Why Should the Logic of Discovery Be Revived? A Reappraisal

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Abstract Three decades ago Laudan posed the challenge: Why should the logic of discovery be revived? This paper tries to answer this question arguing that the logic of discovery should be revived, on the one hand, because, by Gödel's second incompleteness theorem, mathematical logic fails to be the logic of justification, and only reviving the logic of discovery logic may continue to have an important role. On the other hand, scientists use heuristic tools in their work, and it may be useful to study such tools systematically in order to improve current heuristic tools or to develop new ones. As a step towards reviving the logic of discovery, the paper follows Aristotle in asserting that logic must be a tool for the method of science, and outlines an approach to the logic of discovery based on the analytic method and on ampliative inference rules.

1 Introduction

In the last century, scientific discovery has been generally held to be beyond the scope of rationality. The received view has been that scientific discovery is the unique product of intuition.

Thus Planck states that the creative scientist "must have a vivid intuitive imagination, for new ideas are not generated by deduction, but by an artistically creative imagination" [1, p. 109]. Einstein states that "there is no logical path to" the basic laws of physics, "only intuition, resting on sympathetic understanding of experience, can reach them" [2, p. 226]. Indeed, only "the intuitive grasp of the essentials of a large complex of facts leads the scientist" to a "basic law, or several such basic laws" [3, p. 108]. Reichenbach states that "the act of discovery escapes logical analysis" [4, p. 231]. The "scientist who discovers a theory" cannot "name a method by means of which he found the theory," and can only say "that he saw

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intuitively which assumption would fit the facts" (ibid., p. 230). Discovery "is a process of intuitive guessing and cannot be portrayed by a rational procedure controlled by logical rules" since "there are no such rules" [5, p. 434].

The received view has its foundation in the romantic theory of the scientific genius going back to Novalis, who states that discoveries "are leaps—(intuitions, resolutions)" and products "of the genius—of the leaper *par excellence*" [6, p. 28]. The genius brings forth numerous living thoughts, and "whoever is able to bring forth numerous living thoughts, is called a genius" (ibid., p. 194). (See also [7]).

Contrary to the romantic theory of the scientific genius, it can be argued that discovery can be pursued through rational procedures, specifically heuristic procedures. Although the latter offer no complete guarantee to reach a solution, they restrict the search space thus easing the search for a solution.

The importance of heuristic procedures is stressed by Lakatos who, in his early work, seems genuinely interested in "the logic of mathematical discovery" [8, p. 4]. But the heuristic rules he provides are not genuine discovery rules (see [9]). On the other hand, in his later work, by 'logic of discovery' or 'methodology' Lakatos "no longer means rules for arriving at solutions, but merely directions for the appraisal of solutions already there. Thus" logic of discovery or "methodology is separated from heuristics" [10, I, p. 103, Footnote 1].

The aim of this paper is to reexamine the question of discovery, seen as pursued through rational procedures. Specifically, the paper considers some of the ways in which the question of discovery has been dealt with in the past. It follows Aristotle in asserting that logic must be a tool for the method of science, and outlines an approach to the logic of discovery based on the analytic method and on ampliative inference rules, that is, inference rules where the conclusion is not contained in the premises.

2 Attempts to Develop the Logic of Discovery in the Modern Age

In the seventeenth century it was widely held that the then current logic paradigm, Scholastic logic, was inadequate to the needs of the new science. Thus Bacon stated that Scholastic logic "is useless for the discovery of sciences;" it "is good rather for establishing and fixing errors (which are themselves based on vulgar notions) than for inquiring into truth; hence it is more harmful than useful" [11, I, p. 158]. Descartes stated that Scholastic logic "contributes nothing whatsoever to the knowledge of truth" [12, X, p. 406]. It "does not teach the method by which something has been discovered" (ibid., VII, p. 156). Therefore, it "is entirely useless for those who wish to investigate the truth of things" (ibid., X, p. 406).

The dissatisfaction with Scholastic logic gave rise to several attempts to develop logics of discovery alternative to Scholastic logic. The resulting logics of discovery, however, have serious limitations.

