Chapter 1
Introduction

1.1 Mixed Hodge Modules

Let us briefly review a stream in the Hodge theory. It is concerned with the functoriality with respect to various operations, and it was finally accomplished with the most great generality by the theory of mixed Hodge modules due to M. Saito. The author regrets that this review is quite restricted by his personal interest, and that it is not exhaustive. The readers can find more thorough reviews in [21, 62] and the recent book "Hodge Theory" edited by E. Cattani, F. El Zein, P.A. Griffiths, and L.D. Tràng (Princeton University Press). We also refer to [72, 74, 75] for the introduction to mixed Hodge modules.

A variation of Hodge structure on a complex manifold X is a pair of $\mathbb{Q}$-local system $L_{\mathbb{Q}}$ of finite rank and a Hodge filtration F. Here, a Hodge filtration is a decreasing filtration of holomorphic subbundles of $L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_X$ indexed by integers satisfying the Griffiths transversality. Although we may replace $\mathbb{Q}$ with other algebras such as $\mathbb{Z}$ and $\mathbb{R}$, we omit such details here. A variation of Hodge structure is called pure of weight w if each restriction $(L_{\mathbb{Q}}, F)_{|Q}$ $(Q \in X)$ is a pure Hodge structure of weight w. It is called polarizable if moreover it admits a polarization, i.e., there exists a flat $(-1)^w$-symmetric pairing S of $L_{\mathbb{Q}}$ such that each $S_{|Q}$ $(Q \in X)$ is a polarization of the pure Hodge structure $(L_{\mathbb{Q}}, F)_{|Q}$. A variation of mixed Hodge structure is a variation of Hodge structure $(L_{\mathbb{Q}}, F)$ with a weight filtration W of $L_{\mathbb{Q}}$ which is an increasing filtration indexed by integers, such that $\mathrm{Gr}^W_w(L_{\mathbb{Q}})$ $(w \in \mathbb{Z})$ with the induced Hodge filtration F are pure of weight w. The variation of mixed Hodge structure is called graded polarizable if each $\mathrm{Gr}^W_w(L_{\mathbb{Q}}, F)$ is polarizable. In this paper, we almost always impose the polarizability (resp. the graded polarizability) to variations of pure (resp. mixed) Hodge structure. So, we often omit the adjective "graded polarizable".

The notion of polarized variation of Hodge structure was originally discovered by P. A. Griffiths as *something* on the Gauss-Manin connections associated to smooth projective families of varieties. This can already be regarded as one of the most basic

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and interesting cases of the functoriality of Hodge structure for the push-forward by any smooth projective morphism. The seminal work of Griffiths opened several interesting research projects, for example, the study of polarized variation of Hodge structure with singularity, which we will return later.

Inspired by the dream of motives, P. Deligne discovered the notion of mixed Hodge structure, and he proved a deep theorem which ensures that the cohomology group of any complex algebraic variety is naturally equipped with a mixed Hodge structure. This can be regarded as one of the most important cases of the functoriality of mixed Hodge structure. He also proved the functoriality in various cases. For example, if we are given a smooth projective family of varieties $f : \mathcal{Y} \longrightarrow S$ and a graded polarizable variation of mixed Hodge structure $(L_{\mathbb{Q}}, F, W)$ on $\mathcal{Y}$, then it was proved that the local system $R^i f_*(L_{\mathbb{Q}})$ is equipped with the naturally induced Hodge filtration F and weight filtration W and that $(R^i f_* L_{\mathbb{Q}}, F, W)$ is graded polarizable variation of mixed Hodge structure. He also observed crucial properties of the induced variation of mixed Hodge structure, including the Hard Lefschetz Theorem in the pure case. His insight has been quite influential on the subsequent works. (See [11, 14–16] for instance for more details on his work).

It is natural to ask what happens in the other more general cases. For example, if we are given a polarizable variation of Hodge structure $(L_{\mathbb{Q}}, F)$ on a quasi projective variety Y which is not extendable on any projective completion of Y, it is asked whether the cohomology group $H^p(Y, L_{\mathbb{Q}})$ or its variant may have mixed or pure Hodge structure. It is also natural to ask what happens for the singular family of varieties. Finally, all of these questions were answered by the functoriality of mixed Hodge modules. But, historically, it was first investigated with the L^2-method on the basis of the study of the asymptotic behaviour of polarized variations of pure Hodge structure and admissible variations of mixed Hodge structure. The study of asymptotic behaviour is also fundamental as the foundation of the theory of mixed Hodge modules.

As mentioned, the work of Griffiths naturally lead to the study of polarized variations of pure Hodge structure with singularity. An extremely important contribution was done by Schmid [77]. Let $X := \{(z_1, \dots, z_n) \mid |z_i| < 1\}$ and $D := \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Let $(L_{\mathbb{Q}}, F)$ be a polarizable variation of pure Hodge structure on $X \setminus D$. For simplicity, we assume that the local monodromy automorphisms around any irreducible components of D are unipotent. The nilpotent orbit theorem of Schmid ensures that, around any $P \in D$, the polarized variation of Hodge structure can be approximated by an easier one called a nilpotent orbit. It is not only interesting itself but also the most important foundation for the further investigation. One of the important consequences is that $(L_{\mathbb{Q}}, F)$ given on $X \setminus D$ naturally induces a nice object on X. Namely, let (V, ∇) be the Deligne extension of $L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{X \setminus D}$ on X, i.e., V is the locally free $\mathcal{O}_X$-module with a logarithmic connection ∇ such that (1) $(V, \nabla)_{|X \setminus D} = L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{X \setminus D}$, (2) the residues of ∇ are nilpotent. (See [12] for more details on the Deligne extension). Then, the nilpotent orbit theorem implies that F is extended to a filtration of V by holomorphic subbundles. In the case $n = \ell = 1$, the study of singular polarized variation of pure Hodge structure was accomplished by his SL(2)-orbit theorem, which ensures that the polarized variation of Hodge structure can be approximated by an easier one called an SL(2)-orbit. As a consequence, in the one variable case, he obtained that the weight filtration of the nilpotent part of the local monodromy controls the growth order of the norms of flat sections with respect to the Hermitian metric associated to the polarization. He also obtained the polarized mixed Hodge structure from the asymptotic data around the singularity, which is called the limit mixed Hodge structure. Note that, for the polarized variation of pure Hodge structure associated to a degenerating family of smooth projective varieties, the asymptotic behaviour was intensively studied by Steenbrink with a different method [86].

