

# Chapter 2

## Fractional Systems for Control

### 2.1 Fractional Control

Fractional systems have received a great attention recently, both from an academic and an industrial viewpoint [2, 20, 21, 59, 62, 74, 85, 87, 114, 116] because of their increased flexibility (with respect to integer-order systems), which allows a more accurate modeling of complex systems [14, 91, 145], and the achievement of more challenging control requirements [29, 45, 71, 74, 123].

The main idea about fractional control is to design control systems where either the system or the controller, or both of them are fractional dynamic systems [31, 140, 141]. Regarding the design of control systems, many contributions are related to the synthesis of robust control systems (see, for example, [12, 87, 88, 91, 94, 114]). Among these method, it is surely worth mentioning the CRONE control [88]. A great effort has been also focused on the synthesis of fractional-order proportional-integral-derivative controllers (see, for example, [13, 25, 30, 42, 72, 73, 97, 99, 116, 136, 139, 147]). Among the other relevant research topics, it is worth citing the approximation of a fractional system with integer ones [62, 89, 135, 148] and the stability of a fractional (control) systems [2, 107, 124, 125]. All the previous methods share a common mathematical object for the design of fractional control systems: fractional linear time-invariant (LTI) systems. Exactly as in the integer case, also in the fractional one, linear systems are undoubtedly the fundamental tool in control design. Bearing in mind this idea, the rest of this chapter is dedicated to the analysis of fractional LTI systems.

### 2.2 The Two Parameters Mittag-Leffler Function

Before approaching LTI systems, a fundamental class of functions is introduced: the two parameters Mittag-Leffler functions. These special functions play for fractional LTI systems the same role that exponential functions play for integer ones.

The Mittag-Leffler function (MLF) is defined as [115]

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad \alpha > 0, \beta > 0. \quad (2.1)$$

MLF are generalization of the exponential function, indeed it holds that [115]:

$$\begin{aligned} E_{1,1}(z) &= e^z \\ &\vdots \\ E_{1,m} &= \frac{1}{z^m} \left( e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right). \end{aligned} \quad (2.2)$$

A further interesting generalization is the function introduced by Podlubny in [117] and hereafter addressed as Podlubny function (PF):

$$\varepsilon_k(t, \lambda; \alpha, \beta) = t^{k\alpha+\beta-1} \frac{d^k}{d(\lambda t^\alpha)^k} E_{\alpha,\beta}(\lambda t^\alpha), \quad (2.3)$$

being the  $k$ th derivative of (2.1) [108]

$$\frac{d^k}{dz^k} E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{(i+k)z^i}{\Gamma(\alpha(i+k) + \beta)}. \quad (2.4)$$

### 2.2.1 Laplace Transforms

Now, the Laplace transform of the functions defined in Sect. 2.2 is introduced. The Laplace transform of (2.1) is [108, 115]

$$\mathcal{L} \left[ t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) \right] = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \quad (2.5)$$

while the Laplace transform of (2.3) is

$$\mathcal{L} [\varepsilon_k(t, \lambda; \alpha, \beta)] = \frac{k! s^{\alpha-\beta}}{(s^\alpha - \lambda)^{k+1}}. \quad (2.6)$$

MLFs and PFs are fundamental tools to solve fractional LTI systems. Indeed, the previous equations look like a generalization to the fractional case of simple and multiple poles (and actually they are). This concept will be fully clarified through the rest of the chapter.

## 2.3 Fractional LTI Systems

In general, a fractional LTI system is represented in the external form by a fractional differential equation with the following structure

$$\sum_{k=0}^n a_{k0} D_t^{\beta_k} y(t) = \sum_{k=0}^m b_{k0} D_t^{\alpha_k} u(t), \quad \beta_k, \alpha_k \in \mathbb{R}^+, \quad a_k, b_k \in \mathbb{R}. \quad (2.7)$$

By means of (1.28)–(1.30), it is immediate to see that, independently from the adopted definition, the transfer function of (2.7) is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{k=0}^m b_{k0} s^{\alpha_k}}{\sum_{k=0}^n a_{k0} s^{\beta_k}}, \quad \beta_k, \alpha_k \in \mathbb{R}^+, \quad a_k, b_k \in \mathbb{R}. \quad (2.8)$$

In general, a fractional transfer function is the ratio of two fractional polynomials, i.e., polynomials whose exponents are real numbers.

