Universal Logic as a Science of Patterns

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Abstract This article addresses Béziau's (Sorites 12:5–32, 2001) vision that universal logic should be capable of helping other fields of knowledge to build *the right logic for the right situation*, and that for some disciplines *mathematical abstract conceptualization* is more appropriate than *symbolic formalization*. Hertz's (Math. Ann. 87(3–4):246–269, 1922) diagrams of logical inference patterns are formalized and extended to present the universal logic conceptual framework as a comprehensible science of patterns. This facilitates those in other disciplines to develop, visualize and apply logical representation and inference structures that emerge from their problématique. A family of protologics is developed by resemantifying the sign for deduction, $\rightarrow$, with inference patterns common to many logics, and specifying possible constraints on its use to represent the structural connectives and defeasible reasoning. Proof-theoretic, truth-theoretic, intensional and extensional protosemantics are derived that supervene on the inference patterns. Examples are given of applications problem areas in a range of other disciplines, including the representation of states of affairs, individuals and relations.

Keywords Universal logic · Inference patterns · Protologic · Protosemantics · Structural connectives · Paraconsistency · Default reasoning · Applied logic

Mathematics Subject Classification (2000) Primary 03B22 · Secondary 03A05

1 Introduction

Twenty years ago, Jean-Yves Béziau [9] proposed that the notion of *deduction* in any logical system should be studied within an integrative conceptual framework that presupposes no particular axioms. He termed this framework *universal logic* by analogy to Birkhoff's [23] *universal algebra*. This approach encompasses all logical systems and facilitates specifying each of their particular axioms whether they are common to many logics, or peculiar to a few.

Logical systems have proliferated since the 1920s when Hilbert, Frege and Russell extended the inference patterns of Aristotle's syllogistic [7, 33] to provide formal foundations for mathematics. Béziau's proposal situates a wide range of historical and ongoing studies of such systems within a coherent framework, and suggests significant directions for further research. His publications expounding [11, 13, 14] and illustrating [10, 12, 19] the approach, and the fora he has provided through editing books [15, 18], conference proceedings [21, 22], and the *Logica Universalis* journal, have inspired a research community collaborating within the universal logic paradigm.

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Béziau [14, p. 14] presents the study of logical structures as a mathematical discipline in its own right, a Bourbakian [25] *mother system* "having the same status as algebraic, topological and order structures". However, like other sub-disciplines of mathematics, it also provides foundational capabilities for many other disciplines. A universal logic conceptual framework contributes to these disciplines by providing techniques for tailoring logical systems to address the precise purposes for which they are required. It helps build "the right logic for the right situation" [11, p. 19], making it possible to avoid the wholesale import of inappropriate axioms of a logical system that may introduce artifacts by going beyond those deriving from a conceptual analysis of the problématique.

In order to support other disciplines, Béziau [11, p. 23] has suggested that universal logic might be presented as a formal but comprehensible conceptual framework that avoids unnecessary symbolism, that, for example, "the definitions philosophers need involve mathematical abstract conceptualization rather than symbolic formalization." However, most universal logic research has naturally adopted the symbolism of formal logic and this, like mathematical symbolism in general, can be a significant barrier to understanding [66].

This article addresses the question of whether the conceptual framework of universal logic may be formalized and presented in a way that minimizes the use of technical terminology and mathematical symbols. Its objective is to preserve formal rigor while providing a useful and comprehensible tool for those applying logic to problems of representation and inference in non-mathematical disciplines. It adopts the perspective that views mathematics as a *science of patterns* [95], and presents possible inference patterns in a logical system as two-dimensional configurations of an arrow symbolizing deduction.

2 Logic as a Science of Patterns

The objective of this section is to develop the conceptual framework of universal logic in a simple and comprehensible form as a science of patterns. The primitive notion of *deduction* is represented as a process of recognizing a pattern within a structure that licenses the addition of deletion of part of that structure. This process is itself represented by metastructures termed *inference patterns*, a collection of which will be said to constitute a *protologic* [79].

2.1 Foundations for a Universal Protologic

The genesis of what has come to be termed universal logic was in the philosophy of Hilbert that he derived by reflecting on his experience in proving his basis theorem [68] and rationally reconstructing Euclidean geometry [69], and the resultant controversy with Gordan [83, p. 18] and Frege [49, pp. 1–24] about his innovations in the logical foundations of mathematics.

Hilbert [70] evolved a new conceptual framework for *axiomatic thinking* that involved reconstructing a formalized discipline by abstraction to a minimal set of independent axioms, each having a meaningful interpretation in that discipline. His methodology introduced notions such as *logical existence* [49, p. 12] being equivalent to lack of contradiction, and *ideal elements* [71] being introduced in order to simplify the axioms even though they were not part of the original system.

Hilbert [70, p. 413] emphasized that the axiomatic foundations of logic underpinned those of other disciplines and themselves needed to be made secure. When he and his colleagues at Göttingen, such as Hertz, Bernays and Gentzen, worked on this problem it was natural for them to adopt his principles of axiomatic thinking and deconstruct logical deduction as a minimal collection of axioms, introducing ideal elements as necessary to simplify them, and focusing on freedom from contradiction and the complete reconstruction of expected inferential outcomes.

The objective was to provide logical foundations for mathematics and the axiomatic method rather than to characterize all possible logics. However, there were already two contending logics for mathematics, classical and intuitionistic [58], and it was natural for Hertz [67] in applying Hilbert's axiomatic thinking to logic to abstract as much as possible and consider inference patterns between arbitrary sentences. He even abstracted from the notion of inference itself, stating that there is no need to specify "what the symbol $\rightarrow$ linking the characters $a \rightarrow b$ or the word 'if' in the corresponding linguistic formulation means" [67, p. 247].

Bernays further extends Hertz's level of abstraction when he used it to exemplify Hilbert's philosophy of mathematics by reducing $\rightarrow$ to a *sign* rather than a *symbol*: "If the hypothetical relationship 'if A then B' is symbolically represented by $A \rightarrow B$, then the transition to the formal position is that we abstract from that the meaning of the symbol $\rightarrow$ and take the linkage by the 'sign' $\rightarrow$ itself as the primary consideration." [8, p. 333].

In her analyses of the evolution of written language to include non-phonetic technical material involving mathematical and logical symbology, Krämer [75] has characterized such extreme abstraction as complete *desemantification*. Dutilh Novaes [39] adopted this terminology and situated the desemantification of logic historically through her analysis of the use of the qualifier *formal* in the logical literature. She [40, §6.1.2] introduces the term *resemantification* to describe the process of reintroducing expected features of a desemantified system.

The following sections develop a universal protologic by commencing with deduction as a desemantified sign, $\rightarrow$, and incrementally resemantifying it by introducing common logical constraints as inference patterns represented by structures based on $\rightarrow$. The additional term *semantification* is used to distinguish various extra-logical interpretations that add meaning without changing the underlying logical system, including: those intrinsic to the logical system, such as truth values (Sect. 4.2), and intensional (Sect. 4.3) and extensional (Sect. 4.4) reconstruction.

2.1.1 Protosemantics

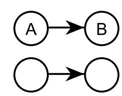
Whilst the protologic is itself a mathematical abstraction, it is intended to have practical applications and terms have been adopted to name abstract patterns that reflect those in the literature and seem natural to those patterns. However, these terms are strictly technical and none of their possible connotations beyond their formal definitions are used to draw inferences. That is, the protologic semantifies the terms and not the terms the protologic.

For example, $A \rightarrow B$ is read as 'A includes B' and two complementary technical terms are also introduced to provide an abstract protosemantics. The term *content* is used to reference that which might be included. That is, A → B may be read as 'the content of A includes the content of B.' This terminology is consistent with that of those who have proposed that logical entailment be explicated as *meaning* [41], *sense* [90] or *content* [26] *containment*, and enables one to assess when that notion is appropriate and when it is not.

The term *context*, is used to reference the effect of inclusion, of being within the scope of the meaning. That is, A → B may be read as 'the context of B encloses the context of A.' This terminology is consistent with Aristotle's [1, 1061b30] use of the term *qua* in his *Metaphysics* to introduce a context, for example, to consider physical objects 'qua moving' rather than 'qua bodies.' However, any connotations of the terms 'content' and 'context' derive explicitly from the constraints placed upon the use of →, rather than from *a priori* intuitions.

2.1.2 Arrows, Links and Graphs

Béziau [9, p. 85] introduces a minimal resemantification of a *deduction* sign by going back to the Latin roots of the term and interpreting it as *leading away* from premise to conclusion, that is, we may regard the arrow in A → B as a directed connective leading from the symbol 'A' at the tail to the symbol 'B' at the head. The resulting structure may be termed a *link* constituted by the triple (A, →, B). It may also be described as a *link out of* A constituted by the pair (→, B) or a *link in to* B constituted by the pair (A, →).



A link is naturally represented in graphical form as two labeled nodes with an arrow between them or, more generically, as an arrow between two anonymous nodes. It is assumed that labels constitute unique identifiers from some family of identifiers with an equality relation, and that an anonymous node has an implicit label unique to that node.

The resemantification involved in labeling nodes is the assumption that the entities linked by → may be identified, distinguished and equated. Node labels may be chosen to suggest possible connotations but these make no formal contribution to the logical structure represented by the links. They are logically meaningful only to the extent that the linkage structure represents such connotations.

Nodes with identical labels will be taken to represent the same node shown more than once, and may be merged to form a canonical graph-like structure with no duplication of nodes. This allows structures to be split into substructures, possibly overlapping, that can be merged to reconstitute the original structure. It also allows a structure to be merged with one of several other structures, each representing an alternative component, for example, a different 'ontology' or 'theory.'

Figure 1 shows a number of links merged into a graphical structure, S, with no duplication of nodes.

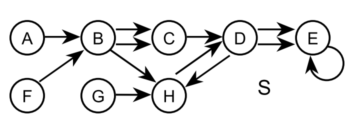


Fig. 1 Graphical structure specified by multiple links

Note that the graph representation adds no additional information to that of a linear representation as the multiset,

$$\mathrm{S} \equiv [\mathrm{A} \to \mathrm{B}, \mathrm{B} \to \mathrm{C}, \mathrm{B} \to \mathrm{C}, \mathrm{C} \to \mathrm{D}, \mathrm{D} \to \mathrm{E}, \mathrm{D} \to \mathrm{E}, \mathrm{E} \to \mathrm{E}, \mathrm{F} \to \mathrm{B}, \mathrm{B} \to \mathrm{H},$$
$$\mathrm{G} \to \mathrm{H}, \mathrm{H} \to \mathrm{D}, \mathrm{D} \to \mathrm{H}]$$

It is only a more perspicuous representation of the structure represented by multiple links involving the same nodes.

Figure 2 shows the same links merged into three overlapping substructures, S1, S2 and S3, representing the same structure in a modular way.

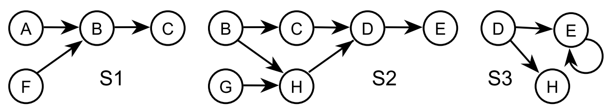


Fig. 2 Same structure specified modularly as three substructures

The linear representations are shown below and S is their multiset sum:

$$\mathrm{S1} \equiv [\mathrm{A} \to \mathrm{B}, \mathrm{B} \to \mathrm{C}, \mathrm{F} \to \mathrm{B}]$$
$$\mathrm{S2} \equiv [\mathrm{B} \to \mathrm{C}, \mathrm{C} \to \mathrm{D}, \mathrm{D} \to \mathrm{E}, \mathrm{B} \to \mathrm{H}, \mathrm{H} \to \mathrm{D}, \mathrm{G} \to \mathrm{H}]$$
$$\mathrm{S3} \equiv [\mathrm{D} \to \mathrm{E}, \mathrm{E} \to \mathrm{E}, \mathrm{D} \to \mathrm{H}]$$
$$\mathrm{S} \equiv \mathrm{S1} \uplus \mathrm{S2} \uplus \mathrm{S3}$$

The resemantification of the arrow sign so far is sufficient to support the basic structures of graph theory. However, as evident in the examples, there are no constraints to preclude parallel arrows between the same nodes or loops from a node to itself. Hence, in graph-theoretic terms, the examples given are not strictly *directed graphs* but rather *nets* [65, pp. 4–7] or *directed pseudographs* [4, p. 4].

There is also, as yet, no support for the inferential processes of logic. However, in the continuing resemantification inference pattern are introduced such that $\to$ represents a partial order underlying a logical system (Sect. 2.5).

2.1.3 Inference Patterns and Invariance Under Logical Interpretations

While it is convenient to use the terminology of graph theory to describe the nets that represent collections of links, logical theory focuses on the *dynamics of change* in such collections. This is not a primary concern of graph theory [64].

The logical dynamics of nets will be captured in terms of *inference patterns* in which an abstract subnet of a particular form is recognized as licensing the addition or deletion of one or more links while leaving any ‘logical interpretation’ of the net invariant. These notions are formalized in the following sections.

A *logical interpretation* of a net is defined to be an inference-preserving conservative translation [29] of the net into statements of a logical system supporting some notion of inference. To state that it is *invariant* under a change to the net is to assert that, within that logical system, the translation of the changed net will have the same inferences as that of the original net. Two nets with the same logical interpretations will be termed *logically equivalent*.

This is a constraint upon appropriate translations, that the target logical system must implement within its own framework the inference patterns of the protologic, and this must be verified for each translation. It is also intended that the inference patterns in the protologic can be understood in their own right and that the graphical language can be used to represent the form of knowledge structures and the dynamics of inference.