The higher dimensional case was accomplished by the definitive works by Cattani and Kaplan [7], Cattani et al. [8, 9], Kashiwara [31] and Kashiwara and Kawai [35]. In the above situation, for each point $P \in D$, we obtain the limit mixed Hodge structure with the induced bi-linear form from the asymptotic data around P, which is a polarized mixed Hodge structure in several variables. It turned out that the limit mixed Hodge structure controls the behaviour of $(L_{\mathbb{Q}}, F)$ around P. They obtained a generalization of the norm estimate. They also obtained a rather strong constraint on the nilpotent parts of the local monodromy along the loops around $\{z_i = 0\}$ $(i = 1, \dots, \ell)$. Moreover, they proved various interesting properties of polarized mixed Hodge structures, which are significant for their study on the L^2-cohomology group associated to any polarized variation of Hodge structure. Although it requires much more preparation to describe their results precisely, which we do not intend here, they are quite impressive.

As for singular graded polarizable variations of mixed Hodge structure, it was one of the main issues to clarify what condition should be imposed at the boundary. Thanks to the studies of Kashiwara [32], Steenbrink and Zucker [87] and Zucker [89], it turned out that the admissibility condition is the most appropriate one. Let us recall it in the case that $X = \{(z_1, \dots, z_n) \mid |z_i| < 1\}$ and $D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$ as above. Let $(L_{\mathbb{Q}}, F, W)$ be a graded polarizable variation of mixed Hodge structure on $X \setminus D$. For simplicity, suppose that the monodromy g_i along the loops around $\{z_i = 0\}$ are unipotent. Let $N_i := \log g_i$. Let (V, ∇) be the Deligne extension of $L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{X \setminus D}$, which is naturally equipped with the flat filtration W. We should impose that the filtration F is extended to a filtration of V by holomorphic subbundles such that $\mathrm{Gr}^F \mathrm{Gr}^W(V)$ is a locally free $\mathcal{O}_X$-module. We should also impose the existence of a relative monodromy weight filtration $M(N_i; W)$ of N_i with respect to the induced filtration W on the space of the multi-valued flat sections of $L_{\mathbb{Q}}$. It was introduced by Steenbrink-Zucker in the one variable case, by Kashiwara in the higher dimensional case. Note that Kashiwara clarified many issues to ensure that the condition is good. (The condition here is not equal but equivalent to that in [32], by results in [32].) Moreover, Kashiwara introduced and studied *infinitesimal mixed Hodge modules*, which is the "mixed version" of polarizable mixed Hodge structures. He constructed some natural filtrations, as a generalization of some filtrations in [87], which are crucial in the study on mixed Hodge modules.

As mentioned, one of the motivations in the study of singular variations of Hodge structure was to establish the functoriality of Hodge structure, as a generalization of the results of Deligne. Let X be a smooth projective variety with a normal crossing hypersurface D. Let $(L_{\mathbb{Q}}, F)$ be any polarizable variation of pure Hodge structure on $X \setminus D$. One of the issues in those days was to show that there exists a natural pure Hodge structure on the intersection cohomology group of $L_{\mathbb{Q}}$. If $(L_{\mathbb{Q}}, F)$ has no singularity at D, then it follows from the result of Deligne. In the singular case, the contribution of Zucker [88] is quite important. He studied the issue in the case $\dim X = 1$, and he proved that the intersection cohomology group is isomorphic to the L^2-cohomology group if $X \setminus D$ is equipped with a Poincaré like metric. He also developed the L^2-harmonic theory for singular polarized variation of pure Hodge structure on projective curves. As a result, he obtained a naturally induced pure Hodge structure on the intersection cohomology of $L_{\mathbb{Q}}$. In the higher dimensional case, Cattani-Kaplan-Schmid and Kashiwara-Kawai established it by making good use of their results on the asymptotic behaviour of polarized variation of Hodge structure, and by generalizing the method of Zucker. As for the mixed case, for an admissible variation of mixed Hodge structure on curves, Steenbrink-Zucker proved that the various naturally associated cohomology groups have mixed Hodge structure, based on their results on the asymptotic behaviour.

This stream of research for functoriality was eventually accomplished with much more great generality by the theory of mixed Hodge modules due to M. Saito. A cohomology theory can be regarded as a part of the theory of six functors on the derived categories of some type of sheaves. Briefly, the theory of mixed Hodge modules ensures that the derived functors for $\mathbb{Q}$-perverse sheaves on complex algebraic varieties can be enriched by mixed Hodge structures. (In this introduction, we consider only polarizable pure Hodge modules and graded polarizable mixed Hodge modules, we omit the adjectives "polarizable" or "graded polarizable".)

