Chapter 2
First-Species Model

Here we present the full first-species model for a $2k$-tone equal temperament. In particular, we explain what a *counterpoint dichotomy* and a *contrapuntal symmetry* is, how the rules of counterpoint are deduced from these concepts, and how to do the relevant calculations.

2.1 Dichotomies

A significant part of [57] proves that modules are appropriate mathematical objects for the aim of capturing essential features of musical objects; we will also take them as the natural ambient of our theory. The reader can consult the mathematical appendix, Section A.4, for a summary about elementary module theory. In what follows, we will always suppose that the module M is defined over a commutative ring R.

Definition 2.1. *Let M be R-module. A* marked dichotomy *X is a subset of M such that* $|X| = |X^c|$.

We denote with $MiD(M)$ the set of all marked dichotomies of M. Note that when M is finite, its cardinality must be an even number for our definition to make sense. We shall also use this definition in case M is any set since the module structure does not really matter.

Example 2.1. In the $\mathbb{Z}$-module $\mathbb{Z}$, the set of even integers is a marked dichotomy. If $\mathbb{Z}_2$ is the field of two elements, then $X = \{(0,0),(0,1)\}$ is a dichotomy in the $\mathbb{Z}_2$-module $\mathbb{Z}_2^2$.

Remember that the $\mathbb{Z}$-module $\mathbb{Z}_{2k}$ parametrizes pitch classes in the equally tempered $2k$-chromatic scale. We are interested in the intervals between pitch classes, i.e., the differences $x-y$ of pitch classes x and y in $\mathbb{Z}_{2k}$. Mathematically speaking, the result $x-y$ is again an element of $\mathbb{Z}_{2k}$, but in musical terms it is understood as an interval. To make this subtlety more evident, we consider the module $\varepsilon.\mathbb{Z}_{2k}$ of intervals in $\mathbb{Z}_{2k}$, whose elements are intervals in the pitch class module $\mathbb{Z}_{2k}$. The choice of the notation will become clear soon.

Definition 2.2. *A* marked dichotomy of intervals *is a marked dichotomy of the* $\mathbb{Z}$-*module* $\varepsilon.\mathbb{Z}_{2k}$.

A marked interval dichotomy is sometimes denoted by (X/X^c), in order to make explicit the complement.

As we mentioned in the first chapter, we will regard some transformations of a given dichotomy X as essentially equivalent to the original. This idea is captured by the notion of a group acting on a set: in short, this means we have a set X endowed with a group of symmetries G that transform X. The *orbit* of an element $k \in X$ with respect to the action of G is the set of all possible outcomes of a symmetry applied to k (more formal definitions can be found in Section A.3 of the mathematical appendix).

The first of these transformations for marked interval dichotomies that we will consider is the simple act of switching from the one part of a dichotomy to its complement, which defines the following action of $\mathbb{Z}_2$ over $MiD(M)$:

$$\begin{aligned} ?^c : \mathbb{Z}_2 \times MiD(M) &\longrightarrow MiD(M), \\ (i,X) &\longmapsto X^{ic}; \end{aligned}$$

here X^{ic} stands for the operation of taking i times the complement of X.

The process of reversing the roles of consonance and dissonance is not foreign to musical practice. On a small scale, we have seen in the first chapter this change specifically for the interval of a fourth and in the recent idea of dissonant counterpoint [20, p. 35].

Another kind of transformation comes from the action of the general affine group $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$, which is the group that consists of these bijective functions

$$\begin{aligned} T^u \cdot v : \mathbb{Z}_{2k} &\longrightarrow \mathbb{Z}_{2k}, \\ x &\longmapsto T^u \cdot v(x) = u + v(x), \end{aligned}$$

where $u \in \mathbb{Z}_{2k}$ defines a shifting or transposition, and $v \in \mathrm{GL}(\mathbb{Z}_{2k})$ is a linear isomorphism on $\mathbb{Z}_{2k}$. Please note that we are doing a slight abuse of notation, writing $v(x)$ for the multiplication $v.x$ by the invertible element $v \in {\mathbb{Z}_{2k}}^{\times}$. It is worthwhile to state that for the twelve-tone scale there are four invertible elements—1, 5, 7 and 11—as is well known from pitch class theory.

Another way of understanding $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$ is as the set $\mathbb{Z}_{2k} \times {\mathbb{Z}_{2k}}^{\times}$ with the group operation

$$(u_1, v_1) * (u_2, v_2) = (u_1 + v_1 u_2, v_1 v_2).$$

Thus the composition of $T^{u_1} \cdot v_1$ and $T^{u_2} \cdot v_2$ can be expressed in the following form

$$T^{u_1} \cdot v_1 \circ T^{u_2} \cdot v_2 = T^{u_1 + v_1 u_2} \cdot v_1 v_2.$$

The subgroup $\overrightarrow{\mathrm{SL}}(\mathbb{Z}_{2k})$ of $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$, consisting of the elements $T^u \cdot \pm 1$, is very familiar to musicians, for it consists of the (musical) transpositions and inversions, or combinations of both. The action of the supergroup $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$ perhaps is not as intuitive, unless we abandon a one-dimensional view of the intervals and we visualize them over the surface of a torus, at least for the classical case of $\mathbb{Z}_{12}$.