Thus, Bacon's logic is based on the use of tables involving a process of exclusion and rejection. But this process may either leave open too many possibilities, or none at all, therefore it may not lead to a necessary conclusion. To remedy this weakness, Bacon makes a plea for giving "the intellect permission" at this point to "try an interpretation of nature in the affirmative" [11, I, p. 261]. That is, Bacon makes a plea for giving the intellect permission to formulate a hypothesis, for example, a hypothesis about the nature of heat. But he does not really derive such hypothesis from an examination of the tables, rather, he borrows it from one of the views on heat which were discussed at the time. In fact, Bacon's method provides no means to formulate hypotheses.

On the other hand, Descartes' logic is ultimately based on intuition. According to Descartes, even deduction is based on intuition, because a deduction consists of a number of simple deductions, and "a simple deduction of one thing from another is performed by intuition" [12, X, p. 407]. Now, there are no rules for intuition. For such reason, Descartes states that his logic "cannot go so far as to teach us how to perform the actual operations of intuition and deduction, because these are the simplest and most primitive of all" (ibid., X, p. 372). Thus Descartes' method provides no means to formulate hypotheses.

That Bacon's and Descartes' logics have serious limitations does not mean, as Blanché states, that in their work "there is strictly nothing that deserves to be retained for the history of logic" [13, p. 174]. On the contrary, from their work we can learn that a logic of discovery can be based neither on a process of exclusion and rejection nor on intuition.

3 The Limitations of Mathematical Logic

The attempts to develop logics of discovery alternative to Scholastic logic fade in the nineteenth century and come to a definite end with Frege.

According to Frege, "the question of how we arrive at the content of a judgment should be kept distinct from the other question, Whence do we derive the justification for its assertion?" [14, p. 3]. Logic cannot be concerned with the former, the question of discovery, because it is a psychological question, "not a logical one" [15, p. 146]. It can be concerned only with the latter, the question of justification. For one cannot "count the grasping of the thought as knowledge, but only the recognition of its truth" (ibid., p. 267). In order to give a justification of a judgment, we must determine "upon what primitive laws it is based" [16, p. 235]. Then we must deduce the judgment from them. This will provide the required justification, since to deduce is "to make a judgment because we are cognisant of other truths as providing a justification for it" (ibid.). There are "laws governing this kind of justification," the laws of deduction, and "the goal of logic" is to study these laws, because they are the "laws of valid inference" (ibid.).

On this basis, mathematical logic has been developed as the logic of justification and as the study of the laws of deduction, avoiding the question of discovery. This attempt, however, has not been very successful. First, by Gödel's first incompleteness theorem, for any consistent sufficiently strong deductive theory T, there are sentences of T which are true but indemonstrable in T. In order to demonstrate them, one must discover new axioms. Therefore, mathematical logic cannot avoid the question of discovery. Secondly, by Gödel's second incompleteness theorem, for any consistent sufficiently strong deductive theory T, the sentence canonically expressing the consistency of T is not demonstrable by absolutely reliable means. Therefore, mathematical logic cannot be the logic of justification. Finally, by the strong incompleteness theorem for second-order logic, there is no set of deductive rules capable of deducing all second-order logical consequences of any given set of formulas. Therefore, mathematical logic is inadequate to the study of deduction. (For details, see [17], Introduction and Chap. 12).

Contrary to Frege, who considers the question of discovery a psychological one, Hilbert tries to trivialize it. On the one hand, he assumes that there is no question of discovering the axioms, because "the axioms can be taken quite arbitrarily" [18, p. 563]. They are only subject to the condition that "the application of the given axioms can never lead to contradictions" [19, p. 1093]. For "if the arbitrarily given axioms do not contradict one another, then they are true, and the things defined by the axioms exist. This for me is the criterion of truth and existence" [20, pp. 39-40]. On the other hand, Hilbert assumes that the question of discovering demonstrations of mathematical propositions from given axioms is a purely mechanical business. Since the axioms are arbitrarily given, this question—namely, the "decidability in a finite number of operations—is the best-known and the most discussed; for it goes to the essence of mathematical thought" [21, p. 1113].

Hilbert's attempt, however, fails because, on the one hand, by Gödel's second incompleteness theorem, it is impossible to show by absolutely reliable means that the application of the given axioms can never lead to contradictions. On the other hand, by the undecidability theorem, for any consistent sufficiently strong deductive theory, there is no mechanical procedure for deciding whether or not a mathematical proposition can be demonstrated from the axioms of the theory. Therefore, the question of the decidability of a mathematical question in a finite number of operations has a negative answer.

