

Chapter 2

Mathematical Preliminaries

We begin by formalizing various notions of visibility and what is visible to an observer in mathematical terms. These notions serve as a basis for the subsequent development.

2.1 Objects Under Observation

Let the *world space* $\mathcal{W}$ be a specified subset of the n -dimensional Euclidean space $\mathbb{R}^n$, $n \in \{2, 3\}$ in which a vector x is an ordered n -tuple $(x_1, \dots, x_n)$ of real numbers x_i , $i = 1, \dots, n$. The scalar product between two vectors x and x' in $\mathbb{R}^n$ is denoted by $\langle x, x' \rangle$. The representation of a vector x with respect to a given bases $B = \{e_1, \dots, e_n\}$ for $\mathbb{R}^n$ is denoted by the column vector $[x]$. When ambiguity does not occur, the bracket notation $[\cdot]$ is dropped for brevity. We assume that an *object under observation* $\mathcal{O}$ is a compact (closed and bounded) connected subset of $\mathcal{W}$ with boundary $\partial\mathcal{O}$. The observation of the object $\mathcal{O}$ is made from *point-observers* or *observation points* on an *observation platform* $\mathcal{P}$ corresponding to a given subset of $\mathcal{W}$. In what follows, we shall focus our attention mainly on the practically important case where $\mathcal{P} \subset \overline{\mathcal{O}^c}$ (the closure of the complement of $\mathcal{O}$ relative to $\mathcal{W}$). We assume that $\mathcal{P}$ is *transparent* in the sense that it can be penetrated by the line segment or an arc connecting an observation point in $\mathcal{P}$ and an observed point in $\mathcal{O}$. For *opaque* or non-transparent objects, only the boundary $\partial\mathcal{O}$ can be observed by point-observers in $\mathcal{P}$. The difficulty associated with the visibility problems to be considered generally depends on the dimensions and geometric properties of $\mathcal{O}$ and $\mathcal{P}$. A few cases of special interest are given in the sequel.

Case (i) $\mathcal{W} = \mathbb{R}^2$

- (a) $\mathcal{O}$ is a plane curve which can be represented by $G_f = \{(x, f(x)) \in \mathbb{R}^2 : x \in \Omega\}$, the graph of a real-valued C_m -function $f = f(x)$ defined on a compact interval $\Omega \subset \mathbb{R}$ (i.e. $f(\cdot) \in C_m(\mathbb{R}; \Omega)$, $m \geq 1$, the space of all real-valued functions having continuous derivatives on Ω up to the m th order). The observations of the

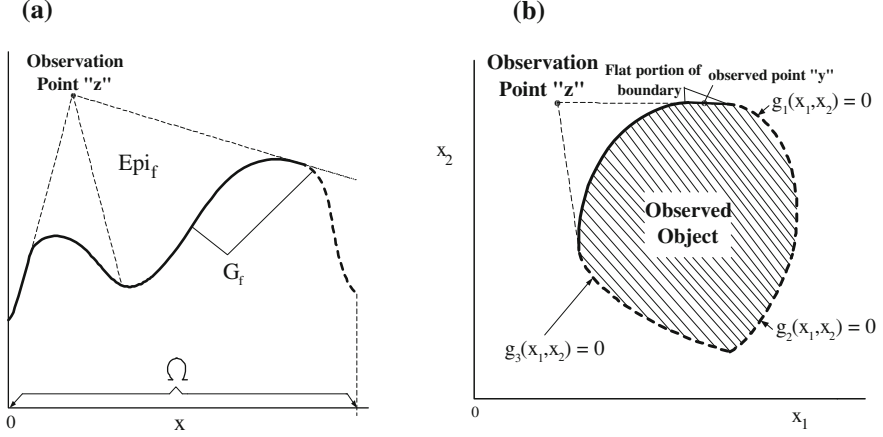


Fig. 2.1 Cases (i)(a) and (i)(b). The solid and dashed curves correspond to points of the object that are visible and invisible from the observation point z respectively

curve are made from points $z = (x, w) \in \mathcal{P} \subset \text{Epi}_f \stackrel{\text{def}}{=} \{(x', w') \in \Omega \times \mathbb{R} : w' \geq f(x')\}$, the *epigraph* of f (See Fig. 2.1a). We may also consider observations from points below the curve G_f , i.e. $z \in \{(x', w') \in \Omega \times \mathbb{R} : w' < f(x')\}$.

- (b) $\mathcal{O}$ is an opaque object represented by a compact connected subset of $\mathcal{W}$. It can be described by $\bigcap_{k \in \{1, \dots, K\}} \{x \in \mathcal{W} : g_k(x) \leq 0\}$, and has interior points and a piecewise smooth boundary $\partial\mathcal{O}$, where the g_k 's are specified functions in $C_m(\mathbb{R}; \mathcal{W})$, $m \geq 1$. The point-observers or observation points lie in $\mathcal{O}^c$ (See Fig. 2.1b).

Case (ii) $\mathcal{W} = \mathbb{R}^3$

- (a) $\mathcal{O}$ is an opaque smooth surface described by G_f , the graph of a real-valued C_m -function ($m \geq 1$) $f = f(x)$, $x = (x_1, x_2) \in \Omega$, a compact connected subset of $\mathbb{R}^2$ (See Fig. 2.2). The observation points $z = (x, w)$ are located in $\text{Epi}_f \subset \mathbb{R}^3$.
- (b) $\mathcal{O}$ is an opaque solid body represented by a nonempty compact connected subset of $\mathbb{R}^3$ with interior points and a piecewise smooth boundary $\partial\mathcal{O}$ which may be expressed as a level set of a real-valued continuous function $g = g(x)$ defined on $\mathbb{R}^3$, i.e. $\mathcal{O} = \{x \in \mathbb{R}^3 : g(x) = \alpha\}$ or $g^{-1}(\alpha)$, where $\alpha \in \mathbb{R}$. As in Case (i)(b), the object $\mathcal{O}$ may also be described by $\bigcap_{k \in \{1, \dots, K\}} \{x \in \mathcal{W} : g_k(x) \leq 0\}$, where the g_k 's are specified functions in $C_m(\mathbb{R}; \mathcal{W})$. The observation points lie in $\mathcal{O}^c$.

In physical situations, the object under observation in Case (ii)(a) may correspond to the spatial profile of a terrain (e.g. Mars surface) under observation by a robot camera. Thus the region below the surface (i.e. the complement of Epi_f relative to $\Omega \times \mathbb{R}$) is irrelevant. The object under observation in Case (ii)(b) may correspond to an asteroid as described in Chap. 1, a nano-particle, or a product under inspection in

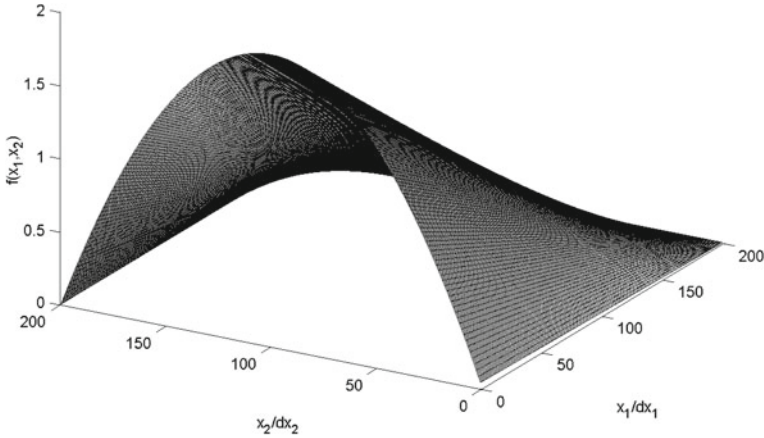


Fig. 2.2 The object $\mathcal{O}$ under observation is an opaque surface in $\mathbb{R}^3$ described by G_f . The observation points are located above the surface or in Epi_f .

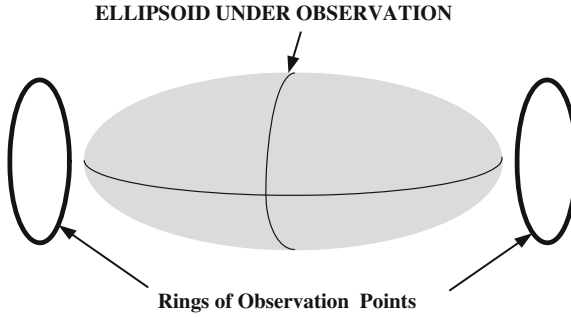


Fig. 2.3 $\partial\mathcal{O}$ is an ellipsoid, and $\mathcal{P}$ consists of two rings

a manufacturing plant. Figure 2.3 shows an example where $\partial\mathcal{O}$ is an ellipsoid, and the observation platform $\mathcal{P}$ is composed of two rings.

Case (iii) $\mathcal{W} = \mathbb{R}^n$, $n = 2, 3$

Cases (i) and (ii) do not involve the time of observation. Moreover, the shapes of the objects $\mathcal{O}$ under observation are assumed to be time-invariant over the observation period. This assumption may be invalid in many physical situations. For example, if $\mathcal{O}$ is a closed bounded subset of Earth's solid surface, its shape may vary significantly over the observation period during an Earth quake. Another example is the heating or cooling of a solid body $\mathcal{O} \subset \mathbb{R}^3$ whose shape may change significantly over the observation period. In these cases, we may characterize the objects under observation by an *one-parameter (time t) family of objects* $\mathcal{O}_t$, i.e. $\mathcal{F}_{\mathcal{O}} = \{\mathcal{O}_t : t \in I_T\}$, where I_T is a specified observation time-interval.

Example 2.1 $\mathcal{F}_O$ is an one-parameter family of 2-dimensional surfaces in $\mathbb{R}^3$ described by $\mathcal{O}_t = \{(x_1, x_2, f(t, x_1, x_2)) \in \mathbb{R}^3 : sf(t, x_1, x_2) = \sin(2\pi t) \sin(\pi x_1) \sin(2\pi x_2); (x_1, x_2) \in \Omega\}$, where $t \in I_T = [0, 10]$ and $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$.

Example 2.2 $\mathcal{F}_O$ is an one-parameter family of pulsating spherical balls in $\mathbb{R}^3$ described by $\mathcal{O}_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq (1 + 0.25 \sin(t))^2\}$, $t \in I_T = [0, 2\pi]$.

2.2 Notions of Visibility [1, 2]

First, we consider objects $\mathcal{O}$ with time-invariant shapes as described in Cases (i) and (ii). We introduce the notion of visibility of a point y on $\partial\mathcal{O}$ from an observation point z , assuming no other objects or obstacles are in the world space $\mathcal{W}$. Unless stated otherwise, the object under observation is assumed to be opaque, and the observation point $z \in \overline{\mathcal{O}^c}$. The visibility of the point under observation is based on line-of-sight from the observation point z .

Definition 2.1 A point $y \in \partial\mathcal{O}$ is *visible* from an observation point $z \in \overline{\mathcal{O}^c}$, if (i) $L(y, z) \subset \mathcal{W}$, and (ii) $L(y, z) \cap \text{int}(\mathcal{O}) = \emptyset$ (the empty set), where $L(y, z)$ is the line segment joining points y and z described by $\{x \in \mathbb{R}^3 : x = \lambda y + (1 - \lambda)z, 0 < \lambda < 1\}$, and $\text{int}(\mathcal{O})$ denotes the interior of $\mathcal{O}$. The set $\mathcal{V}(z)$ of all points $y \in \partial\mathcal{O}$ that are visible from a point $z \in \overline{\mathcal{O}^c}$ is called the *visible set* of z . The complement of $\mathcal{V}(z)$ relative to $\partial\mathcal{O}$ is called the *invisible set* of z .