This is possible for the following reason: The main theorem for finitely generated abelian groups (see the mathematical appendix, Section A.3) implies that we have an isomorphism

$$\tau : \mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4$$

because of the prime decomposition $12 = 3 \cdot 2^2$. In particular, we choose the isomorphism $x \mapsto \tau(x) = (x \mod 3, -x \mod 4)$, whose inverse is given by $(x_3, x_4) \mapsto 4x_3 + 3x_4$. Under τ, the major third is mapped to $(1,0)$, while the minor third is sent to $(0,1)$. The graph with $\mathbb{Z}_3 \times \mathbb{Z}_4$ as vertices and edges joining any two vertices that differ in exactly one minor third or one major third is a toroidal[1] graph, which we call the *torus of thirds* (Figure 2.1).

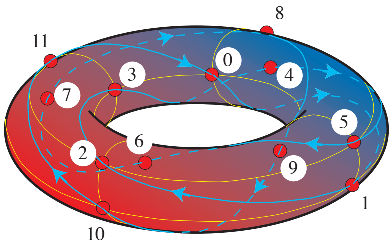

Fig. 2.1 The torus of thirds.

The periodicity of the octave can be readily captured by the group of complex numbers of unit module (the circle S^1) or the clock arithmetic. Since the underlying set of a torus is $S^1 \times S^1$, now we can see a double periodicity of the intervals: One cycle goes in steps of minor thirds, the other in steps of major thirds. With this picture in mind, we may visualize the multiplication by 5 as a reflection along the equatorial plane of the torus; by 7 as a reflection along a plane perpendicular to the toroidal plane; by 11, as a combination of the previous two. Translations can be seen to act as rotations in the toroidal or poloidal directions.

The actions of the complement and $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$ can be considered simultaneously since they commute. This means we have an action of the direct product $\mathbb{Z}_2 \times \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$ on $\varepsilon.\mathbb{Z}_{2k}$. The orbits of this joint action are called *dichotomy classes*.

[1] This means that the graph can be embedded in the surface of a torus.

Definition 2.3. *A marked dichotomy of intervals X is* autocomplementary *if its complement belongs to its orbit or, in other words, if its dichotomy class coincides with its marked dichotomy class. The dichotomy X is said to be* rigid *if the only symmetry that leaves it invariant is the identity, and if it is both autocomplementary and rigid it is called* strong.

If X is a member of a strong dichotomy class and p is such that $p(X) = X^c$, then p is unique. Let q be another symmetry such that $q(X) = X^c$. Then

$$p^{-1}(q(X)) = p^{-1}(X^c) = X$$

which means that $p^{-1}q = T^0 \cdot 1$, where $T^0 \cdot 1$ is the identity of $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$, and thus $p = q$ pre-multiplying both sides by p. Note also that from this equality we conclude that p is an involution, i.e., $p^2 = T^0 \cdot 1$. Being unique, p is called the *polarity* of (X/X^c).

Example 2.2. In $\mathbb{Z}_{12}$, the marked dichotomy $X_1 = (\{0,1,3,6,8,11\}/\{2,4,5,7,9,10\})$ is not autocomplementary. On the other hand, $X_2 = (\{0,1,3,7,8,11\}/\{2,4,5,6,9,10\})$ is autocomplementary, because

$$\begin{aligned} T^9 \cdot 7(X_2) &= \{7\cdot 0+9, 7\cdot 1+9, 7\cdot 3+9, 7\cdot 7+9, 7\cdot 8+9, 7\cdot 11+9\} \\ &= \{9,4,6,10,5,2\} = X_2{}^c. \end{aligned}$$

The set X_2, nevertheless, is not strong because the symmetry $T^8 \cdot 5$ leaves it invariant. The dichotomy $K = \{0,3,4,7,8,9\}$ is strong, because

$$\begin{aligned} T^2 \cdot 5(K) &= \{5\cdot 0+2, 5\cdot 3+2, 5\cdot 4+2, 5\cdot 7+2, 5\cdot 8+2, 5\cdot 9+2\} \\ &= \{2,5,10,1,6,11\} = K^c \end{aligned}$$

and it is laborious, but direct, to check that apart from $T^0 \cdot 1$ no other symmetry leaves K invariant. Thus $p = T^2 \cdot 5$ is the polarity of K. This last example is of capital importance for this book, since K is the set of intervals from the tonic (within an octave) of the classical Renaissance consonances: prime (or unison), minor third, major third, perfect fifth, minor sixth and major sixth. The rest of the intervals (again, within a octave) are (of course) dissonances and are related to the consonances, via the polarity, in the following respective order: major second, perfect fourth, minor seventh, minor second, tritone, major seventh.