4 The Psychology of Discovery

Frege's view that logic cannot be concerned with the question of discovery because it is a psychological one, finds correspondence in Poincaré's proposal for a psychology of discovery.

According to Poincaré, "mathematical discovery" consists "in making new combinations" with concepts "that are already known" and in selecting "those that are useful" [22, pp. 50–51]. This is the action of the unconscious mind, which selects useful combinations on the basis of the "feeling of mathematical beauty" (ibid., p. 59). Once useful combinations have been selected, "it is necessary to verify them" (ibid., p. 56). This is the action of the conscious mind.

Poincaré's proposal for a psychology of discovery, however, is unconvincing.

First, if discovery consisted only in making new combinations with concepts that are already known, there should be some primitive concepts out of which all combinations of concepts would be made. Then, as Leibniz first pointed out, it would be possible to assign characters to primitive concepts, and form new characters for all other concepts, by means of combinations of such characters. The resulting characters would provide a universal language for mathematics, because it would be possible to express all mathematics concepts in terms of them. But this conflicts with Tarski's undefinability theorem, by which there cannot be a theory T capable of expressing all mathematical concepts, in particular, the concept of being a true sentence of T. Therefore, there cannot be a universal language for mathematics.

Secondly, the feeling of mathematical beauty, while sometimes useful, is generally unreliable as a means of selection of useful combinations. For example, the feeling of mathematical beauty led Galileo to stick to Copernicus' circular orbits for planets, which contrasted with observations, rejecting Kepler's elliptical orbits, which agreed with them. As another example, the feeling of mathematical beauty led Dirac to stick to his own version of quantum electrodynamics, which made predictions that were often infinite and hence unacceptable, rejecting renormalization, which led to accurate predictions.

5 The Need for a Rethinking of Logic

The failure of mathematical logic to provide a justification for truths already known, and the failure of the psychology of discovery to provide an account of the process of discovery, suggest that a rethinking of logic is necessary.

In particular, it is necessary to put a stop to the divorce of logic from method due to mathematical logic. Tarski states that there is "little rational justification for combining the discussion of logic and that of the methodology of empirical sciences" [23, p. xiii]. Consequently, as Aliseda points out, nowadays "logic (classical or otherwise) in philosophy of science is, to put it simply, out of fashion" [24, p. 21]. This contrasts with Aristotle's logic, which was developed as a tool for the method of science.

Because of the divorce of logic from method, mathematical logic has had little impact on scientific research. If logic is to play any significant role in science, an alternative logic paradigm is necessary. In particular, contrary to Scholastic logic and mathematical logic, in the alternative logic paradigm logic must be developed as a tool for the method of science and as the logic of discovery.

This is opposed by Laudan. He states that "the case has yet to be made that the rules governing the techniques whereby theories are invented (if such rules there be) are the sorts of things that philosophers should claim any interest in or competence at" [25, p. 182]. Therefore Laudan poses the challenge: "Why should the logic of discovery be revived?" (ibid.).

This question can be answered by saying that the logic of discovery should be revived, on the one hand, because, by Gödel's second incompleteness theorem, mathematical logic fails to be the logic of justification, and only reviving the logic of discovery, logic may continue to have an important role. On the other hand, scientists use heuristic tools in their work, and it may be useful to study such tools systematically in order to improve current heuristic tools and to develop new ones.

This means that logic must be developed as a tool for the method of science. But what is the method of science? Contemporary answers to this question involve two methods which, in their original form, were stated in antiquity: the analytic method and the analytic–synthetic method, where the latter includes the axiomatic method as its synthetic part.

6 The Analytic Method

The analytic method was first used by the mathematician Hippocrates of Chios and the physician Hippocrates of Cos and was first explicitly formulated by Plato. (On the original form of the method, see [17], Sects. 4.9, 4.13 and 4.18).

The analytic method is the method according to which, to solve a problem, one looks for some hypothesis from which a solution to the problem can be deduced. The hypothesis is obtained from the problem, and possibly other data already available, by some non-deductive rule, it need not belong to the same field as the problem, and must be plausible, that is, in accord with experience. But the hypothesis is in its turn a problem that must be solved, and is solved in the same way. That is, one looks for another hypothesis from which a solution to the problem posed by the previous hypothesis can be deduced, it is obtained from the latter problem, and possibly other data already available, by some non-deductive rule, it need not be of the same kind as the problem, and must be plausible. And so on, ad infinitum.

