

Chapter 2

Classical Mechanics and Poisson Structures

In this chapter, we will briefly recall the Hamiltonian formulation of classical mechanics, focusing in particular on its algebraic aspects. In this framework, a classical system will be described by a commutative algebra of functions (classical observables) with the Poisson bracket as a Lie bracket.

We will discuss in detail the properties of the Poisson bracket and introduce the tensor formulation of Poisson structures on manifolds, as the Poisson bracket plays a fundamental role in classical mechanics and in deformation quantization. We will mainly focus on the algebraic rather than geometrical properties of Poisson manifolds, the latter being less important for the theory of deformation quantization. Furthermore, we will introduce the reader to the basic notions needed for the formulation of the formality theory, i.e. formal power series, formal Poisson structures and equivalence classes of formal Poisson structures.

2.1 Hamiltonian Mechanics and Poisson Brackets

This section aims to give a brief introduction to classical mechanics, starting with Newton's laws and heading towards the Hamiltonian approach, with a particular attention to the role of the Poisson bracket. The interested reader is referred to the classical literature on the subject, as e.g. [1, 2, 6] for an exhaustive treatment.

We start by discussing the motion of a point particle of mass m in the Euclidean space \mathbb{R}^n . The position of the particle is described by the vector $q := (q^1, \dots, q^n) \in \mathbb{R}^n$. The vector q is generally parametrized by the variable $t \in \mathbb{R}$. We say that $q(t)$ is the position of the particle at time t .

The velocity $v(t)$ of the particle at time t is defined as

$$v(t) := \dot{q}(t) = \frac{dq}{dt}(t), \quad (2.1)$$

where we used Newton's notation $\dot{q}(t)$ to denote the total derivative w.r.t. the time. Similarly, the acceleration is defined as

$$a(t) := \ddot{q}(t) = \frac{d^2 q}{dt^2}(t). \quad (2.2)$$

The evolution in time of the particle position is described by the n functions $q^i(t)$, $i = 1, \dots, n$, which are solutions of the set of Newton's equations

$$m \ddot{q}(t) = F(q^1, \dots, q^n), \quad (2.3)$$

where $F := (F_1, \dots, F_n)$ denotes the force acting on the particle, together with the initial conditions

$$\begin{aligned} q(0) &= q_0, \\ v(0) &= v_0. \end{aligned} \quad (2.4)$$

From here on, we assume that the force is conservative, i.e. it can be written in terms of the gradient of a some function $V : \mathbb{R}^n \rightarrow \mathbb{R}$

$$F_i = -\frac{\partial V}{\partial q^i}, \quad i = 1, \dots, n. \quad (2.5)$$

The function V is generally called potential.

As will be seen in the following, in the Hamiltonian formalism, the system of second order differential equations (2.3), in the n variables (q^1, \dots, q^n) , becomes a first order system in the $2n$ variables $(q, p) := (q^1, \dots, q^n, p_1, \dots, p_n)$. The variables p_i , are called conjugated momenta and they are defined as

$$p_i = m \dot{q}^i. \quad (2.6)$$

Indeed, using this definition, Newton's equations (2.3) can be rewritten as

$$\begin{aligned} \dot{q}^i(t) &= \frac{p_i(t)}{m}, \\ \dot{p}_i(t) &= -\frac{\partial V}{\partial q^i}(q(t)), \end{aligned} \quad (2.7)$$

and the initial conditions (2.4) read

$$\begin{aligned} q(0) &= q_0, \\ p(0) &= p_0. \end{aligned} \quad (2.8)$$

Introducing the Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$H(q, p) := \sum_{i=1}^n \frac{p_i^2}{2m} + V(q), \quad (2.9)$$

which represents the energy of the system as a function of the position q and the momentum p , the set of equations (2.7) can be rewritten as the well-known Hamilton's equations:

$$\begin{aligned} \dot{q}^i(t) &= \frac{\partial H}{\partial p_i}(q, p), \\ \dot{p}_i(t) &= -\frac{\partial H}{\partial q^i}(q, p). \end{aligned} \quad (2.10)$$

The $2n$ -dimensional space of all the possible positions q and momentum p of a single particle is called phase space and coincides with $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$. A real-valued smooth function f on the phase space, i.e. $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, is called classical observable.

