Preface

Despite its long history and its stunning experimental success, the mathematical foundation of perturbative quantum field theory (pQFT) is still a subject of ongoing research. This book aims at presenting some of the most recent developments in the field, and at reflecting the diversity of the approaches and the tools that have been invented and that are used. Some of the leading experts as well as newcomers in the field present their latest advances in the attempt for a better understanding of quantum, but also classical field theories.

The chosen material is, however, far from complete. As mentioned in the first foreword, the idea for this book grew out of a school in Les Houches on the subject, most lecturers agreeing to write a contribution. This then was complemented by selecting some of the customary young-participant-presentations to contribute, too, as well as by two, three additional invited articles. And, as mentioned in the second foreword, even though the book is aimed both at mathematicians and physicists, it is more oriented toward the mathematical developments. Here, it is maybe a pity that for example Nekrasov's lectures about the path integral on $N = 2$ supersymmetric gauge theories did not find entry into the present addition. But there are many more promising directions, which did not, at least one of which shall be mentioned below. Maybe this can be a reason to come back to the enterprise at a later point again, summarizing also those aspects, and possibly updating the ones which are contained in the present edition.

On this occasion, we use the opportunity to thank Jürg Fröhlich for his valuable physical insight arising from decades of own original work on the forefront of the subject and his complementary remarks on the physics that is involved in the mathematical descriptions, even if it is partially only in terms of some keywords due to lack of "space-time". In the winter school, there was in addition an inspiring opening lecture of another great person in the field, Ludwig Faddeev, who commented on his perspective on the still open one-million dollar Clay problem "Yang–Mills Existence and Mass Gap". His lecture had been published already in a similar form elsewhere, so that we briefly summarize it here only—also since it suits well to explain the problematic of the subject.

The task to win the prize may sound deceptively simple: essentially, one is asked to prove that Yang–Mills theory (for a semi-simple compact structure group G) *exists* (in its quantum version) and that there is a minimal mass for the spectrum of particles. Deceptively simple, since from the physical perspective there is absolutely *no* doubt that this theory exists (at least for $G = SU(2)$ or $G = SU(3)$); it is one of the corner stones of the standard model of elementary particles, verified experimentally to an incredible precision, as also emphasized by Fröhlich in his foreword. However, as pointed out by Faddeev in his lecture (but also independently by R. Jackiw), the problematic becomes already more evident if one notices that the underlying classical Yang–Mills theory is conformal, i.e., scale-invariant in four space-time dimensions. One way of seeing this is that the overall coupling constant does not carry any physical units in precisely this dimension. On the other hand, any mass of a particle to be specified in a physical theory needs to refer to some standard mass (like 1 kg). The definition of the theory does not carry any such a mass (or, equally, length) scale on the outset of the problem, i.e., in its classical formulation in terms of an action functional.

According to Faddeev, the remedy can lie only in the usually so unloved infinities encountered typically in interacting quantum field theories (QFTs). Those infinities that plagued the foundators of the theory, subsequently were handled with increasing success in a more or less well-founded theory of perturbative renormalization, but which still cost many contemporary students of theoretical and mathematical physics a large number of unpleasant hours; the latter fact is the case, since in particular in standard physics lectures on QFTs, often *the* experimentally verified *end is used to justify the* mathematical *means*, with a mathematical argumentation that either appears inconsistent or otherwise at least arbitrarily ad-hoc. On the other hand, the necessary regularization of the theory on the quantum level will introduce a length-scale, and in this way there can be hope that the resulting quantum Yang–Mills theory can yield a minimal mass in a well-defined way.