In the analytic method there are no principles, everything is a hypothesis. The problem and the other data already available are the only basis for solving the problem. A user of the analytic method is like Machado's walker: "Walker, your footsteps | are the road, and nothing more. | Walker, there is no road, | the road is made by walking" [26, p. 281].

The analytic method involves not only an upward movement, from problems to hypotheses, but also a downward movement, from hypotheses to problems, because one must examine the consequences of hypotheses in order to see whether they include a solution to the problem and are plausible.

The above statement of the analytic method is a revised version of Plato's original formulation. It differs from Plato's formulation in three respects.

1. Plato gives no indication as to how to find hypotheses to solve problems.
2. Plato only asks to examine the consequences of hypotheses in order "to see whether they are in accord, or are not in accord, with each other" (Plato, *Phaedo*, 101 d 4–5). This does not guarantee that they are in accord with experience.
3. Plato asks that knowledge be most certain and infallible, which leads him to conclude that "as long as we have the body and our soul is contaminated by such an

evil, we will never adequately gain the possession of what we desire, and that, we say, is truth" (ibid., 66 b 5–7). Thus we cannot reach knowledge during life.

The above statement of the analytic method is not subject to these limitations.

As to (1), it specifies that hypotheses are found by non-deductive rules.

As to (2), it asks that hypotheses be plausible, that is, in accord with experience.

As to (3), it does not ask that knowledge be most certain and infallible, but only plausible. This does not exclude that in the future new data may emerge with which the hypothesis is not in accord. (For more on the analytic method, see [17], Chap. 4).

7 The Analytic–Synthetic Method

The analytic–synthetic method was stated by Aristotle and is the basis of Aristotle's logic.

The analytic–synthetic method is the method according to which, to solve a problem of a given field, one must find premises for that field from which a solution to the problem can be deduced. According to Aristotle, the premises are obtained from the conclusion "either by syllogism or by induction" (Aristotle, *Topica*, Θ 1, 155 b 35–36). Moreover, the premises must be plausible, in the sense that they must be in accord with experience. If the premises thus obtained are not principles of the field in question, one must look for new premises from which the previous premises can be deduced. The new premises are obtained from the previous premises either by syllogism or by induction and must be plausible. And so on, until one arrives at premises which are principles of the field in question. Then the process terminates. This is analysis.

At this point one tries to see whether, inverting the order of the steps followed in analysis, one may obtain a deduction of the conclusion from the principles of the field in question, which must be known to be true by absolutely reliable means. This is synthesis. When synthesis is successful, this yields a solution to the problem.

8 Use of Non-deductive Rules

Both in the analytic and the analytic–synthetic method hypotheses are obtained by means of non-deductive rules. In the case of the analytic–synthetic method, this requires an explanation. As we have seen, according to Aristotle, in such method hypotheses are obtained from the conclusion either by syllogism or by induction. What does Aristotle mean by saying that they are obtained by syllogism?

Syllogism can be seen in a twofold manner: either as a means of obtaining conclusions from given premises, so as a means of justification, or as a means of obtaining premises for given conclusions, so as a means of discovery. According to a widespread view, for Aristotle syllogism is a means of justification, because he "shares with modern logicians the notion that central to the study of logic is examining the formal conditions for establishing knowledge of logical consequence" [27, p. 107]. This view is based on the first 26 Chapters of the first book of *Prior Analytics* where Aristotle describes the morphology of syllogism.

But in Chap. 27 Aristotle states: "Now it is time to tell how we will always find syllogisms on any given subject, and by what method we will find the premises about each thing. For surely one ought not only to investigate how syllogisms are constituted, but also to have the ability to produce them" (Aristotle, *Analytica Priora*, A 27, 43 a 20–24). In order to produce them, one must indicate "how to reach for premises concerning any problem proposed, in the case of any discipline whatever" (ibid., B 1, 53 a 1–2). That is, one must indicate, for any given conclusion, how to reach for premises from which that conclusion can be deduced.