Problem 2.1. *Verify in detail the claims of Example 2.2 (by the use of a computer, if possible).*

It is not obvious that autocomplementary, rigid, or strong dichotomies exist in $\mathbb{Z}_{2k}$. But, whenever we have an involutary element p of $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$ without fixed points, we can easily construct an autocomplementary dichotomy. Indeed, we first pick an arbitrary element $x_1 \in \mathbb{Z}_{2k}$, next we choose $x_2 \in \mathbb{Z}_{2k}$ distinct from x_1 and $p(x_1)$. Then we take x_j different from the elements

$$x_1, \dots x_{j-1}, p(x_1), \dots p(x_{j-1})$$

until we reach $j = k$. Thus $X = (\{x_1, \dots x_k\}/\{p(x_1), \dots p(x_k)\})$ is an autocomplementary marked dichotomy.

Lemma 2.1. *For any integer k and $0 \leq s \leq 2k$ odd, the automorphism $T^s \cdot -1 \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$ is involutary and has no fixed points.*

Proof. Since

$$(T^s \cdot -1) \circ (T^s \cdot -1) = T^s \circ T^{-s} \cdot 1 = T^0 \cdot 1 = 1$$

the involutarity is settled. If $T^s \cdot -1(x) = s - x = x$ for some x, then $2x - s = 0$. This means that $2x - s$ is divisible by $2k$, which contradicts that $2x - s$ is odd. Consequently, $T^s \cdot -1$ does not have fixed points. □

For $\mathbb{Z}_2$ it is obvious that $(\{1\}/\{0\})$ and $(\{0\}/\{1\})$ are its only strong marked interval dichotomies. For $\mathbb{Z}_4$, there are no strong dichotomies. It suffices to check this for

$$X_1 = (\{0,1\}/\{2,3\}), X_2 = (\{0,2\}/\{1,3\}), X_3 = (\{0,3\}/\{1,2\}),$$

since all of them are autocomplementary. Nevertheless, $T^1 \cdot -1(X_1) = X_1$, $-1(X_2) = X_2$ and $T^{-1} \cdot -1(X_3) = X_3$, so none of them are rigid.

Now we can show that there exists at least one strong interval dichotomy in $\mathbb{Z}_{2k}$ for $k \geq 3$. We should keep in mind that the invertible elements of $\mathbb{Z}_{2k}$ are odd because they are coprime with $2k$.

Proposition 2.1. *Let $k \geq 3$. The dichotomy*

$$(X/Y) = (\{-1, 2k-2, 2k-4, \dots 4, 2\}/\{0, 1, 3, \dots 2k-5, 2k-3\})$$

in $\mathbb{Z}_{2k}$ (which is obtained from the automorphism $T^{-1} \cdot -1$) is strong.

Proof. In general, the proposed dichotomy is clearly autocomplementary, with isomorphism $T^{-1} \cdot -1$. In order to prove its rigidity, we will show that for any automorphism $T^u \cdot w$, except the identity, at least one element of the marked interval dichotomy is mapped to an element in the complement. If $u = 0$, then $w(-1) = -w \neq -1$ is odd and therefore $w(-1) \in X^c$. If $u \neq 0$ is even, then $X \ni -w^{-1}u \neq 0$ is even and $T^u \cdot w(-w^{-1}u) = u - u = 0 \in X^c$. If u is odd, $T^u \cdot w(2) = u + 2w$ is odd. If it belongs to X^c, we are done. Otherwise, $T^u \cdot w(2) = u + 2w = -1$. It is impossible that $T^u \cdot w(4) = u + w = -1$, for it would imply that $2w = 0$ and $2 = 0$, contradicting that $k \geq 3$. □

It is worthwhile to emphasize that strong dichotomies exist for any even cardinality except 4. This means that, in terms of this model, we cannot formulate a counterpoint theory for the 4-tone equally tempered scale. This scale can be interpreted as the tones of a diminished 7th chord in a twelve-tone equally tempered scale.

Remark 2.1. *The dichotomy*

$$(\{11,10,8,6,4,2\}/\{0,1,3,5,7,9\}),$$

constructed for $k = 6$ *in Proposition 2.1, belongs to the dichotomy class of*

$$\Delta_{78} = (\{0,1,2,4,6,10\}/\{3,5,7,8,9,11\})$$

in Mazzola's list [57, appendix L].

2.2 Counterpoint Dichotomies

Now we need to extend the module of intervals so we can capture the idea of cantus firmus and discantus. In order to do so, we consider the ring of dual numbers

$$\mathbb{Z}_{2k}[\varepsilon] = \{a+\varepsilon.b : a,b \in \mathbb{Z}_{2k}, \varepsilon^2 = 0\},$$

and two dual numbers $a+\varepsilon.b, c+\varepsilon.d$ are equal if and only if $a = c$ and $b = d$. We define the sum and multiplication of dual numbers by

$$\begin{aligned}(a+\varepsilon.b)+(c+\varepsilon.d) &= (a+c)+\varepsilon.(b+d),\\ (a+\varepsilon.b)(c+\varepsilon.d) &= ac+\varepsilon.ad+\varepsilon.bc+\varepsilon^2.bd\\ &= ac+\varepsilon.(ad+bc).\end{aligned}$$

Within the dual number interpretation of counterpoint, a *contrapuntal interval* is a dual number $b+\varepsilon.b$, where the first component b, represents the cantus firmus, while the second one, b, represents the distance between the cantus firmus and the discantus. This justifies our election of the notation $\varepsilon.\mathbb{Z}_{2k}$ for the set of intervals: They are just the ε-components of contrapuntal intervals.