Given a generic observable f , it is natural to ask how it evolves in time. Denoting by

$$f_t(q, p) = f(q(t), p(t)), \quad (2.11)$$

the value of the observable at the generic time t , where $q(t)$ and $p(t)$ are solutions of (2.10) with $q(0) = q_0$, $p(0) = p_0$, we have that

$$\begin{aligned} \frac{df_t}{dt}(q, p) &= \frac{d}{dt}f(q(t), p(t)) \\ &= \frac{\partial f_t}{\partial q^i} \frac{dq^i}{dt} + \frac{\partial f_t}{\partial p_i} \frac{dp_i}{dt} \\ &= \frac{\partial f_t}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f_t}{\partial p_i} \frac{\partial H}{\partial q^i}. \end{aligned} \quad (2.12)$$

In the above expression, we used the Einstein notation on the sum over repeated indices. We will use this convention throughout these notes. The expression obtained above in (2.12) can be written in a more convenient manner, by introducing the canonical Poisson bracket. This is defined, for two arbitrary functions f and g on the phase space \mathbb{R}^{2n} , as

$$\{f, g\} := \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}. \quad (2.13)$$

In this notation, Hamilton's equations read

$$\frac{df_t}{dt} = \{H, f_t\}. \quad (2.14)$$

A given observable f , constant along all solutions $(q(t), p(t))$ of Hamilton's equations (2.10), i.e. $f(q(t), p(t)) = f(q_0, p_0)$ for any $t \in \mathbb{R}$, where $q_0 = q(0)$ and $p_0 = p(0)$, is called (mostly in physics) a constant of motion. It is clear from (2.14) that the Hamiltonian H is always a constant of motion (conservation of the energy of the system).

The above discussion can be generalized to the case in which the phase space is a generic smooth manifold. As will be seen, a classical physical system can be described by the algebra of functions on a given phase space endowed with a Poisson bracket. Because of their importance in the formulation of both quantum and classical mechanics, the Poisson structures will be the main topic in the rest of this chapter.

2.2 Poisson Manifolds

Let M be a smooth manifold. The set $C^\infty(M)$ of real-valued smooth functions on M describes the set of observables. It is a commutative algebra with addition, scalar multiplication and pointwise multiplication given by

$$\begin{aligned} (\alpha f)(x) &= \alpha f(x), \\ (f + g)(x) &= f(x) + g(x), \\ (f \cdot g)(x) &= f(x)g(x), \end{aligned} \quad (2.15)$$

for any $f, g \in C^\infty(M)$, $\alpha \in \mathbb{R}$ and $x \in M$.

A Poisson bracket can be defined on $C^\infty(M)$ as follows

Definition 2.1 The bracket operation denoted by

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) \quad (2.16)$$

is called Poisson bracket if it satisfies the following properties

1. $\{f, g\}$ is bilinear with respect to f and g
2. $\{f, g\} = -\{g, f\}$ (skew-symmetry)
3. $\{h, fg\} = f\{h, g\} + \{h, f\}g$ (Leibniz rule)
4. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity)

for any $f, g, h \in C^\infty(M)$.

The Poisson bracket makes $C^\infty(M)$ into a Lie algebra, as it satisfies bilinearity, skew-symmetry and the Jacobi identity. It follows that the algebra of observables

$C^\infty(M)$ is a commutative algebra with Poisson bracket as Lie bracket. In other words, it is a Poisson algebra, as defined below.

Definition 2.2 A Poisson algebra is a commutative algebra A with a bracket $\{\cdot, \cdot\}$ making it into a Lie algebra such that it satisfies the Leibniz rule.

