Preface

Historical Remarks: Mathematics and Physics

Physics, as we know it, began in the sixteenth and seventeenth century, when in the study of natural phenomena empirical observation and mathematical modeling were for the first time systematically and successfully combined. This is exemplified in the person of Galilei, and even more so in Newton, who laid the foundations of our picture of the physical world and who was equally great as a mathematician and as an empirical observer and investigator.

For a long time mathematics and physics formed in an obvious way a single integrated subject. Think of Archimedes, Newton, Lagrange, Gauss, more recently Riemann, Cartan, Poincaré, Hilbert, von Neumann, Weyl, Birkhoff and many others. Lorentz, the great Dutch physicist, was offered a chair in mathematics at the university of Utrecht, simultaneously with a chair in physics at a second Dutch university. He chose the latter and became in 1878 Professor of Theoretical Physics at Leiden university, the first in this subject in Europe.

All this is a thing of the past. From the 1950s onward physics and mathematics parted company, or rather mathematics underwent a drastic change in the way it was formulated, largely due to the Bourbaki movement. It became more abstract, more "formal". In this respect, it should be noted that none of the great Bourbaki mathematicians, Weil, Dieudonné, Grothendieck, for example, had the slightest interest in physics. Another cause was the growing specialization in all of science and, more recently, the increasing publication pressure leading to much narrowly focused short-term research. The language of mathematics is now very different from that of physics.

In the gap between physics and mathematics, mathematical physics as a distinct discipline came into being. Journals were established: Journal of Mathematical Physics (1960), Communications in Mathematical Physics (1965), Letters in Mathematical Physics (1975). Separate conferences were held. The International Association of Mathematical Physics was founded in 1976. All this was and still is very welcome, not because it might lead to a new specialization, which would not be a good thing, but as bridges between mathematics and physics. This book is a modest attempt to contribute to this.

Notwithstanding the present distance between the fields, the interaction between mathematics and physics remains of great importance. Modern theoretical physics cannot exist without advanced mathematics. On the other hand, many ideas in mathematics still have their origin in physics—often in a heuristic form.

The importance of the connection between mathematics and physics is no longer reflected in the curriculum of most universities—certainly not in the Dutch universities. Physics students have to learn a great deal of standard mathematics in their first and second year, mainly calculus and linear algebra, but they are not exposed to the more advanced parts of modern mathematics; its abstract language remains strange in spirit to them, even though they sometimes pick up and learn to use methods based on it. Mathematics, on the other hand, is taught as a self-contained subject, which can be studied for its own sake, without any reference to, or knowledge of physics. Words like "classical mechanics" or "quantum mechanics" have no meaning for most present-day mathematics graduates. Moreover, after a few years of training in rigorously formulated mathematics they will find the loose language of standard physics textbooks very hard to understand.

Comparable Books

There has always been a strange asymmetry. Numerous books have been written in the past, explaining to physicists advanced topics from mathematics, such as functional analysis, differential geometry, Lie groups and Lie algebras, and algebraic topology. There has been much less in the other direction; books on physics written for mathematicians are relatively rare, even though one would think that there is a need for such books.

However, in the last decade a number of books of this sort have appeared. There is one book that may not have been written with this intent, but it nevertheless clearly stands out in this field. It is Roger Penrose's *The Road to Reality* [1], an amazing book which gives in almost 1,100 pages a wonderful overview of the physical world seen through mathematical eyes. The books that I am about to mention, as well as my own book, can in a certain sense all be seen as providing additional material or more detailed or slightly alternative versions of subjects or aspects discussed in the book of Penrose.

Books that have recently appeared:

– L.A. Takhtajan (2008)
  Quantum Mechanics for Mathematicians [2].
– L.D. Faddeev, O.A. Yakubovskii (2009)
  Lectures on Quantum Mechanics for Mathematics Students [3].
– K. Hannabuss (1997)
  An Introduction to Quantum Theory [4].