Condition (ii) implies that $L(y, z)$ does not penetrate into the interior of $\mathcal{O}$, and $L(y, z) \cap \partial\mathcal{O}$ may be nonempty. In the case where diffraction phenomenon occurs, we may replace $L(y, z)$ in Definition 2.1 by a smooth arc (in a given class) connecting the points y and z . Figure 2.1 shows the visible and invisible sets of a point z for observing a plane curve and a compact connected set in $\mathbb{R}^2$ indicated by solid and dashed curves respectively. Note that the boundary of the object in Fig. 2.1b has a flat portion. Thus, for the indicated observation point z and observed point y , $L(y, z) \cap \partial\mathcal{O}$ is nonempty. The visible and invisible sets of an observation point for an opaque solid object in $\mathbb{R}^3$ are illustrated by Fig. 2.4.

Remark 2.1 When a point-observer z has finite viewing aperture such as a camera, we may define the visible set of the point-observer z as

$$\mathcal{V}(z) = \{y \in (\mathcal{C}_z \cap \partial\mathcal{O}) : \lambda z + (1 - \lambda)y \notin \mathcal{O} \text{ for all } \lambda \in]0, 1[\}, \quad (2.1)$$

where $\mathcal{C}_z$ is the *cone of visibility* associated with the point-observer z at its vertex, and has a finite viewing-aperture angle as illustrated in Fig. 2.5. In the case of a camera, its visible set may be enlarged by rotating the camera about the vertex of its viewing-aperture cone. This approach may be useful when the object under observation does not change its shape significantly during the rotation period. Throughout this book,

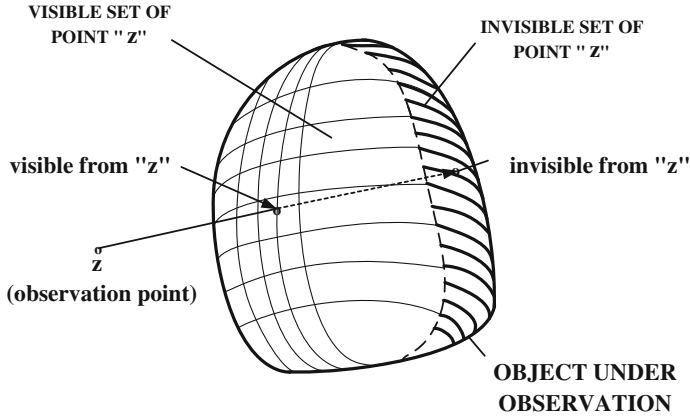
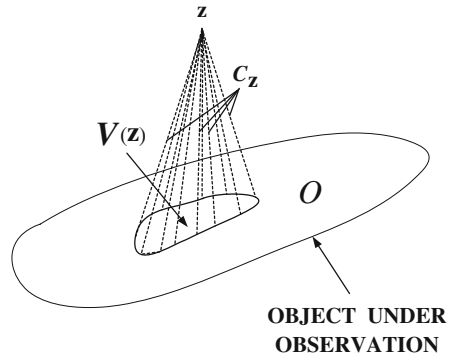


Fig. 2.4 Visible and invisible sets of an observation point for an opaque solid object in $\mathbb{R}^3$

Fig. 2.5 Visible sets of a point-observer with finite viewing aperture angle



unless stated otherwise, no restriction is imposed on the viewing aperture of a point-observer.

Remark 2.2 In Definition 2.1, visibility is defined in terms of line-of-sight observations. In practical situations, observations may be accomplished using more complex devices. For example, the observation of an object $\mathcal{O}$ may be made using a reflective surface R_f as illustrated in Fig. 2.6. Here, a point $y \in \partial\mathcal{O}$ is visible from a point-observer $z \in \overline{\mathcal{O}^c}$, if there exists a ray composed of incident and reflected rays connecting z and y . The direction of the reflected ray is determined by the usual Law of Reflection in optics (i.e. angle of incidence equals the angle of reflection). In this work, we only consider line-of-sight observations.

Figure 2.7 shows the visible sets from two different observation points for Case (ii)(a) in which the observed object is a surface in $\mathbb{R}^3$. We may associate this case with a mountaineer (the observer) climbing up a mountain surface. When the mountaineer is at position “a”, his visible set is limited to only one side of the mountain surface. But when the climber reaches position “b”, his visible set suddenly expands to both

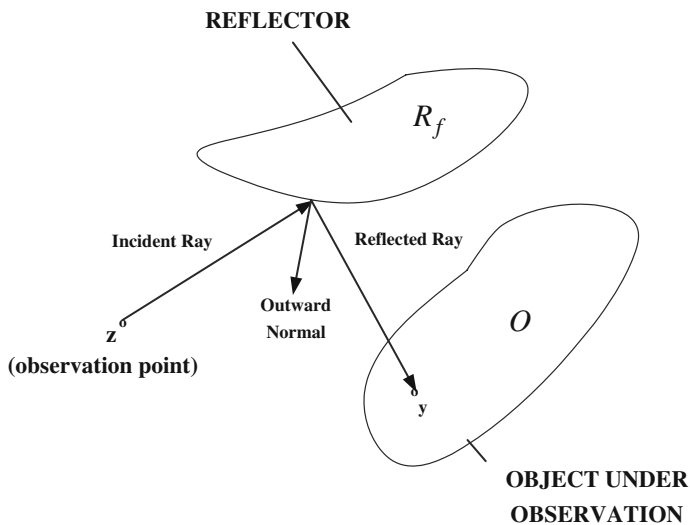


Fig. 2.6 Observation of object O from a point-observer via a reflective surface R_f

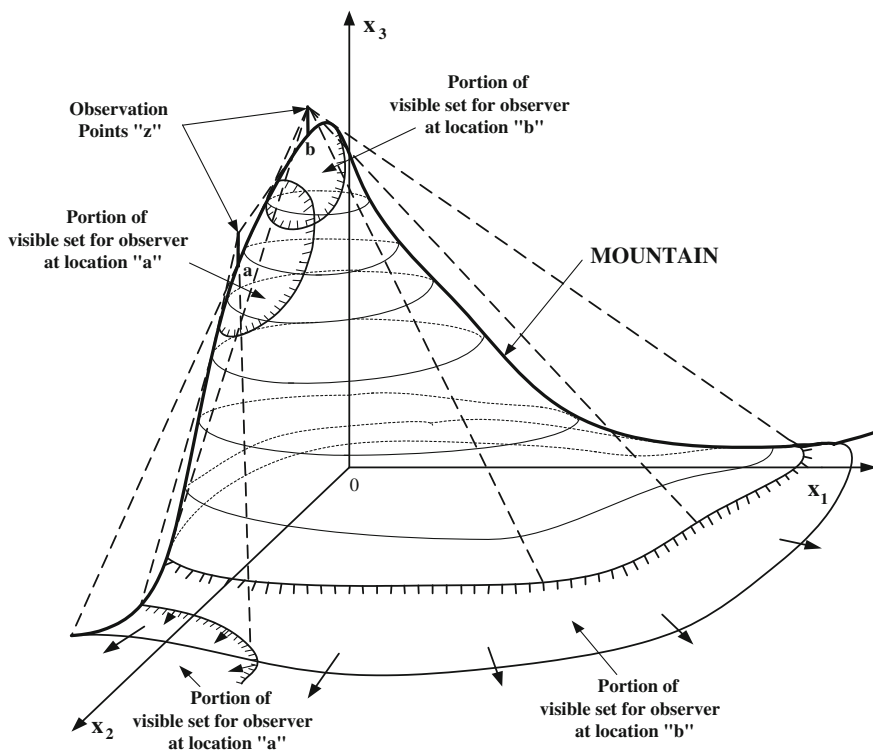


Fig. 2.7 Visible sets of an observer at two different locations "a" and "b"

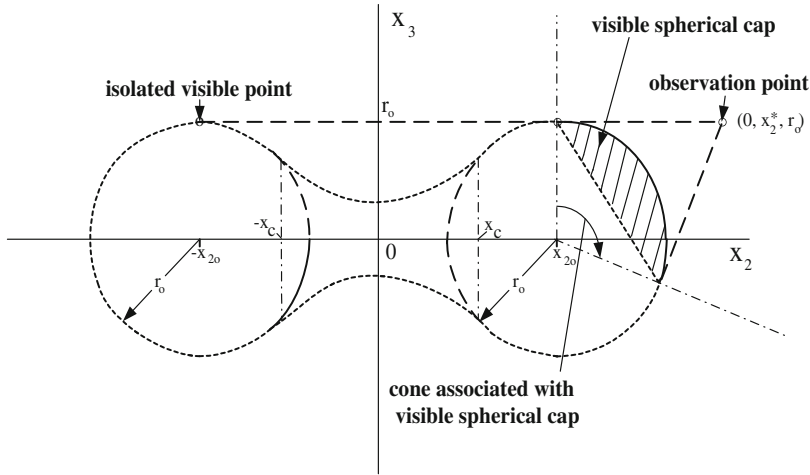


Fig. 2.8 Projection of $\mathcal{V}((0, x_2^*, r_o))$ onto the (x_2, x_3) -plane

sides of the mountain surface as if the world is below his feet (an exhilarating moment experienced by many mountaineers!).

In general, $\mathcal{V}(z)$ may be composed of disconnected sets, isolated points and arcs as illustrated by the following example:

Example 2.3 Let $B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be the orthonormal bases for $\mathbb{R}^3$. A point $x \in \mathbb{R}^3$ can be specified by the ordered triplet (x_1, x_2, x_3) or its representation with respect to B given by the column vector $[x_1, x_2, x_3]^T$. Consider a solid body $\mathcal{O}$ in $\mathbb{R}^3$ whose boundary surface $\partial\mathcal{O}$ is formed by revolving the plane curve $\mathbf{C}$ composed of three circular arcs about the x_2 -axis (see Fig. 2.8), where $\mathbf{C} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0; x_3 = \sqrt{r_o^2 - (x_2 - x_{2o})^2} \text{ if } |x_2| > x_c; x_3 = \sqrt{r_c^2 - x_2^2} + x_{3o} \text{ if } |x_2| \leq x_c\}$, with r_o and $r_c = \sqrt{x_{2o}^2 + x_{3o}^2} - r_o > 0$ being the radii of curvature of the circular arcs, and $x_c = x_{2o}r_c / \sqrt{x_{2o}^2 + x_{3o}^2}$. For the observation point $z = (x_1, x_2, x_3) = (0, x_2^*, r_o)$ with $x_2^* > x_{2o} > 0$, its visible set is given by $\mathcal{V}(z) = \hat{\mathcal{C}} \cup \{(0, -x_{2o}, r_o)\}$, where $\hat{\mathcal{C}}$ is a spherical cap associated with the sphere centered at $x = (0, x_{2o}, 0)$ with radius r_o , described by

$$\begin{aligned} \hat{\mathcal{C}} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + (x_2 - x_{2o})^2 + x_3^2 \\ &= r_o^2; (x_2 - x_{2o}) + r_o x_3 / (x_2^* - x_{2o}) \geq r_o^2 / (x_2^* - x_{2o})\}. \end{aligned} \quad (2.2)$$

When more than one object or obstacles are present in the world space $\mathcal{W}$, there may exist observation points from which no points on $\partial\mathcal{O}$ are visible.