Example 2.3. In the twelve-tone scale, $2+\varepsilon.7 \in \mathbb{Z}_{2k}[\varepsilon]$ represents a fifth over D. Thus the cantus firmus has the note D and the discantus has the note A, for $7 = 9-2$ or, equivalently, $9 = 2+7$. Observe that this interpretation of the discantus as being the sum of the two components of the interval $2+\varepsilon.7$ defines what is called the "sweeping" counterpoint in literature. For the hanging interpretation, namely that the discantus would be $2-7 = 2+5 = 7$, we refer to Section 2.2.1 where the relations between sweeping and hanging counterpoint are discussed.

As with $\mathbb{Z}_{2k}$, we have the general affine group

$$\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k}[\varepsilon]) = \{T^{a+\varepsilon.b}\cdot(c+\varepsilon.d) : c+\varepsilon.d \in \mathbb{Z}_{2k}[\varepsilon]^{\times}\}$$

that acts on $\mathbb{Z}_{2k}[\varepsilon]$. Note that the set ${\mathbb{Z}_{2k}}^{\times}[\varepsilon]$ of invertible dual numbers consists of the dual numbers $a+\varepsilon.b$ with $a \in {\mathbb{Z}_{2k}}^{\times}$.

Since $\mathbb{Z}_{2k}[\varepsilon]$ has an even number of elements, dichotomies make sense for this ring, and thus we can talk about *marked counterpoint dichotomies* and *counterpoint dichotomy classes*.

Example 2.4. To each marked interval dichotomy (X/Y), we can associate a marked counterpoint dichotomy $(X[\varepsilon]/Y[\varepsilon])$ called the *induced counterpoint dichotomy* by X, where

$$X[\varepsilon] = \{a+\varepsilon.b : a \in \mathbb{Z}_{2k}, b \in X\} := \mathbb{Z}_{2k} + \varepsilon.X.$$

Indeed

$$\begin{aligned} X[\varepsilon]^c &= \{a+\varepsilon.b : a \in \mathbb{Z}_{2k}, b \notin X\} \\ &= \{a+\varepsilon.b : a \in \mathbb{Z}_{2k}, b \in X^c = Y\} \\ &= \mathbb{Z}_{2k} + \varepsilon.Y = Y[\varepsilon]. \end{aligned}$$

If (X/Y) is a marked strong dichotomy, the induced counterpoint dichotomy $(X[\varepsilon]/Y[\varepsilon])$ is autocomplementary but not necessarily strong.

Problem 2.2. *Provide an example of a marked strong dichotomy such that $X[\varepsilon]$ is not strong.*

Problem 2.3. *Check that if $T^u \cdot v$ is an autocomplementary symmetry of X, then $T^{\varepsilon.u} \cdot v$ is an autocomplementary symmetry of the induced counterpoint dichotomy $X[\varepsilon]$.*

The following proposition will prove to be fundamental for the computational aspects of counterpoint.

Proposition 2.2. *Let X be a marked strong dichotomy with polarity $p = T^u \cdot v$, and let $x \in \mathbb{Z}_{2k}$ be a cantus firmus. There exists exactly one symmetry $p^x[\varepsilon] \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k})$ (which will be called the* induced polarity of $X[\varepsilon]$*) such that*

1. *it is an autocomplementary function of $X[\varepsilon]$,*
2. *leaves $x+\varepsilon.\mathbb{Z}_{2k}$ (the set of all the counterpoint intervals with cantus firmus x) invariant.*

It is given by

$$p^x[\varepsilon] = T^{x(1-v)+\varepsilon.u} \cdot v \tag{2.1}$$

and satisfies the following translational formula

$$p^{x+y}[\varepsilon] = T^x \circ p^y[\varepsilon] \circ T^{-x}. \tag{2.2}$$

Proof. It is straightforward to check that (2.1) satisfies the three requirements, so we only need to prove its unicity. Let $z \in X$. We are looking for some $u_1 + \varepsilon.v_1$ and $u_2 + \varepsilon.v_2$ such that the symmetry

$$p^x[\varepsilon] = T^{u_1+\varepsilon.v_1} \cdot (u_2 + \varepsilon.v_2)$$

satisfies

$$p^x[\varepsilon](x+\varepsilon.z) = x+\varepsilon.w.$$

This last equality states that $p^x[\varepsilon]$ sends an interval with cantus firmus x to another interval with the same cantus firmus. If we perform the calculations