From this it is clear that, for Aristotle, syllogism is primarily a means of discovery, specifically, a means for finding premises to solve problems. For this reason Aristotle says that, while "arguments are made from premises," the "things with which syllogisms are concerned are problems" (Aristotle, *Topica*, A 4, 101 b 15–16). Consistently with this view, in Chaps. 27-31 of the first book of *Prior Analytics* Aristotle describes a heuristic procedure for finding premises to solve problems. The medievals called this procedure *inventio medii* [discovery of the middle term] because it is essentially a procedure for finding the middle term of a syllogism, given the conclusion. (For a detailed description of this procedure, see [17], Sect. 7.4).

9 The Analytic–Synthetic Method and Modern Science

The originators of modern science adopted the analytic–synthetic method as the method of science. Contrary to a widespread opinion, the core of the Scientific Revolution of the seventeenth century was not a revolutionary change in the scientific method, but rather a change in the goal of science with respect to Aristotle. While, for Aristotle, the goal of science was to penetrate the true and intrinsic essence of natural substances, for Galileo it was only to know certain properties of natural substances, mathematical in character.

Indeed, Galileo famously stated: "Either, by speculating, we seek to penetrate the true and intrinsic essence of natural substances, or we content ourselves with coming to know some of their properties [*affezioni*]" [28, V, p. 187]. Trying to penetrate the essence of natural substances is "a not less impossible and vain undertaking with regard to the closest elemental substances than with the remotest celestial things" (ibid.). Therefore, we will content ourselves with coming to know "some properties of them," mathematical in character, "such as location, motion, shape, size, opacity, mutability, generation, and dissolution" (ibid., V, p. 188). While we cannot know the essence of natural substances, "we need not despair of our ability" to come to know such properties "even with respect to the remotest bodies, just as those close at hand" (ibid.).

But, while changing the goal of science, the originators of modern science adopted the analytic–synthetic method as the method of science. For example, Newton states: “In natural philosophy, the inquiry of difficult things by the method of analysis, ought ever to precede the method” of synthesis, or “composition” [29, p. 404]. Analysis “consists in making experiments and observations, and in drawing general conclusions from them by induction, and admitting of no objections against the conclusions, but such as are taken from experiments, or other certain truths” (ibid.). Synthesis or composition “consists in assuming the causes discovered, and established as principles, and by them explaining the phenomena proceeding from them, and proving the explanations” (ibid., p. 404–405). Newton’s own propositions “were invented by analysis,” then he composed, that is, wrote synthetically, what he had “invented by analysis” [30, p. 294].

10 Disadvantage of the Analytic–Synthetic Method

Despite its role as the method of modern science, the analytic–synthetic method has a serious disadvantage: it is incompatible with Gödel’s incompleteness theorems.

By Gödel’s first incompleteness theorem, for any consistent sufficiently strong principles of a given field, there are truths of that field which cannot be demonstrated from those principles. Their demonstration may require principles of other fields. Conversely, the analytic–synthetic method requires that every truth of a given field be deducible from principles of that field. Therefore, the analytic–synthetic method is incompatible with Gödel’s first incompleteness theorem.

On the other hand, by Gödel’s second incompleteness theorem, for any consistent, sufficiently strong principles, the principles cannot be known to be true by absolutely reliable means. Conversely, the analytic–synthetic method requires that principles be known to be true by absolutely reliable means. Therefore, the analytic–synthetic method is incompatible with Gödel’s second incompleteness theorem.

Since the analytic–synthetic method is incompatible with Gödel’s incompleteness theorems, the scientific method cannot be identified with it.

11 Advantages of the Analytic Method

Contrary to the analytic–synthetic method, the analytic method is compatible with Gödel’s incompleteness theorems. The latter even provide evidence for it.

The analytic method is compatible with Gödel’s first incompleteness theorem. In such method the solution to a problem is obtained from the problem, and possibly other data already available, by means of hypotheses which are not necessarily of the same field as the problem. Since Gödel’s first incompleteness theorem implies that solving a problem of a certain field may require hypotheses of other fields, Gödel’s result provides evidence for the analytic method.

The analytic method is also compatible with Gödel's second incompleteness theorem. In such method the hypotheses for the solution to a problem, being only plausible, are not absolutely certain. Since Gödel's second incompleteness theorem implies that no solution to a problem can be absolutely certain, Gödel's result provides evidence for the analytic method.

Not only the analytic method is compatible with Gödel's incompleteness theorems, but has several other advantages. (On the latter, see [17], Sects. 4.10, 5.17). In view of this, it seems reasonable to claim that the scientific method can be identified with the analytic method.