Remark 2.3 In many practical situations, attention may be focused on certain special subsets of the visible sets of the observation points rather than the entire visible sets. These subsets may be characterized by special properties. For example, let $\mathcal{V}(z)$ be the visible set of a point $z \in \mathcal{P}$ for observing a 2D-surface $\mathcal{O} = \{(x, f(x)) \in \mathbb{R}^3 : x \in \Omega \subset \mathbb{R}^2\}$, where f is a real-valued continuous function of $x \in \Omega$. The subsets $\tilde{\mathcal{V}}(z)$ of $\mathcal{V}(z)$ of special interest correspond to certain specified level-sets of $\mathcal{V}(z)$, i.e. $\tilde{\mathcal{V}}(z) = \{(x, f(x)) \in \mathcal{V}(z) : f(x) = c, x \in \Omega\}$, where c is a specified real number. Thus, we may describe this situation by defining a set-valued mapping Υ on $\mathcal{P}$ into $2^{\mathcal{V}(z)}$ such that $z \rightarrow \tilde{\mathcal{V}}(z)$.

Let $n(x)$ denote the outward unit normal at a point $x \in \partial\mathcal{O}$, and $T_x(\partial\mathcal{O})$ the corresponding tangent plane at x . We assume that the boundary of $\mathcal{O}$ is describable by the level set of a smooth function $f = f(x)$ defined on a compact set $\Omega \subset \mathcal{W}$, i.e. $\partial\mathcal{O} = \{x \in \Omega : f(x) = \alpha\}$ for some real number α . Then, the outward unit normal at x is defined by $n(x) = \nabla f(x) / \|\nabla f(x)\|$, where $\nabla f(x)$ denotes the gradient of f at $x \in \partial\mathcal{O}$. A special observation platform is the dilated $\partial\mathcal{O}$ such that at any point $x \in \partial\mathcal{O}$, the corresponding observation point $z \in \mathcal{P}$ is at a given height $h(x) > 0$ along $n(x)$ above x , where $h = h(x)$ is a specified positive continuous function of x defined on $\partial\mathcal{O}$. The corresponding observation platform (denoted hereafter by $\mathcal{P}_h$) is described by $\{z \in \mathcal{W} : z = x + h(x)n(x), x \in \partial\mathcal{O}\}$, a dilation of $\partial\mathcal{O}$ in the directions of the surface outward normals with increment $h(\cdot)$. When h is a positive constant, we have the practically important case of a *constant-height observation platform*. Note that since $\partial\mathcal{O}$ is a compact set in $\mathcal{W}$, so is the observation platform $\mathcal{P}_h$.

Given an object $\mathcal{O}$ in $\mathcal{W}$, and a set $\mathcal{P}$ of observation points z , the visible set of $\mathcal{P}$ (denoted by $\mathcal{V}(\mathcal{P})$) is given by $\bigcup_{z \in \mathcal{P}} \mathcal{V}(z)$.

Definition 2.2 The object $\mathcal{O}$ is said to be *partially visible from* $\mathcal{P}$, if $\mathcal{V}(\mathcal{P})$ is a nonempty proper subset of $\partial\mathcal{O}$. If $\mathcal{V}(\mathcal{P}) = \partial\mathcal{O}$, then $\mathcal{O}$ is said to be *totally visible from* $\mathcal{P}$.

First, consider the case where $\mathcal{P} = \mathcal{O}^c$. A basic question pertaining to total visibility can be posed as follows: Given an object $\mathcal{O} \subset \mathcal{W}$, does there exist an observation point $z \in \mathcal{O}^c$ at a finite distance from $\mathcal{O}$ such that $\mathcal{O}$ is totally visible? If not, is total visibility of $\mathcal{O}$ attainable by a finite set of point-observers $z^{(i)} \in \mathcal{O}^c$, $i = 1, \dots, N$, each located at a finite distance from $\mathcal{O}$? Here, the distance between $z^{(i)}$ and $\mathcal{O}$ is defined by

$$\rho(z^{(i)}, \mathcal{O}) \stackrel{\text{def}}{=} \inf\{\|z^{(i)} - x\| : x \in \mathcal{O}\}, \quad i = 1, \dots, N, \quad (2.3)$$

where $\|\cdot\|$ denotes the Euclidean norm.

The following proposition provides the answer to the foregoing question for observed objects $\mathcal{O}$ corresponding to Cases (i)(a) and (ii)(a):

Proposition 2.1 *For an object $\mathcal{O}$ corresponding to a plane curve or a smooth surface described in Cases (i)(a) or (ii)(a) in Sect. 2.1 respectively, there exists an observation point $z \in \text{Epi}_f$ from which $\mathcal{O}$ is totally visible.*

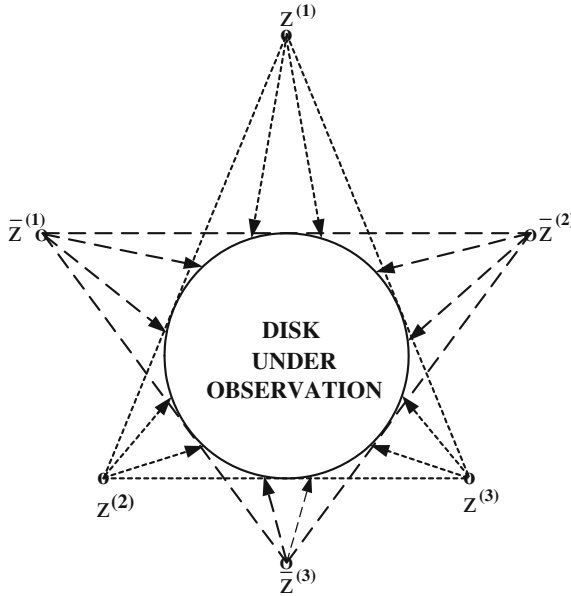


Fig. 2.9 Total visibility of a circular disk is attainable by a set of three distinct observation points located at the vertices of any triangle that inscribes the disk, e.g. $\{z^{(1)}, z^{(2)}, z^{(3)}\}$ and $\{\bar{z}^{(1)}, \bar{z}^{(2)}, \bar{z}^{(3)}\}$

A proof for the above result is included in the proof of Lemma 3.1 in Chap. 3. This result is consistent with intuition that a smooth surface or terrain without folds is totally visible if the observation point is sufficiently high above the surface.

Next, consider Case (i)(b) in Sect. 2.1 where the object $\mathcal{O}$ is described by a compact connected subset of $\mathcal{W} = \mathbb{R}^2$ with interior points and a piecewise smooth boundary $\partial\mathcal{O}$. To fix ideas, let the object $\mathcal{O}$ under observation be a circular disk. Evidently, total visibility of the disk or its circular boundary $\partial\mathcal{O}$ can be attained by three point-observers located at the vertices of any triangle that inscribes the disk (see Fig. 2.9). Moreover, the visible set of a point-observer at a finite distance from the disk is always a proper subset of a boundary semi-circle. Therefore total visibility of a circular disk cannot be attained using less than three point-observers that are at finite distances from the disk. Now, one may ask whether at least three point-observers are necessary for total visibility of any compact simply connected object in $\mathbb{R}^2$. A little reflection leads one to conclude that for some compact connected objects in $\mathbb{R}^2$, only two point-observers are required for total visibility. For example, a n -sided polygon requires only two point-observers at finite distances outside the polygon for total visibility. However, its boundary is piecewise smooth.

Proposition 2.2 *For an object $\mathcal{O}$ described by a compact connected subset of $\mathbb{R}^2$ with interior points and a piecewise smooth boundary described in Cases (i)(b) in Sect. 2.1, at least two point-observers located in $\mathcal{O}^c$ are required for total visibility of $\mathcal{O}$.*

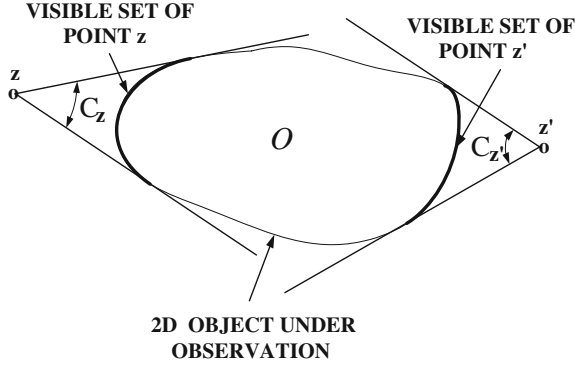


Fig. 2.10 Illustration of at least two point observers are needed for total visibility

Proof Since $\mathcal{O}$ is compact, its diameter $\text{diam}(\mathcal{O}) \stackrel{\text{def}}{=} \max\{\|x - x'\| : x, x' \in \mathcal{O}\}$ is finite. For any observation point $z \in \mathcal{O}^c$, we can construct a cone $\mathcal{C}_z \subset \mathbb{R}^2$ with its vertex at z such that $\mathcal{O} \subset \mathcal{C}_z$ and the boundary lines of $\mathcal{C}_z$ are tangent to $\partial\mathcal{O}$ as illustrated by Fig. 2.10. Clearly, it is impossible to attain total visibility of $\mathcal{O}$ by any single observation point $z \in \mathcal{O}^c$. Thus, we can partition $\partial\mathcal{O}$ into two subsets, $\mathcal{V}(z)$ and $\partial\mathcal{O} - \mathcal{V}(z)$. The total visibility of the latter subset requires at least one other point-observer. $\square$

Proposition 2.3 *Let $\mathcal{O}$ be an object under observation described by a compact connected subset of $\mathbb{R}^n$, $n \in \{1, 2, 3\}$, such that $\partial\mathcal{O}$ is totally visible from a given observation platform $\mathcal{P} \subset \overline{\mathcal{O}^c} \subset \mathbb{R}^n$. Let T be any nonsingular linear mapping from $\mathbb{R}^n$ onto itself. Then, $\partial(T\mathcal{O})$ is totally visible from $T\mathcal{P}$.*

Proof From the assumption that $\partial\mathcal{O}$ is totally visible from a given observation platform $\mathcal{P}$, for any point $y \in \partial\mathcal{O}$, there exists a point $z \in \mathcal{P}$ such that the line segment $\mathbf{L}(z, y) \cap \text{int}(\mathcal{O}) = \emptyset$ (the empty set). Now, for any nonsingular linear transformation T on $\mathbb{R}^n$ onto itself, the image of $\mathbf{L}(z, y)$ under the mapping T is the line segment $\mathbf{L}(Tz, Ty)$ which does not intersect $\text{int}(T\mathcal{O})$. By definition, the point $Ty \in \partial(T\mathcal{O})$ is visible from $Tz \in T\mathcal{P}$. It follows that $\partial(T\mathcal{O})$ is totally visible from $T\mathcal{P}$. $\square$

The significance of Proposition 2.3 is that once the total visibility of an object $\mathcal{O}$ from a given platform $\mathcal{P}$ is established, one can generate families of objects with different shapes via nonsingular linear transformations T such that the pairs $\{T\mathcal{O}, T\mathcal{P}\}$ have the total visibility property.