– F. Strocchi (2005)
An Introduction to the Mathematical Structure of Quantum Mechanics [5].
– G. Teschl (2009)
Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators [6].
– Jonathan Dimock (2011)
Quantum Mechanics and Quantum Field Theory. A Mathematical Primer [7].
– Brian C. Hall (2013)
Quantum Theory for Mathematicians [8].

All these books have some overlap with each other and also with my book. Let me give here the principal points in which my book is different from the books in this list.

- One of the main ideas put forward in my book is that, by using for the description of a physical system the algebra of its observables as basic notion, one can give a uniform formulation of both classical and quantum physics. The algebra of classical observables is commutative, that of quantum observables noncommutative. The states of a system are in both cases the positive normalized linear functionals on the algebra of observables. This idea is also a basis for [3, 5]. Book [3] expresses it in an elegant but very general manner; it is, however, a short book with not much explicit mathematical detail. Book [5] has more such details, but the mathematical framework used for this basic idea is much too narrow. My discussion in Chap. 12 has most of the mathematical details, as far as they are known in the literature at the present time. I, moreover, apply this algebraic framework in Chap. 13 to connecting the various approaches to quantization.
- Most modern applications of quantum theory depend on statistical quantum physics. In [4, 7] there are short and excellent sections on this, but there is nothing in the other books. My book gives in Chap. 11 an extensive discussion of quantum statistical physics, after a review of classical statistical physics in Chap. 10.
- Book [2] describes the mathematics of quantum theory. A superb book, advanced, too difficult even for the average graduate student in mathematics. It explains the mathematics, but not the way it is used in quantum theory. Book [6], another excellent book, gives an account of the mathematics of the Schrödinger equation, an important special topic in quantum theory. My book gives, somewhat different from [2, 6], quantum theory as seen through mathematical eyes, but still as a complete physical theory.
- "Quantization", is the nonunique procedure which, starting from a given classical theory, constructs a corresponding quantum theory. Historically, there is, for example, Born-Jordan quantization and Weyl quantization, in principle different, but for which at present no experimental situations exist that distinguish between the two. Nevertheless, the theoretical problem of the nonuniqueness of quantization remains.

  A specific quantum theory may be seen as a deformation of a classical theory, with Planck's $\hbar$ as deformation parameter. Note that $\hbar$ is a constant of nature, but

it has a dimension and takes different numerical values for different unit systems. For a macroscopic system of units the value of $\hbar$ is vanishingly small: the classical limit. See Sect. 1.4.4 for a few remarks on dimensions in physics. Deformation quantization has been studied along various lines. There is for instance strict deformation quantization (Marc Rieffel) and formal deformation quantization (Moshe Flato et al.). Strangely enough there is almost no communication between these different schools. None of the books mentioned pays attention to the general problem of quantization, its nonuniqueness and the relation between the different approaches. My Chapter 13 is devoted to all this.
- Almost immediately after the beginning of quantum mechanics in the 1920s, work began on a relativistic version of the theory, culminating in Dirac's great 1928 paper. See Chap. 15, Ref. [6]. It has since then developed into relativistic quantum field theory, today the main theoretical tool for elementary particle physics, very successful, although many interesting and difficult mathematical problems remain. None of the books mentioned above, except [7], has anything on relativistic quantum theory. In my book, Chap. 15 is devoted to relativistic quantum mechanics and Chap. 16 gives a brief introduction to relativistic quantum field theory and elementary particle physics.
- The approach to quantum theory in my book is "axiomatic", which means that I first give the underlying mathematical structure and then the interpretation of the mathematical notions in physical terms, supported by and enlivened with explicit examples. This is the opposite of the historical direction followed by the more traditional textbooks. Most modern books, such as mentioned in my list of references, do the contrary, and present the theory as it is now, which is good, in particular because they do this very well, but they leave out any sort of historical context, refer only to other modern textbooks, never mention original sources. The history of quantum theory is, however, interesting as an intellectual background. Therefore, I believe that some knowledge of it should be a part of a general physics education. For this reason, I supply in my book more historical details than most of the books mentioned.