12 An Example

An example of use of the analytic method is the solution of Fermat's problem: Show that there are no positive integers x, y, z such that $x^n + y^n = z^n$ for $n > 2$.

Ribet showed that this problem could be solved using the hypothesis of Taniyama and Shimura, 'Every elliptic curve over the rational numbers is modular' (see [31]. But the hypothesis in question was in turn a problem that had to be solved. It was solved by Wiles using hypotheses from various mathematics fields. And so on.

13 The Paradox of Inference

In the analytic method, hypotheses are obtained by non-deductive rules, thus by rules which are not valid, that is, not truth preserving. That non-deductive rules cannot be valid follows from the so-called 'paradox of inference' (originally stated in [32], p.173).

According to the paradox of inference, if in an inference rule the conclusion is not contained in the premises, the rule cannot be valid; and if the conclusion does not possess novelty with respect to the premises, the rule cannot be ampliative; but the conclusion cannot be contained in the premises and also possess novelty with respect to them; therefore, an inference rule cannot be both valid and ampliative.

Since non-deductive rules are ampliative, from the paradox of inference it follows that they cannot be valid.

14 Objections Against the Use of Non-deductive Rules

Against the claim that the hypotheses for solving problems are obtained by the analytic method using non-deductive rules, some objections have been raised. I will consider two of them.

1. *Peirce's objection*. According to Peirce, the hypotheses for solving problems are obtained by abduction: "The surprising fact, C, is observed. But if A were true, C would be a matter of course. Hence, there is reason to suspect that A is true" [33, 5.189]. For example, Galileo observes the surprising fact C: 'Four bodies change their position around Jupiter'. But if A: 'Jupiter has satellites', were true, C would be a matter of course. 'Hence', Galileo concludes, 'there is reason to suspect that A is true'. According to Peirce, not only hypotheses for solving problems are obtained by abduction, but "all ideas of science come to it by way of abduction" (ibid., 5.145). Thus "abduction must cover all operations by which theories and conceptions are engendered" (ibid., 5.590).

This objection, however, is unjustified because abduction is of the form

$$\frac{C \quad A \rightarrow C}{A}$$

Thus the conclusion, A, already occurs as a part of the premise, $A \rightarrow C$, and hence is not really new. Peirce himself states that "A cannot be abductively inferred, or if you prefer the expression, cannot be abductively conjectured, until its entire content is already present in the premiss, 'If A were true, C would be a matter of course'" (ibid., 5.189). And he admits that "quite new conceptions cannot be obtained from abduction" (ibid., 5.190). This conflicts with his claim that abduction yields something new since it must cover all operations by which theories and conceptions are engendered.

Actually, a new hypothesis is not generated by abduction, but rather by the process that yields the premise, $A \rightarrow C$, thus a process prior to the application of abduction. As Franfurt points out, "clearly, if the new idea, or hypothesis, must appear in one of the premisses of the abduction, it cannot be the case that it originates as the conclusion of such an inference; it must have been invented before the conclusion was drawn" [34, p. 594].

Peirce claims that abduction is "what Aristotle meant by *apagogè*" [35, p. 140]. He assumes that Aristotle's *apagogè* is "the inference of the minor premiss" of a certain syllogism "from its other two propositions" [33, 7.251]. This, however, is unjustified because it contrasts with Aristotle's statement that "it is *apagogé* when it is clear that the first term belongs to the middle and unclear that the middle belongs to the third, though nevertheless more convincing than the conclusion" (Aristotle, *Analytica Priora*, B 25, 69 a 20–22). In order to support his claim, Peirce assumes that Aristotle's original text was corrupted, and the first editor filled the corrupted text "with the wrong word" (Peirce [35, p. 140]. But this is unconvincing, because Aristotle's text is perfectly intelligible as it stands. Peirce himself ends up acknowledging that his assumption is a "doubtful theory" [33, 8.209]. Rather than with *apagogè*, abduction can be compared with Aristotle's procedure of *inventio medii* mentioned in Sect. 8, but the comparison is unfavourable to abduction because, while in abduction the conclusion, A, already occurs as a part of the premise, $A \rightarrow C$, and hence is not really new, in *inventio medii* the middle term does not occur in the conclusion, and hence is new.