To formulate a mathematically well-defined and conceptually convincing regularization and renormalization scheme is one of the tasks of a mathematical approach to quantum field theory (QFT). But it goes even further: one wants the theory to satisfy a minimal number of axioms that seem to be enforced by compatibility with for example special relativity, running in part under the name of (Einstein) "locality" in this context. More precisely, basic considerations require a number of properties any "physically acceptable" QFT should satisfy. One version of such a set of axioms is the one formulated by Wightman. It contains for example (projective) equivariance of the quantum fields of the theory with respect to the action of the Poincare group (the isometry group of Minkowski space in four dimensions). Later, it was permitted also to trade in Euclidean four-space for the physical Minkowski space; the idea of the so-called Osterwalder–Schrader axioms being then that mathematically the theory is easier to define and the physical interpretation results in a second step by an appropriate analytic continuation, called Wick rotation in physics. The formulation of the Clay prize requires to define 4d quantum Yang–Mills theory with a rigor of *at least* such axioms.

The only problem here is that up to now there is not a *single* known interacting quantum field theory in four dimensions satisfying such a typical set of axioms; there are only examples of such theories in two or three space-time dimensions, which have, however, no physical significance and are (to be) considered as so-called "toy-models" only. As Max Kreuzer from the Technical University of Vienna used to say, torturing herewith some of the more mathematical-conceptually oriented students (all the more since the statement is true, at least from a physicist perspective): "The only theories satisfying the Wightman axioms are free theories." A free theory is one that physically corresponds essentially to a single particle travelling alone through empty space not subject to any interactions and thus not subject to any experimental observations or tests. Clearly, this is highly dissatisfying, all the more, since the formulated axioms, in one or the other form, seem more or less unavoidable from a point of view of principles governing our *contemporary* understanding of quantum field theory.

At this point, we want to mention one of the unfortunate omissions of this volume, all the more since it contains a glimpse of hope for possibly finding an interacting QFT in four dimensions after all. The omission comes from a recent direction motivated by String Theory (but not only!) to consider QFTs on so-called noncommutative space-times. In fact, the idea is already quite old and pursues the goal that the "fuzzyness" of the underlying space resulting from non-commuting space(-time) coordinates could cure the problem of the UV-(or "high energy"/"small distance") divergencies of QFTs mentioned already above. In the simplest setting, the commutator of the coordinates is a constant matrix Θ, corresponding to the deformation quantization of a constant Poisson tensor (in flat space). The resulting product of functions on space-time can then be described by the Moyal product of Θ. In this way, the classical action functional of the theory under investigation is replaced by one that is an infinite formal power series in Θ, reducing to the original functional for $\Theta = 0$. It is then *this* new functional to be used for the "quantization", i.e., as a starting point of the construction of a pQFT.

Although first considerations indeed show improvement of the UV-behavior, it turns out that the problem is not solved in many cases (keyword "UV/IR"-mixing) and the original hype on the study of such theories seems to have decreased over the last years again. However, there is one proposal, the so-called Grosse–Wulkenhaar model, that resists many of the problems of other theories considered in this context and now even gives some hope to lead to a well-defined interacting QFT in four dimensions (although it is still too early to make this statement, there are at least several indications that look promising). One important issue to address at this point is that certainly the introduction of the tensor Θ on Minkowski or Eucildean space spoils its covariance. However, in a simultaneous limit sending Θ as well as the volume (made finite for an IR-regularization) to ∞, it was shown to lead to a covariant and local theory on Euclidean fourspace for this model. Reinterpreting thus this matrix Θ as another way of regularizing the theory, one is led to an apparently consistent, non-trivial quantum version of the φ^4-theory in four dimensions. The Wick rotation to Minkowski signature is a problem still under investigation on the day of this writing, while preliminary computer simulations in this direction seem promising.

The remarks of the introduction up to here aimed at a complementary argumentation to the one of Fröhlich of why one would wish to have a mathematically well-founded theory of quantum fields describing known physics at high energies. Even on the level of perturbation theory, i.e., in terms of formal power series, the situation concerning physically relevant theories in this context is far from satisfactory. Theories of physical relevance are in some sense of quite a different nature than those of relevant mathematical impact: while the first ones are characterized by so-called "propagating degrees of freedom", the latter ones are mostly of "topological" nature. Essentially or at least in a first approximation, the difference lies in the dimension of the (generic part of the) moduli space of (classical) solutions to the Euler–Lagrange equations of the theory modulo its gauge symmetries. For physically relevant theories, this needs to be an infinite-dimensional space, reflecting the fact that physically observable excitations describing elementary particles can be generated locally everywhere in space-time, while for topological models this space is usually finite-dimensional. On the quantum level, the latter type of theories are then called topological quantum field theories (TQFTs).