The foregoing elementary observations suggest the following constructive approach to the determination of point-observers for a special class of simply connected compact objects in $\mathbb{R}^2$ that are related to a circular disk by a continuous radial deformation as illustrated in Fig. 2.11. Note that there exist compact connected objects in $\mathbb{R}^2$ that are not transformable from a circular disk via radial deformations (see Fig. 2.12 for a few examples). For this special class of objects $\mathcal{O}$, we assume that their boundary curves $\partial\mathcal{O}$ can be expressed in the form:

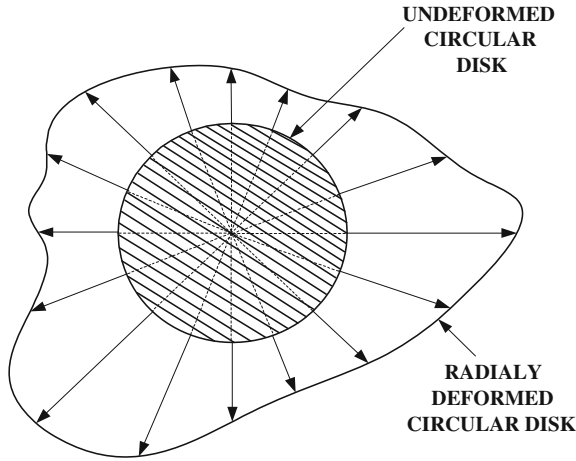


Fig. 2.11 Sketch of a radially deformed circular disk in $\mathbb{R}^2$

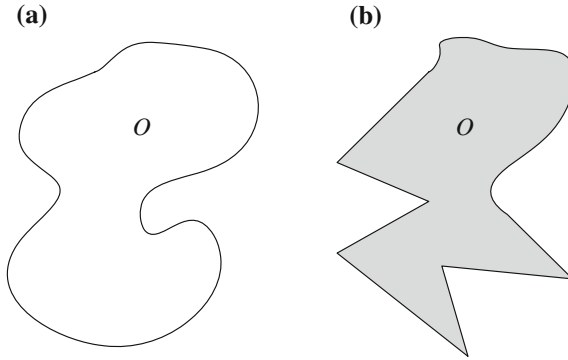


Fig. 2.12 Examples of compact connected objects in $\mathbb{R}^2$ that are not transformable from a circular disk via radial deformations. **a** object with smooth boundary; **b** object with nonsmooth boundary

$r = r(\theta)$, where r (the radial distance from the disk center) is a positive C_m -function ($m \geq 1$) of the angle $\theta \in [0, 2\pi[$. Thus, for each point $x \in \partial\mathcal{O}$, there exists a unique outward unit normal $n(x)$ with its corresponding tangent line described by $L(x) \stackrel{\text{def}}{=} \{x' \in \mathbb{R}^2 : \langle n(x), x' - x \rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^2$. Suppose we can find a pair of adjacent points $a, b \in \partial\mathcal{O}$ such that their corresponding tangent lines $L(a)$ and $L(b)$ intersect at a point $z \in \mathcal{O}^c$, then the arc $\mathcal{A}(a, b) \subset \partial\mathcal{O}$ connecting a and b is visible from z (see Fig. 2.13), provided that $L(a)$ and $L(b)$ are lines of support to $\mathcal{O}$ with the *separation property*: $\mathcal{O}$ lies in the negative half-spaces $H^-(a) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 : \langle n(a), x - a \rangle \leq 0\}$ and $H^-(b) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 : \langle n(b), x - b \rangle \leq 0\}$ respectively. Moreover, $L(a) \cap \mathcal{O} = \{a\}$ and $L(b) \cap \mathcal{O} = \{b\}$. Under the foregoing conditions, we can determine the point-observers for this class of objects by first

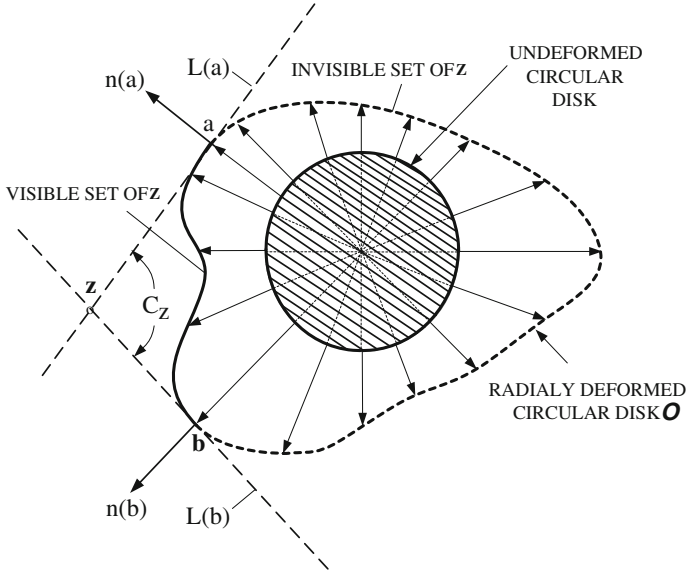


Fig. 2.13 Construction of point-observers via intersecting lines of support to $\mathcal{O}$

determining the intersection points z of the lines of support for $\mathcal{O}$ for points along $\partial\mathcal{O}$, and then select a subset of these points z to attain total visibility.

Remark 2.4 There exist objects in the foregoing special class that do not have intersecting lines of support with the desired separation property. In Fig. 2.13, the object $\mathcal{O}$ has only one pair of points on $\partial\mathcal{O}$ whose supporting lines have the desired separation property. Therefore total visibility of $\mathcal{O}$ cannot be attained by the point of intersection of these lines. If $\mathcal{O}$ is convex with a smooth boundary, then the existence of such lines of support is guaranteed by the Hahn-Banach Theorem (Separation form) [3]. Figure 2.14 shows an object in the above-mentioned class for which total visibility is attainable by three point-observers determined by the proposed approach.

Now, consider Case (ii)(b) in Sect. 2.1 where the observed object $\mathcal{O}$ is a solid body in $\mathbb{R}^3$. Following the line-of-thinking in the foregoing discussion, let $\mathcal{O}$ be a closed spherical ball $\tilde{\mathcal{B}}(0; R) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\| \leq R\}$ centered at the origin of $\mathbb{R}^3$ with radius R . For any point-observer $z \in \tilde{\mathcal{B}}(0; R)^c$ with $R < \|z\| < \infty$, its visible set $\mathcal{V}(z)$ is the spherical cap $\hat{\mathcal{C}}(z)$ described by

$$\hat{\mathcal{C}}(z) \stackrel{\text{def}}{=} \mathcal{S}(0; R) \cap H^+(R^2/\|z\|), \quad (2.4)$$

where $H^+(R^2/\|z\|)$ denotes the half-space $\{x \in \mathbb{R}^3 : \langle x, z/\|z\| \rangle \geq R^2/\|z\|\} = \{x \in \mathbb{R}^3 : \langle x, z \rangle \geq R^2\}$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product. We note that as $\|z\| \rightarrow \infty$, $\hat{\mathcal{C}}(z)$ tends to a hemisphere of $\mathcal{S}(0; R)$.

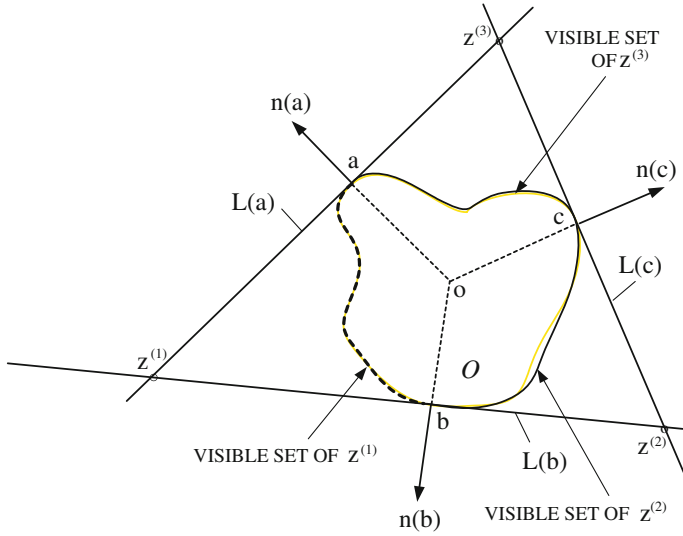


Fig. 2.14 Example of a 2D object which is totally visible by a set of three point-observers obtained by intersecting lines of support to $\mathcal{O}$

In view of the result for total visibility of a circular disk in $\mathbb{R}^2$, we may conjecture that $\bar{B}(0; R)$ or its boundary sphere $S(0; R) = \{x \in \mathbb{R}^3 : \|x\| = R\}$ is totally visible from a set of four point-observers at the vertices $z^{(i)}$ of a regular tetrahedron with edge a inscribing $S(0; R)$ as shown in Fig. 2.15. The vertices of the tetrahedron are given by

$$\begin{aligned}
 z^{(1)} &= (0, 0, h - R) = (0, 0, 3)R, \\
 z^{(2)} &= (0, -2L/3, -R) = (0, -2\sqrt{2}, -1)R, \\
 z^{(3)} &= (a/2, L/3, -R) = (\sqrt{6}, \sqrt{2}, -1)R, \\
 z^{(4)} &= (-a/2, L/3, -R) = (-\sqrt{6}, \sqrt{2}, -1)R,
 \end{aligned} \tag{2.5}$$

where $h = a\sqrt{6}/3$ is the altitude and $L = a\sqrt{3}/2$ the slant height of the regular tetrahedron. $R = h/4 = a\sqrt{6}/12$ is the radius of the inscribed sphere $S(0; R)$ and $a = 2\sqrt{6}R$ is the edge of the regular tetrahedron. To verify the validity of the foregoing conjecture, we first compute the visible sets $\mathcal{V}(z^{(i)})$ described by spherical caps:

$$\begin{aligned}
 \hat{\mathcal{C}}(z^{(1)}) &= S(0; R) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq R/3\}, \\
 \hat{\mathcal{C}}(z^{(2)}) &= S(0; R) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -2\sqrt{2}x_2 - x_3 \geq R\}, \\
 \hat{\mathcal{C}}(z^{(3)}) &= S(0; R) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{6}x_1 + \sqrt{2}x_2 - x_3 \geq R\}, \\
 \hat{\mathcal{C}}(z^{(4)}) &= S(0; R) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -\sqrt{6}x_1 + \sqrt{2}x_2 - x_3 \geq R\}.
 \end{aligned} \tag{2.6}$$

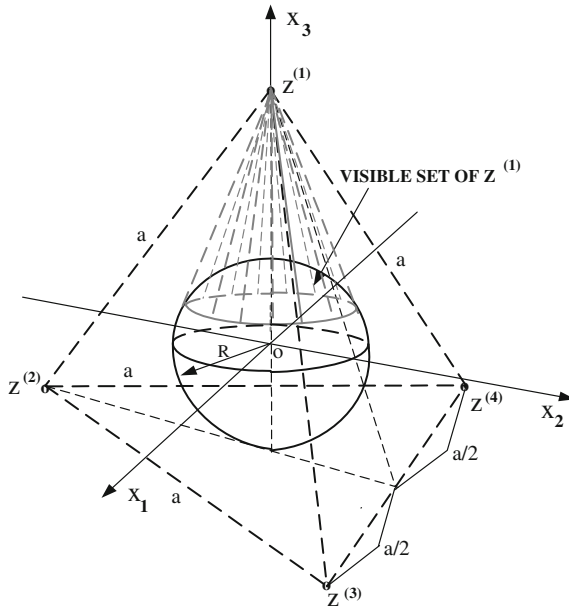


Fig. 2.15 Four point-observers at the vertices of a regular tetrahedron inscribing a spherical ball with radius R