The following theorem shows the behavior of fractional polynomial at their roots.

**Theorem 2.1** *Let  $P(s)$  be a fractional polynomial*

$$P(s) = \sum_{i=1}^n a_i s^{\gamma_i}, \quad a_i \in \mathbb{R}, \quad \gamma_i \in \mathbb{R}^+, \quad (2.9)$$

*whose zeros are  $z_1, \dots, z_m$ . It holds that*

$$\lim_{s \rightarrow z_i} \frac{P(s)}{(s - z_i)^{k_i}} \quad (2.10)$$

*exists finite, for some  $k_i \in \mathbb{N}$  for  $i = 1, \dots, m$  provided that  $z_i \neq 0$ .*

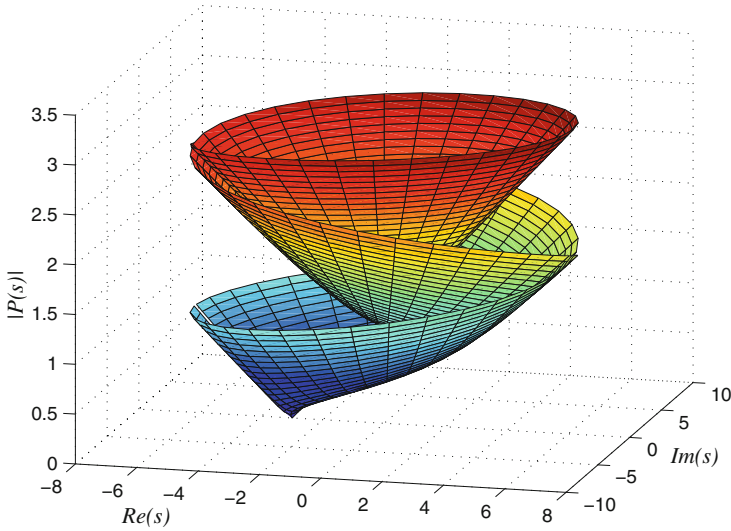
*Proof* By means of the substitution  $t = s - z_i$  (2.10) can be rewritten as

$$\lim_{t \rightarrow 0} \frac{P(t + z_i)}{(t)^{k_i}}, \quad i = 1, \dots, m. \quad (2.11)$$

By series expansion,  $P(s)$  can be represented around  $z_i$  as

$$P(t) = \sum_{i=1}^n a_i \sum_{m=1}^{l_i} \frac{d^m P(t + z_i)}{d(t)^m} \Big|_{t=0} t^m + o(t^{l_i}) \text{ for } t \rightarrow 0, \quad l_i \in \mathbb{N}; \quad (2.12)$$

since the function  $P(s)$ , evidently, is not constant in the complex plane, for each zero  $z_i$ , there exists  $l_i$  such as  $\sum_{i=1}^n a_i \sum_{m=1}^{l_i} \frac{d^m P(t+z_i)}{d(t)^m} \Big|_{t=0} \neq 0$ . Choosing  $k_i$  equal to the smallest  $l_i$  such as this condition holds, (2.11) exists finite.



**Fig. 2.1** The multivalued function  $P(s) = 1 + s^{1/3}$

**Definition 2.1** The integer  $k_i$  is called order of the zero  $z_i$ .

Calculating the zeros of a fractional polynomial is, in general, a complex task and it is beyond the scope of this book. From now on their knowledge is assumed. However, it is worth stressing that when dealing with commensurate-order fractional polynomials (see Sect. 2.4), the computation of the roots becomes trivial [135].

Another interesting characteristic of fractional polynomials (and fractional transfer functions) is that they are multivalued functions [107] as in Fig. 2.1, where a representation of the multivalued behavior of  $(1 + s)^{1/3}$  is shown. It is worth stressing that only the first Riemann sheet has a physical meaning [41]; it is defined as [107]

$$\Omega = \{re^{j\phi} : r > 0, -\pi < \phi < \pi\}. \quad (2.13)$$

Hence, from now on, the analysis will be limited to this surface.