Starting from this definition we can introduce the concept of Poisson manifolds in a very natural way as follows

Definition 2.3 A Poisson manifold is a smooth manifold M equipped with a bracket $\{\cdot, \cdot\}$ on its function space $C^\infty(M)$, such that the pair $(C^\infty(M), \{\cdot, \cdot\})$ is a Poisson algebra.

The reader can find a more detailed discussion on Poisson manifolds and their geometrical properties e.g. in [3, 8, 9, 11].

A Poisson manifold can be redefined, in a more modern way, in terms of bivector fields. This formulation is necessary for the description of Kontsevich's theory of deformation quantization of Poisson manifolds. In order to rewrite the definition of Poisson manifolds we need to take a step back.

Let us consider a generic bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ satisfying the conditions (1)–(3) of Definition 2.1, i.e. bilinearity, skew-symmetry and Leibniz rule. The Leibniz rule implies that, for a given function f on $C^\infty(M)$, the map $g \mapsto \{f, g\}$ is a derivation. Thus, there is a unique vector field X_f on M , called Hamiltonian vector field, such that for any $g \in C^\infty(M)$ we have

$$X_f(g) = \{f, g\}. \quad (2.17)$$

Here

$$X_f(g) = \langle dg, X_f \rangle, \quad (2.18)$$

where dg is the differential of the function $g \in C^\infty(M)$ and $\langle \cdot, \cdot \rangle$ is the pairing between one-forms and vector fields.

In the following, we will express Poisson structures in terms of bivector fields satisfying certain conditions. Recall that $\wedge^2 TM$ is the space of bivector of M : it is a vector bundle over M . A (smooth) bivector field π on M is, by definition, a smooth section of $\wedge^2 TM$, i.e. a map $\pi : M \rightarrow \wedge^2 TM$, which associates to each point $m \in M$ a bivector $\pi(m) \in \wedge^2 T_m M$. We denote by $\Gamma(\wedge^2 TM)$ the space of sections on $\wedge^2 TM$.

Given a bivector field π , one can define a bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ as

$$\{f, g\} := \pi(df, dg) = \langle df \otimes dg, \pi \rangle, \quad (2.19)$$

which satisfies the conditions (1)–(3) of Definition 2.1. It is important to remark that, at this stage, this is not a Poisson bracket, because the Jacobi rule is not a priori satisfied. We sketch the conditions which guarantee this bracket to be a Poisson bracket. A bivector field π such that the bracket defined in Eq. (2.19) satisfies the Jacobi identity is called Poisson tensor or Poisson bivector field. In a local system of coordinates (x_1, \dots, x_n) , Eq. (2.19) can be expressed as

$$\{f, g\} = \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \quad (2.20)$$

where π^{ij} are smooth functions on the local chart and are defined by

$$\pi^{ij} = \{x^i, x^j\} = -\pi^{ji}. \quad (2.21)$$

This implies that the bivector field π is locally given by

$$\pi = \frac{1}{2} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}; \quad (2.22)$$

using the local expression (2.20) for the Poisson bracket, we can easily compute the terms of the Jacobi identity:

$$\begin{aligned} \{\{f, g\}, h\} &= \pi^{ij} \frac{\partial}{\partial x^i} \left(\pi^{kl} \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial x^l} \right) \frac{\partial h}{\partial x^j} \\ &= \pi^{ij} \pi^{kl} \left(\frac{\partial f}{\partial x^i \partial x^k} \frac{\partial g}{\partial x^l} + \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial x^i \partial x^l} \right) \frac{\partial h}{\partial x^j} \\ &\quad + \pi^{ij} \frac{\partial \pi^{kl}}{\partial x^i} \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial x^l} \frac{\partial h}{\partial x^j}. \end{aligned} \quad (2.23)$$

Similarly we get $\{\{g, h\}, f\}$ and $\{\{h, f\}, g\}$. From the skew-symmetry (2.21) follows that the expressions without derivatives of π^{ij} are invariant under switching $\{ij\} \leftrightarrow \{kl\}$, thus the sum of those three terms yields zero. For this reason, the Jacobi identity reads

$$\pi^{hi} \frac{\partial}{\partial x_h} \pi^{jk} + \pi^{hj} \frac{\partial}{\partial x_h} \pi^{ki} + \pi^{hk} \frac{\partial}{\partial x_h} \pi^{ij} = 0. \quad (2.24)$$