Aims of this Book: Potential Readers

The writing of this book was inspired by the general observation that the great theories of modern physics have simple and transparent underlying mathematical structures—usually not emphasized in standard physics textbooks—which makes it easy for mathematicians to understand their basic features. Someone who is familiar with modern differential geometry, in particular Riemannian geometry, will easily grasp the essentials of general relativity. The same is true for familiarity with Hilbert space theory in understanding quantum theory.

This determines, up to a point, the ideal readership for this book. It is a text book on quantum theory, an important topic from modern physics, meant in the first place for advanced undergraduate or graduate students in mathematics, interested in modern physics, and in the second place for physics students with an interest in the mathematical background of physics, unhappy with the level of rigor in standard physics courses. More generally, it should be a useful book for all mathematicians interested in—and sometimes puzzled by—modern physics, and all physicists in search of more mathematical precision in the basic concepts of their field.

Courses that might be given on the basis of this book could be:

– A mathematical approach to quantum theory.
– Quantum theory for mathematicians.

On my book together with one or more other books;

– Modern mathematical physics.
– Modern physics for mathematicians.

New fundamental physical theories are usually not derived from first principles or in a straight manner from earlier theories. Instead they arise as a complex of suggestions, from physical intuition and experiments, mathematical hypotheses, leading finally to a system of precisely formulated postulates, the validity of these guaranteed by the experimental success of the theory that it describes. At the end of this we have the theory in its best possible form, a clear mathematical model, based on a few assumptions, “axioms”, together with a physical interpretation, which leads to statements that can be tested experimentally. In this I am much in sympathy with Dirac’s point of view that true understanding of a physical theory comes from understanding the beauty of its basic mathematical structure. This is also one of the underlying themes of this book. See for a quote of Dirac on this [9], p. xv, and for a lecture that he gave on the relation of physics and mathematics [10].

Structure of the Book

The main body of the book consists of two parts:

1. **Part I: The Main Text**. This contains the main story of the book, in 17 chapters, from Chap. 1 “Introduction”, to Chap. 17 “Concluding Remarks”.
2. **Part II: Supplementary Material**. This consists of a series of chapters with supplementary material. They could have been called appendices, but are in fact called chapters, and run from Supp. Chap. 18 “Topology” to Supp. Chap. 27 “Algebras, States, Representations”.

After this comes what in Book Trade jargon is called “back matter”, the Subject Index, a List of Authors Cited, and an Index of Persons.

The dual readership that I have in mind—both mathematicians and physicists—determines, for better or worse, the structure and format of this book. Readers with a mathematical background or with a background in physics will reach the core material of the book coming from opposite directions. The series of supplementary chapters—mainly mathematical—contain much that is already familiar to mathematicians. But modern mathematicians are specialists; a differential geometer, for example, may not have all necessary details of functional analysis instantly available. So the Supp. Chaps. 18–27 may serve for them as reminders and also will serve to establish the notation and terminology. For a reader from physics on the other hand, the mathematical material in these chapters may be not enough. However, the general ideas there, are more important than the details. In any case, in the supplementary chapters, most of the important mathematical concepts are first introduced in an intuitive manner, and then formulated in an appropriately rigorous and precise way. It is, moreover, my experience that the physics students interested in the mathematical aspects of physics are usually bright and quick learners, who can be expected to find their way using the references that I give. Note, finally, that these supplementary chapters also make the book reasonably self-contained.

Summary of the Contents

The book begins in Chap. 1 with an introduction that gives a very brief overview of present-day physics from a historical point of view.

Quantum physics is more fundamental than classical physics, and therefore does not rely—in principle at least—on concepts of classical physics. In practice, however, almost all quantum theoretical models have been suggested by corresponding classical models. This is in particular true for quantum mechanics, which from its beginning was conceived as obtained by "quantization" of classical mechanics in its Hamiltonian formulation. This origin is still clearly visible in the structure and the terminology of modern quantum mechanics. For this reason Chap. 2 is devoted to an exposition of classical mechanics, with some additional justification for giving this classical subject so much space in Sect. 2.1. Maybe at least the first half of it should be read before proceeding to the chapters on quantum theory that follow. The necessary mathematics can be found in Supp. Chap. 20 "Manifolds".

