Geometric Contributions to the Analysis of 2-2 Wire Driven Cranes

Manfred Husty and Paul Zsombor-Murray

Abstract A 2-2 wire driven crane is the simplest wire driven manipulator. Sticking strictly to its planar motion behaviour we give some geometric insight into the possible equilibrium poses using either coupler curves of four-bar mechanisms or kinematic mapping.

Keywords Wire driven cranes · Four-bar coupler curve · Kinematic mapping

1 Introduction to the Two-Cable Planar Platform

In a recent paper Merlet (2013) Merlet states that there are only two other papers, Carricato and Merlet (2010) and Michael et al. (2009), which deal with the simplest wire driven manipulator namely the 2-2 wire driven crane. Such a two-cable planar platform is characterized by fixed supports at given points A, B connected respectively to points D, E on the platform. G is its mass centre. Triangle DEG is given as are the cable segment lengths r_A, r_B that connect A to D and B to E. The problem is to find positions of D, E, G with respect to the frame of A and B as the platform hangs in static equilibrium under the influence of a gravity force vector acting vertically downward on G, such that cables do not sustain compressive load.

In Merlet (2013) and Carricato and Merlet (2010) it is mentioned that the number of solutions to this problem is closely related to the coupler curve that is traced by the point G in the coupler motion of the coupler system DEG. But in both papers this seemingly obvious fact is not exploited geometrically. It is merely observed that a solution of the problem could be found by finding horizontal tangency points on the coupler curve. Closer inspection of the geometric properties of the coupler

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curve on the other hand will show that this is only a sufficient but not necessary condition. It is further mentioned that 12 is the maximum number of solutions to the equilibrium problem. It is not known if 12 equilibrium poses can be reached with cables not sustaining compressive load. A possible reflection of the solutions caused by a prohibited motion of the manipulator in three space will not be pursued in this paper, but it is obvious that allowing such "flip-over" will double the number of solutions.

It is believed that the two approaches to a solution to the above mentioned problem, outlined herein and all based on purely geometric principles, enjoy certain advantages compared with available solutions based on static equilibrium of forces and energy minimization. They additionally provide some geometric insight into how the analysis of spatial wire driven systems may be approached.

The paper is organized as follows. In Sect. 2 some algebraic properties of coupler curves are recalled and used for a simple proof to formally enumerate the solutions of the equilibrium problem. It will also be shown that horizontal tangency is only a sufficient condition for solution. In Sect. 3 the representation of the mechanical system in a three dimensional parameter space via kinematic mapping will be used to derive a solution method that does not depend on the horizontal tangency property of the coupler curve. Section 4 concludes the paper, in giving some ideas about solving spatial wire problems.

2 Algebraic Properties of Coupler Curves

The number of tangents from an arbitrary point in the plane to a planar algebraic curve of degree n is called the class ν of the curve. If the curve has no singularities then the class is $\nu = n(n-1)$. If the curve has simple singularities such as double points, cusps or isolated double points (acknodes) then the class is computed according to the Plücker formula Salmon (1879), p. 65 which states

$$\nu = n(n-1) - 2d - 3r, \tag{1}$$

where n is the degree of the curve, d is the number of double points and r is the number of cusps.

Using coupler curves in the solution of the 2-2 wire problem one must find horizontal tangents, which means one has to find all tangents from a point at infinity, e.g. the one that closes the horizontal x-axis, to the curve. Therefore the class of the coupler curve is needed. In Wieleitner (1908) the class of a general coupler curve is determined to be 12. This number is computed as follows: a sextic curve can have at most $\frac{1}{2}(6-1)(6-2) = 10$ double points. The coupler curve has the two circular points as triple points each with three different tangents, therefore they each count as three double points. Furthermore the coupler curve has three double points on the focal circle (see e.g. Bottema and Roth (1990), p. 338ff), therefore nine double points are known. Substituting this number into the Plücker formula

