Directional Metric Entropy and Lyapunov Exponents for Dynamical Systems Generated by Cellular Automata

Maurice Courbage and Brunon Kamiński

Abstract The deterministic dynamics of a spatially extended physical or chemical or biological system may be complex, as in the case of turbulent flows in contrast with the simple motion of laminar fluids. The complexity of extended dynamical systems has been described in many ways using several characteristics and several models. An important role, In investigating the complexity of dynamical systems, entropy and quantities connected with it plays an important role. In the smooth dynamics the Lyapunov exponents are quantities of this type.

1 Introduction

The deterministic dynamics of a spatially extended physical or chemical or biological system may be complex, as in the case of turbulent flows in contrast with the simple motion of laminar fluids. The complexity of extended dynamical systems has been described in many ways using several characteristics and several models. An important role In investigating the complexity of dynamical systems, entropy and quantities connected with it plays an important role. In the smooth dynamics, the Lyapunov exponents are quantities of this type.

Cellular automata (CA-systems or CAs) on infinite lattice are the simplest extended dynamical systems: they represent an infinite number of interacting elements on the lattice. Entropy theory of CAs is an object of actual research. The topological entropy of some class of CA has been studied [11, 12, 15].

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Another important models are the lattice dynamical systems like coupled map lattices [17], some of them have positive topological entropy [1, 6].

The topological entropy of extended systems has also been studied for some partial differential equations as complex Ginzburg–Landau equation [7].

In chaotic dynamical systems, the entropy corresponds to the rate of exponential proliferation of the orbits that become distinguishable with time. It is related to sensitivity to initial conditions through Lyapunov exponents. For spatially extended systems the entropy corresponds to the rate of exponential proliferation of the spatial bounded patterns that become distinguishable with time. Apart from entropy, the Lyapunov exponents play an important role which are analogues of the Lyapunov exponents from the smooth dynamics. How to describe the instability of spatial patterns? One way is to consider the propagation of localised perturbations of the spatial patterns. The CA model is well adapted to display such relation [29]. Shreshevsky introduced a rigorous concept of Lyapunov exponents for CA and studied its relation with the entropy of CA [22].

Moreover, other phenomena are related to propagation of those perturbations when the system undergoes a translation motion with some velocity, like a convection in fluids. This is the case for example in the so-called convective instability. A related property of a spatially extended dynamical system is described when the system is observed in a moving frame. The theory amounts to consider the composition of the dynamics with spatial translations. The corresponding complexity is the so-called space-time directional entropy and the corresonding instability is descibed by the space-time directional Lyapunov exponents. This will be the object of our considerations here in dynamical systems generated by cellular automata. In lattice dynamical systems, the topological directional entropy has been studied [2].

We present our considerations in six sections.

In the second section we give a list of basic notions and examples.

The third section will be devoted to entropy (topological and measure–theoretical) of CAs.

In the fourth section we give a review of various concepts of Lyapunov exponents. We will focus on the Shereshevsky concept and also on the directional Lyapunov exponents.

The fifth section contains the inequality connecting entropy and Lyapunov exponents. One can think about it as an analogue of the well known Ruelle inequality from the smooth dynamics.

In the last section there are some remarks concerning the problem of a generalization of the notion of a Lyapunov exponent (directional Lyapunov exponent) to the multidimensional case and its relation to entropy.

2 Basic Notions and Examples

By a CA-triple we mean a triple $(S, [l, r], F)$, where S is a finite set (alphabet), $[l, r] \subset \mathbb{Z}$ is an interval (window) and $F : S^{[l,r]} \to S$ a fixed map (local rule).

A dynamical system $\mathcal{X} = (X, \mathcal{B}, \mu, f)$ is called a CA-system (or a system generated by a cellular automation) if there exists a CA-triple $(S, [l, r], F)$ such that the configuration space $X = S^{\mathbb{Z}}$, $\mathcal{B}$ is the smallest σ-algebra containing cylindric sets, $f : X \to X$ is a transformation defined by

$$(fx)_n = F(x_{n+l}, \dots, x_{n+r}), \quad x \in X, \ n \in \mathbb{Z}.$$

We say in the sequel that f is a CA-transformation.

Apart from f, we consider the left shift transformation $\sigma : X \to X$ defined as follows:

$$(\sigma x)_n = x_{n+1}, \quad x \in X, n \in \mathbb{Z}.$$

The set function μ on $\mathcal{B}$ is a probability measure invariant with respect to f and σ.