2. *Popper's objection*. According to Popper, hypotheses for solving problems cannot be obtained by non-deductive rules, in particular by induction, because we are not "justified in inferring universal statements from singular ones, no matter how numerous; for any conclusion drawn in this way may always turn out to be false" [36], p. 4). Generally, "there is no method of discovering a scientific theory" [37, p. 6]. Hypotheses are not obtained by a rational procedure but are rather "the result of an almost poetic intuition" [38, p. 192].

This objection, however, is unjustified because it is based on the following two tacit assumptions: (a) Knowledge must be known to be true; (b) Induction must provide a justification for the hypotheses it yields. Now, (a) is unwarranted because, by Gödel's second incompleteness theorem, it is impossible to know by absolutely reliable means if hypotheses are true. On the other hand, (b) is unwarranted because induction is only used to generate hypotheses, not to justify them, their justification depends on their accord with experience. By denying that induction may be used to obtain hypotheses, and generally that any method for obtaining hypotheses exists, Popper has no other option than saying that hypotheses are the result of an almost poetic intuition.

15 Scientific Knowledge and Certainty

If the analytic method is the method of science, then scientific knowledge is the result of solving problems by the analytic method. When solving problems is successful, this yields scientific knowledge.

Then scientific knowledge cannot be absolutely certain, because it is based on hypotheses that can only be plausible, and plausibility does not guarantee truth or certainty. On the other hand, plausibility is the best we can achieve, because there is no special source of knowledge capable of guaranteeing absolute truth or certainty. As Plato says, human beings can only "adopt the best and least refutable of human hypotheses, and embarking on it as on a raft, risk the dangers of crossing the sea of life" (Plato, *Phaedo*, 85 c 8–d 2).

That knowledge cannot be absolutely certain does not mean, as the sceptics claim, that knowledge is impossible, but only that absolutely certain knowledge is impossible.

16 Provisional Character of Solutions to Problems

As we have seen, in the analytic method solving a problem is a potentially infinite process, so no solution is final. At any stage, the inquiry may bring up new data incompatible with the hypothesis on which the solution is based. Then the hypothesis, while not being rejected outright, is put on a waiting list, subject to further inquiry. Even when the inquiry does not bring up incompatible data, the hypothesis remains a problem to be solved, and is solved by looking for another hypothesis, and so on. Thus any solution is provisional.

It might be objected that this does not account for the fact that mathematical results are final and forever. This objection, however, is unjustified because even solutions to mathematical problems are provisional. As Davis states, even in mathematics “a solved problem is still not completely solved but leads to new and profound challenges,” and “discovering a sense in which” this is the case “is one important direction that mathematical research takes” [39, p. 177]. The analytic method suggests such a sense. A solved problem is still not completely solved since no hypothesis is absolutely justified. Any hypothesis which provides a solution to the problem is liable to be replaced with another hypothesis when new data emerge. Already Kant observed that “any answer given according to principles of experience always begets a new question which also requires an answer” [40, p. 103].

Quite generally, there is no final solution to problems. As Russell says, “final truth belongs to heaven, not to this world” [41, p. 3].

17 The Rules of Discovery

As we have seen, in the analytic method hypotheses are found by means of non-deductive rules. Of course, finding hypotheses is not a sufficient condition for discovery, because the latter requires hypotheses to be plausible. Nevertheless, finding hypotheses is a necessary condition for discovery and, in that sense, one may say that non-deductive rules are rules of discovery.

The latter, however, are not a closed set, given once for all, but rather an open set which can always be extended as research develops. Each such extension is a development of the analytic method, which grows as new non-deductive rules are added. In any case, the rules of discovery will at least include various kinds of induction and analogy, generalization, specialization, metaphor, metonymy, definition, and diagrams.

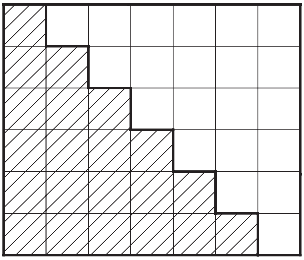
In what follows I will give an example of use of one of such rules in solving a problem. Such example is meant in the spirit of Pólya: “I cannot tell the true story how the discovery did happen, because nobody really knows that. Yet I shall try to make up a likely story how the discovery could have happened. I shall try to emphasize” the “inferences that led to it” [42, I, pp.vi–vii]. That nobody really knows the true story how the discovery did happen is due to the fact that discoverers generally do not reveal their way to discovery. They do so not because, as Descartes suggests, they conceal it “with a sort of pernicious cunning” [12, X, p. 376]. They do so rather because either they are not fully aware of how they arrived at their discoveries—awareness requires a good capacity for introspection—or feel uneasy to reveal that their way to discovery was not rigorously deductive.