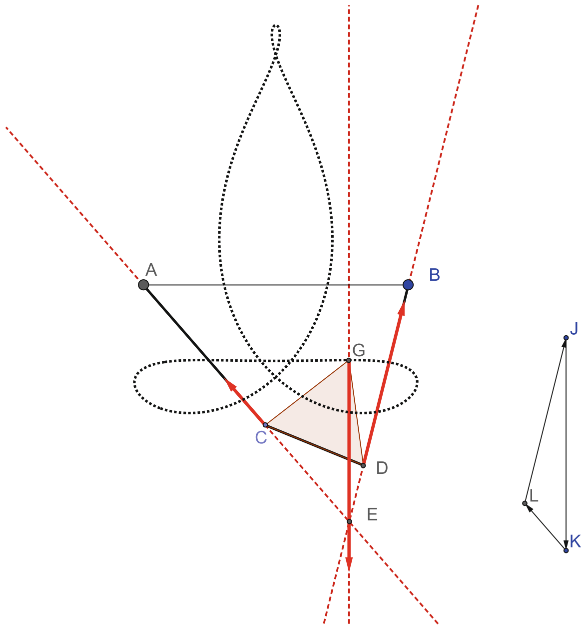

Fig. 1 Coupler curve with four double points

yields $\nu = 6(6-1) - 2 \cdot 9 = 12$. The class of the coupler curve gives therefore a simple proof that the maximum number of equilibria of a 2-2 wire system is 12. It is quite unlikely that this number can be reached with all wires in tension. Necessary conditions on the coupler curve for 12 equilibrium poses of the 2-2 wire system with all wires in tension would be:

- The coupler curve must be unicursal (it must consist of one branch). The conditions on the design parameters to fulfill this condition can be found e.g. in Bottema and Roth (1990, p. 413–418).
- The coupler curve must be on one side of the base.
- The points with horizontal tangents to the coupler curve must be between the two vertical lines passing through fixed support points A and B.

The number of coupler curve points with horizontal tangents can diminish under special design parameters. A special case arises when the four-bar is a folding four-bar. Then the coupler curve possesses a fourth double point (see Bottema and Roth (1990, p. 421)) and is rational (because it has the maximum number of simple singularities). The fourth double point is obtained by the coupler point when the four-bar is in the folded position. The Plücker formula yields $\nu = 6(6-1) - 2 \cdot 10 = 10$. Figure 1 shows the coupler curve traced by the point G with four double points. The point G is in an equilibrium position. On the right side is the force diagram. It is easily seen that this design, because of symmetry, has five equilibrium poses.

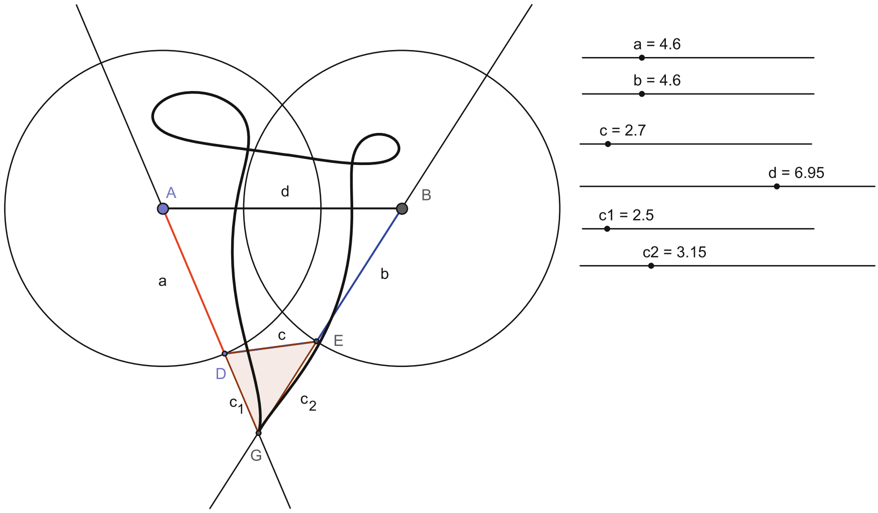

Fig. 2 Equilibrium pose at a cusp of the coupler curve

Fewer solutions arise when the coupler curve has higher order singularities or the four-bar has special design parameters (e.g. equal opposite sides of the four-bar).