For total visibility, the visible sets of the four observation points must cover the sphere, i.e. $\bigcup_{i=1}^4 \hat{C}(z^{(i)}) = \mathcal{S}(0; R)$. Evidently, the sphere $\mathcal{S}(0; R)$ cannot be partitioned into four *disjoint* spherical caps. Consider $\hat{C}(z^{(1)})$ whose lower boundary is the circle: $\mathcal{C}_1 \stackrel{\text{def}}{=} \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 8R^2/9, x_3 = R/3\}$. It can be verified by straightforward computation that

$$\begin{aligned} \mathcal{C}_1 \cap \hat{C}(z^{(2)}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 8R^2/9; x_2 \leq -\sqrt{2}R/3; x_3 = R/3\}, \\ \mathcal{C}_1 \cap \hat{C}(z^{(3)}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 8R^2/9; x_1 \geq 0; x_2 \geq -\sqrt{2}R/3; x_3 = R/3\}, \\ \mathcal{C}_1 \cap \hat{C}(z^{(4)}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 8R^2/9; x_1 \leq 0; x_2 \geq -\sqrt{2}R/3; x_3 = R/3\}. \end{aligned} \quad (2.7)$$

The arcs $\mathcal{C}_1 \cap \hat{C}(z^{(i)})$, $i = 2, 3, 4$ are subsets of the circle $\mathcal{C}_1$ (see Fig. 2.16), and $\bigcup_{i=2}^4 (\mathcal{C}_1 \cap \hat{C}(z^{(i)})) = \mathcal{C}_1$. Evidently, since the tetrahedron is regular, we can repeat the foregoing computations for $\mathcal{C}_j \cap \hat{C}(z^{(i)})$ for $j \neq 1$ and $i \neq j$ and arrive at a similar conclusion, where $\mathcal{C}_j$ is the circle associated with the spherical cap corresponding to the visible set of $z^{(j)}$. Thus, we have verified the truth of our conjecture that total visibility is attainable by the four observation points $z^{(i)}$, $i = 1, \dots, 4$ at the vertices of any inscribing regular tetrahedron. At this point, we may ask whether total visibility is attainable by other 4-point-observer configurations. Since the visible set of a point-observer tends to a hemisphere of $\mathcal{S}(0; R)$ as the distance between the

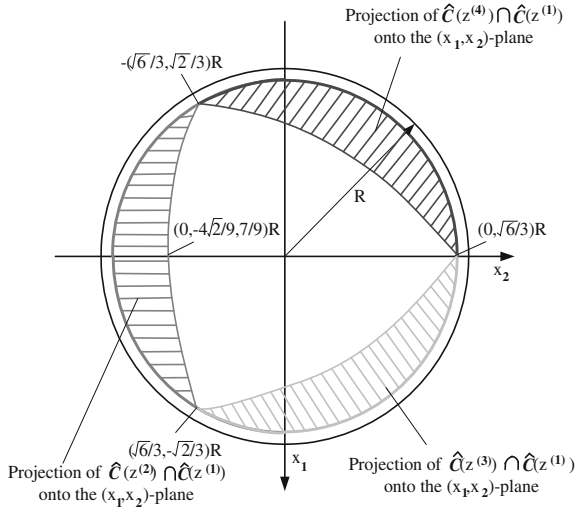


Fig. 2.16 Projection of $\hat{C}(z^{(1)}) \cap \hat{C}(z^{(i)})$, $i = 2, 3, 4$ onto the (x_1, x_2) -plane

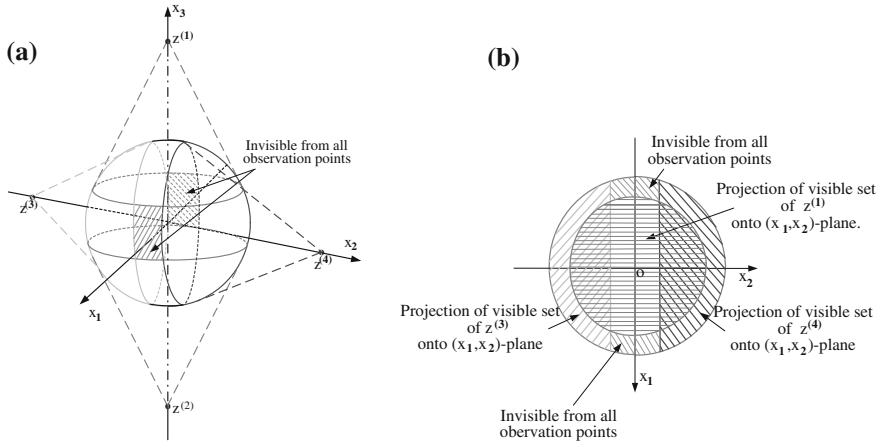


Fig. 2.17 Visible sets of a sphere attainable by four point observers. **a** Point-observer configuration; **b** Projections of visible sets of $z^{(i)}$, $i = 1, \dots, 4$ onto the (x_1, x_2) -plane

observation point z and $S(0; R)$ tends to infinity, it follows that the 4-point-observers at the vertices of any *dilated* inscribing regular tetrahedron (i.e. equal extension of the point-observers along the directions of the vertices) also leads to total visibility.

We may consider other 4-point-observer configurations such as the one shown in Fig. 2.17. Here, the non-overlapping portions of the visible sets of all four observation points have the same surface area. We note there remain two patches on the sphere that are invisible from any point-observer. The areas of these invisible patches tend to zero as the distances between the point-observers and the sphere tend to infinity. Evidently,

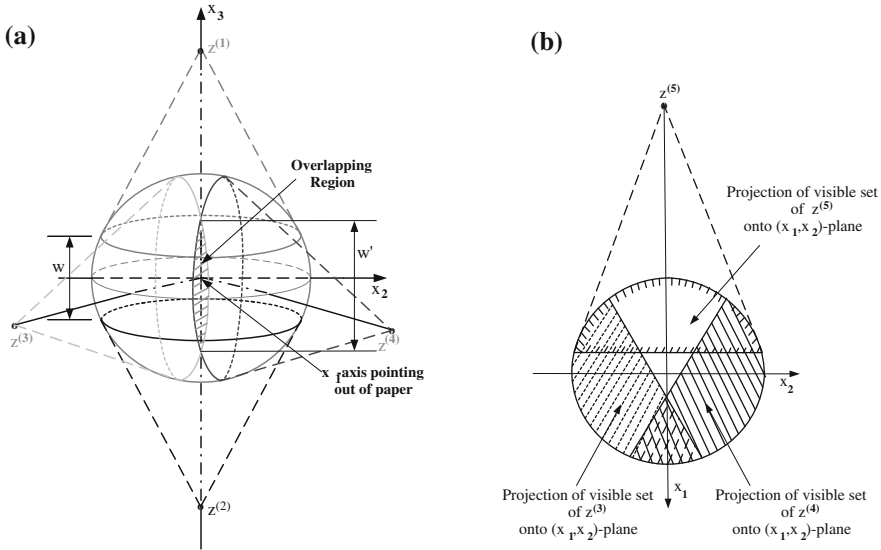


Fig. 2.18 Total visibility of a sphere attainable by five-point observers. **a** Point-observer configuration; **b** Projections of visible sets of $z^{(i)}$, $i = 3, 4, 5$ onto the (x_1, x_2) -plane

total visibility of the sphere is not attainable by this nonsymmetric 4-point-observer configuration.

Consider another nonsymmetric configuration consisting of five point-observers. We shall show by construction that total visibility is attainable by a 5-point-observer configuration shown in Fig. 2.18. It consists of two point-observers $z^{(1)}$ and $z^{(2)}$ located symmetrically about the origin on the x_3 -axis at $(0, 0, \pm d)$. Their visible sets are spherical caps separated by a ring with width w . This ring can be made visible by three point-observers whose visible sets overlap each other as indicated by the projection of the cutting planes associated with the spherical caps onto the (x_1, x_2) -plane as shown in Fig. 2.18b. To attain total visibility, the overlapping height w (see Fig. 2.18a) must be greater than w' . This can be accomplished by setting the point-observers $z^{(3)}$ and $z^{(4)}$ sufficiently far apart.

Remark 2.5 We note that Proposition 2.2 remains valid for objects in $\mathcal{W} = \mathbb{R}^3$ described in Case(ii)(b) in Sect. 2.1. Also, for certain objects, total visibility is attainable by two point-observers. For example, only two point-observers are required for total visibility of the object formed by a cone and a portion of a spherical ball as shown in Fig. 2.19.

In applications, a basic question is: Given $\mathcal{P}$ and $\mathcal{O}$, is $\mathcal{O}$ totally visible from $\mathcal{P}$? In general, $\mathcal{P}$ may be a finite set of observation points or the union of disjoint subsets of $\mathcal{O}^c$. In what follows, we assume that the observation points z are restricted to a transparent observation platform $\mathcal{P}$ in $\mathcal{O}^c$. For $\mathcal{P} = \mathcal{P}_h$, the point $x \in \partial\mathcal{O}$ described by $x = z - h(x)n(x)$ is always visible from z . Hence $\mathcal{V}(z)$ is nonempty. Thus, we may

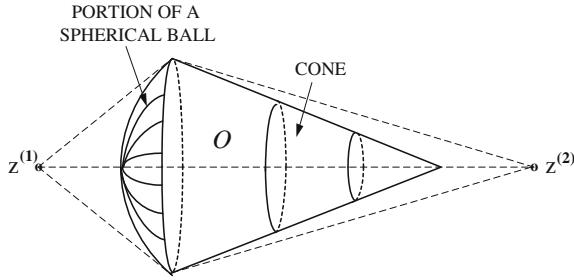


Fig. 2.19 3-D object formed by a cone and a portion of a spherical ball. Total visibility is attained by two point-observers $z^{(1)}$ and $z^{(2)}$

regard $z \rightarrow \mathcal{V}(z)$ as a set-valued mapping on $\mathcal{P}_h$ into $2^{\partial\mathcal{O}}$, the space of all nonempty compact subsets of $\partial\mathcal{O}$. In general, this mapping is discontinuous with respect to the metric ρ on $\mathcal{P}_h$, and Hausdorff metric ρ_H on $2^{\partial\mathcal{O}}$ defined by

$$\rho_H(A, B) \stackrel{\text{def}}{=} \max\{\max\{\rho(x, B) : x \in A\}, \max\{\rho(x', A) : x' \in B\}\}, \quad (2.8)$$

where

$$\rho(x, B) \stackrel{\text{def}}{=} \inf\{\|x - x'\| : x' \in B\}, \quad (2.9)$$

in which $\|\cdot\|$ denotes the norm for $\mathbb{R}^n$. We observe that for a given smooth $\partial\mathcal{O}$, the corresponding $\mathcal{P}_h$ may become non-smooth for some $h(\cdot) > 0$. Consider again Example 2.1. Evidently, for $h < r_c$, the radius of curvature of the middle section of $\mathcal{O}$, the constant height observation platforms $\mathcal{P}_h$ are smooth. When $h = r_c$, $\mathcal{P}_h$ has a kink or cusp at the circle $x_1^2 + x_3^2 = x_{3o}^2$. For $h > r_c$, there exist more than one point in $\partial\mathcal{O}$ that are at height h from the observation platform $\mathcal{P}_h$. In this case, the height of an observation point $z \in \mathcal{P}_h$ from $\partial\mathcal{O}$ is defined as $h = \inf\{\|z - y\| : y \in \partial\mathcal{O}\}$. Figure 2.20 shows the cross-section of $\mathcal{P}_h$ in the $x_1 = 0$ plane for Example 2.3 with $x_{2o} = 5$, $x_{3o} = 8$, $r_o = 4$ and various values of $h > 0$. In this example, there exist two or more distinct points $x^{(i)} \in \partial\mathcal{O}$ such that the outward rays $\mathcal{R}(x^{(i)}) = \{x \in \mathbb{R}^3 : x = x^{(i)} + \lambda n(x^{(i)}), \lambda \geq 0\}$ along the outward normal $n(x^{(i)})$ intersect at a point $\hat{x}$ corresponding to $h = r_c$, and this point is equidistant from the points $x^{(i)}$. This property motivates the following definition (see Fig. 2.21):

Definition 2.3 A point $\hat{x} \in \mathbb{R}^3$ is an *outward-normal intersection point* of a surface $\partial\mathcal{O}$, if there exist two or more distinct points $x^{(i)} \in \partial\mathcal{O}$ such that the rays $\mathcal{R}(x^{(i)}) = \{x \in \mathbb{R}^3 : x = x^{(i)} + h n(x^{(i)}), h \geq 0\}$ along the outward unit normal $n(x^{(i)})$ intersect at some point $\hat{x}$.