The space X is equipped with a metric d defined as follows. For $x, y \in X, x \neq y$, $x_0 = y_0$, we put

$$N(x, y) = \sup\{n \in \mathbb{N} \mid x_i = y_i, |i| \leq n\}$$

and then we define

$$d(x, y) = \exp(-N(x, y)).$$

In the case $x \neq y$, $x_0 \neq y_0$ we put $d(x, y) = 1$ and $d(x, y) = 0$ for $x = y$.

It is clear that the metric that gives the Tichonov topology on X and $\mathcal{B}$ is the Borel σ-algebra. Moreover, f is continuous, σ is a homeomorphism and $f \circ \sigma = \sigma \circ f$. The classical result of Hedlund [14] says that any continuous transformation f of X commuting with σ is a CA-transformation.

Another interesting, classical result of Hedlund from [14] says that the surjectivity of f is equivalent to the fact that the uniform Bernoulli measure ν is invariant with respect to f and σ.

Now, we recall some classes of CAs, which will be applied in the sequel.

Let $\mathcal{X}$ be a CAs defined by a CA-triple $(S, [l, r], F)$.

2.1 Permutative CAs

A CAs $\mathcal{X}$ (or f) is said to be permutative with respect to the ith coordinate, $l \leq i \leq r$, if the restriction of the local rule $F : S^{[l,r]} \to S$ to this coordinate is a permutation of S. $\mathcal{X}$ is said to be left (right) permutative, if it is permutative w.r.t the lth (rth) coordinate. Finally, $\mathcal{X}$ is bipermutative if it is left and right permutative.

Now, we mention again a classical result of Hedlund ([14]) which says that any permutative (left or right) CA-transformation is onto and so ν is invariant w.r.t f.

2.2 Coven CAs

We take the alphabet $S = \{0, 1\}$, the window is of the form $[0, r]$, $r \geq 2$ and the local rule F is defined as follows. Let $B \in S^{[0,r]}$, $B = [b_0, \dots, b_r]$ be fixed and aperiodic, that is, there is no $1 \le p \le r-1$ such that $b_i = b_{i+p}$, $1 \le i \le r-p$. We define

$$F(s_0, \dots, s_r) = b_0 + \prod_{i=1}^{r} (s_i + b_i).$$

It is clear that the Coven systems are left permutative and so ν is invariant with respect to f.

2.3 Linear CAs

Let the alphabet S be the m-adic ring $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ with the addition $+$ and the multiplication $\cdot$. We take an arbitrary window $[l, r]$ and a local rule F which is linear, that is,

$$F(s_l, \dots, s_r) = a_l s_l + \dots + a_r s_r,$$

where $a_i \in S$ is fixed, $s_i \in S$, $l \le i \le r$. The configuration space X becomes a compact abelian group if we equip it with the coordinate wise addition and the transformation f is an endomorphism. It is known [16] that f is an epimorphism (i.e. it is onto) iff $\gcd(m, a_l, \dots, a_r) = 1$.

Let $m = p_1^{k_1} \cdots p_h^{k_h}$, $h \ge 1$ be the prime decomposition of m. We shall use in the sequel the sets P_i, $1 \le i \le h$ defined by

$$P_i = \{0\} \cup \{j \in [l, r] \mid \gcd\,(a_j, p_i) = 1\}.$$

We put

$$L_i = \min P_i, \quad R_i = \max P_i, \quad 1 \le i \le h.$$

3 Entropy of CAs

One of the basic invariants applied to describe the complexity of dynamical systems is entropy (topological and measure theoretic).

For definitions of these entropies we refer the reader to the book of Walters [27]. The topological (measure theoretic) entropy of a transformation f will be denoted by $h(f)$ $(h_\mu(f))$.

First let us remark that the topological entropy of a CAs is finite. This follows at once from the variational principle and the fact that for any f-invariant measure μ we have (see for example [8])

$$h_\mu(f) \le \max(|l|, |r|) \log p, \quad \text{if } l \cdot r \ge 0$$

and

$$h_\mu(f) \le (r - l) \log p, \quad \text{if } l \cdot r < 0,$$

where $p = \#S$.

There is a small number of classes of CAs for which entropy is calculated.