In other words,

Proposition 2.1 *The bivector field $\pi \in \wedge^2 TM$ defines a Poisson bracket in the local coordinates $\{x_i\}_{i=1}^n$ if and only if it satisfies the condition (2.24).*

This condition can be rephrased in an invariant formalism, by introducing the Schouten-Nijenhuis bracket of π . We briefly recall the notion of for a generic multivector field and we prove that π is a Poisson tensor if and only if the Schouten-Nijenhuis bracket $[\pi, \pi]_S$ is vanishing.

The definition of bivector field can be immediately generalized as follows. A k -th multivector field X on a smooth manifold M is a section of the k -th exterior power $\wedge^k TM$ of the tangent bundle TM . In local coordinates $\{x_i\}_{i=1}^n$, the multivector field $X \in \Gamma(\wedge^k TM)$ can be written as

$$X = \sum_{i_1 \dots i_k=1}^n X^{i_1 \dots i_k}(x) \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_k}}, \quad (2.25)$$

where the coefficients $X^{i_1 \dots i_k}(x)$ are smooth functions on M . We denote by $\mathfrak{X}^k(M)$ the space of sections $\Gamma(\wedge^k TM)$; notice that $\mathfrak{X}^0(M) = C^\infty(M)$. It is well-known that, for any vector field $X \in \mathfrak{X}^1(M)$, there is a well defined Lie bracket on vector fields given in terms of Lie derivative \mathcal{L}_X :

$$[X, Y] := \mathcal{L}_X Y \quad \forall Y \in \mathfrak{X}^1(M). \quad (2.26)$$

This definition can be also applied to the case in which the second argument is a function:

$$[X, f] := \mathcal{L}_X f = \sum_{i=1}^n X^i \frac{\partial f}{\partial x_i}. \quad (2.27)$$

We can extend this bracket to an operation

$$[\cdot, \cdot]_S : \mathfrak{X}^k(M) \otimes \mathfrak{X}^l(M) \rightarrow \mathfrak{X}^{k+l-1}(M) \quad (2.28)$$

defined by

$$[X_1 \wedge \dots \wedge X_k, Y_1 \wedge \dots \wedge Y_l]_S := \sum_{i=1}^k \sum_{j=1}^l (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge X_k \wedge Y_1 \wedge \dots \wedge \widehat{Y}_j \wedge \dots \wedge Y_l, \quad (2.29)$$

where the hat denotes the absence of the corresponding term.

Proposition 2.2 *The operation $[\cdot, \cdot]_S$ defined in (2.29) is the unique well-defined \mathbb{R} -bilinear local type extension of the Lie derivative \mathcal{L}_X and satisfies*

1. $[X, Y]_S = (-1)^{kl} [Y, X]_S$
2. $[X, Y \wedge Z]_S = [X, Y]_S \wedge Z + (-1)^{(k+1)l} Y \wedge [X, Z]_S$
3. $(-1)^{k(m-1)} [X, [Y, Z]_S]_S + (-1)^{l(k-1)} [[Y, Z]_S, X]_S + (-1)^{m(l-1)} [Z, [X, Y]_S]_S = 0$

for three multivectors X, Y and Z of degree resp. k, l and m .