Horizontal tangents to the coupler curve of at G are only a sufficient but not a necessary condition for an equilibrium pose of a 2-2-wire system. This is shown in Fig. 2. In case of a cusp in the coupler curve the two arms of the four-bar intersect at the coupler point. Therefore there are forces that balance the gravity force of G trivially.

The equation of the coupler curve is most elegantly derived using complex coordinates in the plane. Following Wunderlich (1970) the equation can be written[1]

$$[\overline{n}(z-d)P - \overline{m}zQ][n(\overline{z}-d) - m\overline{z}Q] + c^2R^2 = 0, \tag{2}$$

where

$$P = z\overline{z} + m\overline{m}c^2 - a^2, \qquad Q = (z-d)(\overline{z}-d) + n\overline{n}c^2 - b^2$$
$$R = (\overline{m}-m)z\,\overline{z} + (\overline{m}nz - m\overline{n}\,\overline{z})d. \tag{3}$$

Expanding (2), by setting $z = x + iy, \overline{z} = x - iy, m = m_1 + im_2, \overline{m} = m_1 - im_2$ ($i^2 = -1$, complex unit) yields the equation of the coupler curve $F(x, y) = 0$ in Cartesian coordinates. The points where the coupler curve has horizontal tangents are given by the intersection of the two curves

[1] This equation assumes that both base points A, B are on the x-axis. The general situation, with B not on the x-axis can be achieved by applying a simple rotation about the origin.

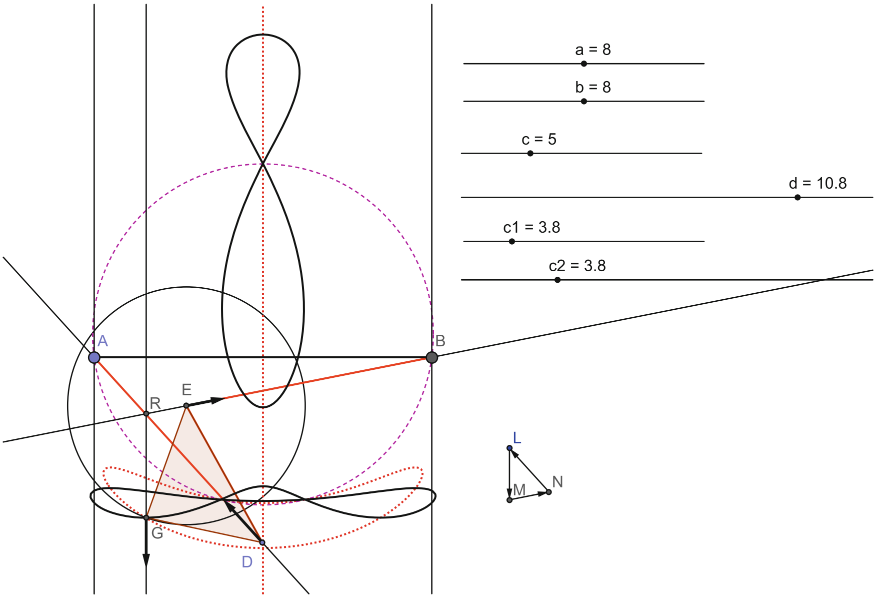

Fig. 3 Equilibrium pose with curve $\frac{\partial F}{\partial x} = 0$

$$F(x, y) = 0, \qquad \frac{\partial F}{\partial x} = 0.$$

Figure 3 shows a two wire system in a equilibrium pose. The red dotted curve represents $\frac{\partial F}{\partial x} = 0$. G is at an intersection point of both curves. The plot shows that this system has six real equilibrium poses with positive forces in the wires.[2] Note that also the double points are on the intersection of both curves. The double points can be easily found because they are, as mentioned above, also on the focal circle. The focal circle is given by the equation $R = 0$ (Eq. 3).