## 2.4 Commensurate Fractional LTI Systems

An important class of fractional LTI systems is the set of commensurate systems.

**Definition 2.2** A fractional LTI system (2.8) is said to be commensurate if all the exponents are integer multiples of the same real number  $\nu$ . The derivative order  $\nu$  is called commensurate order of the system.

The transfer function of a fractional commensurate LTI system always has the following form:

$$H(s) = \frac{\sum_{k=0}^m b_k s^{kv}}{s^{nv} + \sum_{k=0}^{n-1} a_k s^{kv}}, \quad v \in \mathbb{R}^+, \quad a_k, b_k \in \mathbb{R}, \quad (2.14)$$

It is worth noting that every system whose exponents are rational numbers is commensurate. Moreover, when a system has irrational exponents, they can always be approximated with the desired precision degree as ratios of natural numbers.

For many practical applications, commensurate fractional LTI systems are much more diffused than incommensurate one. First, if the exponents are numerically expressed using a finite number of digits, a fractional LTI system is commensurate, but this is almost always the case, unless the system results to be incommensurate from an analytical procedure. Moreover, they are far more mathematically treatable compared to the incommensurate ones. For this second reason, this chapter, which is intended as an introduction to fractional systems in order to drive the reader through for the rest of the book, is focused on commensurate fractional LTI systems. Furthermore, some theoretical results e.g., stability criteria) are still missing for incommensurate systems. Nevertheless, when a given result also exists for incommensurate systems, references to the literature will always be present. Finally, it has to be stressed that, from a mathematical point of view, incommensurate systems are much more interesting and challenging than commensurate ones.

One of the main advantage of commensurate systems is that they can be factorized via partial fraction expansion exactly as integer systems can. Indeed, it is always possible to associate to a given commensurate system an integer-order one by means of the substitution  $s^v = w$  into (2.14):

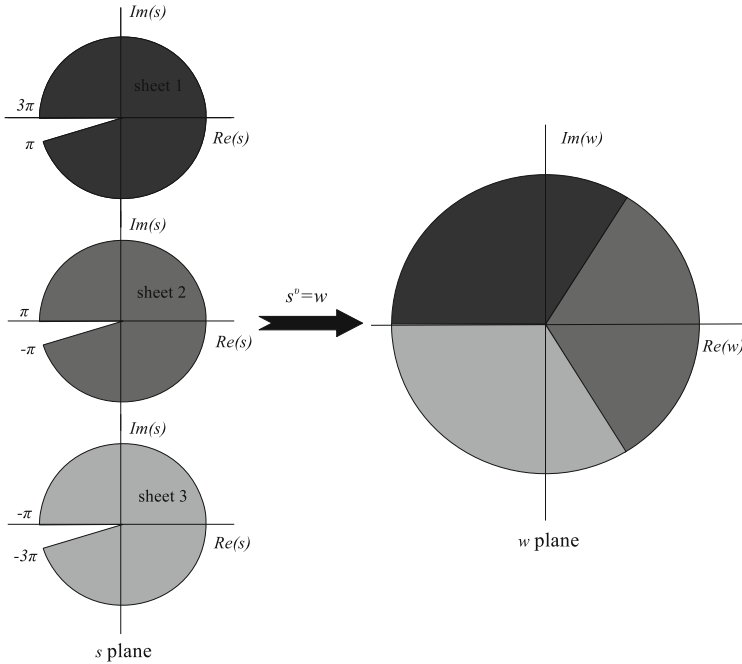
$$\tilde{H}(s) = \frac{\tilde{Y}(w)}{\tilde{U}(w)} = \frac{\tilde{b}(w)}{\tilde{a}(w)} = \frac{\sum_{k=0}^m b_k w^k}{w^n + \sum_{k=0}^{n-1} a_k w^k}. \quad (2.15)$$

A function such as (2.14) is, in general, a multivalued function [107] and the substitution operated to obtain the previous equation maps all the Riemann sheets onto the  $w$  plane, as Fig. 2.2 shows.