Very roughly, a Hodge module on a complex manifold X consists of a $\mathbb{Q}$-perverse sheaf $P_{\mathbb{Q}}$ with a Hodge filtration F on the regular holonomic $\mathscr{D}_X$-module M corresponding to $P_{\mathbb{Q}}$, i.e., (1) $\mathrm{DR}_X(M) \simeq P_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, (2) F is an increasing filtration of M by coherent $\mathcal{O}_X$-modules indexed by integers such that $F_j(\mathscr{D}_X) \cdot F_i(M) \subset F_{i+j}(M)$, where $\mathscr{D}_X$ denotes the sheaf of holomorphic differential operators on X with the filtration F by the order of operators. In his highly original and genius work, Saito invented the appropriate definitions of pure and mixed conditions for such filtered $\mathscr{D}$-modules, and he established their fundamental properties. The most important theorems in the theory are the functoriality with respect to six operations, and the description of pure and mixed Hodge modules.

For the functoriality in the pure case, he proved the Hard Lefschetz Theorem. Namely, let $f : X \longrightarrow Y$ be a projective morphism of smooth projective varieties. Let $(P_{\mathbb{Q}}, F)$ be any polarizable pure Hodge module of weight w on X. Let $f_{\dagger}^i P_{\mathbb{Q}}$ denote the i-th cohomology perverse sheaf of the push-forward of $P_{\mathbb{Q}}$ with respect to f. Then, $f_{\dagger}^i P$ is equipped with a naturally induced Hodge filtration $f_{\dagger}^i F$, so that $f_{\dagger}^i(P_{\mathbb{Q}}, F)$ is a polarizable pure Hodge module of weight $w + i$. Moreover, for the morphisms $L : f_{\dagger}^i P_{\mathbb{Q}} \longrightarrow f_{\dagger}^{i+2} P_{\mathbb{Q}}$ ($i \in \mathbb{Z}$) induced by the first Chern class of a relatively ample line bundle, the morphisms $L^i : f_{\dagger}^{-i} P_{\mathbb{Q}} \longrightarrow f_{\dagger}^i P_{\mathbb{Q}}$ ($i \geq 0$) are isomorphisms. This theorem is a generalization of the classical and important theorem of Beilinson-Bernstein-Deligne-Gabber on perverse sheaves of geometric origin [4].

As for the functoriality in the mixed case, he constructed the six operations together with the nearby and vanishing cycle functors for the derived category of mixed Hodge modules on algebraic varieties, which are compatible with those for the derived category of perverse sheaves.

Because the definitions of pure and mixed Hodge modules are complicated, it is important to know what objects are contained in the categories. Saito proved that a polarizable (resp. graded polarizable) variation of pure (resp. mixed) Hodge structure on X naturally gives a pure (mixed) Hodge module on X, as expected. Hence, the simplest variation of pure Hodge structure $\mathbb{Q}_X$ naturally gives a pure Hodge module. (But, note that while the variation of pure Hodge structure is of weight 0, the pure Hodge module is of weight $\dim X$.) Therefore, if a perverse sheaf $P_{\mathbb{Q}}$ on X is obtained from $\mathbb{Q}_Y$ on a complex algebraic manifold Y by successive use of six functors, it naturally underlies a mixed Hodge module. If a perverse sheaf is of geometric origin, then it naturally underlies a pure Hodge module. In particular, the category of mixed Hodge modules contain many objects. Moreover, he proved the more general results for the description. In the pure case, he proved the following.

- Let $Z \subset X$ be a closed irreducible complex analytic subvariety. Let $U \subset Z$ be a complement of a closed analytic subset, such that U is smooth. Let $\iota : U \longrightarrow X$ be the inclusion, and set $d_U := \dim U$. Let $(L_{\mathbb{Q}}, F)$ be any polarizable variation of Hodge structure on U. Then, the perverse sheaf $\iota_{*!}L_{\mathbb{Q}}[d_U]$, which is the minimal extension of $L_{\mathbb{Q}}[d_U]$ on X, is naturally equipped with the Hodge filtration F so that $(\iota_{*!}L_{\mathbb{Q}}[d_U], F)$ is a polarizable pure Hodge module.
- Conversely, any polarizable pure Hodge module is the direct sum of such minimal extensions.

Hence, for example, suppose that we are given a polarizable variation of Hodge structure $(L_{\mathbb{Q}}, F)$ on $X \setminus D$, where X is a complex manifold, and D is a normal crossing hypersurface. We obtain the pure Hodge module $(P_{\mathbb{Q}}, F)$ on X, obtained as the minimal extension of $(L_{\mathbb{Q}}, F)$, as in the above description. For the canonical map a_X of X to a point, the i-th cohomology of the push-forward $a^i_{X\dagger}(P_{\mathbb{Q}})$ is naturally equipped with the Hodge filtration by the functoriality of the pure Hodge modules. It means that the intersection cohomology of $L_{\mathbb{Q}}$ is equipped with a naturally induced pure Hodge structure, which implies the theorem of Zucker, Cattani-Kaplan-Schmid and Kashiwara-Kawai.

In the mixed case, Saito established the following:

- Let X, Z, U, ι and d_U be as above. Let $(L_{\mathbb{Q}}, F, W)$ be an admissible variation of mixed Hodge structure. Then, the perverse sheaves $\iota_* L_{\mathbb{Q}}[d_U]$ and $\iota_! L_{\mathbb{Q}}[d_U]$ are naturally equipped with the Hodge filtrations $\tilde{F}$ and the weight filtrations $\tilde{W}$ such that $(\iota_\star L_{\mathbb{Q}}[d_U], \tilde{F}, \tilde{W})$ $(\star = *, !)$ are mixed Hodge modules.
- Conversely, any mixed Hodge modules on X are locally obtained as the "gluing" of admissible variation of mixed Hodge structures.

It implies that, for example, we have a natural mixed Hodge structure on various cohomology groups associated to an admissible variation of mixed Hodge structure.

Remark 1.1.1 The theory of pure and mixed Hodge modules can be regarded as a counterpart of the theory of pure and mixed ℓ-adic sheaves on algebraic varieties over finite fields [4], which has been influential in various fields of mathematics including number theory and representation theory. See a very nice book [27] for more details on the philosophical background of Hodge modules, and for applications of the theory of Hodge modules to representation theory. □

Remark 1.1.2 See also a more recent work [10] for another approach to the functoriality of Hodge structures.