Chapter 3 is one of the central chapters of this book. It discusses the general principles of quantum theory. Its emergence against a certain historical background and its subsequent evolution is sketched in Sects. 3.1–3.3. After some general remarks in Sect. 3.4, a preview is given in Sect. 3.5 of the modern axiomatic formulation of quantum theory, based on the mathematical ideas of von Neumann. This is what will be called the first level of the axiomatization of quantum theory, introducing the notions of Hilbert space vectors as "pure" quantum states and self-adjoint operators as observables. After this more in detail the notions of states and observables in Sects. 3.6 and 3.7, time evolution in Sect. 3.8, and finally symmetries in Sect. 3.9. A second and a third level of axiomatization will appear in later chapters. The mathematical structure of quantum theory is explained in Supp. Chap. 21 "Functional Analysis: Hilbert Space". Helpful for this are also Supp. Chap. 18 "Topology", Supp. Chap. 19 "Measure and Integral", and for probabilistic aspects Supp. Chap. 22 "Probability Theory".

Next, the quantum mechanical description of a single particle moving in a potential is formulated, with Heisenberg's uncertainty relation as important theme in Chap. 4, and the behavior of wave packets describing the motion of a particle in Chap. 5. Chapter 6 treats the particular case of the one dimensional harmonic oscillator, the simplest and at the same time one of the most important nontrivial basic examples of quantum mechanics, the idea of which appears as a first approximation in a wide range of realistic physical situations. The hydrogen atom, and more generally, a particle in a centrally symmetric potential, the discussion of which led historically to quantum mechanics, is discussed in Chap. 7. It provides a good example of the application of symmetry principles in quantum theory. A further typically nonclassical addition to atomic physics is the notion of *spin*. This appears in Chap. 8.

Chapter 9 discusses many-particle systems, in particular systems of identical particles in which typical quantum phenomena appear, such as the Pauli exclusion principle. The second half of this chapter is devoted to the so-called Fock space formalism, or—as it is often called, somewhat unfortunately—"second quantization", a powerful and elegant way—especially in its heuristic form—to describe simultaneously systems of arbitrary many identical particles. Mathematical aspects of the Fock space formalism are explained in Supp. Chap. 23 "Tensor Products".

In the Chaps. 10 and 11 statistical physics is discussed. One of the main themes that is emphasized throughout this book, is that quantum theory is in an essential way a *probabilistic* theory. Therefore 'statistical' in the title of Chap. 11 may seem a bit surprising. The explanation lies in the fact that in the quantum theoretical description of large systems consisting of very many small subsystems there is an additional layer of statistical behavior, very analogous to, but different from what one has in the classical probabilistic description of large systems. It is hard to understand where the basic features of the formalism of quantum statistical physics come from if one does not know something of the classical theory. This will, therefore, be reviewed first in Chap. 10. After that, Chap. 11 will give the general theoretical framework together with some interesting applications of quantum statistical physics. Quantum statistical physics means, that after the first level of quantum theory with Hilbert space vectors as "pure" states, explained in Chap. 3, a second level appears, with a generalized system of axioms, and density operators in an ambient Hilbert space as "mixed" quantum states. For these chapters Supp. Chap. 22 "Probability Theory" is again useful.

Chapter 12 develops a central theme of this book. It presents an algebraic formalism, in which both classical and quantum physical systems fit in a natural manner. First the mathematical framework, then examples of concrete physical systems. This leads to a third level of the axiomatization of quantum theory, algebraic quantum theory, with as primary object an abstract algebra of observables, then states as linear functionals on this algebra, and finally a Hilbert space in which this algebra is represented as operator algebra, a representation which is dependent on the choice of the state functional. In the last section of this chapter, there is a comparison between von Neumann's operator formalism and the lattice formulation of quantum theory by Birkhoff, with the connection of these two by Gleason's theorem. The necessary material on algebras and their representations can be found in Supp. Chap. 27 "Algebras, States, Representations".

