negative Q cannot exist for a network containing only passive components. When $Q = 1/2$, the system is critically damped.

2.5.1.2 The Generalized Q factor

When a gain stage, Fig. 2.9b, is present in the electrical network, it can be useful to define the generalized Q factor. The differential equations in this case are not the same for the parallel and the series network:

$$\frac{d^2 v_P(t)}{dt^2} + \omega_n \left(\frac{1}{Q} - \sqrt{\frac{L}{C}} G \right) \frac{dv_P(t)}{dt} + \omega_n^2 v_P(t) = 0 \tag{2.72}$$

$$\frac{d^2 v_S(t)}{dt^2} + \omega_n \left(\frac{1}{Q} - \sqrt{\frac{L}{C}} G \right) \frac{dv_S(t)}{dt} + \omega_n^2 (1 - GR_s) v_S(t) = 0 \tag{2.73}$$

Although both equations have an analytical solution, things can be simplified by defining the generalized Q a follows:

$$Q_G = \left(\frac{1}{Q} - \sqrt{\frac{L}{C}} \cdot G \right)^{-1} \tag{2.74}$$

or more in general:

Definition 2.4 For a general second-order feedback system equation of the form (consisting of an amplifier and a feedback network):

$$\frac{\omega_n \cdot K \cdot s + L}{s^2 + \frac{\omega_n}{Q} \cdot s + \omega_n^2} \cdot I = U \tag{2.75}$$

where Q is the Q factor of the feedback network, ω_n is the natural angular frequency of the network and K and L are constants and I and U are the Laplace transform of the applied current and the resulting output voltage.

- $G = U/I$ is defined as the transconductance of the feedback amplifier.

- The generalized Q factor Q_G of the network is defined as:

$$Q_G = \left(\frac{1}{Q} - K \cdot G\right)^{-1} \tag{2.76}$$

When making use of this definition, the differential equations of the series and of the parallel network, both including a feedback amplifier, are written as[2]:

$$\frac{d^2 v_P(t)}{dt^2} + \omega_n \frac{1}{Q_G} \frac{dv_P(t)}{dt} + \omega_n^2 v_P(t) = 0 \tag{2.77}$$

$$\frac{d^2 v_S(t)}{dt^2} + \omega_n \frac{1}{Q_G} \frac{dv_S(t)}{dt} + \omega_n^2 \left(1 - \frac{1}{Q^2} + \frac{1}{QQ_G}\right) v_S(t) = 0 \tag{2.78}$$

The generic solutions to these equations are then rather straightforward:

$$\begin{aligned} v_P(t) = A \cdot e^{\frac{-\omega_n \cdot t}{2Q_G}} \cdot e^{\frac{\sqrt{1-4Q_G^2}}{2Q_G}\omega_n t} \\ + B \cdot e^{\frac{-\omega_n \cdot t}{2Q_G}} \cdot e^{-\frac{\sqrt{1-4Q_G^2}}{2Q_G}\omega_n t} \end{aligned} \tag{2.79}$$

$$\begin{aligned} v_S(t) = A \cdot e^{\frac{-\omega_n \cdot t}{2Q_G}} \cdot e^{\frac{\sqrt{1-4Q_G^2\left(1-\frac{1}{Q^2}+\frac{1}{QQ_G}\right)}}{2Q_G}\omega_n t} \\ + B \cdot e^{\frac{-\omega_n \cdot t}{2Q_G}} \cdot e^{-\frac{\sqrt{1-4Q_G^2\left(1-\frac{1}{Q^2}+\frac{1}{QQ_G}\right)}}{2Q_G}\omega_n t} \end{aligned} \tag{2.80}$$

where A and B are constants depending on the start conditions. The generalized Q factor Q_G plays the same role in a network with active components as the Q factor does in a passive network. Different from the Q factor, the generalized Q factor can be negative. For the parallel network, it is clear that the transconductance amplifier behaves as a parallel negative resistance which cancels the parallel resistor. When the resistor is in series with the inductor, the resistor cannot be canceled completely. The angular frequency at constant amplitude ($Q_G = \infty$) is in this case equal to $\omega_n \cdot \sqrt{1 - 1/Q^2}$ which is *lower* than in the case of a decaying/increasing oscillation amplitude. For the parallel network, the angular frequency at constant amplitude is ω_n which is somewhat *higher* than in the case of a decaying/increasing amplitude. The fact that the oscillation frequency depends on the amplitude stability shows the importance of a stable amplitude regulation.

[2] Since Q_G depends on G, GR_s can be substituted by $1/Q^2 - 1/(QQ_G)$.

2.5.2 *Stability of an Oscillator Signal*

The stability of an oscillator can have several meanings. In Sect. 2.4, a time-domain representation of an oscillator signal was defined (2.34). This equation both contains a non-constant amplitude function and a phase function. Systematic as well as random variations apply to both the amplitude as the phase. As previously shown, these variations are also visible in the output spectrum of the oscillator. However, it is impossible to separate amplitude from phase effects in the output spectrum. Since amplitude variations can easily be removed using a clipping amplifier, the focus is mostly on the phase variations. Note however, as shown previously, that an influence exists between the amplitude and the phase variations.

Apart from the difference between amplitude and phase errors, fluctuations in an oscillator signal are often divided into a *long-term* and a *short-term* contribution. This mostly corresponds to respectively *systematic* and *random* fluctuations in the oscillator signal. Random variations are mostly called noise, which will elaborately be discussed in Chap. 3. Systematic variations, caused by for instance temperature and supply voltage changes, have a slower impact on the oscillator frequency and amplitude. In the context of this work, the term 'frequency stability' is used to refer to these so-called PVT effects, elaborately discussed in Chap. 4. As will be seen in Chap. 3, low-frequency colored noise sources can also result in long-term systematic variations and are therefore an exception to the proposed division.

2.6 Conclusion

In this chapter, the basic principles of an oscillator together with the representations and properties of an oscillator signal have been discussed. Starting from the phase space description of a dynamic system, the minimum circuit requirements for and necessary components of an oscillator have been identified. From this, the difference between harmonic and relaxation oscillators has been explained. The discussion on oscillator properties has clarified that pointing out general design rules to obtain a stable oscillator is a complex process. The principles of oscillator noise and frequency stability will therefore be discussed in respectively Chaps. 3 and 4. In these chapters, the previously defined parameters to characterize and represent an oscillator signal will frequently be used.

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