For linear CAs, the topological entropy has been calculated by D'amico, Manzini and Margara in [12]. Namely, if the window $[l, r]$ of a linear CAs is symmetric $(l = -r)$, then the topological entropy is equal to

$$h(f) = \sum_{i=1}^{h} k_i (R_i - L_i) \log p_i$$

where $k_i,\ p_i,\ R_i,\ L_i,\ 1 \le i \le h$ have been defined in Example 2.3. If f is an epimorphism then [27]

$$h_\nu(f) = h(f).$$

Another class of CAs for which entropy is calculated are bipermutative systems. Afraimovich and Shereshevsky have shown [23] that any such system with a symmetric window $[-r, r]$ is topologically conjugated with the one-sided shift σ_+ acting on the space Y of all one-sided sequences with values in $\mathbb{Z}_{p^{2r}}$. Therefore

$$h(f) = h(\sigma_+) = 2r \log p, \quad p = \#S.$$

It is also shown that

$$h(f) = h_{\nu_+}(f)$$

where ν_+ is the uniform Bernoulli measure on Y.

Let us mention about entropy of the Coven transformation. As shown in [11] the topological entropy of it equals

$$h(f) = \log 2.$$

On the other hand, considering the Coven system on $X = \{0, 1\}^{\mathbb{Z}}$ Tisseur proved in [25] that $h_\nu(f) = 0$.

It is worth to recall another result of Tisseur given in [25]. Namely, if a CAs has equicontinuity points belonging to the support of the measure μ of the system, then $h_\mu(f) = 0$.

We close our considerations concerning entropy of one-dimensional CAs with the result given in [15], which says that the topological entropy of an arbitrary CAs is uncomputable.

It is worth to mention that for two-dimensional CAs, in contrast to the one-dimensional case, the topological entropy can be infinite. Namely, it is proved in [12] that for linear systems the topological entropy equals 0 (if the system is equicontinuous) or ∞ (if the system is sensitive to initial conditions).

In order to account for the space-time complexity of CAs, Milnor defined in [19] the concept of directional entropy (topological and measure-theoretic) which is a generalization of the above entropies.

For a definition and basic properties of the topological directional entropy we refer the reader to [4]. We denote by $h_{\vec{v}}(f)$ the topological directional entropy of f in the direction vector $\vec{v}$.

A very important property of the above entropy which is useful for the calculation or estimation of it is the continuity w.r.t the direction.

Smilie in [24] has shown that the topological directional entropy is not continuous.

We shall use in the sequel the measure-theoretic directional entropy and therefore we give now the definition of it and its elementary properties. Our definition is a slight modification of the Milnor one.

We consider an action Φ of the semigroup $\mathbb{Z} \times \mathbb{N}$ on X defined as follows

$$\Phi^g = \sigma^m \circ f^n, \quad g = (m,n) \in \mathbb{Z} \times \mathbb{N}.$$

Let μ be a probability measure invariant with respect to σ and f. Let $\mathcal{P}$ denote the family of all finite measurable partitions of X. For any $P \in \mathcal{P}$ and a bounded set $A \subset \mathbb{R} \times \mathbb{R}^+$ we put

$$P(A) = \bigvee_g \Phi^g P,$$

where g runs over the set $A \cap (\mathbb{Z} \times \mathbb{N})$.

For any $P \in \mathcal{P}$ we denote by $H_\mu(P)$ the entropy of P w.r.t μ. Now, we fix a vector $\vec{v} \in \mathbb{R}^2$ and we put

$$h^\mu_{\vec{v}}(f, P) = \sup_A \overline{\lim_{t \to \infty}} \frac{1}{t} H_\mu(P(A + [0,t)\vec{v})),$$

where the sup is taken over all bounded subsets $A \subset \mathbb{R} \times \mathbb{R}^+$.

The directional entropy of f in the direction $\vec{v}$ is defined by

$$h^\mu_{\vec{v}}(f) = \sup_{P \in \mathcal{P}} h^\mu_{\vec{v}}(f, P).$$

It is clear that $h^\mu_{\vec{v}}(f)$ is a generalization of the Kolmogoroff–Sinai entropy, namely

$$\text{If } \vec{v} = (p,q) \in \mathbb{Z} \times \mathbb{N}, \text{then } h^\mu_{\vec{v}}(f) = h_\mu(\sigma^p \circ f^q). \tag{1}$$

In particular,

$$h^\mu_{\vec{e}}(f) = h_\mu(f), \quad \vec{e} = (1,0).$$

An easy consequence of the definition is also the homogeneity of the mapping $\vec{v} \to h^\mu_{\vec{v}}(f)$, i.e.

$$h^\mu_{\alpha\vec{v}}(f) = \alpha h_{\vec{v}}(f), \quad \alpha \geq 0. \tag{2}$$

Park in [21] positively solved the Milnor problem [20] proving that

$$\text{the mapping } \vec{v} \to h^\mu_{\vec{v}}(f) \text{ is continuous.} \tag{3}$$

The above properties allowed us to get in [8] the following estimation of the directional entropy from above.