One of the most famous examples of a TQFT, if not the most famous one among mathematicians, is the Chern–Simons theory. The major breakthrough was made by Witten, who observed that one could recover link and three-manifold invariants *via* the path integral quantization of the Chern–Simons classical action functional. The so-called A- and B-models are other famous examples of TQFTs. They are related by mirror symmetry to one another, a notion originating from physical intuition, relating seemingly different, but in the end equivalent quantum string theories. Mirror symmetry and its relation to enumerative and algebraic geometry became a major research area of pure mathematics by itself in the mean time.

A generalization of the A- and B-model is the Poisson sigma model (PSM), celebrating its twentieth anniversary this year. It was discovered in the context of toy models of coupled gravity and Yang–Mills theories defined on two-dimensional space-time manifolds Σ (Ikeda and Schaller–Strobl). Already at this very beginning it was realized that the quantization of the PSM is intimately related to the quantization of the target Poisson manifold—applying a particular non-perturbative quantization scheme to this theory, the integrality condition of geometric quantization pops up for the symplectic leaves of the Poisson target (cf also Alekseev–Schaller–Strobl). However, only in an unparalleld work of Kontsevich it was observed that already the perturbative quantization of the PSM on a trivial world-sheet topology solves the by then longstanding problem of deformation quantization of Poisson manifolds, leading him to his famous formality theorem (several steps of this procedure were retraced in a series of works by Cattaneo-Felder).

This is a good example of the use of (T)QFTs in mathematics: one trades in the apparently simpler problem of quantization of a Poisson structure on $\mathbb{R}^n$ for the quantization of a *field theory* the target of which is this Poisson manifold $M = \mathbb{R}^n$. This now is an infinite dimensional space, the functional being defined over vector bundle morphisms from $T\Sigma$ to T^*M. Moreover, one needs to factor out an infinite-dimensional gauge group, the quotient yielding in general a complicated, singular, but in this case finite-dimensional space. However, it turns out that the application of *standard* techniques developed in the context of perturbative QFTs with gauge symmetries leads to formulas relevant to the finite dimensional target space that otherwise proved resistant over decades for being invented directly!

The flow is expected to also go into the other direction, however, i.e., one expects to learn from TQFTs and related mathematics for how to sharpen our approaches for the construction of physically more relevant QFTs. It is in this spirit instance that Tamarkin wrote a 100 pages paper only about the renormalization of the PSM—in a standard physics approach the perturbative renormalization of such a topological model would be dealt with in at most a few paragraphs. The functorial approach to TQFTs, as developed also at the examples of topological strings (like the A- and B-model), led to an axiomatic definition of them in terms of the Atyah–Segal axioms. In the lectures of Fredenhagen about the formulation of pQFTs on curved space-times of Lorentzian signature one finds a reformulation of standard QFT axioms closely related to such a functorial perspective.

For the present, as mentioned rather mathematically oriented volume on QFT (cf. also the foreword of Fröhlich), this is maybe one of the main perspectives from our editors' side to its contributions: the hope that, *on the long run*, topological models and mathematics in general can have something to say about (also physically relevant) QFTs. It is thus not so surprising that one out of in total four parts to this book is devoted to mathematics around the Chern–Simons theory. Subsequent to Witten's work, Reshetikhin and Turaev proposed a rigorous mathematical construction of a (nonperturbative!) quantization of the Chern–Simons theory in terms of quantum groups and modular tensor categories. And despite this great achievement, there are many questions that remain open in the context of Chern–Simons theory, both of computational and theoretical nature.