Chapter 13 treats quantization in the spirit mentioned above. I try in particular to give an overall picture of various approaches and procedures. In Chap. 14 scattering theory is discussed; the time-dependent and the time-independent formalism, together with the relation between the two. This chapter also contains a section on perturbation theory. Chapters 15 and 16 are devoted to relativistic quantum theory. For both chapters Supp. Chap. 21 "Functional Analysis: Hilbert Space" will be useful. Chapter 15 describes the not wholly successful attempts to find relativistic single particle wave equations, starting from nonrelativistic Schrödinger quantum mechanics, leading finally to relativistic quantum field theory, introduced in Chap. 16. In this chapter, there is also a brief review of elementary particle physics, its history, and its culmination in the so-called Standard Model. Chapter 17 contains concluding remarks, in particular some highly personal remarks on the present and future state of physics—with a very short review of the problems of modern cosmology and of the fascinating subject that can be characterized by the catchword "Einstein-Podolsky-Rosen and All That"—on the difference between physics and science fiction, and finally on sociological aspects of modern physics.

For everything that has to do with symmetry Supp. Chap. 24 "Lie Groups and Lie Algebras" will be useful.

Most of the supplementary chapters explain mathematics, in the first place to physicists. There are two exceptions, where heuristic notions that are very popular in physics textbooks are explained to mathematicians. Supp. Chap. 25 "Generalized Functions" (i.e., Dirac's δ-function and its derivatives), Supp. Chap. 26 Dirac's Bra-Ket notation.

Of course, studying a chapter from the main body of the text should not start with reading the chapter that gives supplementary material, basic as this may be for understanding; it should be used as much as possible alongside and in parallel with the main text.

The Reference System

There is an extensive system of cross-references, between the chapters and between the chapters and the appendices. Three types of internal references can be distinguished.

1. Almost all chapters have a section with numbered references to books and articles relevant to the chapter, denoted as [1, 2], etc. From within a chapter these are cited as, for instance, See Ref. [2].
2. From now on all the chapters from Part I, together with their sections and subsections, will be denoted with a prefix "Sect." (without parentheses), for instance Sect. 16.9.2 for Subsection 16.9.2 of Chapter 16 . References from one chapter to a second chapter, pointing out, for instance, to a similar situation, will have the form "See for a similar situation Sect. 2.6.1".
3. Chapters, together with their sections and subsections, from Part II will be denoted by the number, preceded by "Supp", for example, as Supp. Sect. 21.9.2.

A reference to a mathematical explanation might read as "For an explanation of this, see Supp. Sect. 21.9.2".
Examples, problems, theorems, etc., will be denoted with parentheses, for instance as (2.3.2,a) **Problem**, (20.5.2,a) **Theorem**.

Various Remarks

1. Most of the chapters and some of the appendices contain exercises—problems to test the understanding of the reader. They may also provide additional information.
2. *For the readers from physics*: The phrase "if and only if" is often abbreviated as "iff", as is fairly common in mathematical texts. Various standard symbols are used in the mathematical formulas, such as:
   - $\in$: "is an element of", and $\notin$: "is not an element of"
   - $\subset$: "is a subset of", and $\supset$: "contains"
   - $\prec$: "precedes"
   - $\leq$: "less or equal than", and $\geq$: "greater or equal than"
   - $\neq$: "is not equal to"
   - $\equiv$: "is equivalent with", and $\sim$: "is similar to"
   - $\forall$: "for all", and $\exists$: "there exists"
   - $\rightarrow$: "maps to (sets to sets)"
   - $\mapsto$: "maps to (elements of sets to elements of sets)"
   - $\Rightarrow$: "implies", and $\Leftarrow$: "is implied by"

   Etc.
3. Much useful material can be found on the internet. Whenever available I give web addresses for items in the reference list, even though some of these may be ephemeral. For almost any notion from mathematics and physics there exist Wikipedia articles, usually extensive and competent, even though some double-checking with similar sources such that of "mathworld.wolfram" and "nLab" is recommended.