An inference pattern may be seen as an *analytic invariant* of a net in that it can dynamically expand a net by adding logical inferences implicit in its links, and hence can also delete them as being superfluous, contracting the net, possibly to a minimal form. Additions or deletions that maintain the net logically invariant will be termed *conservative expansions* or *conservative contractions*, respectively.

2.2 Inclusion Inference Pattern

A significant example of resemantification through a logical inference pattern is that *transitivity* of → which is common to its usage in most logical systems and has been taken to be a characteristic feature of 'a logic' [101]. Figure 3 shows Hertz's [67, Fig. 1] diagram to specify the transitivity of → on the left, and, in the center, its representation by an *inclusion inference pattern* in a net.

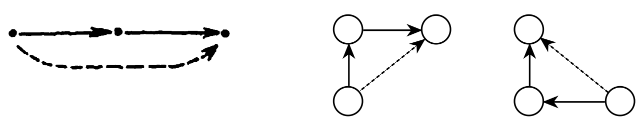


Fig. 3 Inclusion inference pattern

A metalogical distinction has been made in the inference pattern by showing the pattern-defining arrows as solid lines and the inferred arrow as a dotted line. The inference pattern indicates that if the pattern-defining links are found then an inferred link may be added, or any existing inferable link may be deleted, without changing the logical interpretation of the net of which the pattern is part.

The same pattern is drawn differently on the right to show that the inclusion inference pattern might be visualized either as the copying, or inheritance, of a horizontal inclusion link downwards from the head to the tail of a vertical inclusion link, or of a reverse horizontal inclusion link upwards from the tail to the head of a vertical inclusion link.

If x, y and z are links, S a multiset of links, and ≡ indicates identical logical interpretations, an equivalent linear representation might be:

$$\text{for any x, y, z, S,} \quad [\text{S}, \text{x} \to \text{y}, \text{y} \to \text{z}] \equiv [\text{S}, \text{x} \to \text{y}, \text{y} \to \text{z}, \text{x} \to \text{z}]$$

One advantage of using the two-dimensional structure of the page to provide a non-linear graphical presentation is that, as Shönfinkel [99, p. 17] has noted, the introduction of variable names is distracting because they serve merely to link multiple occurrences of the same logical entity. This linkage may be specified formally in a net without requiring an artificial label for a generic node.

However, the graphic representation has introduced no additional constructs beyond those of a conventional linear representation. It is a formal specification with no dependence on visual intuitions that avoids a proliferation of symbols.

2.2.1 Cycles and Node Equivalence

A cycle of mutual inclusion between two nodes, as shown in Fig. 4 left, may be treated as a single symmetric link with the directional arrow heads omitted as shown in the center.

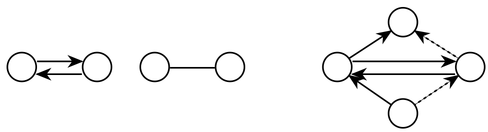


Fig. 4 Cycles and node equivalence within cycle

As shown on the right, such a cycle induces an equivalence between nodes because the inclusion inference pattern implies that any nodes which includes, or is included in, one node includes, or is included in, the other node.

This generalizes to larger cycles because the inclusion inference pattern implies that all nodes in a longer cycle are also linked pairwise in simple cycles, and hence all nodes in a cycle are equivalent (Fig. 5)—Hertz's [67, Definition 10] *web*.

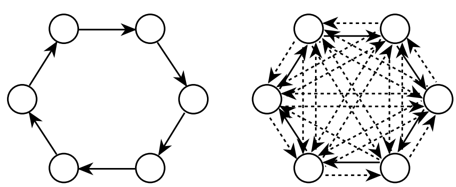


Fig. 5 Inference of node equivalence within a longer cycle

It will be noted in introducing other arrows and their link types that, as they are defined in terms of $\rightarrow$, the node equivalence induced by a cycle generalizes to commonality of links of any type. Any nodes in a cycle have the same incoming and outgoing links of all types and are equivalent in this respect, and their labels may be regarded as aliases for one another under this equivalence.

2.2.2 Equivalence and Loop Inference Patterns

Another commonly expected logical constraint on $\rightarrow$ is that should it be *reflexive* so that every node has an associated loop (as illustrated by node E in Figs. 1 and 2). Rather than being introduced as an *ad hoc* assumption, reflexivity can be derived from a more specific requirement that nodes equivalent to any node may be added or deleted without change of logical interpretation. This corresponds to the *equivalence inference pattern* shown on the left of Fig. 6, which may be explained in terms of nodes always being able to have more than one label.

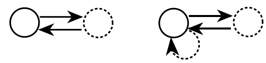


Fig. 6 Equivalence inference pattern and derived loop

As shown on the right, application of the inclusion inference pattern to the equivalence inference pattern implies that the node has an arrow to itself, a *loop*. These results may be used to derive the *loop inference pattern* corresponding to reflexivity through the inference sequence shown in Fig. 7.

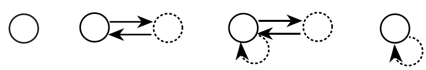


Fig. 7 Derivation of loop inference pattern

From left to right: the equivalence inference pattern is used to add an equivalent node to an isolated node; the inclusion inference pattern is used to infer that the original node has a loop; and the equivalence inference pattern is used to delete the equivalent node resulting in the loop inference pattern on the right.

Reflexivity is significant in the theoretical development of logics but irrelevant to practical applications where, whatever an inclusion link represents, it is unlikely to be useful to infer that a node includes, or is included in, itself—the epitome of a circular argument.

2.3 Repetition Inference Pattern

Another significant example of resemantification through a logical inference pattern is that for adding or deleting parallel arrows between nodes. For many logical interpretations multiple links of the same type in the same direction between two nodes do not affect the logical interpretation of a net. Their addition or deletion is a conservative expansion or contraction.

This could be implemented by requiring a collection of links to be a *set* rather than a multiset or by requiring the graph to represent a *relation* rather than a net. However, in some logical systems such as linear logic [59], the repetition of a statement may be significant, and the lack of significance in other logics is best represented explicitly as an *repetition inference pattern* (Fig. 8).

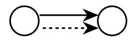


Fig. 8 Inclusion repetition inference pattern

2.4 Protoconjunctions and Protodisjunctions

When a node includes, or is included by, two or more other nodes, it is common in interpretations of nets in other logical formalisms to shorten the representation by collecting the labels of the other nodes separated by conjunction or disjunction symbols, respectively. For example, a description logic [3] translation of the nets in Fig. 9 might be:

$$\mathrm{A} \sqsubseteq (\mathrm{B} \sqcap \mathrm{C}) \qquad (\mathrm{A} \sqcup \mathrm{D}) \sqsubseteq \mathrm{B}$$

Fig. 9 Protoconjunction and protodisjunction

However, the nets might equally well be translated without the introduction of conjunction and disjunction symbols as:

$$\mathrm{A} \sqsubseteq \mathrm{B} \qquad \mathrm{A} \sqsubseteq \mathrm{C} \qquad \mathrm{D} \sqsubseteq \mathrm{B}$$

The use of conjunction and disjunction symbols in the first translation, introduced only to collect terms rather than compose them to define an ideal node, will be characterized as specifying a *protoconjunction* or *protodisjunction*. Protoconjunctions and protodisjunctions satisfying certain extremal conditions to define an additional ideal node as a composition of links will be termed *structural conjunctions* (Sect. 5.1) and *structural disjunctions* (Sect. 5.2), respectively. Similar considerations lead to a distinction between *protonegations* and *structural negations* (Sect. 2.8.2).

The significance of these distinctions is that the semantics, ontological commitments and inferential complexity of the protoconnectives is substantially simpler than that of the structural ones, and that the representation of many significant generic knowledge schemata requires only the protoconnectives [51]. However, the linguistic usage of ‘and,’ ‘or’ and ‘not’ does not give a clear indication of the type of connective intended and neither do most logical symbolisms.

2.5 The Resemantified → as a Preorder or Partial Order

Given any net, if one conservatively contracts it by merging it to canonical form, using the inclusion inference pattern to equivalence the nodes in a cycle, the equivalence inference pattern to delete all but one node in the cycle, and the loop inference pattern to delete any loops, the resultant net is a *directed graph* [4, §1.2].

It can be further contracted by using the inclusion inference pattern to delete all links that can be inferred from it resulting in the *transitive reduction* [4, p. 177] of the cycle-free graph as the *minimal canonical form* of the original net (unique up to the node label chosen to represent all the nodes in a cycle). Figure 10 shows S_{min}, the minimal canonical form of the net specified in Fig. 2.

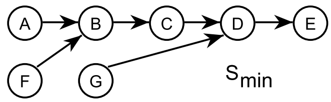


Fig. 10 Net from Fig. 2 reduced to a minimal canonical form

The transitive, reflexive relation defined by the repetition, inclusion and equivalence inference patterns indicates that, if $\rightarrow$ complies with these patterns, it specifies a *preorder* on the nodes. If the node equivalence of Sect. 2.2.1 is taken to define equality of nodes then the preorder becomes a *partial order*.

A preorder has been taken by Straßurger [101] to answer the question "what is a logic?" As the remaining constructions in this article are defined in terms of the preorder it might seem that the preceding sections have already resemantified $\rightarrow$ adequately to specify a logical system. The sign $\rightarrow$ has become the symbol for an *abstract copula* [37, p. 104] capturing the essential features of deduction.

However, the preorder alone offers limited representation and reasoning capabilities. The additional metalogical definitions that follow significantly extend the representation and inference capabilities of the protologic. Because they are always available as constructions within the protologic (as constraints on the usage of $\rightarrow$), one could argue that they are inherent in the order relation. One could also argue that their definition is a significant additional resemantification of the sign $\rightarrow$ leading to a richer notion of what it is to be a logic.

2.6 Definition of an Exclusion Arrow and Link

In representing logical inference, what is most obviously missing in the structures discussed so far is the notion that they may be structurally unsound or *inconsistent*. Every net based on the inclusion link, and every net derived from it using the various inference patterns, is a legitimate logical structure. Nodes and links may be added to any net without logical constraint. There is support for the notion that derived links are logically necessary but none for the notion that some potential links may be logically impossible.

The *exclusion arrow* $\nrightarrow$ supports the addition of a different type of link to a net that constrains what further links may be added to it. An *exclusion link* is defined in terms of the inclusion arrow:

Definition An *exclusion link* may be constituted with $\nrightarrow$ iff any node that includes the tail is excluded from also including the head.

The meta-inference pattern for this definition is shown in Fig. 11. A red cross is used as a metalogical symbol to indicate that an inclusion link is prohibited.

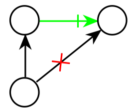


Fig. 11 Exclusion link definition

The tail node of an exclusion link is said to *exclude* the head node, and the head node to be *excluded by* the tail node. The head node may also be termed *opposite to*, *contrary to* or *incompatible with* the tail node.

2.6.1 Exclusion Inference Patterns

Three inference patterns may be derived from the metalogical definition of the exclusion link. First, it complies with a repetition pattern, the *exclusion repetition inference pattern* (Fig. 12 left).

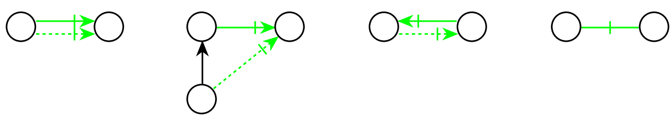


Fig. 12 Exclusion inference patterns

Second, because any node that includes the lower node also includes that to which it is linked (inclusion pattern), it cannot have a link to the node on the right. Hence the lower node satisfies the exclusion link definition and has an exclusion link to the node on the right. The resulting pattern is the *exclusion inference pattern* (Fig. 12 second from left).

Third, any node that includes the right hand node cannot also include the left hand node since the exclusion link definition would be contravened. Hence there is a also an exclusion link from right to left, resulting in the *exclusion symmetry inference pattern* (Fig. 12 second from right). This pattern marks a major difference in the inferential dynamics of the two link types, that the exclusion link is symmetric and can be represented by the single undirected link on the right.

The undirected link has repetition and exclusion inference patterns derived from those of the exclusion link (Fig. 13 left). The right two nets show a *reverse exclusion inference pattern* that also follows from the exclusion inference and symmetry patterns. The variant on the far right shows how this may also be seen as a *reverse inclusion inference pattern* such that an exclusion link to the tail of an inclusion link may be inferred from one to the head.

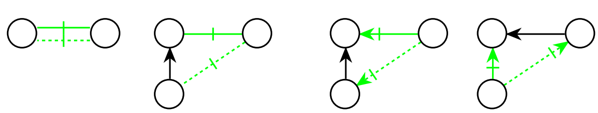


Fig. 13 Exclusion link and reverse exclusion/inclusion patterns

The exclusion inference pattern implies that if two nodes are in a cycle then any node which excludes, or is excluded by one node, excludes, or is excluded by the other. This generalizes the equivalence between nodes in a cycle defined in Sect. 2.2.1 to exclusion as well as inclusion links.

From the perspective of meaning containment, an exclusion link between two nodes signifies that the contents of the two nodes are *incompatible* and their contexts are *disjoint*. That is, some part of the content of one node cannot be included with some part of the content of the other node, and no node may be enclosed in the context of both the nodes.

From an inferential perspective the directed exclusion link can always be replaced by the undirected one with no change in logical interpretation; exclusion is always mutual exclusion. However, from a semantic perspective the metalogical distinction between the *specification* of the net and the *inferences* ensuing from that specification may be significant. There may, for example, be significance to the notion that the specified direction of the arrows depicts how a new node *depends* on existing nodes. Hence the symmetry of the exclusion link is, for some purposes, best represented as an inference rather than an intrinsic feature.