$$\begin{aligned} p^x[\varepsilon](x+\varepsilon.z) &= T^{u_1+\varepsilon.v_1}\cdot(u_2+\varepsilon.v_2)(x+\varepsilon.z) \\ &= (u_1+\varepsilon.v_1)+(u_2+\varepsilon.v_2)(x+\varepsilon.z) \\ &= (u_1+\varepsilon.v_1)+(u_2x+\varepsilon.(u_2z+v_2x)) \\ &= (u_1+u_2x)+\varepsilon.(v_1+u_2z+v_2x). \end{aligned}$$

then we deduce, by comparing the first components of the dual numbers, that we must have

$$u_1+u_2x = x,$$

and thus

$$u_1 = x-u_2x = x(1-u_2).$$

On the other hand, by the autocomplementarity, we must have $w \in Y$. This means that, for any t

$$v_1+u_2z+v_2t = u_2z+(v_1+v_2t) = T^{v_1+v_2t}\cdot u_2(z) \in Y,$$

and the strength of X implies that $u_2 = v$ and $v_1+v_2t = u$. Since t is arbitrary, the last equality holds for $t = 0$, so $v_1 = u$. Now, if $t = 1$, then $u+v_2 = u$, which implies $v_2 = 0$, and we are done. □

2.2.1 Musical Meaning of the Operations with Counterpoint Intervals

Sums and multiplications of counterpoint intervals may seem of questionable musical significance. To prove that it is not so, let us exemplify the situation with $\mathbb{Z}_{12}[\varepsilon]$ and its group of affine symmetries

$$\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{12}[\varepsilon]) = \{T^{a+\varepsilon.b}\cdot(u+\varepsilon.v) : u = 1,5,7,11\}.$$

Lemma 2.2. *Let $u \in \mathbb{Z}_{12}{}^{\times}$. The following identity holds*

$$(u+\varepsilon.v) = u(1+\varepsilon.1)^s$$

for every $v \in \mathbb{Z}_{12}$, where s is any integer within the class of uv.

Proof. Since $u^2 = 1$ in $\mathbb{Z}_{12}$, then $u = u^{-1}$ and $(u+\varepsilon.v) = u(1+\varepsilon.uv)$, so we only have to prove that $1+\varepsilon.uv = (1+\varepsilon)^s$ with $s \in \mathbb{Z}$ in the class of uv. In fact, $1+\varepsilon.w = (1+\varepsilon)^{s'}$, with s' in the class of w. For $w = 0$, we have the calculation $1+\varepsilon.0 = 1 = (1+\varepsilon.0)^0$.

If this is true for w, then

$$\begin{aligned}(1+\varepsilon)^{s'+1} &= (1+\varepsilon.1)^{s'}(1+\varepsilon.1)\\ &= (1+\varepsilon.w)(1+\varepsilon.1)\\ &= 1+\varepsilon.w+\varepsilon.1 = 1+\varepsilon.(w+1),\end{aligned}$$

and it is clear that $s'+1$ belongs to the class of $w+1$, and we are done by induction. □

By Lemma 2.2, we may write

$$T^{a+\varepsilon.b}\cdot(u+\varepsilon.v) = T^{a}T^{\varepsilon.b}\cdot u(1+\varepsilon.1)^{s}$$

for an element $\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{12}[\varepsilon])$. Thus, to understand the effects of interval multiplication, it suffices to examine the four symmetries

$$s_1 = T^{a}, s_2 = T^{\varepsilon.b}, s_3 = T^{0}\cdot u, s_4 = T^{0}\cdot(1+\varepsilon.1).$$

1. For s_1, the identity $T^{a}\cdot(x+\varepsilon.y) = (x+a)+\varepsilon.y$ expresses that the whole interval is transposed by a.
2. For s_2, according to $T^{\varepsilon.b}\cdot(x+\varepsilon.y) = x+\varepsilon.(y+b)$, it causes the interval y to be transposed by b, leaving the cantus firmus unchanged. This kind of operation is common in double counterpoint.
3. When it comes to s_3, we have to consider three cases apart from the identity. Perhaps $u = 11 = -1$ is the most natural because it inverts the interval; it also reflects the cantus firmus with respect the tonic, but it can be restored later via a translation T^{x}. This operation is also common in double counterpoint. The remaining cases $u = 5, 7$ are not as natural unless we resort again to the torus of thirds as described in Section 2.1.
4. The symmetry obtained from s_4 suggests an interesting transformation, since

$$(1+\varepsilon.1)(a+\varepsilon.b) = a+\varepsilon.a+b,$$

which means that the discantus is transposed by the same interval between the tonic and the cantus firmus, but without moving the cantus firmus itself. This operation can be iterated, generating a cycle of counterpoint intervals that return to the original one, because the underlying ring is cyclic with respect to addition.