This operation is called Schouten-Nijenhuis bracket. Notice that, in particular, for any $f \in C^\infty(M)$ and $X \in \mathfrak{X}^k(M)$,

$$[X, f]_S = -\langle df, X \rangle = \sum_{i=1}^k (-1)^i \mathcal{L}_{X_i}(f) X_1 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge X_k. \quad (2.30)$$

In local coordinates, if

$$\begin{aligned} X &= X^{i_1, \dots, i_n} \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_n}}, \\ Y &= Y^{j_1, \dots, j_m} \frac{\partial}{\partial x_{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{j_m}}, \end{aligned} \quad (2.31)$$

the Schouten-Nijenhuis bracket is given by a $n + m - 1$ contravariant tensor field $[X, Y]_S$

$$\begin{aligned} [X, Y]_S &= X^{l_{i_1}, \dots, l_{i-1} i_{l+1}, \dots, i_n} \frac{\partial Y^{j_1, \dots, j_m}}{\partial x_{l_i}} \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_{l-1}}} \wedge \frac{\partial}{\partial x_{i_{l+1}}} \wedge \\ &\wedge \frac{\partial}{\partial x_{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{j_m}} (-1)^n Y^{l_{j_1}, \dots, j_{l-1} j_{l+1}, \dots, j_m} \frac{\partial X^{i_1, \dots, i_n}}{\partial x_{l_i}} \frac{\partial}{\partial x_{i_1}} \wedge \\ &\wedge \dots \wedge \frac{\partial}{\partial x_{i_n}} \wedge \frac{\partial}{\partial x_{j_1}} \dots \wedge \frac{\partial}{\partial x_{j_{l-1}}} \wedge \frac{\partial}{\partial x_{j_{l+1}}} \wedge \dots \wedge \frac{\partial}{\partial x_{j_m}}, \end{aligned} \quad (2.32)$$

or more succinctly

$$\begin{aligned} [X, Y]_S^{k_2 \dots k_{n+m}} &= \varepsilon_{i_2 \dots i_n j_1 \dots j_m}^{k_2 \dots k_{n+m}} X^{l(i_2 \dots i_n)} \frac{\partial}{\partial x^l} Y^{j_1 \dots j_m} \\ &+ (-1)^n \varepsilon_{i_1 \dots i_n j_2 \dots j_m}^{k_2 \dots k_{n+m}} Y^{l(j_2 \dots j_m)} \frac{\partial}{\partial x^l} X^{i_1 \dots i_n}. \end{aligned} \quad (2.33)$$

Here

$$\varepsilon_{j_1 \dots j_{n+m}}^{i_1 \dots i_{n+m}} \quad (2.34)$$

is the Kronecker symbol: it is zero if $(i_1 \dots i_{n+m}) \neq (j_1 \dots j_{n+m})$, and is 1 (resp., -1) if $(j_1 \dots j_{n+m})$ is an even (resp., odd) permutation of $(i_1 \dots i_{n+m})$.

Remark 2.1 The Schouten-Nijenhuis bracket is naturally preserved by any diffeomorphism $\phi : M \rightarrow N$. Indeed, we recall that

$$\phi_*[X, Y] = [\phi_*X, \phi_*Y] \quad X, Y \in \mathfrak{X}^1(N) \quad (2.35)$$

where ϕ_* is the pushforward of a diffeomorphism $\phi : M \rightarrow N$. It is easy to check, using the definition of the Schouten-Nijenhuis bracket, that this can be extended to

$$\phi_*[X, Y]_S = [\phi_*X, \phi_*Y]_S, \quad (2.36)$$

for any $X, Y \in \mathfrak{X}^k(M)$ and any diffeomorphism ϕ .

The Schouten-Nijenhuis bracket allows us to characterize a Poisson manifold in a very convenient way.

Theorem 2.1 *A bivector field π is a Poisson tensor if and only if the Schouten-Nijenhuis bracket of π with itself vanishes, i.e.*

$$[\pi, \pi]_S = 0. \quad (2.37)$$

It is easy to check, from Eq. (2.33), that in local coordinates

$$[\pi, \pi]_S = \left(\pi^{hi} \frac{\partial}{\partial x_h} \pi^{jk} + \pi^{hj} \frac{\partial}{\partial x_h} \pi^{ki} + \pi^{hk} \frac{\partial}{\partial x_h} \pi^{ij} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}. \quad (2.38)$$

Then Eq. (2.37) is equivalent to the Jacobi rule (2.24).