Remark 2.6 In the above definition, the values of h corresponding to the rays at the intersection point $\hat{x}$ may not be the same. In the special case where h is independent of $x \in \partial\mathcal{O}$, the outward-normal intersection point $\hat{x}$ of a surface is equidistant from

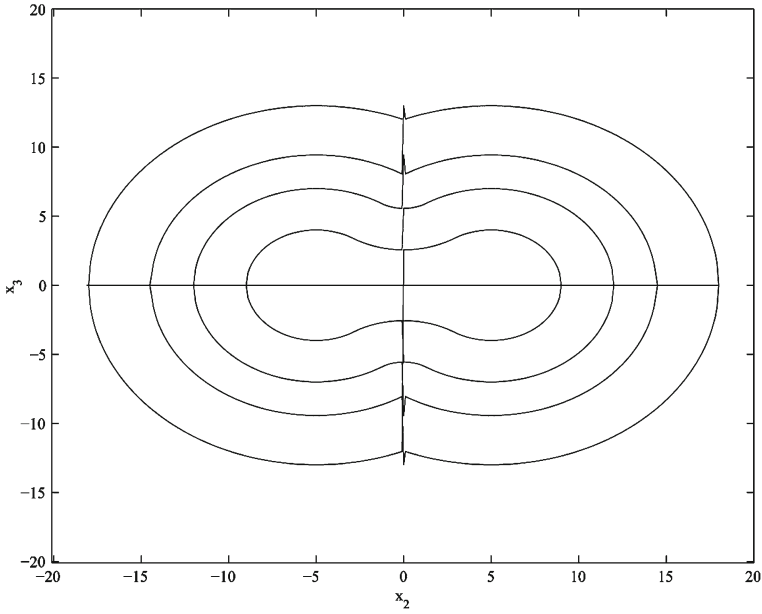
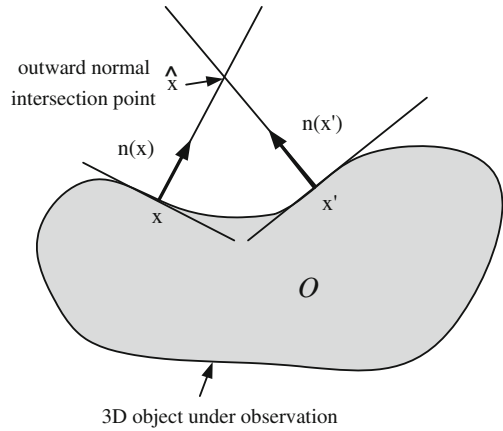


Fig. 2.20 Cross-sections of $\mathcal{P}_h$ in Example 2.3 with $x_{2o} = 5$, $x_{3o} = 8$, $r_o = 4$ for $h = 0$ (innermost curve), 3, $h = r_c = \sqrt{89} - 4$ and $h = 9$ (outermost curve)

Fig. 2.21 A 3-D object with an outward-normal intersection point



the points $x^{(i)} \in \partial\mathcal{O}$ for some constant $h > 0$. Then $\hat{x}$ corresponds to the usual *focal point* of a surface in differential geometry [4]. Note that in geometric optics, the lengths of the rays connecting the focal point and any point $x \in \partial\mathcal{O}$ are equal so that the optical signals emanating from all $x \in \partial\mathcal{O}$ arrived at the focal point are in phase. This requirement is not imposed in the definition of an outward-normal intersection point.

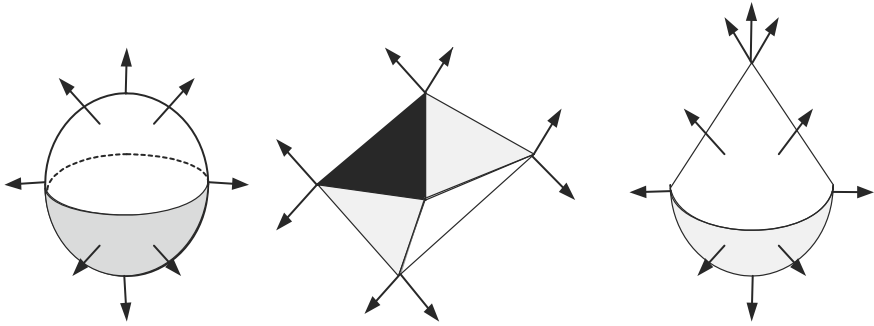


Fig. 2.22 3-D objects with no outward-normal intersection points

Typical 3-D objects in $\mathbb{R}^3$ having no outward-normal intersection points are convex bodies whose boundaries are spheroids, ellipsoids, zonoids and convex polytopes (see Fig. 2.22).

Remark 2.7 For Cases (i)(a) with $\mathcal{W} = \mathbb{R}^2$ or (ii)(a) with $\mathcal{W} = \mathbb{R}^3$ in Sect. 2.1, the object under observation is respectively a plane curve or a surface corresponding to the graph G_f of a smooth function $f = f(x)$ defined on a compact subset of $\mathcal{W}$, and the observation platform $\mathcal{P}$ is at a constant *vertical-height* h_v above the plane curve or surface specified by the graph of $f_{h_v} \stackrel{\text{def}}{=} f + h_v$. Note the distinction between h and h_v . They denote respectively the height along the surface normal $n(x)$ and vertical-height. Clearly, $\mathcal{P}$ corresponds to the elevated G_f which has the same smoothness property as that of G_f for all $h_v > 0$. Thus, the change in smoothness of $\mathcal{P}$ with respect to h_v as discussed here does not occur.

The following prepositions pertaining the smoothness of the observation platform $\mathcal{P}_h$ are evident from Definition 2.3.

Proposition 2.4 *If the surface $\partial\mathcal{O}$ of a compact solid body $\mathcal{O} \subset \mathbb{R}^3$ is smooth, and has no outward-normal intersection points, then the observation platform $\mathcal{P}_h$ and $\partial\mathcal{O}$ have the same smoothness for any $h(\cdot) > 0$.*

Proposition 2.5 *If the surface $\partial\mathcal{O}$ of a compact solid body $\mathcal{O} \subset \mathbb{R}^3$ is smooth, and has a finite number of outward-normal intersection points $\hat{x}^{(i)}$ with corresponding heights $h_i, i = 1, \dots, M$, along the outward normal $n(x^{(i)})$ above $x^{(i)}$, then any observation platform $\mathcal{P}_h$ with constant $h < \min\{h_i, i = 1, \dots, M\}$ and $\partial\mathcal{O}$ have the same smoothness.*

In practical applications, it is of interest to consider the sensitivity and robustness of total visibility with respect to positional perturbations associated with a given set of point-observers. For preciseness, we introduce the following definitions:

Definition 2.4 Given a single point-observer at $z_o \in \mathcal{P}$ from which the observed-object surface $\partial\mathcal{O}$ is totally visible, z_o is said to be *robust with respect to position*

perturbations, if there exists an ϵ -neighborhood of z_o such that total-visibility is invariant, i.e. for each $z' \in (\{z \in \mathcal{W} : \|z - z_o\| \leq \epsilon\} \cap \mathcal{P})$, its visible set $\mathcal{V}(z') = \partial\mathcal{O}$.

For a simple example, consider the case where the object $\mathcal{O}$ is a smooth surface $G_f \subset \mathbb{R}^3$ as described by Case (ii)(a) in Sect. 2.1, with a constant vertical-height observation platform $\mathcal{P}_{h_v} = G_f + h_v$. We mention here that there exists a critical vertical-height h_{vc} such that total visibility is attained for any point-observer $z_o \in \text{Epi}_{f+h_v}$ for any $h_v > h_{vc}$ (A proof of this fact will be given in Chap. 3), i.e. $\mu_2\{\mathcal{V}(z_o)\} = \mu_2\{G_f\}$. Thus, any observation point $z \in \mathcal{P}_{h_v}$ with vertical height $h_v > h_{vc}$ is robust with respect to position perturbations.

The sensitivity of the visible-set measure of a point-observer with respect to position perturbations can be defined in terms of the Euclidean norm of the gradient of the visible-set measure as follows:

Definition 2.5 The sensitivity of the visible-set measure of a point-observer at a given observation point z_o with respect to position perturbations is defined as

$$\sigma(z_o) = \|\nabla_z \mu_2\{\mathcal{V}(z)\}|_{z=z_o}\|. \quad (2.10)$$

Evidently, for the foregoing simple example, $\sigma(z_o) = 0$ for any $z_o \in \mathcal{P}_{h_v}$ with $h_v > h_{vc}$.