Another reason for the importance of the symmetry $1+\varepsilon.1$ is that it allows us to handle voice crossings. For a counterpoint interval $x+\varepsilon.y$, we may define an *orientation*, which can be hanging or sweeping, that indicates how to obtain the discantus tone. If it is sweeping, the discantus is $x+y$, while if it is hanging, it is $x-y$. A little more formally, we have the mappings

$$\begin{aligned}\alpha_{+} : \mathbb{Z}_{12}[\varepsilon] &\longrightarrow \mathbb{Z}_{12},\\ x+\varepsilon.y &\longmapsto x+y,\end{aligned}$$

and

$$\alpha_- : \mathbb{Z}_{12}[\varepsilon] \longrightarrow \mathbb{Z}_{12},$$
$$x + \varepsilon.y \longmapsto x - y.$$

Observe that $-\alpha_+ \circ T^0 \cdot (1+\varepsilon.1)^{-2} = \alpha_-$, since

$$\begin{aligned} -\alpha_+((1+\varepsilon.1)^{-2}(x+\varepsilon.y)) &= -\alpha_+((1+\varepsilon.(-2)(x+\varepsilon.y)) \\ &= -\alpha_+(x+\varepsilon.y-2x) \\ &= -(y-x) = x-y = \alpha_-(x+\varepsilon.y). \end{aligned}$$

Thus, if we have a hanging counterpoint interval $a+\varepsilon.b$, we may write

$$\begin{aligned} \alpha_-(a+\varepsilon.b) &= -\alpha_+((1+\varepsilon.1)^{-2}(a+\varepsilon.b)) \\ &= \alpha_+(-(1+\varepsilon.1)^{-2}(a+\varepsilon.b)), \end{aligned}$$

which means that the sweeping counterpoint interval

$$-(1+\varepsilon.1)^{-2}(a+\varepsilon.b) = -(a+\varepsilon.b-2a) = -a+\varepsilon.(2a-b)$$

can be regarded as equivalent to the original one.

2.3 Counterpoint Symmetries

We have arrived at the crux of the model. As we have discussed in Section 1.3, we can generate tension in the confrontation of a consonance against another by the *deformation* of consonances into dissonances and dissonances into consonances. We can now make this precise, saying that for a symmetry g, the marked interval dichotomy $(gX[\varepsilon]/gY[\varepsilon])$ is a *deformation* of the original dichotomy $(X[\varepsilon]/Y[\varepsilon])$. Thus, if we can interpret a consonant interval $\xi \in X[\varepsilon]$ as a deformed dissonance, i.e., $\xi \in gY[\varepsilon]$, we can "resolve" it in a deformed consonance that is also a consonance, i.e., a member η of $gX[\varepsilon] \cap X[\varepsilon]$.

Definition 2.4. *Let $\{\xi,\eta\}$ be a pair of counterpoint intervals from the dichotomy $(X[\varepsilon]/Y[\varepsilon])$. If there exists at least one symmetry $g \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ such that $\xi \in g.Y[\varepsilon]$ and $\eta \in g.X[\varepsilon]$, then we say that $\{\xi,\eta\}$ is g-polarized.*

Theorem 2.1. *Let $(X[\varepsilon]/Y[\varepsilon])$ a dichotomy and ξ and η two different intervals. Then there exists one symmetry $g \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ such that $\{\xi,\eta\}$ is g-polarized.*

Proof. If already ξ and η belong to the complementary sets defined by $(X[\varepsilon]/Y[\varepsilon])$, the identity $T^1 \cdot 1$ does the job. Suppose first that $\xi = u_1 + \varepsilon.v_1$ and $\eta = u_2 + \varepsilon.v_2$ with $v_1 \neq v_2$. We claim that there exists m such that the symmetry $T^{\varepsilon.m} \cdot 1$ polarizes the pair. If it were not the case, then for any m we would have

$$v_1 + m, v_2 + m \in X.$$

In particular, if we choose an arbitrary a and set $m = a - v_1$, then $a, v_2 - v_1 + a \in X$. Thus $T^{v_2 - v_1} \cdot 1$ leaves $X[\varepsilon]$ invariant, which forces $v_2 - v_1 = 0$ and $v_2 = v_1$, a contradiction.

It remains to examine the case $u_1 \neq u_2$. Now we claim that there exists an m such that $T^{\varepsilon.m} \cdot (1 + \varepsilon.1)$ polarizes the pair. Otherwise, $v_1 + u_1 + m, v_2 + u_2 + m \in X$, and we may choose an arbitrary a to define $m = a - u_1 - v_2$, rendering $T^{u_2 - u_1} \cdot 1$ an automorphism of X. Thus $u_2 = u_1$ and we have a contradiction again. This completes the proof. □

Definition 2.5. *Let $(X[\varepsilon]/Y[\varepsilon])$ be a dichotomy and $\xi \in X[\varepsilon]$ a counterpoint interval. A symmetry $g \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ is* contrapuntal *if*

1. *the interval ξ does not belong to $g(X[\varepsilon])$,*
2. *the symmetry $p^x[\varepsilon]$ is a polarity of $(g(X[\varepsilon])/g(Y[\varepsilon]))$,*
3. *the cardinality of $g(X[\varepsilon]) \cap X[\varepsilon]$ is maximal among the symmetries with the previous two properties.*