1.2 From Hodge Toward Twistor

As mentioned, it is our purpose in this monograph to study a twistor version of mixed Hodge modules. It is C. Simpson who introduced the notion of twistor structure as an underlying structure of Hodge structure. He proposed a principle called Simpson's Meta Theorem, which says that stories of Hodge structures should be generalized to stories of twistor structures.

When he introduced the concept of twistor structure, he was motivated to understand harmonic bundles in a deeper way. Let (V, ∇) be a flat bundle on a complex manifold X with a Hermitian metric h. We have a unique decomposition $\nabla = \nabla^u + \Phi$ into a unitary connection and a self-adjoint section of $\mathrm{End}(V) \otimes \Omega^1$. We have the decompositions into $\nabla^u = \overline{\partial}_V + \partial_V$ and $\Phi = \theta^\dagger + \theta$ into the $(0, 1)$-part and the $(1, 0)$-part. Then, (V, ∇, h) is called a harmonic bundle, if $(V, \overline{\partial}_V, \theta)$ is a Higgs bundle in the sense that $\overline{\partial}_V$ is a holomorphic structure of V and θ is a Higgs field of $(V, \overline{\partial}_V)$. In that case, the metric h is called pluri-harmonic. The concept was discovered by N. Hitchin [26] in the one dimensional case, and by Simpson [78–80] in the higher dimensional case.

One of the most important classes of harmonic bundles is given by polarized variations of Hodge structure. The Hermitian metrics induced by polarizations of polarizable variations of Hodge structure are pluri-harmonic. From the beginning of his study, Simpson was motivated by the investigation of polarized variations of Hodge structure. (See [78–83]). He gave a method to construct polarized variation of Hodge structure by using the Kobayashi-Hitchin correspondence for harmonic bundles. He observed that various properties of polarized variation of Hodge structure are naturally generalized to those for harmonic bundles. For example, he developed the harmonic theory for harmonic bundles as a generalization of that for polarized variation of Hodge structure, and he proved that the push-forward of any harmonic bundle of any smooth projective family of varieties is naturally a harmonic bundle.

To pursue this analogy in a deeper level, he introduced the concept of twistor structure, and observed that harmonic bundles can be regarded as *polarized variations of pure twistor structure [82]*. Thus, he established the analogy between polarized variations of Hodge structure and harmonic bundles in the level of the definitions. This important idea enables us to consider a twistor version of various objects appeared in the Hodge theory.

This is quite efficient in the study of the asymptotic behaviour of singular harmonic bundles, which was studied by Simpson himself and the author. (See [79, 82], [50, 52, 55]). The twistor viewpoint suggests us how to formulate generalizations of results of Cattani, Kaplan, Kashiwara, Kawai and Schmid. Indeed, we obtain a nice object on X from a harmonic bundle on $X \setminus D$, and we also obtain the limit mixed twistor structure, which is quite useful to control the nilpotent part of the residues. However, we would like to mention that there are also some phenomena which do not appear for polarized variation of Hodge structure, such as KMS-structure and Stokes structure, and that the proofs are not necessarily given in parallel ways.

It is also suggested by Simpson's Meta Theorem that we should have a twistor version of the theory of pure and mixed Hodge modules. In the pure case, it was pursued by C. Sabbah and the author. Sabbah prepared the notion of $\mathcal{R}$-triples as an ingredient to define twistor $\mathscr{D}$-modules, which can be regarded as a counterpart of pairs of $\mathbb{R}$-perverse sheaf and filtered $\mathscr{D}$-module. They are suitable even in the case that the underlying $\mathscr{D}$-modules are irregular. Based on Saito's strategy, he gave the appropriate definition of pure twistor $\mathscr{D}$-modules and the framework to prove the Hard Lefschetz Theorem, i.e., the functoriality for projective push-forward. The correspondence between tame harmonic bundles and regular pure twistor $\mathscr{D}$-modules was established in [51–53]. In the wild case, the basic properties were established in [55].

It is interesting to have the correspondence between semisimple holonomic $\mathscr{D}$-modules and pure twistor $\mathscr{D}$-modules on projective varieties, which does not appear in the theory of pure Hodge modules. As a result, we obtain that semisimplicity of algebraic holonomic $\mathscr{D}$-modules is preserved by projective push-forward.

In this monograph, we introduce mixed twistor $\mathscr{D}$-modules, and prove the fundamental properties. For the author, it is one of the ultimate goals of the research for years, driven by Simpson's Meta Theorem.

There are various intermediate objects between twistor structure and Hodge structure such as integrable twistor structure and TERP-structure (see [22–25, 56, 66]). So, we could have variants of mixed twistor $\mathscr{D}$-modules by considering additional structures. Because the twistor structure could be most basic among them, the author hopes that mixed twistor $\mathscr{D}$-modules would play a basic role in the study of generalized Hodge structure on holonomic $\mathscr{D}$-modules.

Remark 1.2.1 See [17] for a philosophical background toward a generalized Hodge theory in the context of irregular singularities. See [37] for a generalized Hodge theoretic aspect of the mirror symmetry.

Remark 1.2.2 See [81] for the functoriality of harmonic bundles with respect to smooth projective morphisms. See also [84] for a twistor structure on the cohomology group of orbifolds.

1.3 Mixed Twistor $\mathscr{D}$-Modules

In the rest of this introduction, let us briefly review the theory of pure twistor $\mathscr{D}$-modules, and explain our issues in the study of mixed twistor $\mathscr{D}$-modules.