2.3 Observation of Complex Objects

So far, we have considered only the observation of an object consisting of a single connected compact set in the world space $\mathcal{W}$. In more general situations, the object $\mathcal{O}$ under observation may correspond to a collection of *disjoint* compact subsets $\mathcal{O}_i, i \in \mathcal{I}$ of $\mathcal{W}$, where $\mathcal{I}$ is a finite or countably infinite index set. The observations of $\mathcal{O}$ are made from points $z \in \mathcal{P} \subset \mathcal{W}$ such that $\mathcal{P} \cap \mathcal{O}$ is empty. In the trivial case where the observation of any object $\mathcal{O}_i$ can be made from a given observation point $z \in \mathcal{P}$ without considering the remaining subsets $\mathcal{O}_j, j \in \mathcal{I} - \{i\}$, then the visible set $\mathcal{V}(z)$ is simply $\bigcup_{i \in \mathcal{I}} \mathcal{V}_i(z)$, where $\mathcal{V}_i(z) \subset \partial\mathcal{O}_i$ denotes the visible set of z with respect to $\mathcal{O}_i$. To illustrate various possible situations involving objects with multiple disjoint subsets, consider a simple example where the object $\mathcal{O} \subset \mathbb{R}^2$ consists of two circular disks $D_i, i = 1, 2$ with different radii r_1 and r_2 as shown in Fig. 2.23. First, consider the observation point $z^{(1)}$ located between the two disks. Evidently, both D_1 and D_2 are partially visible from $z^{(1)}$. From the observation point $z^{(2)}$ (resp. $z^{(3)}$), only D_1 (resp. D_2) is partially visible (as in solar eclipse, where the observation point z is identified with the sun). From the observation $z^{(4)}$, both D_1 and D_2 are partially visible. However, only one point in D_1 is visible from $z^{(4)}$. Finally, both D_1 and D_2 are partially visible from the observation point $z^{(5)}$. Moreover, $\mathcal{V}(z^{(5)})$, the visible set of $z^{(5)}$ is simply $\mathcal{V}_1(z^{(5)}) \cup \mathcal{V}_2(z^{(5)})$, where $\mathcal{V}_i(z^{(5)})$ denotes the visible set of $z^{(5)}$ with respect to D_i , which can be determined independently. This simple example shows

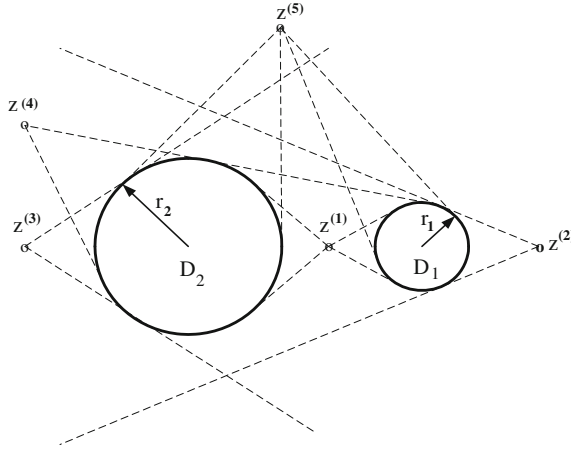


Fig. 2.23 Object composed of two disjoint circular disks in $\mathbb{R}^2$

that the structure of the visible sets for objects with multiple disjoint subsets may be very complex, and their determination may be a computationally intensive task.

The following is an example of an object in $\mathcal{W} = \mathbb{R}^3$: composed of an countably infinite number of disjoint compact subsets:

Example 2.4 Let the object under observation be $\mathcal{O} = \cup_{i \in \mathcal{I}} \mathcal{O}_i$, where $\mathcal{I} = \{1, 2, \dots\}$, and $\mathcal{O}_i$ is a circle with radius r_i , centered at $(x_1, x_2, x_3) = (0, 0, x_{3i})$:

$$\mathcal{O}_i = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = r_i^2, x_3 = x_{3i} = 1/i^2\}, \quad i = 1, 2, \dots \quad (2.11)$$

such that $\mathcal{O}_i \cap \mathcal{O}_j$ is empty for all $i, j \in \mathcal{I}, i \neq j$. The observation platform $\mathcal{P}$ may correspond to a nonempty compact subset of $\mathcal{O}^c$, the complement of $\mathcal{O}$ relative to $\mathcal{W}$. The visible set of an observation point $z \in \mathcal{P}$ is given by

$$\mathcal{V}(z) = \bigcup_{i \in \mathcal{I}} \mathcal{V}_i(z), \quad (2.12)$$

where $\mathcal{V}_i(z)$ is the set of all boundary points of $\mathcal{O}_i$ that is visible from z . The object $\mathcal{O}$ is said to be totally visible from z if $\mathcal{V}(z) = \bigcup_{i \in \mathcal{I}} \partial \mathcal{O}_i$, i.e. the boundary points of every $\mathcal{O}_i$ are visible from z . Evidently, there does not exist an observation point $z \in \mathcal{O}^c$ from which $\mathcal{O}$ is totally visible.

In planetary explorations using mobile robots, one may encounter cavities and tunnel-like structures on the planet surface. Here, the object $\mathcal{O}$ under observation is the surface inside these structures. To describe $\mathcal{O}$ mathematically, consider the simple idealized case in the world space $\mathcal{W} = \mathbb{R}^2$ where $\mathcal{O}$ is the union of the graphs of two real-valued C_1 functions $f_i = f_i(x), i = 1, 2$ defined on the interval $[a, b]$ as shown in Fig. 2.24. In Fig. 2.24a, G_{f_1} and G_{f_2} intersect at $x = b$ where $f_1(b) = f_2(b)$. Thus,

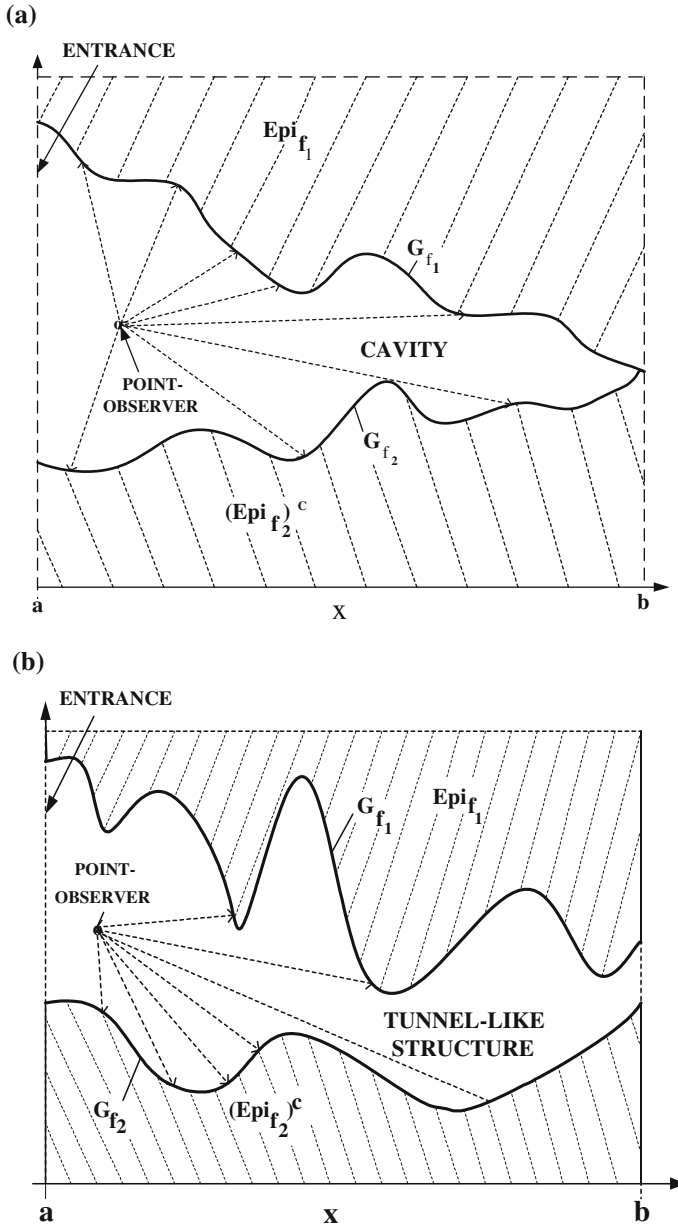


Fig. 2.24 a Cavity and b tunnel-like structures

the region corresponding to $(\text{Epi}_{f_1})^c \cap \text{Epi}_{f_2}$ is an one-dimensional cavity in which the point-observers lie. When G_{f_1} and G_{f_2} do not intersect, we have a tunnel-like structure as shown in Fig. 2.24b. Now, given f_1 and f_2 , it is of interest to determine

whether there exists a point-observer z in the interior of $(\text{Epi}_{f_1})^c \cap \text{Epi}_{f_2}$ such that $\mathcal{O}$ is totally visible from z . Evidently, if the given f_1 and f_2 generate a convex region $(\text{Epi}_{f_1})^c \cap \text{Epi}_{f_2}$, then $\mathcal{O}$ is totally visible from any point-observer in the interior of $(\text{Epi}_{f_1})^c \cap \text{Epi}_{f_2}$. Actually, total-visibility is attainable by a single point-observer even $(\text{Epi}_{f_1})^c \cap \text{Epi}_{f_2}$ is non-convex as demonstrated by the following simple example:

Example 2.5 Let f_1 and f_2 be real-valued functions defined on $\Omega = [-1, 1]$ described by

$$f_1(x) = (1 + x^2); \quad f_2(x) = -f_1(x). \quad (2.13)$$

The object under observation is given by $\mathcal{O} = G_{f_1} \cup G_{f_2}$ as shown in Fig. 2.25. It is evident that $(\text{Epi}_{f_1})^c \cap (\text{Epi}_{f_2}) = \{(x, y) \in \mathbb{R}^2 : |y| \leq f_1(x); x \in \Omega\}$ is non-convex, and $\mathcal{O}$ is totally visible from a single point-observer at the origin.

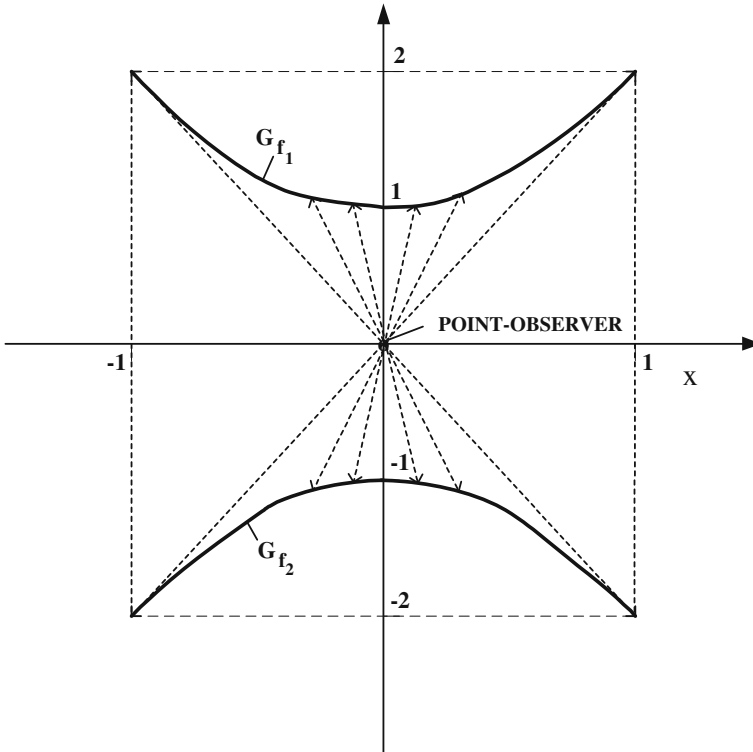


Fig. 2.25 Object under observation in Example 2.5

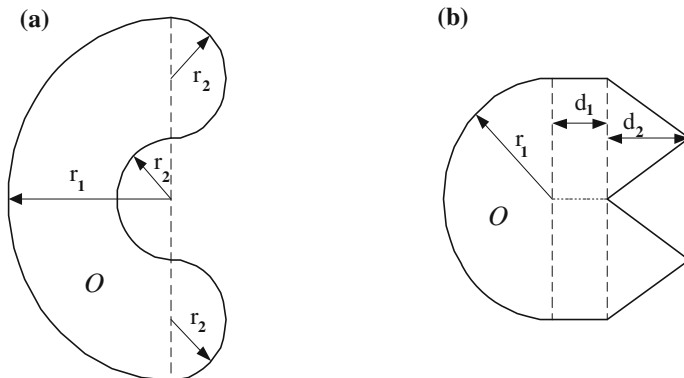


Fig. 2.26 Objects with boundaries composed of **a** circular arcs and **b** both straight-line segments and circular arcs

2.4 Concluding Remarks

In this chapter, we have introduced various notions of visibility associated with line-of-sight observation of an object from a point-observer. These notions are also applicable to target interception by replacing the point-observer with a point source, and the object under observation with a target (e.g. a laser source emitting a beam toward the target). We may rename the “visible set of a point observer” as the “impact set of the point source”. Thus, total visibility of an object from a point-observer corresponds to total impact of the target boundary by a point source, i.e. each point of the target boundary can be impacted by at least one straight beam emitted from the source. In practical applications, it may be of interest to expose a particular part of the target surface to the source. This task can be accomplished only when that particular part lies in the impact set of the source.