In the context of the PSM, on the other hand, one seems still quite far from a nonperturbative quantization. So, this model is not yet really defined as a TQFT—in the sense of the Atiyah–Segal axioms, although there is no serious doubt that such a formulation should exist. However, already now the PSM teaches us at least two more lessons related to the present volume: First, as found by Cattaneo and Felder, the reduced phase space of the PSM, i.e. its Weinstein symplectic quotient, when smooth, carries the structure of a symplectic groupoid (cf. also the contribution of I. Contreras to this volume). And this groupoid is precisely the one that integrates the Lie algebroid T^*M associated to the target Poisson manifold M, a construction suggesting the one needed for the integration problem of general Lie algebroids to Lie groupoids, finally solved by Crainic and Fernandes (in the sense of necessary and sufficient conditions for a smooth integration to exist). This is only one of the examples for a renewed interest in geometrical questions related to field theories already on the *classical* side. Such an understanding of the classical theory is also important in order to identify the difficulties specific to the quantum side when trying to provide rigorous constructions of QFTs. One of the four parts to this book is thus devoted to merely classical or semi-classical investigations of field theories.

Second, the PSM can be viewed as a Chern–Simons theory for the Lie algebroid T^*M: while the integrand of the Chern–Simons theory for an ordinary Lie algebra arises as a transgression of the Pontryagin class, "$tr(F \wedge F) = \mathrm{d}(CS)$", likewise the integrand of the PSM relates to a characteristic 3-form class, "$F^i \wedge F_i = \mathrm{d}(PSM)$" where here F corresponds to the obstruction of the vector bundle morphism $T\Sigma \to T^*M$ to be a Lie algebroid morphism—it has a 1-form part F^i (from the base map) in addition to a standard 2-form part for curvatures. In fact, there is a topological sigma model that reduces to the PSM in two dimensions and includes the Chern–Simons theory in three, and this is the so-called AKSZ sigma model (after Alexandrov–Kontsevich–Schwarz–Zaboronski; cf. the contribution of Bonavolontà–Kotov as well as the introduction of one of us to this book); and even the relation to higher characteristic classes extends to those (Kotov–Strobl, cf. also Fiorenza–Rogers–Schreiber as well as the contribution of Fiorenza–Sati–Schreiber to this volume). In general, there is a—to our mind useful—trend to higher structures in theories of relevance to mathematical physics and this is also reflected partially in the present book.

One of the, from a mathematical point-of-view, most well-understood classes of QFTs which are *not* topological consists of 2-dimensional conformal field theories (CFTs). In this context the axiomatization of the operator product expansion has led to the notion of vertex (operator) and chiral algebras, which are now widely used both in mathematics and physics. There have been several attempts to generalize these and base the axiomatics of perturbative QFT and the renormalization procedure on the operator product expansion: Kontsevich (unpublished), Hollands, and Costello–Gwilliam (see e.g., the contribution of Costello–Scheimbauer to this volume) in the Euclidean context, Fredenhagen et al in the Lorentzian context (cf. the contribution of Fredenhager–Rejzner to this volume). All these approaches share two things: the appearance of a pattern resembling the one of little disk operads, which axiomatizes the physical concept of "locality," and the use of techniques from deformation quantization.

The concept of locality in $2d$ conformal field theory can also be formulated by *defining* a CFT as a functor from a suitable category of cobordisms to vector spaces (cf. Atiyah–Segal) satisfying certain properties. The foundational work of Beilinson–Drinfeld on chiral algebras exhibits a close relation between these two approaches to the concept of locality: namely, *chiral homology* associates a CFT *à la* Atiyah–Segal with any (conformal) vertex algebra. Recently, Lurie defined a topological analog of chiral homology, known as *factorization homology*: it assigns a TQFT to any algebra over the little n-disk operad, or to any E_n-algebra (cf. contributions of Markarian and Tanaka to this volume for an approach to Chern–Simons theory using factorization homology), which can be proven to be *fully extended* (Scheimbauer). Fully extended TQFTs are known, after the *cobordism hypothesis* (Lurie, Baez–Dolan), to be the "most local" TQFT (cf. also the contibutions of Fiorenza–Sati–Schreiber and Cattaneo–Mnev–Reshetikhin to this volume). Factorization/chiral homology can actually be defined for any factorization algebra (Costello–Gwilliam); a new concept that encompasses the ones of E_n-, vertex and chiral algebras, and whose definition was designed to encode the general algebraic structure of local observables of an arbitrary field theory. It has applications that range from conjectures on renormalization of lattice models (cf. the introduction of one of us to this book) to algebraic topology (cf. Ginot's contribution to the last part of this volume).