2.6.2 Inconsistency Inference Pattern

The definition of an exclusion link in Sect. 2.6 introduces a metalogical constraint that an inclusion link is not allowed between two nodes. If this constraint is violated in the net, either by specification or by inference, the outcome will be that two nodes are connected by both inclusion and exclusion links as shown in Fig. 14 (the exclusion link is shown in symmetric form to encompass all cases of parallel inclusion and exclusion links).

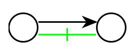


Fig. 14 Inconsistent link

Such a pair of links will be considered as constituting a single link combining conflicting link types and termed an *inconsistent link* from the tail of the inclusion link to its head. The tail node will be termed an *inconsistent node*, and the net of which it is part an *inconsistent net*.

The inconsistency inference pattern shown in Fig. 15 derives from inclusion and exclusion inference patterns. If one node has an inconsistent link to another then any node that includes it also has an inconsistent link to it and to that node, as will any down a path to it. Since any node is down a path from itself, any node with an inconsistent link also has an inconsistent link to itself.

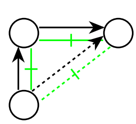


Fig. 15 Inconsistency inference pattern

An inconsistent link in a net, whether specified or inferred, implies that the exclusion link definition has been violated by the specification of a node that includes both the tail and the head of an exclusion link. It derives from the specification of net and indicates a structural inconsistency in that specification.

2.6.3 Paraconsistency

In translations of the protologic to some logical systems, inconsistency cannot be represented and is treated simply as an error in the specified logical structure. In a *paraconsistent* system [20, 91], meaningful inference will generally still be possible, and the protologic supports both possibilities. Whether the inconsistency corresponds to an intrinsically inconsistent entity or to an erroneous specification is an extra-logical issue that does not affect the inference patterns involved in representing inconsistency.

An inconsistent link in a net is *localized* in that it can only propagate through the inclusion and exclusion patterns to a node which includes the tail of its inclusion link. Inconsistency of some links in a net does not 'explode' to inconsistency of all links in the net. Thus, the logical inference schemata that have been defined are paraconsistent in their containment of inconsistency. It is possible to reason normally both about the parts of the net which are not inconsistent and about the propagation of the inconsistencies themselves. The principle of non-contradiction continues to apply, but, if the specification of a net violates it, the adverse impact of the violation is localized to a sub-net.

For example, the net shown in Fig. 16 illustrates the inferences when the definition of the exclusion link specified between nodes A and B is violated by node C which is specified to include both. C is inferred to have inconsistent links to A and B. These propagate to node D which is specified to include C. However, the exclusion link between A and E propagates without inconsistency to C and D, as does the inclusion link from B to F. An inconsistent entity can still exhibit normal patterns of inference for aspects unrelated to its inconsistency, and inconsistency itself propagates in a meaningful way.

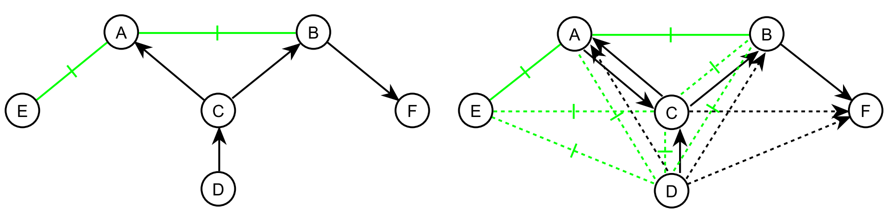


Fig. 16 Localization of inconsistency

2.7 *Generic Inference Patterns for the Kernel Protologic*

The protologic based on the resemantification of $\rightarrow$ with the inference patterns defined so far will be termed the *kernel protologic*. It is constituted through inference patterns constraining $\rightarrow$ to represent the partial order expected of derivation relations together with a further constraint $\nrightarrow$ restricting the placement of $\rightarrow$ to represent inconsistency.

The kernel protologic is extremely simple, yet logically powerful enough to illustrate the dynamics of logical systems that differentiate them from other mathematical structures (Sect. 3), to provide protosemantics underlying a wide range of semantic interpretations of many logics (Sect. 4), and to support the representation of a wide range of common knowledge representation schemata, such as determinables, graded scales, taxonomies and frames [51].

Structural connectives may be defined through constraints on nodes defined in terms of the basic inference patterns, but only very restricted forms of them are needed for much practical reasoning (Sect. 5). Nonmonotonic inference may be represented through a preference relation between nodes (Sect. 6).

The inference patterns for the kernel protologic have similarities that allow them to be condensed to a generic form. For example, the exclusion inference pattern mimics the inclusion inference pattern. If one shows them together (Fig. 17), Fahlman's [44] process of *virtual copying* is apparent. Both inclusion and exclusion links out of the top center node are 'copied' by inference to the lower node so that the links do not need to be shown explicitly but can be treated as 'implicit,' 'virtual' or 'inherited.'

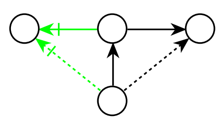


Fig. 17 'Virtual copying' of links

The inference patterns for $\rightarrow$ and $\nrightarrow$ are common to six link types, inclusion, equivalence, exclusion (in both directions and symmetric form), and inconsistent. These links may be treated as parametrized instances of a generic link, symbolized by $\dashrightarrow$, allowing the inference patterns to be represented generically (Fig. 18).

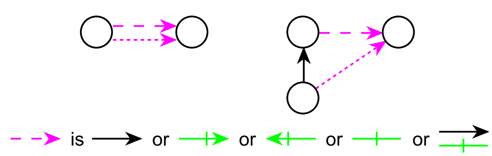


Fig. 18 Generic inference patterns

From an algebraic perspective, the inference patterns for $\dashrightarrow$ signify: left, that the instances represent relations; right, that the relations are idempotent when residuating $\rightarrow$ on the right. That is, in the Boolean algebra of binary relations, $\rightarrow \backslash \dashrightarrow = \dashrightarrow$ for the six substitution instances of $\dashrightarrow$ that are shown. This is the algebraic basis of notions of 'copying' and 'inheritance' and foundational to studies of logics as residuated lattices [53].

2.8 Definition of a Coexclusion Arrow and Link

One may interpret the inference pattern for an inclusion link (Fig. 3 center) as requiring that any node with an inclusion link to the tail also has an inclusion link to the head, and for an exclusion link (Fig. 12 second left) that it has an exclusion link to the head. There are also two obvious complementary link types where an exclusion link to the tail node requires either an inclusion or exclusion link to the head node. The latter is already available as the converse of an inclusion link (Fig. 13 right) but the former is a new link type that will be termed a *coexclusion link* and represented by an arrow with two bars ⇸.[1]

The formal definition of a coexclusion link is:

Definition A *coexclusion link* may be constituted with ⇸ iff any node that excludes the head must include the tail.

This definition is illustrated in graphical form in Fig. 19:

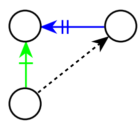


Fig. 19 Coexclusion link definition

From the perspective of meaning containment, a coexclusion link specifies that if the content of the head is excluded then that of the tail is included, and if a node is not within the context of the head then it is within the context of the tail. One might also interpret this definition as specifying that the content of the tail of a coexclusion link is included in the content of the tail of any exclusion link with the same head, and that the tail of a coexclusion link provides a context for the tail of such an exclusion link.

2.8.1 Coexclusion Inference Patterns

Figure 20 shows derived inference patterns for coexclusion links. From left to right, coexclusion links have a repetition pattern, are symmetric allowing the arrow heads to be dropped (subject to semantic considerations) to provide an equivalent line form, and have a downward inheritance pattern similar to that of the reverse inclusion link (Fig. 3).

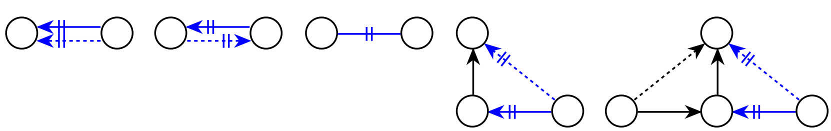


Fig. 20 Coexclusion inference patterns

Figure 21 shows a generic representation of the coexclusion inference patterns.

[1]*Coexclusion* is a neologism reflecting the duality between exclusion and coexclusion links. In the ancient logical literature, the terms *subpares* (Apuleius) and *subcontarias* (Boethius) have been used for a coexclusion relation in the metalogic of the syllogistic.

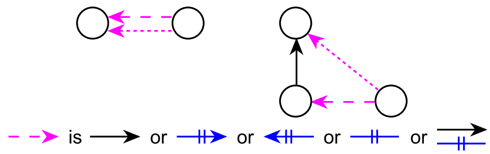


Fig. 21 Generic inference patterns for inclusion and coexclusion

From an algebraic perspective, the inference patterns for ⇠ signify: left, that the instances represent relations; right, that the relations are idempotent when residuating → on the left. That is, in the Boolean algebra of binary relations, ⇠/→ =⇠ for all the substitution instances of ⇠.

2.8.2 Negation Inference Patterns

Exclusion and coexclusion links may coexist between the same nodes without inconsistency, and the combination of exclusion and coexclusion links acts as a structural negation in the protologic. Figure 22 shows some of the possibilities, all of which are logically equivalent but may be semantically distinct in indicating different origins of the negation.

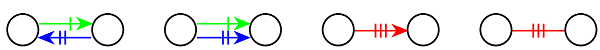


Fig. 22 Structural negation as paired exclusion and coexclusion

The negation aspects of the combination are apparent in the inference patterns shown in Fig. 23. On the left it can be seen that an inclusion link to one node results in an exclusion link to the other, and *vice versa*.

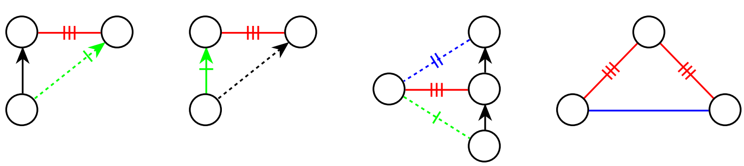


Fig. 23 Structural negation inference patterns

In the center, it can be seen that a node which includes one node excludes the other, and one that is included coexcludes the other. The lower exclusion inference captures Brandon's [27, p. 126] definition that a contradiction is that which is entailed by any contrary, and Dunn [38, p. 10] and Brady's [26, p. 20] that it is a disjunction of all possible contraries. The upper coexclusion inference provides dual definitions: that a negation entails any coexclusion and is the conjunction of all possible coexclusions. The inference pattern may also be seen as splitting a negation into its exclusion and coexclusion components.

On the right is shown the inference pattern for double negation, that if two nodes have negation links to the same node then they are equivalent. The inferred inclusion links derive directly from the coexclusion definition of Fig. 19. This pattern also shows that the negation of a node is unique modulo equivalence.

The exclusion and coexclusion link types may be seen as a *protonegations*: that the inclusion of some content is incompatible with the inclusion of some other; or that the exclusion of some content requires the inclusion of some other.

The latter is a less natural constraint than the former and may be seen as the source of the *semantic fragmentation* [45] that led to Plato and Aristotle's critiques of a bare negation used as if it generated a meaningful concept. For example, that, whilst 'being Greek' is a meaningful concept, use of the term 'barbarian' for 'not being Greek' does not define a concept having a coherent meaning [89, 262d].

2.8.3 Coexclusion Squares and Hexagons of Oppositions

Figure 24 left shows a square net with coexclusion links on the diagonals and an exclusion link on the top side such that inclusion links may be inferred on the left and right sides and a coexclusion link on the bottom side. Similar structures have been studied in the logical literature, both ancient [78] and modern [21, 22], as *squares of oppositions*.

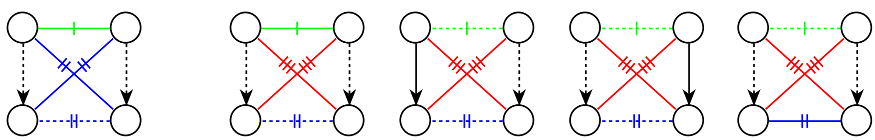


Fig. 24 Coexclusion squares of oppositions

The inferred implications on the left and right of the square are derived directly from the coexclusion definition. The inferred coexclusion link is derived because any node with an exclusion link to one of its nodes will also have an exclusion link to the node above (exclusion inference pattern) and hence an inclusion link to the other node (coexclusion inference pattern) and hence satisfies the coexclusion definition.

In the net on the left, having only coexclusion links as diagonals, the additional inclusion and coexclusion links may be derived from these and the exclusion link at the top. However, if this link is omitted, it cannot be derived by specifying one of the links previously derived.

However, if exclusion links are added to the coexclusion links on the diagonals to make them negations then specifying the link on *any* side allows those on the other three sides to be derived (Fig. 24 right). These are the classic *squares of oppositions* that are common to many logical systems [21].

The net on the left demonstrates that the main phenomena of interest in squares of oppositions derive from the co-exclusion component of the negations on the diagonals, that a node that excludes a node at one end of the diagonal must include that at the other end.

Blanché [24] extended the square of oppositions with nodes to a hexagon of oppositions by adding the disjunction of the top nodes and the conjunction of the bottom ones. His construction has proved significant in a universal logic framework in representing the relations between major constructs in a wide range of logical systems [17, 19], and is interesting to see how it may be represented in the kernel protologic with coexclusion.

Figure 25 left shows the basic square extended to a hexagon with a protoconjunction at the bottom and a protodisjunction at the top, and that one may be inferred from the other. On the right this is extended to a full hexagon of oppositions by adding an exclusion link between the top two nodes of the square and inferring the other links on the sides of the square. As already shown in Fig. 24, any of the four links on the sides of the square could be added to infer the others.