3 Kinematic Mapping

A solution based on planar kinematic mapping (PKM) solves for the positions of D, E, G directly. The notion introduced above is used. It proceeds as follows.

- Point D', expressed in the moving or end effector frame EE, is mapped to D on the circle of radius $r_A = a$ and centred on A expressed in the fixed or base frame FF.
- Similarly, E' is mapped to E on circle, radius $r_B = b$, centred on B.

[2] Numerically stable curve intersection computation can be achieved. A visualization can help to find good starting values for a Newton method.

- A third constraint completes the set of sufficient conditions. G' in EE is transferred to G in FF. G is expressed with the same PKM parameters and is located on the vertical line on H at the intersection of the lines AD and BE.

The circles are given by Eq. 4

$$d_1^2 + d_2^2 - d_0^2 r_A^2 = 0, \quad (e_1 - b_1 e_0)^2 + (e_2 - b_2 e_0)^2 - e_0^2 r_B^2 = 0 \tag{4}$$

where $D\{d_0 : d_1 : d_2\}$, $E\{e_0 : e_1 : e_2\}$, $B\{b_0 : b_1 : b_2\}$ are expressed in homogeneous coordinates. Similarly, G is $G\{g_0 : g_1 : g_2\}$ and $H\{h_0 : h_1 : h_2\}$. The transformation of any (a dummy) point $P' \to P$ is defined in Eq. 5.

$$\mathbf{p} = [\mathbf{X}]\mathbf{p}' : \quad \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} X_0^2 + X_3^2 & 0 & 0 \\ 2(X_0X_2 + X_1X_3) & X_0^2 - X_3^2 & -2X_0X_3 \\ -2(X_0X_1 - X_2X_3) & 2X_0X_3 & X_0^2 - X_3^2 \end{bmatrix} \begin{bmatrix} p_0' \\ p_1' \\ p_2' \end{bmatrix} \tag{5}$$

Transformed coordinates of D, E, G are given in Eq. 6.

$$\mathbf{d} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} X_0^2 + X_3^2 \\ 2(X_0X_2 + X_1X_3) \\ -2(X_0X_1 - X_2X_3) \end{bmatrix}, \mathbf{e} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} X_0^2 + X_3^2 \\ 2(X_0X_2 + X_1X_3) + e_1'(X_0^2 - X_3^2) \\ -2(X_0X_1 - e_1'X_0X_3 - X_2X_3) \end{bmatrix},$$

$$\mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} X_0^2 + X_3^2 \\ 2(X_0X_2 - g_2'X_0X_3 + X_1X_3) + g_1'(X_0^2 - X_3^2) \\ -2(X_0X_1 - g_1'X_0X_3 - X_2X_3) + g_2'(X_0^2 - X_3^2) \end{bmatrix} \tag{6}$$

Before writing the three constraint equations in mapping parameters consider the line segments $AD \equiv a\{A_0 : A_1 : A_2\}$, $BE \equiv b\{B_0 : B_1 : B_2\}$, their intersection $H = a \cap b$ and the vertical line $GH \equiv g\{G_0 : G_1 : G_2\}$.