   Also useful although somewhat dated is the *Encyclopaedia of Mathematics*, a multi-volume work, translated from the Russian, originally published by Kluwer and made now freely available in separate web pages, with as starting page http://eom.springer.de/. Authoritative articles on the philosophical background of modern physics are available from the on-line *Stanford Encyclopedia of Philosophy*, with its home page at : http://plato.stanford.edu/.

   There is a website, *HyperPhysics*, consisting of a large number of well-integrated and cross-referenced small pages, with very clear texts and pictures, covering all aspects of standard quantum mechanics. To be found at: http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html.

   A very useful (printed) general source for information on mathematics, from elementary to advanced, is the *Encyclopedic Dictionary of Mathematics* [11].

Internet sources are ephemere. All the links mentioned in this book were still accessible at the time of writing.

References

1. Roger Penrose.: The Road to Reality. A Complete Guide to the Physical Universe. BCA (2004) (Penrose has written a series of best-selling books. This book has probably a smaller readership; but it may be considered to be his 'magnum opus'.)
2. Takhtajan, L.A.: Quantum Mechanics for Mathematicians. American Mathematical Society (2008) (A tersely written advanced mathematical textbook. After reading it most mathematicians will still not be able to read physics textbooks on quantum mechanics.)
3. Faddeev, L.D., Yakubovskii, O.A.: Lectures on Quantum Mechanics for Mathematics Students.Translated from the Russian. American Mathematical Society (2009) (Written in a clear and engaging style, with a perfect balance between physics and mathematics, and emphasizing an algebraic point of view that unifies classical and quantum mechanics.)
4. Hannabuss, K.: An Introduction to Quantum Theory. Oxford (1997) (A slightly older book based, as the author says, on a course for mathematics students. It is not written with mathematical concepts as a starting point from which the physical theory can be better understood, something that I feel is particularly important in quantum theory. It seems therefore more suitable for physics students who have a more than average interest in mathematical rigor.) (Nevertheless, a solid, very effective, clear and competent book.)
5. Strocchi, F.: An Introduction to the Mathematical Structure of Quantum Mechanics. World Scientific (2005) (A short series of lectures, containing many interesting details. The mathematical framework that the author employs is unsatisfactory.)
6. Teschl, G.: Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators. American Mathematical Society (2009) (A book that is limited to the discussion of the mathematical basis of what essentially is the nonrelativistic quantum mechanics of a single particle in an external field. It is thorough and written in a clear and precise style.)
7. Dimock, J.: Quantum Mechanics and Quantum Field Theory. A Mathematical Primer. Cambridge University Press (2011) (This recent book is written in a similar style as my book, but without the algebraic point of view. It is less comprehensive in its coverage of important aspects of quantum theory. Nevertheless, a fine book, particular in its discussion of mathematical aspects.)
8. Hall, B.C.: Quantum Theory for Mathematicians. Springer (2013) (This even more recent book has the same purpose as my book: explaining quantum theory to a mathematical readership. This implies a certain amount of overlap. There are, however, important differences. The main one is that one of the main themes of my book is to see quantum theory as a whole in an algebraic context, in which it can be compared with a commutative algebraic formulation of classical physics, in the same way as this is briefly but convincingly done in 3. Any sort of discussion of algebraic aspects of quantum theory is absent from Hall's book. But in its format it is an excellent book, a clearly written, straightforward although somewhat traditional account of nonrelativistic quantum mechanics in a mathematical presentation.)
9. Farmelo, G.: It Must be Beautiful: Great Equations of Modern Science. Granta Books, New edition (2003) (During a seminar in Moscow University in 1955, when Dirac was asked to summarize his philosophy of physics, he wrote at the blackboard in capital letters : Physical laws should have mathematical beauty. This piece of blackboard is still on display.)
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11. Ito, K.: Encyclopedic Dictionary of Mathematics. 2nd edn. Translated from the Japanese. MIT Press 1977, paperback edition (1993) (This two-volume work can be strongly recommended. When one is new to any topic in mathematics this is the place where to begin. Too many references in the first edition (1977) were to papers in Japanese, but the second edition is much better in this respect.)

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