Definition 2.6. *Given a dichotomy $(X[\varepsilon]/Y[\varepsilon])$ and an interval $\xi \in X[\varepsilon]$, we say η is an* admitted successor *if ξ is contained in $g(X[\varepsilon])/X[\varepsilon]$ for some contrapuntal symmetry g.*

Example 2.5. Consider the dichotomy

$$(K/D) = (\{0,3,4,7,8,9\}/\{1,2,5,6,10,11\})$$

in $\mathbb{Z}_{12}$ from Example 2.2, the interval $\xi = \varepsilon.9$ (a major sixth over the tonic) and the symmetry $g = T^{\varepsilon.8} \cdot (5 + \varepsilon.4)$. The cantus firmus is $x = 0$, and by Proposition 2.2 we know that a polarity of $(K[\varepsilon]/D[\varepsilon])$ is $p^0[\varepsilon] = T^{\varepsilon.2} \cdot 5$. Let us verify that g is a counterpoint symmetry for ξ. First,

$$\begin{aligned} g(K[\varepsilon]) &= (1 - \varepsilon.4)\mathbb{Z}_{2k} + \varepsilon.T^8 \cdot 5K \\ &= (1 - \varepsilon.4)\mathbb{Z}_{2k} + \varepsilon.T^6 T^2 \cdot 5K \\ &= (1 - \varepsilon.4)\mathbb{Z}_{2k} + \varepsilon.T^6 D, \end{aligned}$$

which means that the elements of $gK[\varepsilon]$ are of the form

$$w + \varepsilon.T^{6-4w} \cdot y$$

for some $y \in D$. This means $\xi \notin gK[\varepsilon]$, since $9 \notin T^6 D = \{0,4,5,7,8,11\}$. Second,

$$\begin{aligned} p^0[\varepsilon](gK[\varepsilon]) &= T^{\varepsilon.2} \cdot 5((1 + \varepsilon.-4)\mathbb{Z}_{2k}\varepsilon.T^6 D) \\ &= (1 - \varepsilon.4)5\mathbb{Z}_{2k} + \varepsilon.T^8 \cdot 5K \\ &= (1 - \varepsilon.4)\mathbb{Z}_{2k} + \varepsilon.T^6 T^2 \cdot 5K \\ &= (1 - \varepsilon.4)\mathbb{Z}_{2k} + \varepsilon.T^6 K = gD[\varepsilon] \end{aligned}$$

which precisely means that $p^0[\varepsilon]$ is a polarity of $(K[\varepsilon]/D[\varepsilon])$. It is not difficult to calculate that

$$\left|(gK[\varepsilon]) \cap K[\varepsilon]\right| = 56$$

and that this is the maximum possible. Thus, g is a counterpoint symmetry.

Take again the dichotomy (K/D). The consonances with cantus firmus w are

$$K_w := w + \varepsilon.K,$$

and the action of $p^0[\varepsilon]$ over them is given by

$$p^0[\varepsilon](K_w) = T^{\varepsilon.2} \cdot 5(w + \varepsilon.K) = 5w + \varepsilon.T^2 \cdot 5K = 5w + \varepsilon.D,$$

which means that this polarity of counterpoint intervals is equivalent to applying the translation T^{4w} to the cantus firmus, and the marked interval polarity to the interval. In this sense, $p^0[\varepsilon]$ is global, since for any consonances with fixed cantus firmus, w acts in the same manner.

This is different for "deformed" consonances by the symmetry g with cantus firmus w:

$$gK_w := (w\varepsilon.\mathbb{Z}_{2k}) \cap gK[\varepsilon]$$

For instance, if g is the counterpoint symmetry of the last example, we have

$$gK_w = w + \varepsilon.T^{8-4w} \cdot 5K$$

and $p^0[\varepsilon]$ does not act by translating gK_w with T^{4w} and applying the polarity to the interval. Instead

$$p^0[\varepsilon](gK_w) = (5w + \varepsilon.\mathbb{Z}_{2k}) \cap (5w + \varepsilon.T^{6-4w}K),$$

thus the way $p^0[\varepsilon]$ "distorts" the consonances depends on the value of the cantus firmus. In this sense, $p^0[\varepsilon]$ is a *local* symmetry for the deformed dichotomy $(gX[\varepsilon]/gY[\varepsilon])$.

2.4 The Counterpoint Theorem

In terms of raw computational power, from this point on we can simply take any interval in a counterpoint dichotomy and test all the available transformations, looking for its counterpoint symmetries, and then we can make a list of its admitted successors to realize the counterpoint theory in an equally tempered $\mathbb{Z}_{2k}$-tone scale. Nevertheless, it is important to study the matter more carefully in order to optimize this naive algorithm and derive further properties of the model.