Evidently, it is possible to operate over (2.15) as an integer transfer function and then reobtaining the corresponding fractional one by means of the substitution  $w = s^v$ . Accordingly, a commensurate LTI system can always be factorized via partial fractions expansion leading to:

$$H(s) = \sum_{i=1}^p \frac{g_i}{(s^v - \lambda_i)^{k_i+1}}, \quad (2.16)$$

where the coefficients  $\lambda_i$  and  $g_i$  can be either real or complex (in the latter case they always appear in conjugate pairs).



**Fig. 2.2** The Riemann sheets for the function  $P(s) = 1 + s^{1/3}$  and their map onto the  $w$  plane

Now, by means of (2.6), it is immediate to see that the impulse response of (2.16) is

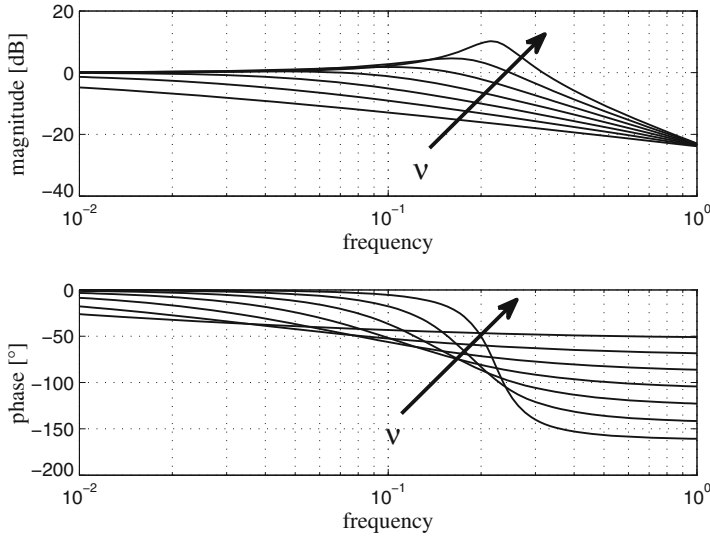
$$\mathcal{L}^{-1}[H(s)] = \eta(t) = \sum_{i=1}^p \frac{g_i}{k_i!} \varepsilon_{k_i}(t, \lambda_i; \nu, \nu). \quad (2.17)$$

Note that, in case of complex coefficients, the PF in (2.16) may be complex valued, but the result of the summation (2.17) is always real valued.

## 2.5 Modes

The previous results highlight that the basic elements of a commensurate LTI system are a kind of fractional generalization of integer simple fractions, hereafter called fractional-first-order (FFO) system:

$$P(s) = \frac{K}{Ts^\nu + 1}. \quad (2.18)$$



**Fig. 2.3** Bode diagram of  $P(s) = \frac{1}{1+15s^v}$  and  $v \in [0.6, 1.8]$

Figures 2.3 and 2.4 show that FFO systems have the interesting properties of being able to describe both monotonic and nonmonotonic behaviors depending on the exponent  $v$ . Note that both the peak frequency  $\bar{\omega}$  and the peak amplitude depend on the derivative order accordingly to [64]

$$\begin{aligned}\bar{\omega} &= -\left(\frac{1}{T}\right)^{\frac{1}{v}} \cos\left(v\frac{\pi}{2}\right) \\ |P(j\bar{\omega})|_{dB} &= -20 \log\left(\sin\left(v\frac{\pi}{2}\right)\right),\end{aligned}\tag{2.19}$$

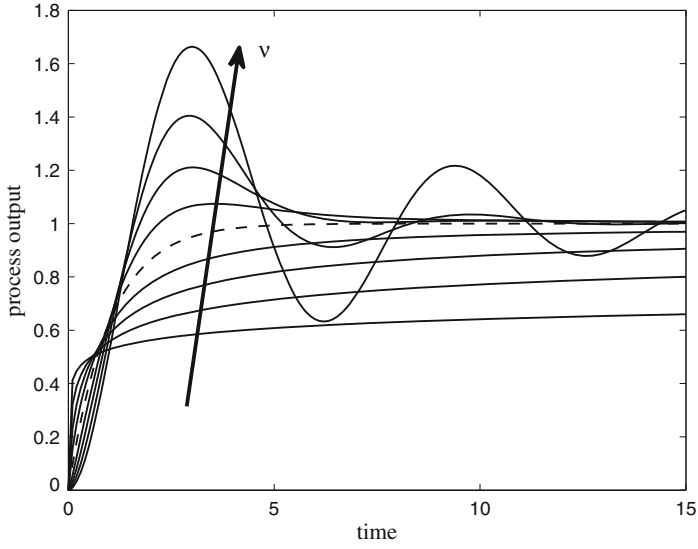
with  $v > 1$ , otherwise (2.18) is not resonant.