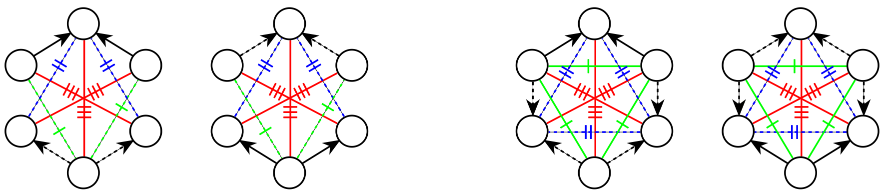


Fig. 25 Inferences in a square extended to hexagon

Thus the logical derivations involved in the Blanché hexagon may be factored through the inclusion, exclusion and coexclusion links involved in their derivations, and only protodisjunction and protoconjunction nodes are required for the top and bottom nodes that he added, not the ideal nodes of structural definitions (Sect. 5).

3 Dynamics of Logical Structures

As noted in Sect. 2.1.3, although logical structures may be represented in graph-theoretic form, logic goes beyond graph theory in its emphasis on the *dynamics of change* in those structures. This encompasses not only the addition and deletion of links in nets through inference patterns but also various ways of restructuring nets, splitting and merging of nets, the specification of additional nodes and links, and the consequences for the logical interpretations of the nets involved. It also involves considerations of effective means of communicating the dynamics to those developing and using knowledge structures represented in nets.

3.1 Minimal and Maximal Canonical Forms

In has been noted (Sect. 2.5) that a net with inclusion links can be reduced to a minimal canonical form having the same logical interpretation. The exclusion and coexclusion inference patterns support a similar deletion of exclusion and coexclusion links that may be regarded as superfluous because they can be inferred from others. However, the symmetry of these links means that the direction in which they are specified is inferentially irrelevant and it is appropriate to treat the symmetric form as canonical in deriving a unique canonical form. Similarly, the repetition inference pattern for all links means that it is appropriate to treat reduction to a single link as canonical for the sake of unicity.

These considerations lead to a minimal canonical form for a net with that is unique up to the choice of node label used to represent all the nodes in a cycle, and has the same logical interpretation as the net from which it was derived. The inclusion, exclusion and coexclusion inference patterns may be used to expand this minimal canonical form to a unique maximal one. Figure 26 illustrates the minimal and maximal canonical forms of a specified net of inclusion and exclusion links.

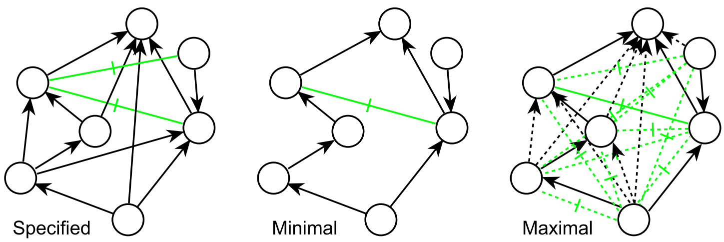


Fig. 26 Example of minimal and maximal canonical forms

The inference patterns of Sect. 2 and their use to expand or contract the net as discussed in this section exemplify net dynamics through the addition or deletion of links. The following sections extend this to expansion and contraction of nets through the addition or deletion of other nets (including single nodes). Since some expansions do not change the logical interpretation of the net, this allows for further expansion beyond the maximal canonical form to a net that is logically equivalent modulo added nodes.

3.2 Consistent and Conservative Imports and Merges

Nets are dynamic, not only through inference patterns that do not change their logical interpretation, but also though the addition of links that are not implicit in the inference patterns and hence change the interpretation, and through the addition of nodes and associated links that may have a variety of effects.

When nodes are added they may already be named and/or linked, and the general case may be considered as one where one net is *imported* by another, either through merging nodes with the same names, or by linking nodes in one net with nodes in the other, or both.

One major consideration is whether the resultant net has additional inconsistent links. An import is termed *consistent* if it does not. Usually the focus is on the import of a consistent net by a consistent net to produce a consistent net, but it can be appropriate to consider situations where the importing or imported net is inconsistent.

Another major consideration is whether the import is *conservative* in that no additional links may be inferred between the nodes in the imported net. This is of concern, for example, in many computational applications where the imported net is intended to be a generic module representing a 'library,' 'ontology' or 'theory.' The intent is to generate additional inferences in the importing net but not in that which is imported [62].

A conservative import may impact the dynamics of the importing net by making inconsistent potential links that could have been added to that net, that is, constraining it more strongly. An import that imposes no such constraints is termed *ultra-conservative*.

An import of one net by another may also be regarded conversely as an import by the second net of the first. If both imports are conservative, or ultra-conservative, it may be termed a conservative, or ultra-conservative, *merger* of the nets.

Since the inference patterns are such that only inclusion or coexclusion links from nodes in one net to nodes in another can generate new links within the first net, importing a net such that there are no inclusion or coexclusion links from the imported net to the importing one is always a conservative import, but generally not ultra-conservative.

Conservative imports are reversible through deletion of the imported net provided this deletion is from the maximal canonical form. This restriction is necessary because the purpose of some imports is to simplify the net by reducing the number of links in the minimal canonical form and, hence, the links removed from the original net must be restored before the imported net is removed (for example, in the *factoring* expansion pattern of Sect. 3.4).

3.3 Single Node and Link Expansion Patterns

Adding an additional anonymous node as an *ideal element* with up to one link to a node in a net is an ultra-conservative merger, and the single node and link expansion patterns of Fig. 27 are always available.

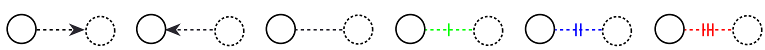


Fig. 27 Single node and link expansion patterns

Thus, nets are extensible by an anonymous node with a single link without logical constraint. However, extra-logical or metalogical constraints may be applied such as a node being specified to be a bottom or top node.

Expansion patterns based on ultra-conservative mergers provide a symmetric contraction pattern, similar to those of the inference patterns, in that the anonymous node may be removed without affecting the logical interpretation of the original net. The addition of a labeled node is conservative if there are no nodes with that label in the net, but generally not conservative if this is not so unless the link is one that may already be inferred.

3.4 Factoring Expansion Patterns

The minimal canonical form (Sect. 3.1) minimizes the number of links whilst keeping the number of nodes constant (treating nodes in a cycle as a single node). There are other forms of restructuring that can further reduce the number of links at the expense of adding additional nodes. For example, one can factor several links with a common tail or head and add an additional node to group them.

Figure 28 left shows a net having 8 nodes and 12 links, with the links at the top having common tails and those at the bottom having common heads. In the center is shown a net with 10 nodes and 10 links where two anonymous nodes have been interpolated to factor out the commonality. The two nets have the same logical interpretation with respect to the nodes in the original net. The import of the anonymous nodes is ultra-conservative, and the original links are implicit, or virtual, in that they may be inferred from the inference patterns.

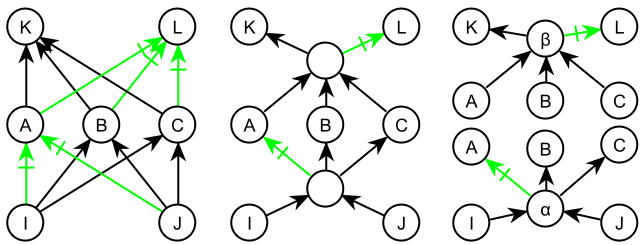


Fig. 28 Example of factoring a net

On the right the factored net has been split into modules and the added nodes have been named. The *interface* nodes between the modules have been duplicated to separate the net into two simpler components.

The factoring pattern is analogous to introducing the middle term of a syllogism, and Vaihinger [104, pp. 212–213] discusses it as an illustration of the introduction in logic of *fictions*, abstract notions that are treated 'as if' they were real. Hertz [67, pp. 248–249] uses this pattern to simplify systems of axioms and notes that the interpolated node may be regarded as an *ideal element*.

Factoring clusters some collection of links in a way that reduces the arrow crossings making common structures more apparent. In an application the resulting sub-structures may well suggest labels for the anonymous nodes that have been created as significant entities in the domain of the application. It generally does not lead to a unique canonical form as there may be several possible factorings having the same logical interpretation as the original net.

The increased perspicuity that can result from factoring is not an artifact of the graphical presentation of the net. If one represents the structures in Fig. 28 as multisets of links then the more modular representation of the net after factoring is also apparent:

$$[\mathrm{I} \nrightarrow \mathrm{A}, \mathrm{I} \rightarrow \mathrm{B}, \mathrm{I} \rightarrow \mathrm{C}, \mathrm{J} \nrightarrow \mathrm{A}, \mathrm{J} \rightarrow \mathrm{B}, \mathrm{J} \rightarrow \mathrm{C}, \mathrm{A} \rightarrow \mathrm{K}, \mathrm{B} \rightarrow \mathrm{K},$$
$$\mathrm{C} \rightarrow \mathrm{K}, \mathrm{A} \nrightarrow \mathrm{L}, \mathrm{B} \nrightarrow \mathrm{L}, \mathrm{C} \nrightarrow \mathrm{L}]$$

$$\big[[\mathrm{I} \rightarrow \alpha, \mathrm{J} \rightarrow \alpha], [\alpha \nrightarrow \mathrm{A}, \alpha \rightarrow \mathrm{B}, \alpha \rightarrow \mathrm{C}],$$
$$[\mathrm{A} \rightarrow \beta, \mathrm{B} \rightarrow \beta, \mathrm{C} \rightarrow \beta], [\beta \rightarrow \mathrm{K}, \beta \nrightarrow \mathrm{L}]\big]$$

$$\big[\big[[\mathrm{I} \rightarrow \alpha, \mathrm{J} \rightarrow \alpha], [\alpha \nrightarrow \mathrm{A}, \alpha \rightarrow \mathrm{B}, \alpha \rightarrow \mathrm{C}]\big],$$
$$\big[[\mathrm{A} \rightarrow \beta, \mathrm{B} \rightarrow \beta, \mathrm{C} \rightarrow \beta], [\beta \rightarrow \mathrm{K}, \beta \nrightarrow \mathrm{L}]\big]\big]$$

The description logic translation makes the *protodisjunctions* and *protoconjunctions* represented graphically by the interpolated nodes more apparent:

$$(\mathrm{I} \sqcup \mathrm{J}) \sqsubseteq \alpha \sqsubseteq (\neg\mathrm{A} \sqcap \mathrm{B} \sqcap \mathrm{C}) \qquad (\mathrm{A} \sqcup \mathrm{B} \sqcup \mathrm{C}) \sqsubseteq \beta \sqsubseteq (\mathrm{K} \sqcap \neg\mathrm{L})$$

From a meaning containment perspective the factoring is innocuous because the logical interpretation of the original net is unchanged; the links between the original nodes are unchanged. The ideal nodes which have been introduced derive content only through their links. Their lack of intrinsic content may be represented logically by making the protoconjunction or protodisjunction structural (Sect. 5), *defining* the ideal nodes in terms of their links. If this is not done and they are named to introduce additional content then extra-logical criteria are involved.

3.5 Existential Status of Added Nodes

Vaihinger's analysis raises the issue of the existential status of added nodes. One may distinguish at least five aspects of the notion of *existence* in the protologic: being a node (*conceptual existence, daseinfrei* [81, p. 51]); being a consistent node (*logical existence*, [49, p. 12]); being specified as an essentially top node representing a maximally generic entity (*categorical existence*); being specified as an essentially bottom node representing a single entity (*singular existence*); and being a bottom node representing an entity within a specified universe of discourse (*situated existence*), such as a state of affairs, phenomenon, event, experience or individual, in some situation, world or time interval specified in logical terms.

The semantification involved in naming ideal elements is not a resemantification of the underlying logical system. It is the extra-logical semantification provided by attaching domain-specific meanings to abstract patterns within that system that represent concepts in the domain.

3.6 Adding a Bottom Node, Probes

A simple but important structure is a *probe*, a bottom node that is added to a net but not regarded as part of it and has an inclusion or exclusion link to some of the nodes constituting the net. Two probes will be termed *distinct* if they have different sets of links.

A probe is *compatible* if it is in maximal canonical form, having all the links that may be inferred from the net. A probe is *admissible* if it is compatible and has no inconsistent links other than those resulting from a link to an inconsistent node; that is, it has no unnecessary inconsistent links. If the net itself is consistent then an admissible probe is also consistent.

Figure 29 illustrates these distinctions for a simple net with a single link between two nodes: P1 is consistent, admissible and compatible; P2 is incompatible because the inferable inconsistent links are not shown; P3 is inconsistent but still incompatible because only one inconsistent link is shown; P4 is compatible but not admissible since the inconsistent links do not derive from an inconsistent node; P5 is admissible because its inconsistent links derive from one in the net.

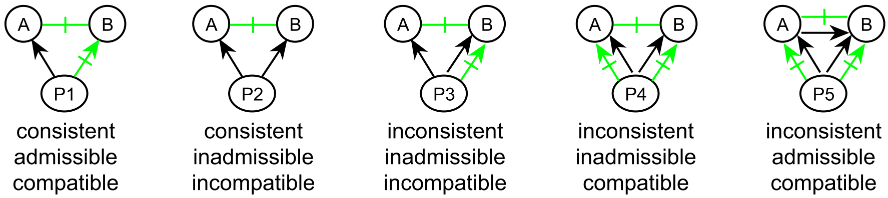


Fig. 29 Examples of different forms of probe

A *full probe* is one with a link to every node in the original net, otherwise it is a *partial probe*. If there are n nodes with distinct labels in the net then there will be 2^n distinct full probes having different combinations of link types, but not all of them may be consistent. A set of all possible distinct admissible probes is said to *saturate* a net.

