Preface

Let us begin with the most basic concept of twistor structure, which was introduced by Simpson [82] as a generalization of the concept of Hodge structure. A twistor structure is a holomorphic vector bundle on $\mathbb{P}^1$, and a (complex) Hodge structure is a $\mathbb{C}$-vector space with two filtrations. The standard Rees construction allows us to obtain an equivalence of Hodge structure and twistor structure with a $\mathbb{C}^*$-action.

Ideally, any notion for Hodge structure can be translated to those for twistor structure. Indeed, by applying the Rees construction, we obtain something on $\mathbb{P}^1$ equipped with a $\mathbb{C}^*$-action. Then, by forgetting the $\mathbb{C}^*$-action, we obtain the counterpart of the notion in the context of twistor structure.

For instance, a twistor structure is called pure of weight m if it is isomorphic to a direct sum of $\mathcal{O}_{\mathbb{P}^1}(m)$. A mixed twistor structure is defined to be a twistor structure V with an increasing filtration W indexed by integers such that $\mathrm{Gr}_m^W(V)$ are pure of weight m. These generalize the notions of pure and mixed Hodge structure. We also have the twistor version of polarizations and Tate twists. They share important properties with their counterparts in the Hodge context. For example, the categories of mixed Hodge structures and mixed twistor structures are abelian. We can compare it with the fact that the category of vector spaces with a filtration is not abelian.

On the basis of the fact that many important features of Hodge structures already appear in the level of twistor structures, Simpson proposed a principle, called Simpson's Meta Theorem. It roughly says that most objects and most theorems in Hodge theory should have their counterparts in the context of twistor structures. This principle leads us to a promising and interesting project, which we call "from Hodge toward twistor". I cannot exaggerate the philosophical significance of his principle.

Let us recall that one of the important branches in the Hodge theory is the study of the functoriality of Hodge structure. Later in Introduction, we will briefly review it. Here, we just remind that the functoriality of Hodge structure was thoroughly established by the extremely deep theory of mixed Hodge modules [69, 73] due to Morihiko Saito. Roughly saying, mixed Hodge modules are regular holonomic $\mathscr{D}$-modules with mixed Hodge structure. Saito's theory ensures that mixed Hodge structures are functorial with respect to standard operations for regular holonomic $\mathscr{D}$-modules.

According to Simpson's Meta theorem, we should have the twistor version of mixed Hodge modules. That is the concept of mixed twistor $\mathscr{D}$-modules which we shall investigate in this monograph. We shall establish their fundamental properties, in particular, the functoriality with respect to the standard operations. That is the goal of this study. Indeed, it is the ultimate goal for me in the project "from Hodge toward twistor".

We should remark that Claude Sabbah introduced the concept of pure twistor $\mathscr{D}$-modules [66, 67], which is a twistor version of pure Hodge modules. The theory of pure twistor $\mathscr{D}$-modules was further studied by Sabbah and myself [52, 55]. We could regard mixed twistor $\mathscr{D}$-modules as the mixed version of pure twistor $\mathscr{D}$-modules. We note that the ingredients for twistor $\mathscr{D}$-modules are not the same as those for mixed Hodge modules. We also note that there are some phenomena which do not appear in the context of Hodge modules. So, although we owe much to the fundamental strategy of Saito in the Hodge case, we also have some additional issues to deal with. It is contrast to the fact that generalization of mixed Hodge structure to mixed twistor structure is technically rather straightforward.

I should mention why it is interesting to have a twistor version of mixed Hodge modules. It is the most important reason that we can apply the theory of mixed twistor $\mathscr{D}$-modules to a wider class of holonomic $\mathscr{D}$-modules possibly with irregular singularities.

Indeed, any algebraic semisimple holonomic $\mathscr{D}$-module underlies a pure twistor $\mathscr{D}$-module [55]. It implies the Hard Lefschetz theorem for algebraic semisimple holonomic $\mathscr{D}$-modules with respect to the push-forward by projective morphisms, which is one of the most interesting results in the study of pure twistor $\mathscr{D}$-modules.

As for the mixed case, we have the mixed twistor $\mathscr{D}$-modules associated with meromorphic functions. Namely, let X be a complex manifold with a hypersurface H. Let f be a meromorphic function on X whose poles are contained in H. We have the $\mathscr{D}_X$-module $L_*(f, H)$ obtained as the $\mathcal{O}_X$-module $\mathcal{O}_X(*H)$ with the flat connection $d + df$. We have the natural mixed twistor $\mathscr{D}$-module over $L_*(f, H)$. By applying the standard operations to such mixed twistor $\mathscr{D}$-modules, we can observe that many important $\mathscr{D}$-modules naturally underlie mixed twistor $\mathscr{D}$-modules.

For instance, some type of GKZ-hypergeometric systems are naturally enriched to mixed twistor $\mathscr{D}$-modules, which also naturally appear in the study of the Landau-Ginzburg models in the toric mirror symmetry. Recently, we applied the degeneration of the mixed twistor $\mathscr{D}$-modules over the GKZ-hypergeometric systems to the study of local mirror symmetry in [60]. See also [59] for an application to the study of Kontsevich complexes.

I hope that our study would be a part of the foundation for the further study on the generalized Hodge theory of holonomic $\mathscr{D}$-modules possibly with irregular singularity. I also hope that it would be a help for readers to get into a technical part of the deep theory due to Saito.

This study grew out of my attempt to understand the works due to Beilinson [2, 3], Kashiwara [32], and Saito [69–71, 73]. The readers can find most essential ideas in their papers. I thank Morihiko Saito for some discussions. I thank Claude Sabbah for numerous discussions on many occasions. I deeply thank Carlos Simpson. It is impossible for me to mention what I owe to him. I just note here that his most fundamental principle (Simpson's Meta-Theorem) invited me to this study. Special thanks go to Mark de Cataldo, Pierre Deligne, Kenji Fukaya, William Fulton, David Gieseker, Akira Kono, Mikiya Masuda, Atsushi Moriwaki, Masa-Hiko Saito, Tomohide Terasoma, Michael Thaddeus, and Kari Vilonen. I would like to express my gratitude to Yves André, Philip Boalch, Helene Esnault, Claus Hertling, Maxim Kontsevich, Thomas Reichelt, Kyoji Saito, Christian Schnell, and Christian Sevenheck, for some discussions. I am grateful to Indranil Biswas for his excellent hospitality during my stay at the Tata Institute of Fundamental Research. I thank Akira Ishii and Yoshifumi Tsuchimoto for their constant encouragement. I appreciate the referees for their valuable comments to improve this monograph.

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This work was partially supported by the Grant-in-Aid for Scientific Research (S) (No. 24224001), the Grant-in-Aid for Scientific Research (A) (No. 22244003), the Grant-in-Aid for Scientific Research (C) (No. 22540078), Japan Society for the Promotion of Science.

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http://www.springer.com/978-3-319-10087-6

Mixed Twistor D-modules
Mochizuki, T.
2015, XX, 487 p., Softcover
ISBN: 978-3-319-10087-6