Example 2.1 A canonical example is given by $M = \mathbb{R}^{2n}$, with coordinates (q^i, p_i) , $i = 1, \dots, n$. The canonical Poisson bracket of functions on the phase space is defined in Eq. (2.13) and the corresponding Poisson bivector field is

$$\pi = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}. \quad (2.39)$$

It is easy to check that the bivector π defined above satisfies Eq. (2.37).

Using this characterization of Poisson manifolds and recalling Remark 2.1, we can say that the set of Poisson structures is acted upon by the group of diffeomorphisms on M , that is

$$\pi_\phi := \phi_* \pi, \quad (2.40)$$

where ϕ_* is the pushforward of $\phi : M \rightarrow M$. Indeed, by Eq. (2.36) we have

$$[\pi_\phi, \pi_\phi]_S = [\phi_* \pi, \phi_* \pi]_S = \phi_* [\pi, \pi]_S = 0. \quad (2.41)$$

This implies that the set of diffeomorphisms $\phi : M \rightarrow M$ defines a gauge group on the set of Poisson structures.

2.3 Formal Poisson Structures

We introduced the Poisson structure on a smooth manifold as a skew-symmetric contravariant bi-tensor which satisfies the Jacobi identity. This structure and its gauge group can be easily extended to formal power series. For this purpose, we briefly recall basic notions and properties of the theory of formal power series.

2.3.1 Formal Power Series

Formal power series are a generalization of power series as formal objects, performed by substituting variables with formal indeterminates. Formal essentially means that there is not necessarily a notion of convergence; formal power series are purely algebraic objects and we essentially use them to represent the whole collection of their coefficients. A detailed discussion on formal power series can be found in [7, 10].

Given a sequence $\{a_n\}_{n \in \mathbb{N}_0}$ of elements on a commutative ring k , a formal power series a is defined by

$$a = \sum_{n=0}^{\infty} t^n a_n \quad (2.42)$$

where t is a formal indeterminate. Two formal power series are equal if and only if their coefficients sequences are the same.

The set of formal power series in t with coefficient in a commutative ring k has also a structure of ring, denoted by $k[[t]]$. Indeed, given two formal power series $a, b \in k[[t]]$, one defines addition of such sequences by

$$a + b = \sum_{n=0}^{\infty} t^n (a_n + b_n), \quad a_n \in k, \quad (2.43)$$

and multiplication by

$$ab = \sum_{n=0}^{\infty} t^n c_n, \quad c_n = \sum_{k=0}^n a_k b_{n-k}, \quad a_n, b_n \in k. \quad (2.44)$$

With these two operations, the set $k[[t]]$ becomes a commutative ring with 0 element $(0, 0, \dots)$, multiplicative identity $(1, 0, 0, \dots)$ and the invertible elements are the series with non-vanishing constant term.

Given a vector space V over the ring k , we denote by $V[[t]]$ the space of formal power series with coefficients in V ,

$$v = \sum_{n=0}^{\infty} t^n v_n, \quad v_n \in V. \quad (2.45)$$

Elements in $V[[t]]$ can also be summed term by term and

$$av = \sum_{n=0}^{\infty} t^n c_n, \quad c_n = \sum_{k=0}^n a_k v_{n-k}, \quad a \in k[[t]], \quad v \in V[[t]]. \quad (2.46)$$

In other words, $V[[t]]$ becomes a $k[[t]]$ -module. The order of a formal power series is defined by the minimum of the set of all non-negative integers n such that $a_n \neq 0$ and is denoted by $o(v)$. If $v = 0$ the order is defined to be $+\infty$. Furthermore, $V[[t]]$ can be endowed with a metric defined by

$$d : V[[t]] \times V[[t]] \rightarrow \mathbb{R} : (v, w) \mapsto d(v, w) := \begin{cases} 2^{-o(v-w)}, & \text{if } v \neq w \\ 0, & \text{if } v = w. \end{cases} \quad (2.47)$$

It induces a Hausdorff topology, called the t -adic topology on $V[[t]]$.