A probe is not unique and there is no way of specifying it to be so within the protologic so far defined. Unicity can be specified in the metalogic by defining it relative to its links as being the *greatest node* in the preorder that has those links. The general form of this additional constraint is analyzed in Sect. 5.1 and, if pushed down to the protologic, is shown to provide a *structural conjunction* [73, Chap. 13].

3.7 Admissible Probes Characterizing Possible Links Between Two Nodes

The four possible probes for the two nodes A and B are:

P0: A⇷ ⇸B P1: A← →B P2: A⇷ →B P3: A← ⇸B

There are 16 possible combinations of consistent probes and these may be used to distinguish between the 16 nets having two nodes linked by all possible types of link or link combination. Figure 30 shows the admissible probes (including the inconsistent probe Pi) for all possible links between A and B. It can be seen that each net has a distinct characteristic set of consistent probes, and hence that such a set may be used to identify the link between the two nodes.

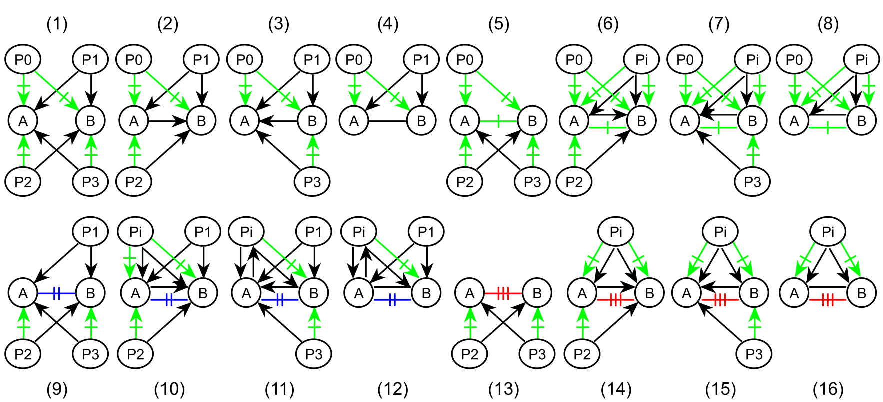


Fig. 30 Admissible probes for sixteen link types

More generally, one can reconstruct the nets from functions, π and $\underline{\pi}$, associating with each node the set of admissible probes that include or exclude that node, respectively. There is an inclusion link from A to B if the set of probes with inclusion links to A, πA, is a subset of that with inclusion links with B, πB, and the set of probes with exclusion links to B, $\underline{\pi}$B, is a subset of that with exclusion links to A, $\underline{\pi}$A, that is $\pi A \subseteq \pi B$ and $\underline{\pi}B \subseteq \underline{\pi}A$. There is an exclusion link from A to B if $\pi A \subseteq \underline{\pi}B$ and $\pi B \subseteq \underline{\pi}A$, and a coexclusion link if $\underline{\pi}A \subseteq \pi B$ and $\underline{\pi}B \subseteq \pi A$.

The conjunction of the two tests is necessary to allow for inconsistent nodes that would otherwise lead to spurious links because a probe has an exclusion link to every exclusion inconsistent node and an inclusion link to every coexclusion inconsistent node. Unless both nodes are inconsistent, the conjunction of the two tests ensures a correct reconstruction.

As shown in Table 1, for the 16 nets of Fig. 30 the links are correctly reconstructed from the relations between the sets of consistent probes. The final example for net (16) is a pseudo inference pattern; all completely inconsistent nodes are both equivalent to and contradictory to one another, playing no useful role in inference processes.

Table 1 Reconstructing links from probe sets

	πA	πB	$\underline{\pi}$A	$\underline{\pi}$B	$\pi A \subseteq \pi B$ $\underline{\pi}B \subseteq \underline{\pi}A$ $A \rightarrow B$	$\pi B \subseteq \pi A$ $\underline{\pi}A \subseteq \underline{\pi}B$ $B \rightarrow A$	$\pi A \subseteq \underline{\pi}B$ $\pi B \subseteq \underline{\pi}A$ $A \mathrel{-\!\!\!|\!\!\!-} B$	$\underline{\pi}A \subseteq \pi B$ $\underline{\pi}B \subseteq \pi A$ $A \mathrel{-\!\!\!\|\!\!\!-} B$
(1)	{1, 3}	{1, 2}	{0, 2}	{0, 3}	false	false	false	false
(2)	{1}	{1, 2}	{0, 2}	{0}	true	false	false	false
(3)	{1, 3}	{1}	{0}	{0, 3}	false	true	false	false
(4)	{1}	{1}	{0}	{0}	true	true	false	false
(5)	{3}	{2}	{0, 2}	{0, 3}	false	false	true	false
(6)	{i}	{i, 2}	{0, i, 2}	{0, i}	true	false	true	false
(7)	{i, 3}	{i}	{0, i}	{0, i, 3}	false	true	true	false
(8)	{i}	{i}	{0, i}	{0, i}	true	true	true	false
(9)	{1, 3}	{1, 2}	{2}	{3}	false	false	false	true
(10)	{i, 1}	{i, 1, 2}	{i, 2}	{i}	true	false	false	true
(11)	{i, 1, 3}	{i, 1}	{i}	{i, 3}	false	true	false	true
(12)	{i, 1}	{i, 1}	{i}	{i}	true	true	false	true
(13)	{3}	{2}	{2}	{3}	false	false	true	true
(14)	{i}	{i, 2}	{i, 2}	{i}	true	false	true	true
(15)	{i, 3}	{i}	{i}	{i, 3}	false	true	true	true
(16)	{i}	{i}	{i}	{i}	true	true	true	true

In Sect. 4.4, it is shown that this technique scales up to nets with any number of nodes and provides extensional semantics for the paraconsistent protologic.

4 Protosemantics

The notion of *semantics* in the logical literature is based on *translations* between a logical system and some other systems, either formal or informal. If one side of the translation is regarded as better founded than the other then it may be used to provide 'foundations' for the other. If not, then both sides may provide insights into aspects of the other. If one system is naturalistic and informal then the more formal system may be seen as providing an *explicatum* of the less formal. Conversely, the naturalistic system may be seen as providing a test of the *adequacy* of its representation by the formal one.

For a universal protologic one would expect there to be a very wide variety of significant translations between it and many other systems, providing protosemantics underlying each of the wide range of approaches to semantics for various logical systems. Some of these have already been mentioned such as the algebraic interpretations in terms of graph theory, residuated relations and preorders. Others have been suggested in terms of potential formalizations of notions of *content* and *context*. The possibility of the reconstruction of nets from probes, bottom nodes that may be regarded as representing states of affairs acting as *protoindividuals*, provides extensional semantics.

This section addresses basic semantic interpretations for the kernel protologic that are common to many logical systems, and later sections examine how they extrapolate to further patterns of inference as they are introduced. Examples of meaningful knowledge structures represented in the protologic are provided in Sect. 7.

4.1 *Proof-Theoretic Semantics*

Gentzen [57, § II 5.1.3] provided proof-theoretic foundations for logical systems when he extended Hertz's [67] approach to the definition of logical systems by defining natural deduction inference schema comprising paired rules for the introduction and elimination of logical constants. He remarks that the introduction rules could be seen as definitions of the constants and the elimination rules as consequences of those definitions.

The protologic has a foundational proof-theoretic semantics though its specification in terms of *inference patterns*, each of which licenses the addition and deletion of a link (including some to an additional node). The patterns form complementary pairs where the addition of a link defines a feature of a logical constant, such as transitivity, and deletion is in harmony with addition, because the link deleted remains 'virtually' present, always available to be reintroduced.

This leads to two levels of *meaning* for the protologic connectives: at the logical level, it has operational semantics in terms of the inference patterns; at the metalogical level, it has conceptual semantics in terms of phenomena induced by the operations such as the *idempotence*, *transitivity*, *reflexivity*, and *symmetry*.

4.2 *Truth-Theoretic Semantics*

The graphical representation of inference used so far is to indicate inferred links and nodes with dotted lines. However, as Fig. 26 right illustrates, in larger nets the inferred links may become difficult to discriminate. The graphic language provides an alternative way of displaying the same information by allowing one or more nodes to be marked and then marking the other nodes to show what kind of specified or inferred link the marked nodes have to them.

A vertical mark placed in a node results in all those it includes being marked with a vertical bar, and those it excludes, or exclude it, by a horizontal bar. A horizontal mark placed in a node results in all those nodes that include it being marked by a horizontal bar, and all those that it coexcludes, or coexclude by it, being marked with a vertical bar.

Figure 31 shows a net in which the center node labeled 'A' has different links to five other nodes, and the effect of marking that node with a vertical mark, a horizontal mark, or both.

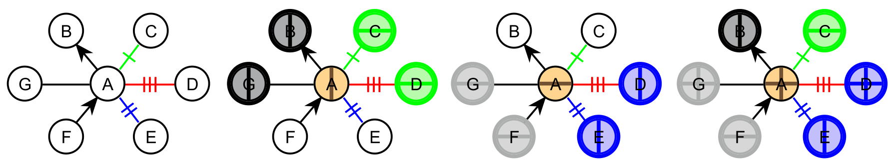


Fig. 31 Propagating marks from one node to linked nodes

The marks are colored similarly to the links to indicate the nature of the source: brown for an asserted mark; black for a vertical bar inferred from an inclusion link; blue for a vertical bar inferred from a coexclusion link; grey for a horizontal bar from an inclusion link; and green for a horizontal bar from an exclusion link (with grey having priority). It is apparent from Fig. 31 that propagation of a vertical mark through negation is through its exclusion component, and a horizontal mark through its coexclusion component.

One can recover a familiar logical vocabulary by terming a vertical mark logically *true* and a horizontal one logically *false*. The inference patterns shown can then be interpreted as: 'if A is true then B and G are true, C and D are false;' 'if A is false then D and E are true, F and G are false;' 'if A is inconsistent (both true and false) then B and E true, C and F are false, D and G inconsistent.'

When a node is marked false all those nodes included in its context are also inferred to be marked false, and this may be viewed as the context determining their *relevance*. One might prefer to say that the node has become *irrelevant* because its context is inapplicable rather than that it is simply false. The grey color of a horizontal bar for falsity distinguishes this type of inference from the other two possibilities of the node being asserted to be false (brown), or being inferred to be so from a node with an exclusion link being inferred false (green).

This explication of logical truth in the protologic is simply an alternative way of representing the inference patterns in a net, a *deflationary* account according to Gupta's definition [63, p. 57]. The complexity of more profound notions of truth [76] beyond those of logic is epistemological, the consideration of issues of the extra-logical justifications for making assertions.

The paraconsistency of the protologic corresponds to it not being bivalent and supporting inference from nodes that are incoherent in being marked both 'true' and 'false.' The inference patterns of Fig. 31 are those of Belnap's [6] *useful four-valued logic* based on 'told true' and 'told false' as independent assertions.

4.3 Intensional Semantics

The set of links having the same tail node provides an *intensional* characterization that node in terms of those of other nodes. A node includes itself, and this link is part of its intension. This corresponds to nodes in the protologic so far developed being *primitive*, constrained by their links to other nodes but not defined by them.

Glashoff [60] has developed an intensional semantics for the categorical syllogistic in which a term, A, is characterized by an *intensional interpretation* comprising a pair of disjoint subsets of a set $\mathcal{O}$, (s(A), $\sigma(A)$), such that the inclusion and exclusion links may be recovered as:

$$\mathrm{A} \to \mathrm{B} \quad \text{iff } \mathrm{s(A)} \supseteq \mathrm{s(B)} \text{ and } \sigma\mathrm{(A)} \supseteq \sigma\mathrm{(B)}$$
$$\mathrm{A} \mathrel{-\!\!\!+} \mathrm{B} \quad \text{iff } \mathrm{s(A)} \cap \sigma\mathrm{(B)} \neq \varnothing \text{ or } \sigma\mathrm{(A)} \cap \mathrm{s(B)} \neq \varnothing$$

He does not exemplify particular interpretations of members of $\mathcal{O}$, but it is reasonable to suppose each member is a 'merkmal,' 'mark,' 'feature,' 'property,' 'quality,' 'characteristic' or a similar term used technically or colloquially for a component of content or meaning, or *atom of intensionality*.

The pair of sets may be reduced to a single set by using a syntactic marker, such as an overbar, to distinguish each member of σ(A) from those of s(A), and merging the two sets to represent the *content* of A, c(A). If content is defined to be *inconsistent* if one pair of members has the same label both marked and unmarked, then one can recover the links as:

$$\mathrm{A} \to \mathrm{B} \quad \text{iff } \mathrm{c(A)} \supseteq \mathrm{c(B)}$$
$$\mathrm{A} \mathrel{-\!\!\!+} \mathrm{B} \quad \text{iff } \mathrm{c(A)} \cap \mathrm{c(B)} \text{ is inconsistent}$$

which nicely characterizes content inclusion as meaning containment, and content exclusion as meaning incompatibility.

Glashoff's construction provides the basis for the mereological explication of the inclusion of content in the kernel protologic, complementary to Euler diagrams as a mereological explication of context.