1.3.1 Pure Twistor $\mathscr{D}$-Modules

For any complex manifold X, the product $\mathbb{C}_\lambda \times X$ is denoted by $\mathcal{X}$. Let $p : \mathcal{X} \longrightarrow X$ be the projection. Let Θ_X be the tangent sheaf of X. We have the sheaf of holomorphic differential operators $\mathscr{D}_\mathcal{X}$ on $\mathcal{X}$. Then, $\mathcal{R}_X$ is the sheaf of subalgebras in $\mathscr{D}_\mathcal{X}$ generated by $\lambda p^* \Theta_X$ over $\mathcal{O}_\mathcal{X}$.

We have two basic conditions on $\mathcal{R}_X$-modules. One is strictness, i.e., flat over $\mathcal{O}_{\mathbb{C}_\lambda}$. The other is holonomicity. Namely, the characteristic variety of any coherent $\mathcal{R}_X$-module $\mathcal{M}$ is defined as in the case of $\mathscr{D}$-modules, denoted by $Ch(\mathcal{M})$. It is a subvariety in $\mathbb{C}_\lambda \times T^*X$. If $Ch(\mathcal{M})$ is contained in the product of $\mathbb{C}_\lambda$ and a Lagrangian subvariety in T^*X, the $\mathcal{R}_X$-module $\mathcal{M}$ is called holonomic.

An $\mathcal{R}_X$-triple is a tuple of $\mathcal{R}_X$-modules $\mathcal{M}_i$ $(i = 1, 2)$ with a sesqui-linear pairing C. To explain what is sesqui-linear paring, we need a preparation. Let $\boldsymbol{S}$ denote the circle $\{\lambda \in \mathbb{C}_\lambda \mid |\lambda| = 1\}$. Let $\sigma : \boldsymbol{S} \longrightarrow \boldsymbol{S}$ be given by $\sigma(\lambda) = -\lambda = -\overline{\lambda}^{-1}$. The induced involution $\boldsymbol{S} \times X \longrightarrow \boldsymbol{S} \times X$ is also denoted by σ.

Let $\mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$ denote the sheaf of distributions on $\boldsymbol{S} \times X$ which are continuous in the $\boldsymbol{S}$-direction in an appropriate sense. This sheaf is naturally a module over $\mathcal{R}_{X|\boldsymbol{S}\times X} \otimes \sigma^* \mathcal{R}_{X|\boldsymbol{S}\times X}$. Then, a sesqui-linear pairing of $\mathcal{R}_X$-modules $\mathcal{M}_i$ $(i = 1, 2)$ is an $\mathcal{R}_{X|\boldsymbol{S}\times X} \otimes \sigma^* \mathcal{R}_{X|\boldsymbol{S}\times X}$-homomorphism $\mathcal{M}_{1|\boldsymbol{S}\times X} \otimes \sigma^* \mathcal{M}_{2|\boldsymbol{S}\times X} \longrightarrow \mathfrak{Db}_{\boldsymbol{S}\times X/\boldsymbol{S}}$. An $\mathcal{R}_X$-triple is called strict (resp. holonomic), if the underlying $\mathcal{R}$-modules are strict (resp. holonomic). The category of pure twistor $\mathscr{D}$-module is constructed as a full subcategory of strict holonomic $\mathcal{R}$-triples.

Let us recall how to impose some conditions on strict holonomic $\mathcal{R}$-triples. In the case of a variation of Hodge structure which is a $\mathbb{Q}$-local system with a Hodge filtration, it is defined to be pure, if its restriction to the fiber over each point is pure. But, for $\mathcal{R}$-triples or even for $\mathscr{D}$-modules, the restriction to a point is not so well behaved. Instead, for holonomic $\mathscr{D}$-modules, there is a nice theory for restriction to hypersurfaces. Namely, we have the nearby and vanishing cycle functors, which describe the behaviour of the holonomic $\mathscr{D}$-modules around the hypersurface in some degree. To define some condition for holonomic $\mathscr{D}$-modules, it seems natural to consider the conditions on nearby and vanishing cycle sheaves inductively, instead of the restriction to a point. Similarly, to define some condition for $\mathcal{R}$-triples, we would like to consider the condition on the appropriately defined nearby and vanishing cycle functors for $\mathcal{R}$-triples. This is a basic strategy due to Saito, and it may lead us to inductive definitions of pure and mixed twistor $\mathscr{D}$-modules, as a variant of pure and mixed Hodge modules.

A strict holonomic $\mathcal{R}$-triple $\mathcal{T}$ is called pure of weight w if the following holds. First, we impose that, for any open subset $U \subset X$ with a holomorphic function g, $\mathcal{T}_{|U}$ is strictly S-decomposable along g. It implies that we have the decomposition $\mathcal{T} = \bigoplus \mathcal{T}_Z$ by strict support, where Z runs through closed irreducible subsets of X. Then, we impose the conditions on each $\mathcal{T}_Z$. If Z is a point, $\mathcal{T}_Z$ is supposed to be the push-forward of a pure twistor structure of weight w by the inclusion of Z into X. In the positive dimensional case, for any open subset of X with a holomorphic function if we take the grading of the weight filtration of the naturally induced nilpotent morphism on the nearby cycle functor along the function, the m-th graded pieces are pure of weight $w+m$. Then, inductively, the notion of pure twistor $\mathscr{D}$-module is defined. Precisely, we should consider the polarizable object. A polarization of $\mathcal{T}$ is defined as a Hermitian sesqui-linear duality of weight w satisfying some condition on positivity, which is also given in an inductive way using the nearby cycle functor.