Exercises

Ex.2.1. Let $\mathcal{W} = \mathbb{R}^2$. For each of the objects $\mathcal{O} \subset \mathcal{W}$ whose boundaries are composed of circular arcs and straightline segments (see Fig.2.26), determine the smallest number and locations of point-observers in $\mathcal{O}^c$ for total visibility of $\mathcal{O}$.

Ex.2.2. Let $\mathcal{W} = \mathbb{R}^2$. The object $\mathcal{O}$ under observation is formed by the union of two circular disks with different radii $r_1, r_2 > 0$.

- (i) Find the visible sets of various observation points $z \in \mathcal{O}^c$ for each of the following cases:
 - (a) two disks are tangent to each other;
 - (b) two disks are disjoint with their centers separated by a finite distance $r_1 + r_2 + d, d > 0$.

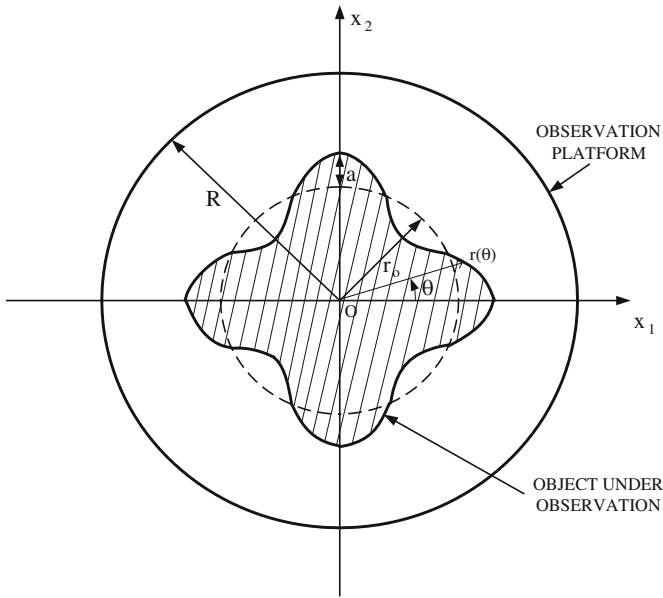


Fig. 2.27 A circular disk with rippled boundary $n = 4$

- (ii) Determine the constant-height observation platforms $\mathcal{P}_h$ for cases (a) and (b) in (i). For each case, examine the smoothness of $\mathcal{P}_h$ as h increases from 0.

Ex.2.3. Let the object under observation $\mathcal{O}$ be a circular disk in $\mathcal{W} = \mathbb{R}^2$ with mean radius r_o and rippled boundary described by $r(\theta) = r_o + a \cos(n\theta)$, $0 \leq \theta \leq 2\pi$, $n = 1, 2, \dots$ (see Fig. 2.27), where a is a positive number $< r_o$. The observation platform $\mathcal{P}$ is a circle centered at the origin with radius $R > r_o + a$.

- (a) Given any $n = 1, 2, \dots$, determine the smallest number and locations of point-observers in $\mathcal{P}$ for total visibility of $\mathcal{O}$.
 (b) Discuss the solution to the problem in (a) as $n \rightarrow \infty$.

Ex.2.4. Extend the notion of a radially-deformed circular disk in $\mathbb{R}^2$ to a radially-deformed sphere in $\mathbb{R}^3$, and develop results pertaining to total visibility using a finite number of point-observers located at finite distances from the radially-deformed sphere.

Ex.2.5. Let $\mathcal{W} = \mathbb{R}^2$. The object $\mathcal{O}$ under observation is composed of three identical circular disks D_i , $i = 1, 2, 3$ with radius r_o , whose centers are at the vertices of an equilateral triangle centered at the origin as shown in Fig. 2.28. The observation platform $\mathcal{P}$ is a circle centered at the origin with radius $R > r_o$. Partition $\mathcal{P}$ into subset sets $\mathcal{P}_j$ from which certain subsets of $\{D_i, i = 1, 2, 3\}$ are partially visible. Is $\mathcal{O}$ totally visible from $\mathcal{P}$.

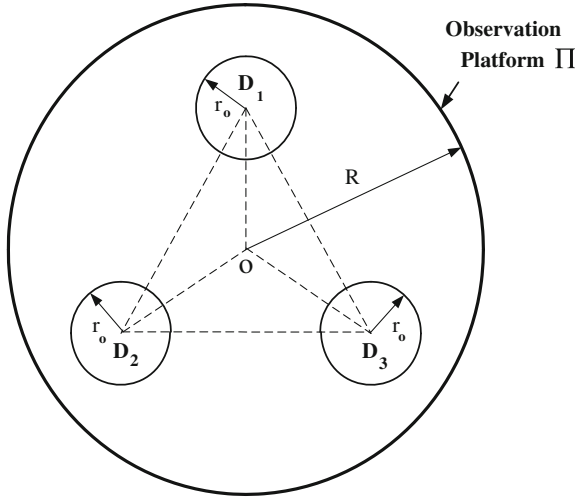


Fig. 2.28 Object under observation for Exercise 2.5

Ex.2.6. Let $\mathcal{W} = \mathbb{R}^3$. The object $\mathcal{O}$ under observation is a rocket engine composed of a spherical ball with radius R attached to a truncated cone with length L and terminal radii r_1, r_2 as shown in Fig. 2.29. Find the visible sets corresponding to various observation points $z \in \mathcal{O}^c$.

Ex.2.7. Let $\mathcal{W} = \mathbb{R}^3$. The object $\mathcal{O}$ under observation is a surface described by $\mathcal{O}_1 \cup \mathcal{O}_2$, where

$$\begin{aligned} \mathcal{O}_1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 + 1)^2 + x_2^2 \leq 1, x_3 = (x_1 + 1)^2 + x_2^2\}; \\ \mathcal{O}_2 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 - 1)^2 + x_2^2 \leq 1, x_3 = (x_1 - 1)^2 + x_2^2\}. \end{aligned} \quad (2.14)$$

The observations are made from the constant vertical-height platform $\mathcal{P}_{h_v}$ defined by

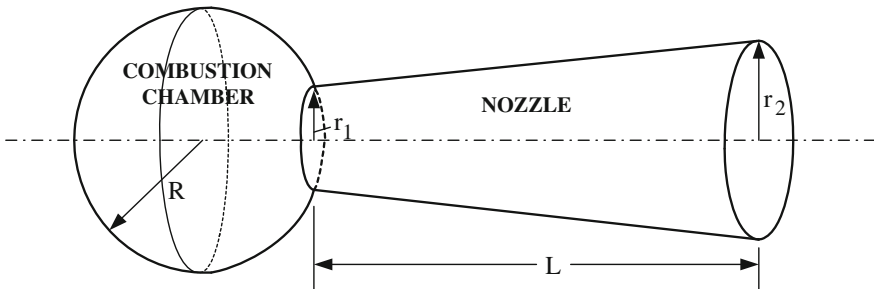


Fig. 2.29 Rocket engine under observation in Exercise 2.6

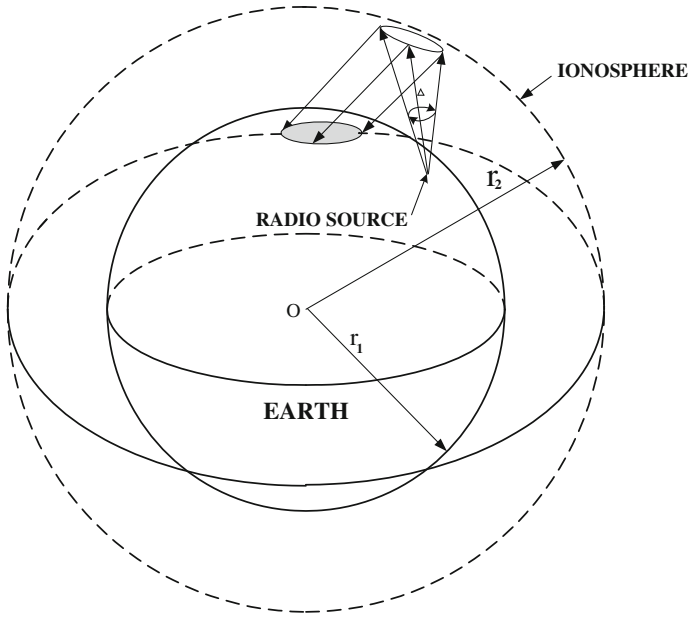


Fig. 2.30 Radio-wave reflection from ionosphere

$$\mathcal{P}_{h_v} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x'_1, x_2 = x'_2, x_3 = x'_3 + h_v, (x'_1, x'_2, x'_3) \in \mathcal{O}\}, \quad h_v > 0. \quad (2.15)$$

- (a) For a given positive vertical height $h_v \leq 1$, determine the visible sets for various observation points $z \in \mathcal{P}_{h_v}$.
- (b) Repeat (a) for vertical height $h_v > 1$.

Ex.2.8. A radio point-source located on the surface of the spherical Earth with radius r_1 emits radio beam with solid angle Δ . The beam is reflected from the ionosphere (a reflective spherical shell with radius r_2) back to Earth (see Fig. 2.30). The reflection obeys the usual law of angle of incidence equals the angle of reflection. Describe in mathematical terms the Earth's surface area that is exposed to the reflected radio beam.

Ex.2.9. Consider the objects $\mathcal{O}$ under observation as shown in Fig. 2.26. Construct an example for each case (a) and case (b) by choosing appropriate C_1 -functions $f_1 = f_1(x)$ and $f_2 = f_2(x)$ such that total visibility of $\mathcal{O} = G_{f_1} \cup G_{f_2}$ requires at least two point-observers in the interior of the region $(\text{Epi}_{f_1})^c \cap \text{Epi}_{f_2}$. Give the coordinates of these point-observers.

Ex.2.10. Determine whether the following statement is (i) true, (ii) false, or (iii) sometimes true and sometimes false? In (iii), give a simple example for each case.

If the convex hull of a nonempty compact subset S of $\mathbb{R}^2$ (the smallest convex set containing S) has interior points, then the boundary of S is totally visible from any interior point of the convex hull of S .

References

1. P.K.C. Wang, A class of optimization problems involving set measures. *Nonlinear Anal. Theor. Methods Appl.* **47**, 25–36 (2001)
2. P.K.C. Wang, Optimal path planning based on visibility. *J. Optim. Theor. Appl.* **117**, 157–181 (2003)
3. D. Luenberger, *Linear Vector Spaces and Optimization* (Wiley, New York, 1979)
4. J.A. Thorpe, *Elementary Topics in Differential Geometry* (Springer, New York, 1979)



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