Theorem 1 *For any $\vec{v} = (x,y) \in \mathbb{R} \times \mathbb{R}^+$ we have*

$$h^\mu_{\vec{v}}(f) \leq \max(|z_l|, |z_r|) \log \#S, \quad \textit{if } z_l \cdot z_r \geq 0$$

and

$$h^\mu_{\vec{v}}(f) \leq |z_r - z_l| \log \#S, \quad \textit{if } z_l \cdot z_r \leq 0,$$

where $z_l = x + ly$ and $z_r = x + ry$.

4 Lyapunov Exponents of CAs

The classical concept of Lyapunov exponents belongs to the theory of smooth dynamical systems. These exponents describe the local instability of orbits of the given dynamical system. Here, one uses the stability in the sense of Lyapunov.

In one-dimensional extended systems, the localised perturbations may propagate to the left or to the right not only as traveling waves, but also as various structures.

Moreover, other phenomena, called convective instability, have been observed in a fluid flow in a pipe, where it has been found that the system propagates a variety of isolated and localised structures (or patches of turbulence) moving down the pipe along the stream with some velocity.

Convective instability has been studied by many authors in various fields (cf. [3]).

Here we consider Lyapunov exponents as quantities describing the maximal velocity of propagation to the right or the left of fronts of permutations in a frame moving with a given velocity. We consider this problem in the framework of one-dimensional cellular automata.

Lyapunov exponents in the theory of cellular automata were first introduced by Wolfram in [29]. The idea was to find a characteristic quantity of the instability of the dynamics of cellular automata analogous to the Lyapunov exponents in the theory of smooth dynamical systems.

The first rigorous mathematical definition of the above exponents was given by Shereshevsky [22] and then modified by Tisseur [25].

In this section we first recall the definition and basic properties of Lyapunov exponents introduced by Shereshevsky and then we give the definition and basic properties of directional Lyapunov exponents as in [9].

Let, as in the previous section, σ be the left shift and f the CA-transformation generated by the local rule F on X.

Let μ be a Borel probability measure on X invariant w.r.t σ and f.

Following Shereshevsky [22] we put

$$W_s^+(x) = \{y \in X \mid y_i = x_i, i \geq s\},$$
$$W_s^-(x) = \{y \in X \mid y_i = x_i, i \leq -s\},$$

$x \in X, s \in \mathbb{Z}$.

It is easy to check that

$$\sigma^a W_c^{\pm}(\sigma^b x) = W_{c\pm a}^{+}(\sigma^{a+b} x), \tag{4}$$

for any $a, b, c \in \mathbb{Z}, x \in X$.

Simple induction argument allows to show that, for any $n \in \mathbb{N}$ it holds

$$f^n(W_0^+(x)) \subset W_{-nl}^+(f^n x), \quad f^n(W_0^-(x)) \subset W_{nr}^-(f^n x). \tag{5}$$

Therefore, for any $n \in \mathbb{N}$ there exists $s \in \mathbb{N}$ with

$$f^n(W_0^{\pm}(x)) \subset W_s^{\pm}(f^n x). \tag{6}$$

Namely, applying the simple observation that the families $(W_n^{\pm}(x))$ are increasing, it suffices to take $s = \max(0, -nl)$ in the case of $W_0^+(x)$ and $s = \max(0, nr)$ in the case of $W_0^-(x)$.

Due to (6) one can introduce the following definition

$$\widetilde{\Lambda}_n^{\pm}(x) = \inf\{s \geq 0 \mid f^n(W_0^{\pm}(x)) \subset W_S^{\pm}(f^n x)\}$$

and next

$$\Lambda_n^{\pm}(x) = \sup_{j \in \mathbb{Z}} \widetilde{\Lambda}_n^{\pm}(\sigma^j x), \quad x \in X, n \in \mathbb{N}.$$

It is clear that

$$0 \leq \Lambda_n^+(x) \leq \max(0, -nl), \quad 0 \leq \Lambda_n^-(x) \leq \max(0, nr), \quad n \in \mathbb{N}. \tag{7}$$

It is shown in [22] the following cocycle inequality

$$\Lambda_{n+m}^{\pm}(x) \leq \Lambda_n^{\pm}(x) + \Lambda_m^{\pm}(f^n x), \quad n, m \in \mathbb{N}, x \in X. \tag{8}$$

Applying it and the Kingman sub-additive ergodic theorem (cf. [18]) one can define a.e. the following functions

$$\lambda^{\pm}(x) = \lim_{n \to \infty} \frac{\Lambda_n^{\pm}(x)}{n}, \quad x \in X.$$

The function λ^+ (resp. λ^-) is called the right (left) Lyapunov exponent of f.