Let $\mathrm{MT}(X, w)$ denote the category of polarizable pure twistor $\mathscr{D}$-modules of weight w on X. Let us recall some of their fundamental properties; (1) The category $\mathrm{MT}(X, w)$ is abelian and semisimple; (2) For objects $\mathcal{T}_i \in \mathrm{MT}(X, w_i)$ with a morphism $f : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ as $\mathcal{R}$-triples such that $w_1 > w_2$, we have $f = 0$; (3) For any projective morphism $f : X \longrightarrow Y$, and for any $\mathcal{T} \in \mathrm{MT}(X, w)$, the i-th cohomology of the push-forward $f_\dagger^i \mathcal{T}$ is an object in $\mathrm{MT}(Y, w+i)$. Moreover, $f_\dagger \mathcal{T} \simeq \bigoplus f_\dagger^i \mathcal{T}[-i]$ in the derived category of $\mathcal{R}$-triples; (4) Let $Z \subset X$ be a closed complex analytic subset. Let $Z_0 \subset Z$ be a closed complex analytic subset such that $Z \setminus Z_0$ is smooth. Then, a wild harmonic bundle on (Z, Z_0) is naturally extended to a pure twistor $\mathscr{D}$-module on X; (5) Conversely, any pure twistor $\mathscr{D}$-modules are the direct sum of such minimal extensions of wild harmonic bundles; (6) Any semisimple algebraic holonomic $\mathscr{D}$-module naturally underlies a polarizable wild pure twistor $\mathscr{D}$-module.

1.3.2 Mixed Twistor $\mathscr{D}$-Modules

To define mixed twistor $\mathscr{D}$-modules, we first consider filtered $\mathcal{R}$-triples $(\mathcal{T}, W)$ such that $\mathrm{Gr}_w^W(\mathcal{T})$ are pure of weight w, where W are locally finite increasing complete exhaustive filtrations indexed by integers. They are too naive, and they play only auxiliary roles. Tentatively, they are called pre-mixed twistor $\mathscr{D}$-modules in this monograph. They have nice functoriality for the push-forward by projective morphisms. However, we need to impose additional conditions for other standard functoriality such as push-forward for open embeddings and pull back. Very briefly, to define mixed twistor $\mathscr{D}$-module, we impose (1) the filtered strict compatibility of W and the V-filtrations, (2) the existence of relative monodromy filtrations on the nearby and vanishing cycle sheaves, (3) the relative monodromy filtrations give the weight filtrations of mixed twistor $\mathscr{D}$-modules with smaller supports. (It will be explained in Chap. 7.)

It is easy to show that mixed twistor $\mathscr{D}$-modules have nice functorial property for nearby and vanishing cycle functors and projective push-forward. However, it is not so easy to show the other functorial properties, for example, the localization

$M \longmapsto M(*H)$ for a hypersurface H. To establish more detailed property, we need a concrete description of mixed twistor $\mathscr{D}$-modules as the gluing of admissible variations of mixed twistor structure.

1.3.3 Gluing Procedure

For perverse sheaves and holonomic $\mathscr{D}$-modules, there are well established theories to glue objects on $\{f = 0\}$ and objects on $\{f \neq 0\}$ [3, 40, 90]. We need such gluing procedure in the context of $\mathcal{R}$-triples. Because of the difference of ingredients, it is not easy to generalize the method of gluing prepared in [73] for Hodge modules to the case of $\mathcal{R}$-triples. Instead, we adopt the excellent method of Beilinson in [3], which reduces the issue to the construction of canonical prolongations $\mathcal{T}[\star t]$ $(\star = *, !)$. (See Chaps. 3–4.)

1.3.4 Admissible Variation of Mixed Twistor Structure

We prepare a general theory for admissible variations of mixed twistor structure (Chaps. 8–9), which is a natural generalization of the theory of admissible variations of mixed Hodge structure. Let X be a complex manifold with a simple normal crossing hypersurface D. Very briefly, it is a filtered $\mathcal{R}$-triple $(\mathcal{V}, W)$ on X with poles along D. We impose the conditions (1) each $\mathrm{Gr}^W_w(\mathcal{V})$ comes from a good wild harmonic bundle, (2) $\mathcal{V}$ has good-KMS structure along D compatible with W, (3) the residues along the divisors have relative monodromy filtrations. It is important to understand the specialization of admissible variations of mixed twistor structure along the divisors. For that purpose, it is essential to study the relative monodromy filtrations and their compatibility. So, as in [32], we study in Chap. 8 the infinitesimal version of admissible variations of mixed twistor structure, called infinitesimal mixed twistor modules. We can show that they have nice properties as in the Hodge case.

Then, we study the canonical prolongation of admissible mixed twistor structure $(\mathcal{V}, W)$ on (X, D) to pre-mixed twistor $\mathscr{D}$-modules on X. Let $D = D^{(1)} \cup D^{(2)}$ be a decomposition. Recall that any good meromorphic flat bundle V on (X, D) is extended to a $\mathscr{D}$-module $V[*D^{(1)}!D^{(2)}]$ on X. We prepare a similar procedure to make a pre-mixed twistor $\mathscr{D}$-module $(\mathcal{V}, W)[*D^{(1)}!D^{(2)}]$ from $(\mathcal{V}, W)$. First, we construct the underlying $\mathcal{R}$-triple $\mathcal{V}[*D^{(1)}!D^{(2)}]$ in Chap. 5. One of the main tasks is to construct a correct weight filtration W on $\mathcal{V}[*D^{(1)}!D^{(2)}]$. By applying the procedure in Chap. 5 to each $W_j\mathcal{V}$, we obtain a naively induced filtration L on $\mathcal{V}[*D^{(1)}!D^{(2)}]$, i.e., $L_j(\mathcal{V}[*D^{(1)}!D^{(2)}]) = W_j(\mathcal{V})[*D^{(1)}!D^{(2)}]$. But, this is not the correct filtration. We need much more considerations for the construction of the correct weight filtration W. It is essentially contained in [32, 73], but we shall give rather details, which is one of the main themes in Chap. 6 and Chaps. 8–10.