Lemma 2.1 *Let V_1 and V_2 be two k -modules and $\Phi : V_1[[t]] \rightarrow V_2[[t]]$ be a $k[[t]]$ -linear map. Then, for any non-negative integer r there is a unique linear map $\Phi_r : V_1 \rightarrow V_2$ such that*

$$\Phi(v) = \sum_{r=0}^{\infty} t^r \sum_{s=0}^r \Phi_s(v_{r-s}) \quad (2.48)$$

for all $v = \sum_{r=0}^{\infty} t^r v_r \in V_1[[t]]$.

If k is a commutative ring, this Lemma can be generalized to the case of k -multilinear maps.

It is important to remark that if A is an algebra over the commutative ring k , the set of formal power series $A[[t]]$ with coefficients in A

$$a = \sum_{n=0}^{\infty} t^n a_n, \quad a_n \in A \quad (2.49)$$

forms an algebra over the ring $k[[t]]$. In fact, elements in $A[[t]]$ can be composed by

$$ab = \sum_{n=0}^{\infty} t^n c_n \quad c_n = \sum_{k=0}^n a_k b_{n-k}, \quad a_n, b_n \in A, \quad (2.50)$$

as $A[[t]]$ is a $k[[t]]$ -module. Notice that $A[[t]]$ is an algebra of the same type of A ; in particular if A is unital associative, $A[[t]]$ will be also unital and associative.

Let U be an open set in \mathbb{R}^n such that $0 \in U$ and let $f \in C^\infty(U)$. We denote by \hat{f} the formal power series

$$\hat{f} = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(0) \quad (2.51)$$

where $f^{(n)}$ is the n -th derivative of the function f . A fundamental property of formal power series is given by the following

Theorem 2.2 (Borel Lemma, first version) *Given a sequence of real numbers $\{a_n\}$ of non-negative integers, there exists a smooth function $f \in C^\infty(U)$ such that*

$$\frac{1}{n!}f^{(n)}(0) = a_n \in \mathbb{C}[[t]]. \quad (2.52)$$

In other words, the mapping from $C^\infty(U)$ to the ring of formal power series $C^\infty(\mathbb{R})[[t]]$ given by $f \mapsto \widehat{f}$ is a \mathbb{R} -linear surjective algebra homomorphism.

The surjectivity of the map defined by is quite hard to prove; on the other hand, the linearity is evident and we have

$$(fg)^{(n)}(0) = \sum_{s=0}^r \binom{r}{s} f^{(s)}(0) g^{(r-s)}(0), \quad (2.53)$$

which implies $\widehat{fg} = \widehat{f}\widehat{g}$. An elementary proof of Borel's lemma can be found in [4] and in [5]. This lemma implies that we can view the formal power series as the (formal) Taylor expansion of a smooth function at zero.

2.3.2 Formal Poisson Structures

In order to extend Poisson structures to formal power series, we need to figure out what a formal multivector field is. Using the notion of formal power series with coefficients on a vector space discussed above, we can introduce the concept of formal vector field as follows.

Definition 2.4 A formal vector field is a formal power series

$$X = \sum_{n=0}^{\infty} X_n t^n, \quad X_n \in \mathfrak{X}^1(M). \quad (2.54)$$

The set of formal vector fields is denoted by $\mathfrak{X}^1(M)[[t]]$. This definition can be immediately extended to multivector fields, thus a formal multivector field is an element in $\mathfrak{X}^k(M)[[t]]$, i.e. a formal power series with coefficients in $\mathfrak{X}^k(M)$. Finally, we can define the extension of Poisson structures to formal power series as follows:

Definition 2.5 A formal Poisson structure is a formal power series

$$\pi_t = \pi_0 + t\pi_1 + t^2\pi_2 + \cdots = \sum_{n=0}^{\infty} t^n \pi_n \in \mathfrak{X}^2(M)[[t]], \quad (2.55)$$

where the π_n 's are skew-symmetric vector fields on M , such that the Schouten-Nijenhuis bracket of π_t with itself vanishes order by order,

$$[\pi_t, \pi_t]_S = 0. \quad (2.56)$$

Given a Poisson manifold (M, π) , the formal Poisson structure can be interpreted as a formal deformation of the structure π by setting $\pi_0 = \pi$; the requirement (2.56) gives k equations, i.e.