We now give a very brief overview on the contents of the book, which starts with an introductory chapter that emphasizes the importance of derived and homotopical (or higher) structures in the mathematical treatment of TQFTs.

Summary of Part I

The first Part is about local aspects of perturbative quantum field theory, with an emphasis on the axiomatization of the algebra behind the operator product expansion and the ideas coming from deformation quantization techniques.

It begins with a Chapter, by Fredenhager–Rejzner, summarizing the approach that was developed for the Lorentzian signature and applicable to also curved (globally hyperbolic) space-times, applying a quantization procedure to QFT by adapting deformation quantization to its setting. It then continues with a contribution, by Costello–Scheimbauer, on partially twisted supersymmetric four-dimensional gauge theories that are studied using the foundational work of Costello and Costello–Gwilliam. The last chapter, written by Wendland, is a short review of Conformal Field Theory, summarizing in particular recent progress made in that field and its relation to the geometry of $K3$ surfaces and *Mathieu moonshine*.

Summary of Part II

The second Part focuses on Chern–Simons (CS) gauge theories.

It begins with a Chapter of Andersen–Kashaev on a construction of $SL(2, \mathbb{C})$ quantum CS theory by means of Teichmüller theory and the quantum dilogarithm of Faddeev. This is followed by a Chapter of Fiorenza–Sati–Schreiber, exhibiting higher structures in a systematic way in the context of an extended prequantum theory of CS-type gauge field theories. The subsequent Chapter consists of two contributions, one by Markarian and one by Tanaka, and deals with the relation between three-dimensional CS theory and factorization homology. Part II is completed by a review of Thuillier about the use of Deligne–Beilinson cohomology for an alternative or deepened understanding of abelian $U(1)$ CS theory.

Summary of Part III

The third Part of this book is devoted to a classical or at most semi-classical analysis of field theories.

It begins with a Chapter of Cattaneo–Mnev–Reshetikhin, introducing some very recent work on the treatment of constraints and boundary conditions in classical field theories, with an emphasis on the BV and BFV formalism. The subsequent contribution, written by Kotov–Bonavolontà, deals with the BV-BRST formalism in the context of AKSZ sigma models, improving previous local results to a global level. The following Chapter of Li-Bland–Ševera provides a beautiful treatment of the (quasi-)Hamiltonian and Poisson geometry of various moduli spaces of flat connections on quilted surfaces, which are relevant in classical Chern–Simons and WZW theories. The final Chapter of this Part aims at understanding the construction of the sympletic groupoid associated to the PSM from the axiomatics of Frobenius algebras.

Summary of Part IV

The fourth Part consists of a single Chapter written by Ginot. It provides a detailed account of the mathematical foundations of Factorization Algebras and Factorization Homology, making extensive use of higher homotopical structures, thus closing the circle opened in the introductory Chapter.

We would like to conclude this preface with a quotation from the Clay Institute's official description of the "Yang–Mills existence and Mass gap" problem as formulated by Arthur Jaffe and Edward Witten:

> *… one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition of quantum gauge theory in four dimensions. Will this change in the 21st century? We hope so*!

We wholeheartedly share this wish, and hope in turn that some of the mathematical concepts presented in this book will help to better understand, one day, quantum field theories in four dimensions.

France

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