Once we obtain canonical prolongations of admissible variations of mixed twistor structures across normal crossing hypersurfaces, we can glue them to obtain pre-mixed twistor $\mathscr{D}$-modules, which are called good pre-mixed twistor $\mathscr{D}$-modules. It is one of the main theorems to show that any good pre-mixed twistor $\mathscr{D}$-module is a mixed twistor $\mathscr{D}$-module (Theorem 10.3.1). Then, by a rather formal argument, we can show that any mixed twistor $\mathscr{D}$-module can be expressed as gluing of admissible variation of mixed twistor $\mathscr{D}$-modules as in Sect. 11.1. We can deduce some basic functorial properties by using this description. See Sects. 11.2–11.4.

Remark 1.3.1 In this monograph, we will often omit "variation of" just for simplification. For example, an admissible variation of mixed twistor structure is often called an admissible mixed twistor structure. □

1.3.5 Duality and Real Structure

1.3.5.1 Duality

In the study of $\mathscr{D}$-modules, the duality functor is fundamental. To define the duality functor for mixed twistor $\mathscr{D}$-modules there are two issues which we should address. Let X be any complex manifold. Let $d_X := \dim X$. Let ω_X be the sheaf of holomorphic d_X-forms. We put $\omega_{\mathcal{X}} := \lambda^{-d_X} p^* \omega_X$. As given in Sect. 13.1, the dual of any coherent $\mathcal{R}_X$-module $\mathcal{M}$ is defined as follows in the derived category of $\mathcal{R}_X$-modules:

$$\boldsymbol{D}\mathcal{M} := \lambda^{d_X} R\mathcal{H}om_{\mathcal{R}_X}\big(\mathcal{M}, \mathcal{R}_X \otimes \omega_{\mathcal{X}}^{-1}\big)[d_X] \tag{1.1}$$

The dual $\boldsymbol{D}M$ of any holonomic $\mathscr{D}$-module M in the derived category is also a holonomic $\mathscr{D}$-module, i.e., the j-th cohomology sheaf of $\boldsymbol{D}M$ vanishes unless $j = 0$. We cannot expect such a property for general holonomic $\mathcal{R}_X$-modules. We need to prove that if $\mathcal{M}$ is an $\mathcal{R}_X$-module underlying a mixed twistor $\mathscr{D}$-module then $\boldsymbol{D}\mathcal{M}$ is also a strict holonomic $\mathscr{D}$-module. This issue already appeared in the Hodge case, and solved by Saito. Even in the twistor case, we can apply Saito's method in a rather straightforward way. (See Sect. 13.2.)

The other issue is the construction of a sesqui-linear pairing for the dual. Let $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$ be the $\mathcal{R}$-triple underlying a mixed twistor $\mathscr{D}$-module on X. We need to construct a sesqui-linear pairing $\boldsymbol{D}C$ of $\boldsymbol{D}\mathcal{M}_1$ and $\boldsymbol{D}\mathcal{M}_2$. This issue did not appear in the Hodge case. It is non-trivial even for sesqui-linear pairings of holonomic $\mathscr{D}$-modules. Let M_i $(i = 1, 2)$ be holonomic $\mathscr{D}$-modules. Let $C : M_1 \otimes \overline{M_2} \longrightarrow \mathfrak{Db}_X$ be a $\mathscr{D}_X \otimes \mathscr{D}_{\overline{X}}$-homomorphism. Here, $\mathfrak{Db}_X$ denotes the sheaf of distributions on X. We need to construct an induced sesqui-linear pairing $\boldsymbol{D}C$ of $\boldsymbol{D}M_1$ and $\boldsymbol{D}M_2$. We obviously have such a pairing, if M_i are smooth, i.e., flat bundles. It is not difficult to construct it in the case of regular holonomic $\mathscr{D}$-modules, thanks to the Riemann-Hilbert correspondence [28, 30, 46–48]. But, at this moment, some additional arguments are required in the irregular case, which will be given in Chap. 12. Once we have such a pairing in the case of $\mathscr{D}$-modules, it is rather formal to construct it in the context of mixed twistor $\mathscr{D}$-modules. (See Sect. 13.3.)

1.3.5.2 Real Structure

Combining the duality functor with some other functors, we can introduce the concept of real structures on mixed twistor $\mathscr{D}$-modules.

We have a contravariant auto equivalence on the category of mixed twistor $\mathscr{D}$-modules, called the Hermitian adjoint. For any $\mathcal{R}_X$-triple $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$, we have the associated $\mathcal{R}_X$-triple $\mathcal{T}^* = (\mathcal{M}_2, \mathcal{M}_1, C^*)$, where $C^*(a, \sigma^* b) := \overline{\sigma^* C(b, \sigma^* a)}$. If $\mathcal{T}$ is equipped with a filtration W, we set $W_j(\mathcal{T}^*)$ as the image of $(\mathcal{T}/W_{-j-1})^* \longrightarrow \mathcal{T}^*$. If $(\mathcal{T}, W)$ is a mixed twistor $\mathscr{D}$-module, then $(\mathcal{T}, W)^*$ is also a mixed twistor $\mathscr{D}$-module. This can be regarded as the enhancement of the operation $M \longmapsto \overline{\mathcal{H}om_{\mathscr{D}_X}(M, \mathfrak{Db}_X)}$ for holonomic $\mathscr{D}_X$-modules, where $\mathfrak{Db}_X$ denotes the sheaf of distributions on X. We also denote $\mathcal{T}^*$ by $\boldsymbol{D}^{\mathrm{herm}}(\mathcal{T})$.