An intensional partition of a net in the kernel protologic (without coexclusion links) is the set of subgraphs each constituted by a single node together with the two sets of nodes at the head of the inclusion and exclusion links of which it is the tail (corresponding to Glassoff's [60, §1.2.2] sets of positive and negative terms). If X is a node in a net,

$$\mathrm{s(X)} = \{\mathrm{Y} : \mathrm{X} \to \mathrm{Y}\}$$
$$\sigma\mathrm{(X)} = \{\mathrm{Y} : \mathrm{X} \nrightarrow \mathrm{Y}\}$$

defines an intensional partition where all links are included, and the recovery of the net is a simple merger of the node sub-graphs equivalent to

$$\mathrm{A} \to \mathrm{B} \quad \text{iff } \mathrm{B} \in \mathrm{s(A)}$$
$$\mathrm{A} \mathrel{-\!\!\!+} \mathrm{B} \quad \text{iff } \mathrm{B} \in \sigma\mathrm{(A)}$$

When the net characterized is in maximal canonical form, Glashoff's recovery of the connectives may be seen as a consequence of the inclusion inference pattern and the exclusion link definition. More complex intensional partitions satisfying these definitions may be developed by following Glashoff and treating node labels as designating non-empty sets which conform with the inclusion and exclusion inference patterns and whose members represent atoms of intensionality.

Such sets may be subsumed within the protologic by representing their members as additional nodes with links that have not been explicitly specified as part of the net. The acceptance that node labels may designate implicit links provides an intensional semantics in terms of the explicit and implicit structure of a net that explicates notions of meaning and content within the protologic without introducing additional constructions. In particular, it recognizes that nodes may have unspecified common content.

Coexclusion adds a third link type to the kernel protologic and the intensional partition of a net may be extended to encompass this by defining a third set, $\omega(\mathrm{X})$, characterizing the coexclusion links of node X such that

$$\omega(\mathrm{X}) = \{\mathrm{Y} : \mathrm{X} \nrightarrow\!\!\!\rightarrow \mathrm{Y}\}$$

The net can be recovered by the merger of the node sub-graphs, equivalent to adding the recovery of coexclusion links through

$$\mathrm{A} \nrightarrow\!\!\!\rightarrow \mathrm{B} \quad \text{iff } \mathrm{B} \in \omega(\mathrm{A})$$

4.4 Extensional Semantics

Extensional semantics in which a logical structure is characterized through the set-theoretic relations between the sets of individuals providing models that are consistent with that structure play a central role in modern logic. They can be studied in the protologic without introducing an additional construct to represent an 'individual' by noting that the only feature required in a suitable entity is that it be impredicable or noninstantiable [61], that is, specified to be a bottom node in the protologic.

Thus, the question becomes one of whether a sufficient collection of bottom nodes may be used to reconstruct the inclusion and exclusion links between the non-bottom nodes in a net. Probes were introduced in Sect. 3.6 as bottom nodes added to investigate a net from outside in order to infer its internal structure as constituted by the links between its nodes.

Extensional semantics for syllogistic systems with essentially the same connectives as inclusion and exclusion links have been developed [34] and the techniques and results are applicable to the protologic. The standard result is that, for a consistent net in the kernel protologic, a set of consistent full probes that *saturates* the net by including all possible distinct examples of such probes may be used to reconstruct the inclusion and exclusion relations in the net [80]. If $\pi\mathrm{A}$ is the set of probes that include node A, then

$$\begin{aligned} \mathrm{A} \rightarrow \mathrm{B} &\quad \text{iff } \pi\mathrm{A} \subseteq \pi\mathrm{B} \\ \mathrm{A} \mathrel{-\!\!\!+} \mathrm{B} &\quad \text{iff } \pi\mathrm{A} \cap \pi\mathrm{B} = \varnothing \end{aligned}$$

The algorithm is basically that of Sect. 3.6 illustrated in Table 1 since $\pi\mathrm{A} \cap \pi\mathrm{B} = \varnothing$ is equivalent to $\pi\mathrm{A} \subseteq \underline{\pi}\mathrm{B}$ if the net is consistent (since $\underline{\pi}$ is then the complement of π relative to the set of admissible probes).

Coexclusion relations may be reconstructed in terms dual to those for exclusion relations: $\mathrm{A} \mathrel{-\!\!+\!\!+} \mathrm{B}$ iff $\underline{\pi}\mathrm{A} \cap \underline{\pi}B = \varnothing$, which is equivalent to $\mathrm{A} \mathrel{-\!\!+\!\!+} \mathrm{B}$ iff $\underline{\pi}\mathrm{A} \subseteq \pi\mathrm{B}$ if the net is consistent.

If the net is inconsistent, there will be one or nodes where the associated set of consistent nodes is empty and the algorithm would infer that such nodes include and exclude all others, that is, the structure of the links leading to inconsistency would be lost. As noted in Sect. 3.6, the algorithm can be extended to recover the full structure from the set of admissible probes some of which may be inconsistent.

The inference of $\mathrm{A} \to \mathrm{B}$ from the subset relation carries over, but that of $\mathrm{A} \mathrel{-\!\!+} \mathrm{B}$ and $\mathrm{A} \mathrel{-\!\!+\!\!+} \mathrm{B}$ from the disjoint relations does not; inconsistency arises because a node has been specified that violates the exclusion or coexclusion link constraints. However, these relations can be inferred through a slight extension to the definitions (noting that $\underline{\pi}$, the set of probes that exclude a node, is no longer the complement of π in inconsistent nets). The extended reconstructions are:

$$
\begin{array}{ll}
\mathrm{A} \to \mathrm{B} & \text{iff } \pi\mathrm{A} \subseteq \pi\mathrm{B} \text{ and } \underline{\pi}\mathrm{B} \subseteq \underline{\pi}\mathrm{A} \\
\mathrm{A} \mathrel{-\!\!+} \mathrm{B} & \text{iff } \pi\mathrm{A} \subseteq \underline{\pi}\mathrm{B} \text{ and } \pi\mathrm{B} \subseteq \underline{\pi}\mathrm{A} \\
\mathrm{A} \mathrel{-\!\!+\!\!+} \mathrm{B} & \text{iff } \underline{\pi}\mathrm{A} \subseteq \pi\mathrm{B} \text{ and } \underline{\pi}\mathrm{B} \subseteq \pi\mathrm{A}
\end{array}
$$

For consistent nets, where $\underline{\pi}$ is the complement of π, these definitions reduces to

$$
\begin{array}{ll}
\mathrm{A} \to \mathrm{B} & \text{iff } \pi\mathrm{A} \subseteq \pi\mathrm{B} \\
\mathrm{A} \mathrel{-\!\!+} \mathrm{B} & \text{iff } \pi\mathrm{A} \cap \pi\mathrm{B} = \varnothing \\
\mathrm{A} \mathrel{-\!\!+\!\!+} \mathrm{B} & \text{iff } \underline{\pi}\mathrm{A} \cap \underline{\pi}\mathrm{B} = \varnothing.
\end{array}
$$

As an illustration of the reconstruction algorithm, consider the set of admissible probes (Table 2) for the net of Fig. 16, the last two probes of which are inconsistent, one having a positive link to C and another to D. Table 3 shows the π and $\underline{\pi}$ sets of probes associated with each node.

Table 2 Admissible probes for the net of Fig. 16

	A	B	C	D	E	F
1	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$
2	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\rightarrow$	$\nrightarrow$
3	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\rightarrow$
4	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\rightarrow$	$\rightarrow$
5	$\nrightarrow$	$\rightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\rightarrow$
6	$\nrightarrow$	$\rightarrow$	$\nrightarrow$	$\nrightarrow$	$\rightarrow$	$\rightarrow$
7	$\rightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$
8	$\rightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\rightarrow$
9	$\nrightarrow \rightarrow$	$\nrightarrow \rightarrow$	$\nrightarrow \rightarrow$	$\nrightarrow$	$\nrightarrow$	$\rightarrow$
10	$\nrightarrow \rightarrow$	$\nrightarrow \rightarrow$	$\nrightarrow \rightarrow$	$\nrightarrow \rightarrow$	$\nrightarrow$	$\rightarrow$

Table 3 Admissible probes associated with each node

	π	$\underline{\pi}$
A	$\{7, 8, 9, 10\}$	$\{1, 2, 3, 4, 5, 6, 9, 10\}$
B	$\{5, 6, 9, 10\}$	$\{1, 2, 3, 4, 7, 8, 9, 10\}$
C	$\{9, 10\}$	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
D	$\{10\}$	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
E	$\{2, 4, 6\}$	$\{1, 3, 5, 7, 8, 9, 10\}$
F	$\{3, 4, 5, 6, 8, 9, 10\}$	$\{1, 2, 7\}$

The subset relations for π and $\underline{\pi}$ reconstruct the maximal canonical form of the net of Fig. 16 as shown in Table 4.

Table 4 Extensional reconstruction of links for the net of Fig. 16

A	$(-\!\!\!\!+, \mathrm{B}), (\leftarrow -\!\!\!\!+, \mathrm{C}), (\leftarrow -\!\!\!\!+, \mathrm{D}), (-\!\!\!\!+, \mathrm{E})$
B	$(-\!\!\!\!+, \mathrm{A}), (\leftarrow -\!\!\!\!+, \mathrm{C}), (\leftarrow -\!\!\!\!+, \mathrm{D}), (\rightarrow, \mathrm{F})$
C	$(\rightarrow -\!\!\!\!+, \mathrm{A}), (\rightarrow -\!\!\!\!+, \mathrm{B}), (-\!\!\!\!+, \mathrm{C}), (\leftarrow -\!\!\!\!+, \mathrm{D}), (-\!\!\!\!+, \mathrm{E}), (\rightarrow, \mathrm{F})$
D	$(\rightarrow -\!\!\!\!+, \mathrm{A}), (\rightarrow -\!\!\!\!+, \mathrm{B}), (\rightarrow -\!\!\!\!+, \mathrm{C}), (-\!\!\!\!+, \mathrm{D}), (-\!\!\!\!+, \mathrm{E}), (\rightarrow, \mathrm{F})$
E	$(-\!\!\!\!+, \mathrm{A}), (-\!\!\!\!+, \mathrm{C}), (-\!\!\!\!+, \mathrm{D})$
F	$(\leftarrow, \mathrm{B}), (\leftarrow, \mathrm{C}), (\leftarrow, \mathrm{D})$

5 Structural Connectives

The focus of prior sections has been on logical structures based on the basic inclusion and exclusion connectives in order to demonstrate the representational and inference capabilities of the foundational logical connectives. This contrasts with specifications of logical systems that take structural connectives as primitive.

Structural connectives may be introduced in the protologic as *ideal elements* [7, 71] represented by the addition of nodes that are extremal relative to a set of links in the order relation associated with $\rightarrow$. Koslow [73, 74] uses this approach to generalize Gentzen's introduction and elimination rules for logical connectives. He considers a general implication relation subject to the normal axioms for a consequence operator [14] and defines the usual structural connectives as extremal structures in the order relation of implication/consequence.

5.1 Structural Conjunction

The inclusion and exclusion links out of a node may be viewed as specifying a *proto-conjunction* (Sect. 2.4) in that, if a probe includes that node then, from the inclusion and exclusion inference patterns, it will also include or exclude all the linked nodes. In the terminology of Sect. 4.2, if the node is marked *true* all the nodes that it includes will be marked *true* and those it excludes *false*.

However, the converse does not apply. One cannot infer that the node is true if some or all of the nodes to which it has outgoing links are marked appropriately. One can, however, define a constraint that specializes a node to be a full structural conjunction through an inference pattern that specifies this.

Definition A node is the *conjunction* of a set of outgoing links iff any node with the same links includes it.

Figure 32 is a graphical representation of this definition. The outgoing links defining the conjunctive node at the bottom left are distinguished by conjunctive variants of the inclusion and exclusion arrows that have a heavier head. Any node, such as that at the bottom right, that has the same set of outgoing links may be inferred to include the conjunctive node.

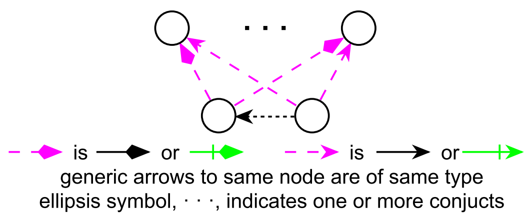


Fig. 32 Structural conjunction inference pattern

If the bottom node on the right is regarded as a probe then the inference pattern implies that any probe that marks the nodes to which the conjunctive node has inclusion links *true*, and those to which is has exclusion links *false*, also marks node the conjunctive node as *true*. These are the converse truth conditions defining a full structural conjunction.

The inference pattern of Fig. 32 parallels Koslow's [73, §13.1] approach to the definition of conjunction but takes into account exclusion as well as inclusion links. It specifies that, in terms of the order relation associated with →, the conjunctive node is *maximal* among all nodes having the outgoing links specified by the heavier arrows.

Arbitrary conjunctive nodes may be freely added as ideal elements to any net but they will not always be consistent; for example, if inclusion links are specified to nodes between which there is an exclusion link. In terms of content containment, these ideal nodes are *defined* by their conjunctive links and have no other content. However, if additional links are added such that they include or exclude additional content then the nodes become *rules* imposing meaning constraints upon the net that reflect extra-logical normative or empirical contingencies.

5.2 *Structural Disjunction*

The inclusion and coexclusion links into a node may be viewed as specifying a *protodisjunction* (Sect. 2.4) in that, if a probe excludes that node then, from the exclusion and coexclusion inference patterns, it will also exclude or include all the linked nodes. In the terminology of Sect. 4.2, if the node is marked *false* all the nodes that it includes will be marked *false* and those it coexcludes *true*.

However, the converse does not apply. One cannot infer that the node is false if all the nodes from which it has incoming links are marked appropriately. One can, however, define a constraint that specializes the node itself through an inference pattern that allows this to be inferred.

Definition A node is the *disjunction* of a set of incoming links iff it includes any node with the same links.

Figure 33 is a graphical representation of this definition. The incoming links defining the disjunctive node at the top left are distinguished by disjunctive variants of the inclusion and coexclusion arrows that have a double head. Any node, such as that on at the top right, that has the same set of incoming links may be inferred to be included in the disjunctive node.