It is worth noting that the peak amplitude only depends on the differential order  $v$ , whereas its frequency also depends on the time constant  $T$ .

For a given input function  $u(t)$ , the corresponding function output  $y(t)$  can be computed as the convolution integral between (2.17) and  $u(t)$  [82, 85, 115], that is:

$$y(t) = \int_0^t \eta(t - \xi) u(\xi) d\xi.\tag{2.20}$$

When the input function is simple, the solution of the input–output problem for a commensurate LTI system becomes trivial. For fractional integrators, by considering (2.5) with  $\lambda = 0$  and  $v = \alpha = \beta$ , it results



**Fig. 2.4** Step responses of  $P(s) = \frac{1}{1+s^\nu}$  and  $\nu \in [0.2, 1.8]$

$$\begin{aligned}
 \mathcal{L}^{-1} \left[ \frac{1}{s^\nu} \right] &= \frac{t^{\nu-1}}{\Gamma(\nu)}, \\
 \mathcal{L}^{-1} \left[ \frac{1}{s^\nu} \mathcal{L}[1(t)] \right] &= \frac{t^\nu}{\Gamma(\nu+1)}, \\
 &\vdots \\
 \mathcal{L}^{-1} \left[ \frac{1}{s^\nu} \mathcal{L}[t^n] \right] &= \frac{t^{\nu+n}}{\Gamma(\nu+n)},
 \end{aligned} \tag{2.21}$$

where  $1(t)$  is the Heaviside function.

For simple fractions (FFO systems), the following expressions can be derived (see (2.5) with  $\nu = \alpha = \beta$  and  $a = \lambda$ ):

$$\begin{aligned}
 \mathcal{L}^{-1} \left[ \frac{1}{(s^\nu \mp a)} \right] &= t^{\nu-1} E_{\nu,\nu}(\pm at^\nu), \\
 \mathcal{L}^{-1} \left[ \frac{1}{(s^\nu \mp a)} \mathcal{L}[1(t)] \right] &= t^\nu E_{\nu,\nu+1}(\pm at^\nu), \\
 &\vdots \\
 \mathcal{L}^{-1} \left[ \frac{1}{(s^\nu \mp a)} \mathcal{L}[t^n] \right] &= t^{\nu+n} E_{\nu,\nu+n}(\pm at^\nu).
 \end{aligned} \tag{2.22}$$

An analogous reasoning can be applied in case of multiple fractions starting from (2.6), yielding

$$\begin{aligned}
 \mathcal{L}^{-1} \left[ \frac{1}{(s^\nu \mp a)^{k+1}} \right] &= \frac{t^{\nu(k+1)-1}}{\Gamma(k+1)} \frac{d^k E_{\nu, \nu}(\pm at^\nu)}{d(\pm at^\alpha)}, \\
 \mathcal{L}^{-1} \left[ \frac{1}{(s^\nu \mp a)^{k+1}} \mathcal{L}[1(t)] \right] &= \frac{t^{\nu(k+1)}}{\Gamma(k+1)} \frac{d^k E_{\nu, \nu+1}(\pm at^\nu)}{d(\pm at^\alpha)}, \\
 &\vdots \\
 \mathcal{L}^{-1} \left[ \frac{1}{(s^\nu \mp a)^{k+1}} \mathcal{L}[t^n] \right] &= \frac{t^{\nu(k+1)+n}}{\Gamma(k+1)} \frac{d^k E_{\nu, \nu+n}(\pm at^\nu)}{d(\pm at^\alpha)}.
 \end{aligned} \tag{2.23}$$

The previous equations can also be expressed in term of PFs, and together with (2.17) and (2.20), completely address the input–output problem of commensurate LTI systems.