Both functions are of course σ-invariant and, by the Kingman theorem also f-invariant.

From (7) we have at once the following estimations

$$0 \leq \lambda^+(x) \leq \max(0, -l), \quad 0 \leq \lambda^-(x) \leq \max(0, r). \tag{9}$$

Example 1 (Permutative CAs) It is easy to check that if f is left permutative, then $\lambda^+ = \max(-l, 0)$ and if it is right permutative, then $\lambda^- = \max(r, 0)$.

Example 2 (Coven CAs) Taking $r = 2$ and $B = 01$ Tisseur has shown in [25] that

$$\lambda^+ = 0, \quad \lambda^- = 2.$$

Example 3 (Linear CAs) D'amico, Manzini and Margara have shown in [12] that

$$\lambda^+ = -\min\{L_1, \ldots, L_s\}, \quad \lambda^- = \max\{R_1, \ldots, R_s\}.$$

Example 4 (Positively Expansive CAs) Finelli, Manzini and Margara have proved in [13] that if f is positively expansive then there exists $c > 0$ s.t. $\lambda^{\pm} \geq c$ a.e.

Now, we recall the concept of Lyapunov exponents given by Tisseur in [25]. The space $X = S^{\mathbb{Z}}$ is equipped with a probability measure μ which is invariant w.r.t the shift σ, ergodic and the support $S(\mu)$ of which is invariant w.r.t f. The configuration space of the CAs is a sub-shift $Y \subset X$ which is f-invariant.

For a fixed $x \in Y$ one considers the sets $\widetilde{W}_s^{\pm}(x)$ defined by

$$\widetilde{W}_s^{+}(x) = W_s^{+}(x), \quad \widetilde{W}_s^{-}(x) = W_{-s}^{-}(x), \; s \in \mathbb{Z}$$

and the functions $I_n^{\pm}(x)$ given as follows

$$I_n^{+}(x) = \min\{s \geq 0 \mid f^i(\widetilde{W}_{-s}^{+}(x) \subset \widetilde{W}_0^{+}(f^i x), 1 \leq i \leq n\}$$

and

$$I_n^{-}(x) = \min\{s \geq 0 \mid f^i(\widetilde{W}_{-s}^{-}(x) \subset \widetilde{W}_0^{-}(f^i x), 1 \leq i \leq n\}.$$

The limits

$$I_\mu^{\pm} = \liminf_{n\to\infty} \frac{1}{n} \int_X I_n^{\pm}(x)\mu(dx)$$

are called the average Lyapunov exponents.

It is proved that $I_\mu^{\pm} \leq \lambda^{\pm}$.

Now, we shall consider the concept of space-time directional Lyapunov exponents which are generalizations of the exponents defined by Shereshevsky. They are defined as averages along a given space-time direction of propagation to the left (resp. right) of a front of right (resp. left) perturbations of a given configuration.

Let, as before, X be the space $S^{\mathbb{Z}}$ of sequences with values in $S = \{0, 1, \ldots, p-1\}$, $p \geq 2$ and let σ and f be the left shift and the CA-transformation, generated by a local rule $F : \prod_{i=l}^{r} S_i \to S$, $S_i = S, l \leq i \leq r$, respectively.

Let $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$ be fixed and let $\alpha(t)$, $\beta(t)$ be integer parts $[at]$ and $[bt]$ of at and bt, respectively.