$$\begin{aligned} [\pi, \pi]_S &= 0, & \text{order } 0 \\ [\pi, \pi_1]_S + [\pi_1, \pi]_S &= 0, & \text{order } 1 \end{aligned} \quad (2.57)$$

and generally, at order $k \geq 2$

$$[\pi, \pi_k]_S = -\frac{1}{2} \sum_{l=1}^{k-1} [\pi_l, \pi_{k-l}]_S. \quad (2.58)$$

A formal Poisson structure on M induces a Lie bracket on $C^\infty(M)[[t]]$ by

$$\{f, g\}_t := \sum_{n=0}^{\infty} t^n \sum_{\substack{i,j,k=0 \\ i+j+k=n}}^n \pi_i(df_j, dg_k), \quad (2.59)$$

where

$$f = \sum_{j=0}^{\infty} t^j f_j \quad \text{and} \quad g = \sum_{k=0}^{\infty} t^k g_k. \quad (2.60)$$

We recall that the gauge group on the set of Poisson structures is given by the diffeomorphisms on M and the action is given by

$$\pi_\phi = \phi_* \pi. \quad (2.61)$$

To extend this action to the set of formal Poisson structures we consider *paths* of formal diffeomorphisms of M which start at the identity Id_M diffeomorphism. More explicitly, consider the one-parameter group of diffeomorphisms ϕ_t on M with $\phi_0 = \text{Id}_M$. Given a Poisson structure π on M , ϕ_t defines a *deformed* Poisson structure by

$$\pi_t = (\phi_t)_* \pi. \quad (2.62)$$

Using Eq. (2.51) we can find the formal version of π_t . Since ϕ_t is the flow of a vector field X on M , we have

$$\frac{d(\phi_t)_*}{dt} = \mathcal{L}_X(\phi_t)_* = (\phi_t)_* \mathcal{L}_X, \quad (2.63)$$

for any t . It follows that

$$\left. \frac{d^n}{dt^n} \right|_{t=0} \pi_t = \left. \frac{d^n}{dt^n} \right|_{t=0} (\phi_t)_* \pi = (\mathcal{L}_X)^n \pi. \quad (2.64)$$

Thus, using Eq. (2.51), the formal power series of π_t is given by

$$\hat{\pi}_t = \sum_{n=0}^{\infty} \left. \frac{d^n}{dt^n} \right|_{t=0} \pi_t = \pi + t\mathcal{L}_X\pi + \frac{t^2}{2}(\mathcal{L}_X)^2\pi + \cdots =: \exp(t\mathcal{L}_X)\pi. \quad (2.65)$$

In other words, the gauge group is given by the formal diffeomorphism $\phi_t = \exp(t\mathcal{L}_X)$. Notice that the structure of a group is given by the formula (BCH):

$$\exp(tX) \cdot \exp(tY) := \exp\left(tX + tY + \frac{1}{2}t[X, Y] + \cdots\right). \quad (2.66)$$

We can generalize the above discussion to the case in which X is a formal vector field and we can define the formal diffeomorphism on M as a $\mathbb{R}[[t]]$ -linear map $\phi_t : \mathfrak{X}^k(M)[[t]] \rightarrow \mathfrak{X}^k(M)[[t]]$ of the form $\phi_t = \exp(\mathcal{L}_X)$ with $X \in t\mathfrak{X}^1(M)[[t]]$. Finally we can define the equivalence class of formal Poisson structures as follows

Definition 2.6 Two formal Poisson structures π_t and $\tilde{\pi}_t$ are said to be equivalent if there exists a formal diffeomorphism such that

$$\pi_t = \exp(t\mathcal{L}_X)\tilde{\pi}_t. \quad (2.67)$$

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