$$\begin{vmatrix} p_0 & p_1 & p_2 \\ 1 & a_1 & a_2 \\ d_0 & d_1 & d_2 \end{vmatrix} = A_0p_0 + A_1p_1 + A_2p_2 = (a_1d_2 - a_2d_1)p_0 + (a_2d_0 - d_2)p_1 + (d_1 - a_1d_0)p_2 = 0 \tag{7}$$

$$\begin{vmatrix} p_0 & p_1 & p_2 \\ 1 & b_1 & b_2 \\ e_0 & e_1 & e_2 \end{vmatrix} = B_0p_0 + B_1p_1 + B_2p_2 = (b_1e_2 - b_2e_1)p_0 + (b_2e_0 - e_2)p_1 + (e_1 - b_1e_0)p_2 = 0 \tag{8}$$

$$\begin{vmatrix} P_0 & P_1 & P_2 \\ A_0 & A_1 & A_2 \\ B_0 & B_1 & B_2 \end{vmatrix} = h_0P_0 + h_1P_1 + h_2P_2 = (A_1B_2 - A_2B_1)P_0 + (A_2B_0 - A_0B_2)P_1 + (A_0B_1 - A_1B_0)P_2 = 0 \tag{9}$$

$$\begin{vmatrix} p_0 & p_1 & p_2 \\ g_0 & g_1 & g_2 \\ h_0 & h_1 & h_2 \end{vmatrix} = G_0p_0 + G_1p_1 + G_2p_2 = (g_1h_2 - g_2h_1)p_0 + (g_2h_0 - g_0h_2)p_1 + (g_0h_1 - g_1h_0)p_2 = 0 \tag{10}$$

Since g is a vertical line the third constraint equation, expressing the redundant situation that these lines intersect on a common point, i.e., $a \cap b \cap g = H$, reduces to Eq. 11. The coefficient of the y-coordinate in an equation of a vertical line must vanish.

$$G_2 = g_0 h_1 - g_1 h_0 = g_0(b_1 e_2 - b_2 e_1)d_1 - g_1[(b_1 e_0 - e_1)d_2 - (b_2 e_0 - e_2)d_1] = 0 \tag{11}$$

Substituting the results from the first two of Eqs. 6 respectively into the pair Eq. 4, dehomogenizing by setting $X_0 = 1$, subtracting the first circle equation from the second, removing a common factor $2(X_0^2 + X_3^2)$ and normalizing on the length interval between D and E so that $e_1' = 1$ yields two quadric constraints, Eq. 12,

$$4(X_1^2+X_2^2)-r_A^2(X_3^2+1) = 0, \quad j_1+j_2X_1+j_3X_2-j_2X_3+j_4X_1X_3-j_2X_2X_3+j_5X_3^2 = 0 \tag{12}$$

where

$$j_1 = (b_1-1)^2 + b_2^2 + r_A^2 - r_B^2, \quad j_2 = 4b_2, \quad j_3 = -4(b_1-1),$$
$$j_4 = -4(b_1+1), \quad j_5 = (b_1+1)^2 + b_2^2 + r_A^2 - r_B^2 \tag{13}$$

Incorporating all the choices of origins, direction and length into Eqs. 7 through 10 and combining this with the first two of Eq. 6 to get h_0, h_1 then using the third of Eq. 6 to get g_0, g_1, directly, reveals Eq. 11 as the quartic, Eq. 14,

$$k_1X_1 + k_2X_2 - 2X_1X_2 + k_3X_1X_3 + k_4X_2X_3 - 2X_1^2X_3 - 2X_2^2X_3$$
$$+ k_5X_1X_3^2 + k_6X_2X_3^2 - 2X_1X_2X_3^2 + k_7X_1X_3^3 + k_8X_2X_3^3 = 0 \tag{14}$$

where

$$k_1 = (b_1-1)g_1', \quad k_2 = (g_1'-1)b_2, \quad k_3 = k_2 + 2(1-b_1)g_2', \quad k_4 = 2(b_1 - b_2g_2') - k_8$$
$$k_5 = 2(b_1 - b_2g_2') - k_1, \quad k_6 = 2(b_1+1)g_2' - k_2, \quad k_7 = -k_2, \quad k_8 = (1+b_1)g_1'$$