Finally, for the sake of completeness, it has to be said that similar results also exists for incommensurate LTI systems, but in this case, the impulse response (the so-called fractional Green's function) cannot be expressed as a finite summation of PF. For the reasons discussed at the beginning of the section, this solution has been dropped here, the reader may conveniently refer to [115] for a complete treatise.

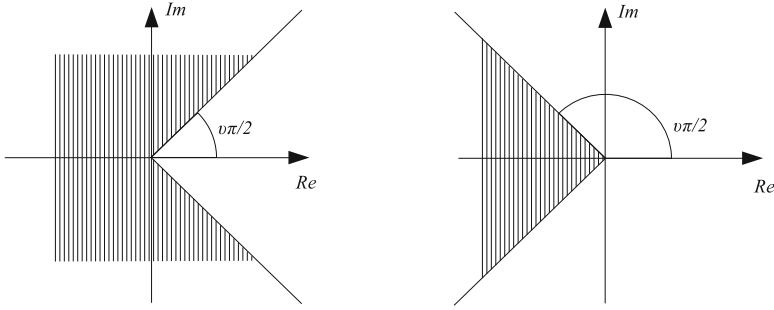
## 2.6 Stability

It is well known that stability is a property of main concern for a control systems. Research on stability of fractional linear system is still an ongoing process and many research works have been proposed in the last years [3, 8, 107, 120]. Unlike for integer system, a simple general stability test only exists for commensurate fractional systems. The following theorem generalizes in a very natural way the well-known stability criteria for integer systems [66, 67, 107].

**Theorem 2.2** *A fractional-order commensurate fractional systems. The following theorem generalizes in a very natural way the well-known systems described by a commensurate transfer function (2.14) with commensurate fractional systems. The following theorem generalizes in a very natural way the well-known order  $\nu < 2$ , is stable if and only if all the poles  $\tilde{p}_i$  of its integer counterpart (2.15) satisfy*

$$|\arg(\tilde{p}_i)| > \nu \frac{\pi}{2}. \tag{2.24}$$

The previous theorem states that the stable region for a fractional commensurate system fractional systems. The following theorem generalizes in a very natural way the well-known is no longer the left half plane (LHP), see Fig. 2.5. This concept is true but sometimes misleading. Indeed (2.24) defines a stable region for the integer



**Fig. 2.5** Stable regions with  $\nu = 0.5$  (left) and  $\nu = 1.5$  (right)

system associate to a commensurate one. This condition corresponds at having all the singularities of the fractional system (that, in view of Lemma (2.1), may be called poles) in the LHP. Indeed, the first Riemann sheet is mapped onto the region defined by  $|\arg(w)| > \nu\pi$  and the right half plane (RHP) is mapped onto the region defined by inequality (2.24).

When the system is incommensurate, stability should be checked computing the roots of the characteristic fractional polynomial (in the first Riemann sheet).

Nevertheless, if it is possible to factorize the system as

$$H(s) = \sum_{i=1}^N \sum_{k=1}^{n_k} \frac{g_i, k}{(s^{\beta_i} + \lambda_i)^k}, \quad \beta_i < 2, \quad (2.25)$$

then, it holds that the system is stable if [107]

$$|\arg(\lambda_i)| < \pi \left(1 - \frac{\beta_i}{2}\right), \quad \forall i. \quad (2.26)$$

Again, note that the previous condition is equivalent to saying that  $H(s)$  has no poles of the first Riemann sheet in the RHP.

As a final remark, it has to be stressed that only the first Riemann sheet is of interest in the stability analysis because the roots that lie on secondary sheets will give solutions that are always monotonically decreasing [67, 146, 158].

## 2.7 Conclusions

An introduction to fractional linear system and fractional control has been provided in this chapter. Basic concepts that will be useful in the next chapters have been explained. A more detailed overview on fractional systems and fractional control can be found in many contributions to the literature, see for example [21, 31, 45, 71, 87, 123, 140, 141].

Advances in Robust Fractional Control

Padula, F.; Visioli, A.

2015, XI, 176 p. 102 illus., 9 illus. in color., Hardcover

ISBN: 978-3-319-10929-9