We put

$$z_l = a + bl, \quad z_r = a + br.$$

Applying (1) and (2) and the fact that σ and f commute we obtain

$$\sigma^{\alpha(t)} f^{\beta(t)} W_0^{+}(x) \subset W^{+}_{-\alpha(t)-\beta(t)\cdot l}(\sigma^{\alpha(t)} f^{\beta(t)} x) \tag{10}$$

and

$$\sigma^{\alpha(t)} f^{\beta(t)} W_0^{-}(x) \subset W^{-}_{\alpha(t)+\beta(t)\cdot r}(\sigma^{\alpha(t)} f^{\beta(t)} x). \tag{11}$$

Similarly, as in the previous section we define

$$\widetilde{\Lambda}^{\pm}_{\vec{v},t} = \inf\{s \geq 0 \mid \sigma^{\alpha(t)} f^{\beta(t)} W_0^{\pm}(x) \subset W_s^{\pm}(\sigma^{\alpha(t)} f^{\beta(t)} x)\}$$

and

$$\Lambda^{\pm}_{\vec{v},t}(x) = \sup_{j\in\mathbb{Z}} \widetilde{\Lambda}^{\pm}_{\vec{v},t}(\sigma^j x).$$

The following inequalities are obvious

$$0 \leq \Lambda^{+}_{\vec{v},t}(x) \leq \max(-\alpha(t) - \beta(t)\cdot l, 0), \tag{12}$$

$$0 \le \Lambda^{-}_{\vec{v},t}(x) \le \max(\alpha(t) + \beta(t) \cdot r, 0). \tag{13}$$

Since

$$\frac{\alpha(t)}{t} \to a, \qquad \frac{\beta(t)}{t} \to b$$

as $t \to \infty$, the functions

$$\lambda^{\pm}_{\vec{v}}(x) = \overline{\lim_{t\to\infty}} \frac{1}{t} \Lambda^{\pm}_{\vec{v},t}(x), \quad x \in X \tag{14}$$

are well defined and

$$0 \le \lambda^{+}_{\vec{v}}(x) \le \max(-z_l, 0), \quad 0 \le \lambda^{-}_{\vec{v}}(x) \le \max(z_r, 0). \tag{15}$$

The function $\lambda^{+}_{\vec{v}}$ (resp. $\lambda^{-}_{\vec{v}}$) is said to be the right (resp. left) space-time directional Lyapunov exponent.

As an easy consequence of (14) is the following

Lemma 1 *For any $x \in X$ the function $\vec{v} \to \lambda^{\pm}_{\vec{v}}$ is positively homogeneous, that is, for any $c \ge 0$ it holds*

$$\lambda^{\pm}_{c\vec{v}}(x) = c\lambda^{\pm}_{\vec{v}}(x).$$

The next property considered by us is the cocycle inequality for the function $\Lambda^{\pm}_{\vec{v},t}$, $t \in \mathbb{N}$.

Lemma 2 *For any $s, t \in \mathbb{N}$, $x \in X$ and $\vec{v} \in \mathbb{R} \times \mathbb{R}^{+}$ we have*

$$\Lambda^{\pm}_{\vec{v},s+t}(x) \le \Lambda^{\pm}_{\vec{v},s}(x) + \Lambda^{\pm}_{\vec{v},t}(f^{\beta(s)}x) + 2|l|.$$

The proof goes similarly as the proof of (8). Lemma 2 enables us to show the following

Proposition 1 *For any $\vec{v} \in \mathbb{R} \times \mathbb{R}^{+}$ and almost all $x \in X$ there exist the limits*

$$\lim_{t\to\infty} \frac{1}{t} \Lambda^{\pm}_{\vec{v},t}(x) = \lambda^{\pm}_{\vec{v}}(x).$$

Proof To prove Proposition 1 one first considers the case $\vec{v} = (a, b)$, $b = 1$. Then the cocycle inequality admits the form

$$\Lambda^{\pm}_{\vec{v},s+t}(x) \le \Lambda^{\pm}_{\vec{v},s}(x) + \Lambda^{\pm}_{\vec{v},t}(f^{s}x) + 2|l|, \tag{16}$$

$s, t \ge 0$. Hence, taking the sequences

$$l^{\pm}_{t}(x) = \Lambda^{\pm}_{\vec{v},t}(x) + |l|, \quad t \in \mathbb{N},$$

we obtain from (16) the inequality

$$l^{\pm}_{s+t}(x) \le l^{\pm}_{s}(x) + l^{\pm}_{t}(f^{s}x), \quad s, t \in \mathbb{N}.$$

Applying the Kingman sub-additive ergodic theorem we see that the following limits exist a.e.

$$\lim_{t\to\infty} \frac{l^{\pm}_{t}(x)}{t} = \lim_{t\to\infty} \frac{\Lambda^{\pm}_{t}(x)}{t} = \lambda^{\pm}_{\vec{v}}(x).$$

The arbitrary case easily reduces to the above by Lemma 1. $\square$

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