Let $j : \mathcal{X} \longrightarrow \mathcal{X}$ be given by $j(\lambda, Q) = (-\lambda, Q)$. For any $\mathcal{R}_X$-triple $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$, the $\mathcal{R}_X$-triple $j^*\mathcal{T} = (j^*\mathcal{M}_1, j^*\mathcal{M}_2, j^*C)$ is naturally defined. If $(\mathcal{T}, W)$ is a mixed twistor $\mathscr{D}_X$-module, then $j^*(\mathcal{T}, W)$ is naturally a mixed twistor $\mathscr{D}_X$-module.

We define the functor $\tilde{\gamma}^*$ on the category of mixed twistor $\mathscr{D}_X$-modules by

$$\tilde{\gamma}^*(\mathcal{T}, W) = j^* \circ \boldsymbol{D} \circ \boldsymbol{D}^{\mathrm{herm}}(\mathcal{T}, W).$$

Namely, for $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$, we set $\tilde{\gamma}^*(\mathcal{T}) = (j^*\boldsymbol{D}\mathcal{M}_2, j^*\boldsymbol{D}\mathcal{M}_1, j^*\boldsymbol{D}C^*)$. Then, a real structure on a mixed twistor $\mathscr{D}$-module $(\mathcal{T}, W)$ is defined to be an isomorphism $\kappa : \tilde{\gamma}^*(\mathcal{T}, W) \simeq (\mathcal{T}, W)$ such that $\tilde{\gamma}^*(\kappa) \circ \kappa = \mathrm{id}$. We show the functorial property of such real structure in Sect. 13.4.

1.3.5.3 Relation with Mixed Hodge Modules

We shall see in Sect. 13.5 that integrable mixed twistor $\mathscr{D}$-modules with real structure are closely related with mixed Hodge modules. Let (P, F, W) be a mixed Hodge module on X. Let M be a regular holonomic $\mathscr{D}_X$-module with an isomorphism $\mathrm{DR}_X M \simeq P \otimes_{\mathbb{Q}} \mathbb{C}$ on which F is defined. It is also naturally equipped with the weight filtration W. The real structure of $\mathrm{DR}_X M$ given by $P \otimes \mathbb{R}$ naturally induces a sesqui-linear pairing $\boldsymbol{D}M \times \overline{M} \longrightarrow \mathfrak{Db}_X$. Let $\mathcal{M}$ be the analytification of the Rees module of (M, F). Then, we have the induced sesqui-linear pairing C of $j^*\boldsymbol{D}\mathcal{M}$ and $\mathcal{M}$. It turns out that the $\mathcal{R}_X$-triple $\mathcal{T} = (j^*\boldsymbol{D}\mathcal{M}, \mathcal{M}, C)$ with the induced filtration W is a mixed twistor $\mathscr{D}$-module. It is equipped with the real structure given by $\kappa = (\mathrm{id}, \mathrm{id}) : \tilde{\gamma}^*\mathcal{T} \simeq \mathcal{T}$. If (P, F, W) is pure of weight w, then the associated $(\mathcal{T}, W)$ is also pure of weight w. Moreover, $(\mathcal{T}, W, \kappa)$ is naturally integrable. So, we obtain a functor from the category of mixed Hodge modules to the category of integrable mixed twistor $\mathscr{D}$-modules with real structure.

We shall give more details on this functor in Sect. 13.5, and observe that it is naturally compatible with various operations. Briefly, it is compatible with the push-forward by projective morphisms, the localizations and the duality, by construction of the functors. As a consequence, the functor is compatible with the six operations in the algebraic setting. Note that the pull back functors for a closed immersion are described in terms of the localizations and the Kashiwara equivalence. We also check in Sect. 13.5.2 the coincidence of the concepts of polarizations for pure Hodge modules and pure twistor $\mathscr{D}$-modules. So, we follow the same rules for signatures and weights in the Hodge context and the twistor context.

Let us look at the simplest variation of pure Hodge structure $(\mathbb{R}_X[d_X], F)$ of weight 0 on a complex manifold X, where F is the Hodge filtration on $\mathcal{O}_X$ given by $F_0(\mathcal{O}_X) = \mathcal{O}_X$ and $F_{-1}(\mathcal{O}_X) = 0$. It naturally gives a pure Hodge module of weight d_X. The analytification of the Rees module of $(\mathcal{O}_X, F)$ is $\mathcal{O}_{\mathcal{X}}$. We have a natural isomorphism $j^*\boldsymbol{D}\mathcal{O}_{\mathcal{X}} \simeq \lambda^{d_X}\mathcal{O}_{\mathcal{X}}$. It turns out that $\mathcal{T}_X := (j^*\boldsymbol{D}\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}, C)$ is isomorphic to the pure twistor $\mathscr{D}$-module $(\lambda^{d_X}\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}, C_0)$ of weight d_X, where $C_0(f, \sigma^* g) = f \cdot \overline{\sigma^*(g)}$. A natural polarization of $\mathcal{T}_X$ is induced by the pairing $\mathbb{R}[d_X] \otimes \mathbb{R}[d_X] \longrightarrow \mathbb{R}[d_X]$ given by $(a, b) \longmapsto (-1)^{d_X(d_X-1)/2}ab$. Let H be a smooth hypersurface of X. Let $\iota_H : H \longrightarrow X$ be the inclusion. We have the exact sequences $0 \longrightarrow \iota_{H\dagger}\mathcal{T}_H \longrightarrow \mathcal{T}_X[!H] \longrightarrow \mathcal{T}_X \longrightarrow 0$ and $0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_X[*H] \longrightarrow \iota_{H\dagger}\mathcal{T}_H \otimes \boldsymbol{T}(-1) \longrightarrow 0$, where $\boldsymbol{T}(-1)$ is the (-1)-th Tate twist. In the algebraic setting, we have $\iota_H^*\mathcal{T}_X = \mathcal{T}_H[1]$ and $\iota_H^!\mathcal{T}_X = \mathcal{T}_H[-1] \otimes \boldsymbol{T}(-1)$.

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