The two quadric constraints, Eq. 12, and the quartic, Eq. 14, indicate that a univariate of degree no greater than 16 in, say, X_3 should be available as a solution to the planar problem. Actually, one may do better. Elimination of X_1 from the two equations Eq. 12 and from the second of that pair and Eq. 14 produces a bivariate pair of equations whose resultant with respect to X_2 yields a univariate of degree 12 in X_3 and the factor $[8\sqrt{2}\{(b_1+1)X_3 - b_2\}]^4$. This elimination can be done entirely with symbolic design parameters but unfortunately not in their compressed form as given by $j_i,\ i = 1, \ldots, 5$ and $k_j,\ j = 1, \ldots, 8$ in Eqs. 12 and 14. To effect backsubstitution so that X_1, X_2 can be calculated uniquely for each of (up to) 12 real values of X_3, X_1 is removed between the two Eqs. 12 and between the left hand equation of Eqs. 12 and 14. These two equations, devoid of X_1, are quadratic in X_2 so the coefficients of X_2^2 in each can be used as multipliers in the other equation and the difference will be linear in X_2. With values of X_2, X_3 available, Eq. 14, which is linear in both X_1 and X_2, will furnish X_1. A numerical example with six real solutions, all of them with compression-free legs, was computed, without loss in generality, with the same design parameters as in Fig. 3. The results, are tabulated below. Their disposition shown in Fig. 4.

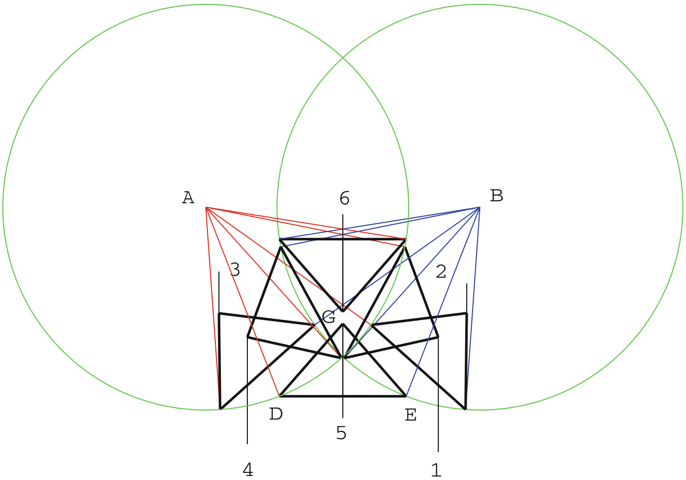

Fig. 4 Six feasible poses

Table 1 Numerical example with six real solutions

Solution	D	E	G
1	(1.5693, −0.31205)	(1.0906, −1.1901)	(1.8325, −1.0250)
2	(1.3024, −0.92906)	(2.0476, −1.5958)	(2.0566, −0.83590)
3	(0.11229, −1.5961)	(0.85751, −0.92923)	(0.10325, −0.83611)
4	(1.0694, −1.1901)	(0.59076, −0.31210)	(0.32754, −1.0251)
5	(-0.5800, −1.4912)	(1.5797, −1.4912)	(1.0802, −2.0634)
6	(1.5797, −0.2522)	(0.5800, −0.2522)	(1.0802, −0.8244)

4 Conclusion

PKM solves the platform pose problem directly without resort to a four-bar coupler curve and its positions of zero slope. The univariate polynomial can be computed symbolically in terms of design parameters but its coefficients are too large to be practical unless a more efficient means of compression can be found. Using a formulation in complex number notation provides a way to examine the coupler curve. Its positions of zero slope can be found by expanding the curve equation to Cartesian coordinates and intersection with the curve's derivative with respect to the x coordinate. The univariate can be computed completely symbolically and reveals apparent solutions at cusps at non-zero slope. It is proposed to extend this work using *spatial* kinematic mapping with a three sphere paradigm. Three additional constraints must be found. The three rank deficient determinants provided by the vertical fourth line define necessary but insufficient conditions (Table 1).

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