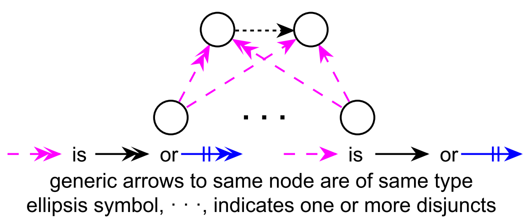


Fig. 33 Structural disjunction inference pattern

If a probe has exclusion links to the tail nodes of the inclusion links, and inclusion links to the tail nodes of the coexclusion links, defining the disjunctive node then it may, without inconsistency, have an exclusion link to a an arbitrary node with inclusion links from these nodes such as that at the top right. Such a node also has an inclusion link from the disjunctive node so that an exclusion link may be inferred from the probe to the disjunctive node. That is, if the tail nodes of the incoming links defining the disjunctive node are all marked to propagate false then the disjunctive node is false. This is the converse condition defining a full structural disjunction.

The definition of structural disjunction specifies that, in terms of the order relation associated with →, the disjunctive node is *minimal* among all nodes having the incoming links specified by the double arrows. This differs from Koslow's [73, §13.1] definition which specifies the maximality of a different construction, but the two definitions are equivalent.

An important application of structural disjunction is to represent *abduction* as inference to those *abducibles* consistent with a state of affairs [47]. If the disjuncts in Fig. 33 are bottom nodes representing possible states of affairs, such as Millikan's [82] *substance templets*, asserting the disjunction true for a particular state of affairs represents the abductive hypothesis that one or more of those templets must fit that state. Disjunctive inference then derives the consequences of this hypothesis.

5.2.1 Material Implication and Other Sentential Calculus Formulae

The inclusion link, $\rightarrow$, is appropriate to represent *entailment*, necessary, rather than material, implication. It specifies relations between terms prescribing their proper usage, conventions of language rather contingencies of a world. In addition, the implicative conditional is not represented in a form subject to inference; it may only be *used* in the protologic, not *mentioned*.

The disjunctive inference pattern may be used to represent a *material implication* as a logical structure in which the implication is itself represented as a node subject to inference. Figure 34 left illustrates the simplest case. If node M is asserted true then at least one of its disjuncts may be inferred to be true. Thus $\neg A \vee B$ is true, the classical interpretation of $A \supset B$ as material implication. However, the conditioning on M means that what is represented is $M \supset (A \supset B)$.

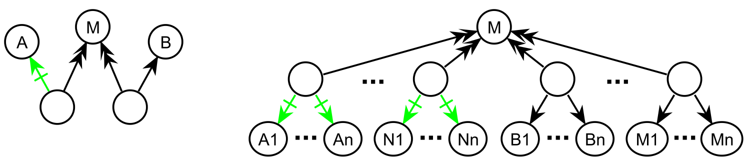


Fig. 34 Representing material implication in the protologic

This non-truth-functional representation in terms of implication rather than equivalence avoids the so-called 'paradoxes of material implication' [42, §2.3]. The only situation in which the truth value of M is determined by those of A and B is that it is inferred false if A is asserted true and B false, that is when the material implication does not hold.

Figure 34 right shows a more general form of material implication where there are multiple disjuncts of nodes with multiple outgoing links. This represents:

$$M \supset \big(\big((A1 \vee \cdots \vee An) \wedge \cdots \wedge (N1 \vee \cdots \vee Nn)\big) \supset \big((B1 \wedge \cdots \wedge Bn) \vee \cdots \vee (M1 \wedge \cdots \wedge Mn)\big)\big)$$

Thus a disjunctive node together with inclusion/exclusion links may be used to represent a material implication between complex clauses with premises in conjunctive normal form and conclusions in disjunctive normal form.

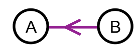
6 Defeasible Inference

Paraconsistency in a logical system is desirable in that an inconsistency is localized and does not undermine reasoning outside that locale. However, such containment alone is inadequate to support reasoning *within* the logical system about the sources and consequences of the inconsistency. These aspects of the inconsistency remain at the metalogical level.

Reasoning about inconsistency can be pushed down to the logical level by introducing nodes representing potential inconsistencies as anomalies, and playing no role unless nodes representing normal states of affairs do not apply. This may be implemented through a *preference relation* between possible states of affairs represented as disjuncts of a disjunction such that, when that disjunction is asserted true, a disjunct is inferred to be false if any preferred disjunct is not false.

If an anomalous state of affairs is represented as a disjunct that is less preferred than the other disjuncts, it will be inferred false and play no role unless all the other disjuncts are false (non-applicable). Then, being the only non-false node, it will be inferred to be true (applicable).

6.1 Preferential Defeasible Inference



The preference relation allows a partial order to be specified between the nodes of a structural disjunction. It is represented textually by the symbol, $<$, with A $<$ B specifying that node B is preferred to node A, and in a net by a preference link as shown on the right.

The associated inference patterns are shown in Fig. 35: left, transitivity of preference; right, exclusion if a preferred node is included or not linked (that is, it is not excluded).

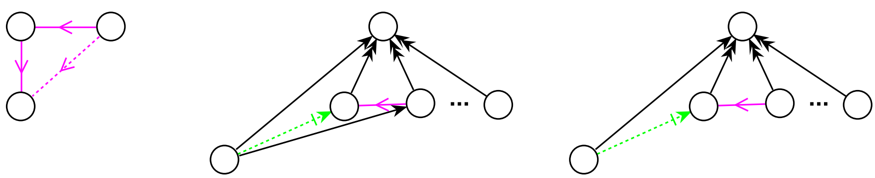


Fig. 35 Preference inference patterns

In terms of contextual semantics, if the context of a node is part of that of a node that has sub-contexts then it is assumed to be disjoint with that of a less preferred sub-context if it is not disjoint with that a more preferred one.

These inference patterns are only applicable once all other inference patterns have been applied. They, and ensuing inferences, are *defeasible* because they are based on the preference relation. The source of the defeasibility is apparent in the inference pattern on the far right which licenses an inference to be made when the truth value of a node is unspecified.

6.1.1 Normal Defaults

The preference inference patterns may be used to represent the *normal defaults* [77, 94] used in applications of nonmonotonic reasoning, that a proposition is true unless there are grounds for it being false. Figure 36 from left to right shows nets representing, from left to right: :A, that A is normally true; A:B, that A normally materially implies B; A:B and ¬B:¬A, Lehman's [77, §6] preferred representation of a default material conditional that also incorporates default *modus tollendo tollens*; and a more complex combination of defaults such that: A is normally true; B false; A normally implies C; but A and B normally implies not C.

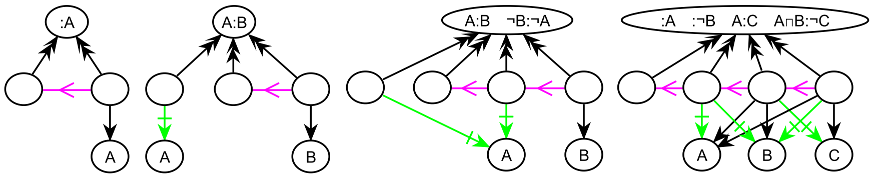


Fig. 36 Normal defaults

Figure 37 from left to right illustrates default reasoning: (1) when no truth values are asserted for A and B, no inferences are made about them (the nodes inferred false by the preference inference patterns are colored mauve); (2) when A is asserted true, B is defeasibly inferred true; (3) if B is also asserted false, this is accepted and an anomaly is inferred (anomalous node is colored red as an inference from the disjunctive definition); (4) if B is asserted false, A is defeasibly inferred false (if the inference is overridden then (3) results again).

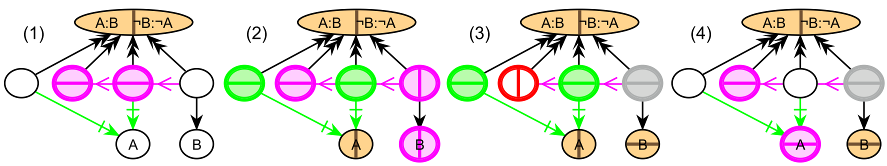


Fig. 37 Inference from normal defaults

This implementation of defeasible reasoning encompasses the test cases in the literature collected in [77], and more complex ones such as Stalnaker's [100] combinations of classical and default inference. It also encompasses abductive, or case-based, reasoning where the disjuncts represent states of affairs templets to be fitted to a particular state of affairs.

7 Some Illustrative Applications

A major objective of this article has been to address Béziau's [11] proposal that universal logic should be able to support other fields of knowledge to build *the right logic for the right situation*. This section provides some brief illustrative example of how the protologics developed above can be applied in other disciplines.

7.1 Merging Biological Taxonomies

Reasoning about taxonomies is important to biological science, for example, in analyses of the consequences of merging taxonomies of the same species from different sources. There are literature studies of the use of powerful theorem provers to reason about taxonomic merging [103], but the same results may be obtained in the kernel protologic in a more perspicuous form.

Figure 38 presents an example of two simple taxonomies, their proposed merging through the articulations shown, and queries arising about the outcome.

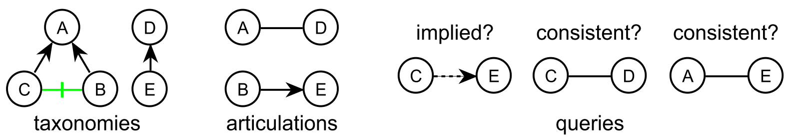


Fig. 38 Some taxonomy articulation queries [103, p. 198]

The three nets on the left of Fig. 39 show the inferred links corresponding to the to the three queries: there is no inferred link from C to E; equivalencing C and D makes the net inconsistent; but doing so for A and E does not.

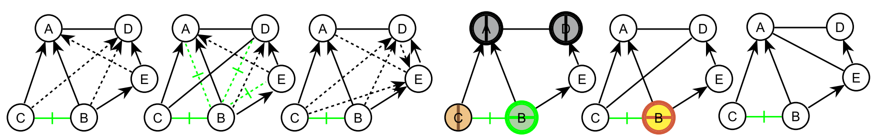


Fig. 39 Inferred responses to the taxonomy articulation queries

The nets on the right provide the same information in terms of true–false marking: if C is marked true there is no propagation to E; if C is equivalenced to D then node D is marked false in all consistent truth-assignments, that is, inconsistent; if A is equivalenced to E there are no such inferences.

The study cited represents the taxonomic data by similar diagrams but treats inferences from them as informal and resolves the queries through logical symbolism and a first-order logic theorem prover. However, the diagrams themselves can be used to provide formally well-founded responses to the queries.

7.2 An Expert System

Figure 40 shows a net implementing a simple expert system [32] that prescribes hard or soft contact lenses for a client having the attributes the four determinables shown at the top left leading to one of three prescriptions shown at the top right.

The solution is specified in terms of conjunctions specifying rules and exceptions: a client whose tear production is normal should be prescribed a hard lens if astigmatic and a soft lens if not; however, there is an exception to the soft prescription if the patient is presbyopic and myopic, and to the hard if hypermetropic and old. The inferences are shown when the state of affairs representing a client at the bottom of Fig. 40 is activated by marking it true.

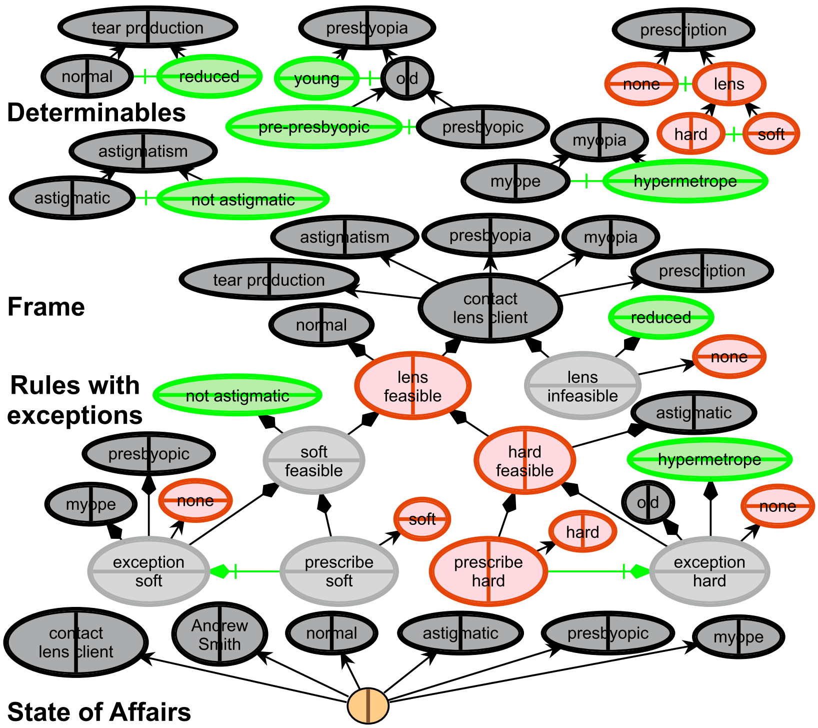


Fig. 40 Inference from rules with exceptions in an expert system

7.3 Defining an Art Object

The problem of defining what it is to be an *art object* has been a continuing issue for the philosophy of art community since ancient times [30, 35]. It is generally accepted that a definition in terms of necessary and sufficient conditions is not appropriate [106], and a variety of other classificatory techniques have been investigated such as *family resemblances* [28] and *cluster concepts* [55].

Gaut [56] has defined the notion of an *art object* as a cluster concept, and one may argue [51] that the definitional aspects of his model lie in the frame constituted by the eleven determinables that he uses to characterize art objects. Figure 41 illustrates the use of the kernel protologic and default reasoning to analyze aspects of his classificatory structure.