The example concerns metaphor. Let $T \triangleright S$ stand for ‘The elements of a domain, T, called the target domain, are to be considered as if they were elements of a domain, S, called the source domain’. Metaphor is an inference of the form: If $T \triangleright S$ and any arbitrary element of S has a certain property, then also any arbitrary element of T will have that property. So it is an inference by the rule:

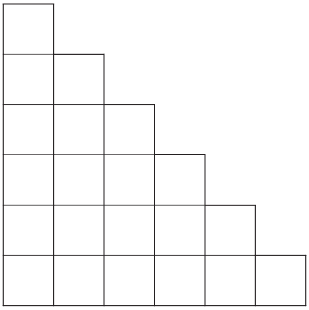
$$\text{(MTA)}\ \frac{T \rhd S \quad a \in T \quad \overset{[x\in S]}{A(x)}}{A(a)}.$$

Here $[x \in S]$ indicates that assumptions of the form $x \in S$ may be discharged by the rule.

An example of use of (MTA) is the Pythagoreans' discovery of the hypothesis $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$. From the following diagram, by (MTA), the Pythagoreans conclude the result.



The argument pattern is as follows. Let T be the domain of positive integers, and S be the domain of triangular figurate numbers, that is, geometric figures of the following form:



Let $T \rhd S$ and $n \in T$. Let $A(x)$: $1 + 2 + \cdots + x = \frac{1}{2}x(x + 1)$. If $x \in S$ then $A(x)$. From this, by (MTA), it follows $A(n)$.

For other examples of rules of discovery and their use in solving problems, see [17], Chaps. 20 and 21.

18 Conclusion

The possibility of the logic of discovery is often denied by arguing that, since there are no inference rules "by which hypotheses or theories can be mechanically derived or inferred from empirical data," the "transition from data to theory requires creative imagination" [43, p. 15]. The "discovery of important, fruitful mathematical theorems, like the discovery of important, fruitful theories in empirical science, requires inventive ingenuity; it calls for imaginative, insightful guessing" (ibid., p. 17). That is, it calls for intuition.

This argument depends on the alternative: Either there are inference rules by which hypotheses can be mechanically inferred from empirical data, or the discovery of hypotheses calls for intuition. Since there are no inference rules by which hypotheses can be mechanically inferred from empirical data, the argument concludes that the discovery of hypotheses calls for intuition.

Now, the second horn of the alternative, that the discovery of hypotheses calls for intuition, is unsatisfactory because it amounts to admitting that, with respect to the discovery of new hypotheses or axioms, "there is no hope, there is, as it were, a leap in the dark, a bet at any new axiom. We are no longer in the domain of science but in that of poetry" [44, p. 169]. This would imply that science is an essentially irrational enterprise.

The first horn of the alternative, that there are inference rules by which hypotheses can be mechanically inferred from empirical data, corresponds to Bacon's assumption that, "in forming axioms, a form of induction, different from that hitherto in use, must be thought out" [11, I, p. 205]. Through it "the mind is from the very outset not left to itself, but constantly guided, so that everything proceeds, as it were, mechanically" (ibid., I, p. 152). Being mechanical, this form of induction is such as "not to leave much to the acuteness and strength of talents, but more or less to equalise talents and intellect (ibid., I, p. 172). Bacon's mechanical induction has had contemporary developments (see [45]). But Bacon's assumption is unwarranted. The purpose of the logic of discovery is not to dispense with the acuteness and strength of talents by use of mechanical rules, but rather to boost the acuteness and strength of talents by means of tools capable of guiding the mind, though not infallibly. Such a boost is provided by heuristic procedures which, rather than constantly guiding the mind so that everything proceeds mechanically, give the mind tools for finding hypotheses.

This shows that the alternative is unjustified. Between the determinism of mechanical rules and the inscrutability of intuition there is an intermediate region, inhabited by heuristic procedures. The latter consist of non-mechanical, non-deductive rules such as various kinds of induction and analogy, generalization, specialization, metaphor, metonymy, definition, and diagrams. These rules permit to find hypotheses by the analytic method without need to call for intuition.

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