At the top are shown: left, four of Gaut's determinables; right, the frame for an art object that represents the relevance of these determinables. Beneath these are shown templets for five types of art of art object including, on the left, that for an 'ideal art object' that includes all the positive determinates. These together capture the essence of Gaut's analysis of art as a cluster concept.

The defaults at the bottom capture expectations when one is told that some entity is an art object: that, by default, it should have the characteristics of an ideal art object. The markings illustrate the reasoning when it is asserted that something is an art object but not created with the intent that it be one, such as Duchamps' *readymades* [36]. The default

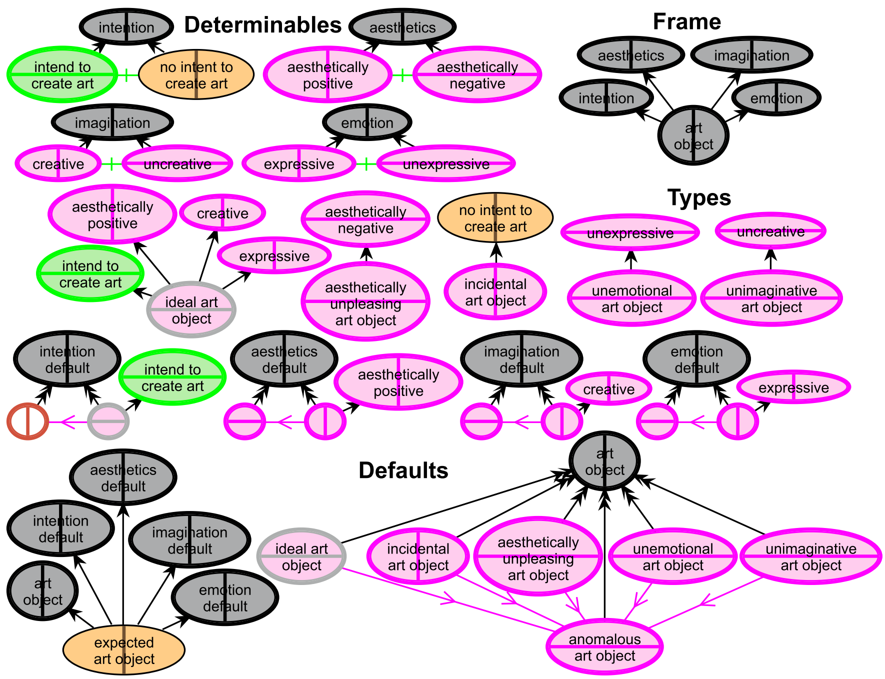


Fig. 41 Default reasoning about art objects

inferences are that it has the other characteristics of an 'ideal art object' but is only an 'incidental art object.'

A similar structure of defaults may be used to represent logical aspects of fiction such as Ryan's [98] *principle of minimal departure* from expectations corresponding to those of a world appropriate to the fictional genre.

7.4 Representing States of Affairs, Individuals and Relations

The notions of *states of affairs* [2, 72] and their constituent *individuals* and *relations* are important to most applications of logical systems but their appropriate definition raises many deep philosophical issues [43] leading to a range of different approaches to their representation. A universal logic framework needs to be able to represent these adequately whilst making no commitment to any in particular.

The protologic as developed above does not require the definition of individuals as primitive entities for any of the technical purposes for which they are normally required. Set-theoretic extensional semantics have been provided in Sect. 4.4 based on *probes* simply defined as bottom nodes in Sect. 3.6. Bottom nodes satisfy Gracia's impredicability criterion for an individual and could be termed *protoindividuals* but they need not satisfy Strawson's [102, p. 214] criteria that they be distinct and reference some other entity. Extensional semantics are a technical feature intrinsic to the protologic involving no other connotations or denotations.

Figure 42 is a simple illustration of the representation of a state of affairs referencing an individual based on the well-known example that Ramsey [92] provided to counter Russell's [97] argument that there was an essential logical distinction between universals and particulars. The anonymous bottom node represents a templet for a state of affairs in which an individual named 'Socrates' exhibits behavior that can be characterized as 'wise.'

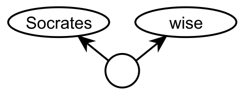


Fig. 42 Representation of an individual

The nature of the Russell–Ramsey debate may be analyzed in terms of Fig. 42. Ramsey's position corresponds to the anonymous node being symmetrically 'some Socrates' entity and 'some wise' entity. However, in its role of representing reference to the state of an individual, it may be distinguished from the other nodes in being an impredicative bottom node corresponding to the focus in Russell's article on singulars as states of affairs.

The bottom node satisfies Gracia's criterion for an individual if it is intended to represent a particular situation. If it is intended as a generic *substance templet* [82] that applies to particular states of Socrates then it does not, but those instances of which it is predicated will satisfy the criterion. That is, impredicability is not an absolute distinction but rather a pragmatic feature of use.

Strawson's criterion of reidentification captures the essence of the intent behind the use of proper names, to track what is imputed to be the 'same' individual in different situations. That is, reference to an individual is intended to be a *rigid designator* and this is also a pragmatic rather than logical distinction [5].

The bottom node will satisfy Strawson's criterion of distinctness if the term 'Socrates' is always used to identify what is intended to be a single individual, but, if it is applicable to several individuals, then additional identifying terms may be required to track the intended individual. That is, proper names may require contextual information to disambiguate their intended application [87, p. 295].

As Engel [43, §3.3.2] notes, Ramsey's argument concerns the logical, rather than ontological, status of particulars and universals. The distinction between a node being used to reference an individual and to ascribe a property to one still needs to be representable, but not necessarily as a logical primitive.

Castañeda [31] models individuation as requiring an *individuator operator* providing indexical access. One can model the individuator operator as one that infers which bottom nodes would make the specified indexical nodes true, and this is implemented in the protologic through the form of abductive inference discussed in Sect. 5.2. That is, given the assignment of truth values to the indexical nodes, one considers the disjunction of all the bottom nodes representing states of affairs that have not been inferred to be false and draws the inferences common to them.

This corresponds to Perry's [88, 93] analysis of proper names as functioning to index *mental files* of information about an individual, it may be that a combination of name and context is required to disambiguate the file access. One may generalize this to any combination of terms, including names, being used to provide a context sufficiently constrained to identify a particular individual, as in Orilia's [84] *contextual descriptivist* account of singular reference.

7.4.1 Venus as Morning Star and Evening Star

To illustrate the representation of states of affairs, Fig. 43 represents the two perceived states of Venus that Frege [48] used to exemplify his distinction between *sense* and *reference*. The planet appears as a bright celestial object near the rising or setting sun at dawn or dusk. Early Greek astronomers regarded the phenomena as arising from different celestial bodies and named them differently [105, n. 23]. As shown, abductive inference over the bottom nodes marks the common features as defeasibly true.

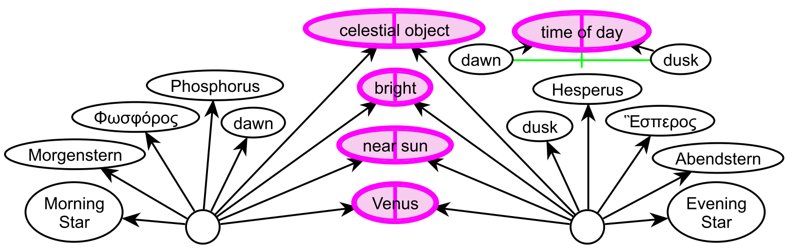


Fig. 43 Representation of perceived states of Venus

The net illustrates the major issues. Many names may be used to reference the same physical entity, and someone may not be aware that they do not refer to different ones; removing the node for Venus illustrates this. Someone may know some of the names and that they have a common referent but not know those names in another language; removing some state-specific names illustrates this.

Abductive inference after marking the nodes representing dawn or dusk as true leads to all the morning star names and Venus being marked true but none of the evening star names, and *vice versa*. That is, it is reasonable to have different names that reference the same entity in different contexts.

7.4.2 Representing Relations and Structural Universals

While the representation of individuals requires no additional constructs in the protologic, that of relations requires distinguishing the states of affair that are related. One version implemented in the protologic is based on Orilia's [85] representation of *neutral relations* [46] in terms of *onto-thematic roles*.

As shown in Fig. 44, relations are represented by nodes having *relational links* represented by arrows having distinctive tails that serve to group relational links of the same type. There may be any number of types of relational links. Nodes with only one type of relational link represent symmetric relations, and those with two or more asymmetric relations. On the left are inference patterns through which relational links interact with inclusion links.

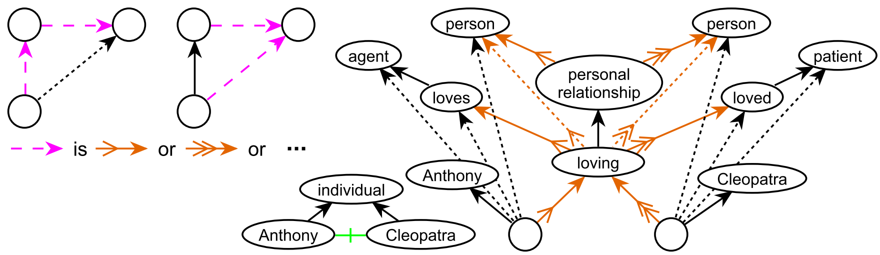


Fig. 44 Relation inference patterns and example relation

At the bottom center 'Anthony' and 'Cleopatra' are typed as two distinct individuals, with the node 'individual' making Castañeda's individuator operator available as needed. On the right, the state of affairs where there is a loving relationship between them is represented as a personal relationship having two thematic roles, 'loves' as *agent* and 'loved' as *patient*. Inferences from the patterns are shown with dotted lines.

This representation avoids many of the known issues of representing relations [46, 54, 85]. There is a single relation and the notion of *converse* is a matter of its linguistic expression. Subordination of relations is represented. The relational arrows have no ontological role but serve only to link the constraints on each component of a relation. The components are distinguished by their thematic roles, not their positions or link types. The roles themselves are simply nodes that may be part of a net representing their logical interrelationships. The generic linguistic roles mark those that include them as relational roles, providing similar functionality to an individuator operator. Additional types of relational arrows may be added to represent other roles such as 'instrument.'

Figure 45 exemplifies Armstrong's [2] *structural universals* in terms of a templet of the states of affairs and their relationships in a water molecule.

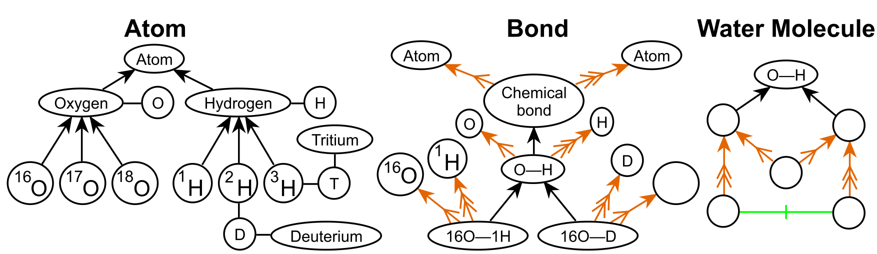


Fig. 45 Relational structure of a water molecule

From left to right: the determinable, atom, is represented with oxygen, hydrogen and their isotopes as determinates; the hydroxyl chemical bond is represented together with some of its forms; the structure constituting a water molecule is represented as a structural universal having two hydroxyl bonds with a common oxygen atom and two distinct hydrogen atoms [86].

8 Conclusions

Over the past two decades, Béziau's [9] notion of *universal logic* as an integrative conceptual framework for all logical systems that presupposes no particular axioms has provided a focal point for a wide range of historical and ongoing studies of foundational issues in logic and the nature and relationship of logical systems, as did Birkhoff's [23] notion of *universal algebra* for foundational studies of algebras.

Béziau also envisioned [11] that the universal logic conceptual framework could help many fields of knowledge build *the right logic for the right situation*, noting that for some disciplines *mathematical abstract conceptualization* is more appropriate than *symbolic formalization*. This idea has so far had less impact although Béziau has exemplified it in his own research, for example, on the diverse interpretations of the square of oppositions and the special issue of *Logica Universalis* on 'Is logic universal?' [16] raising a range of cross-disciplinary issues [52].

One should not expect a significant impact of the conceptual framework of universal logic on other disciplines to develop rapidly because the diffusion of knowledge and techniques between different fields of knowledge is known to be intrinsically slow [96]. In particular, the symbolic formalism of the normal expositions of logic may be an impediment to diffusion and it might help expedite wider adoption if the conceptual framework of universal logic could be presented in a way that is more accessible whilst remaining formally sound.

This article has provided an alternative formalism that avoids mathematical symbolism by extending Hertz's [67] original graphic presentation of the principles of logical deduction to encompass the sequent calculus that he, Gentzen [57], and others developed. The approach is based on past research on the formal representation of semantic networks as a practical tool applicable to the development, representation and application of knowledge structures in diverse applications [50, 51].

One side-effect of the techniques used to avoid mathematical symbolism is that the knowledge structures used in various disciplines become represented in a graphical form that often mimics the informal diagrams used in those disciplines yet allows formal logical inference patterns to be applied.

The conceptual and computational tools illustrated in this article have proved useful in several disciplines, and should, hopefully, contribute to the wider adoption of Béziau's visionary ideas. Universal logic can provide a *mathematical abstract conceptualization* that is logically sound, readily comprehensible and practically useful in the clarification, communication and evaluation of ideas, theories and associated controversies in many fields of knowledge. Facilitating this is a significant and challenging research objective for the universal logic community.

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