2.4.1 Some Preliminary Calculations

Lemma 2.3. *Let (X/Y) be a strong dichotomy and $(X[\varepsilon]/Y[\varepsilon])$ its induced counterpoint dichotomy. The symmetries that leave $X[\varepsilon]$ invariant are of the form*

$$T^{\mathbb{Z}_{2k}} := \{T^w : w \in \mathbb{Z}_{2k}\}.$$

Proof. Every element of $T^{\mathbb{Z}_{2k}}$ is a symmetry of $X[\varepsilon]$, since

$$T^w(y+\varepsilon.x) = (x+w)+\varepsilon.x \in X[\varepsilon].$$

With this in mind, it is clear that $T^{z+\varepsilon.w}\cdot(u+\varepsilon.v)$ is a symmetry if and only if $T^{\varepsilon.w}\cdot(u+\varepsilon.v)$ is a symmetry. In particular, we would need that

$$w+vy+ux \in X$$

for every $y \in \mathbb{Z}_{2k}$ and $x \in X$. Since X is strong, we have $u = 1$ and $w+vy = 0$. In particular, for $y = 0$, we have $x+w \in X$ for every w, and strength implies that $w = 0$. Thus $vx = 0$, which is also valid when $y = 1$, and therefore $v = 0$. □

For the following, we define the group

$$H := T^{\varepsilon.\mathbb{Z}_{2k}}\mathrm{GL}(\mathbb{Z}_{2k}[\varepsilon]) = \{T^{\varepsilon.t}\cdot(u+\varepsilon.v) : u+\varepsilon.v \in \mathbb{Z}_{2k}{}^{\times}\}.$$

Lemma 2.4. *For $g = T^{\varepsilon.t}\cdot(u+\varepsilon.v)$ and $z \in \mathbb{Z}_{2k}$, define*

$$g^{(z)} = gT^{\varepsilon.u^{-2}vz} \in H.$$

Then

$$(g^{(z_1)})^{(z_2)} = g^{(z_1+z_2)} \tag{2.3}$$

and

$$T^z gX[\varepsilon] = g^{(-z)}X[\varepsilon]. \tag{2.4}$$

Proof. The first equality is direct:

$$\begin{aligned}
(g^{(z_1)})^{(z_2)} &= (g^{(z_1)})T^{\varepsilon.u^{-2}vz_2}\\
&= (gT^{\varepsilon.u^{-2}vz_1})T^{\varepsilon.u^{-2}vz_2}\\
&= g(T^{\varepsilon.u^{-2}vz_1}T^{\varepsilon.u^{-2}vz_2})\\
&= gT^{\varepsilon.u^{-2}v(z_1+z_2)}\\
&= g^{(z_1+z_2)}.
\end{aligned}$$

The second equality is a little more involved. First

$$T^z g = T^z \cdot (u+\varepsilon.v) = T^{\varepsilon.t} \cdot (u+\varepsilon.v) T^{z(u^{-1}-\varepsilon.u^{-2}v)}$$
$$= g^{(-z)} T^{zu^{-1}}$$

and thus

$$T^z gX[\varepsilon] = g^{(-z)} T^{zu^{-1}} X[\varepsilon] = g^{(-z)} X[\varepsilon]$$

using Lemma 2.3. □

Corollary 2.1. *For every $g \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k}[\varepsilon])$, there exists a symmetry $h \in H$ such that $gX[\varepsilon] = hX[\varepsilon]$.*

Proof. Given $g = T^{u_1+\varepsilon.v_1} \cdot (u_2 + \varepsilon.v_2)$, define

$$f = T^{\varepsilon.v_1} \cdot (u_2 + \varepsilon.v_2).$$

Then $g = T^{u_1} f$, and by Lemma 2.4

$$gX[\varepsilon] = T^{u_1} f X[\varepsilon] = f^{(-z)} X[\varepsilon],$$

hence $h = f^{(-z)}$ is the required symmetry. □

Lemma 2.5. *Let $\xi = x + \varepsilon.k$, $g \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ and $y \in \mathbb{Z}_{2k}$. If*

$$\xi \notin gX[\varepsilon] \quad \text{and} \quad p^x[\varepsilon] : gX[\varepsilon] \to gY[\varepsilon]$$

where $p^x[\varepsilon]$ is the induced polarity of $X[\varepsilon]$, then

$$T^z \xi \notin T^z gX[\varepsilon] \quad \text{and} \quad p^{z+x}[\varepsilon] : T^z gX[\varepsilon] \to T^z gY[\varepsilon].$$

Furthermore,

$$(T^z gX[\varepsilon]) \cap X[\varepsilon] = T^z \cdot (gX[\varepsilon] \cap X[\varepsilon]).$$

Proof. It is clear that $T^z \xi \notin T^z gX[\varepsilon]$, since $T^z \in T^{\mathbb{Z}_{2k}}$ is bijective. Using the translation formula (2.2), we can write

$$\begin{aligned} p^{z+x}[\varepsilon] T^z gX[\varepsilon] &= p^{x+z}[\varepsilon] T^z gX[\varepsilon] \\ &= T^z p^x[\varepsilon] T^{-z} T^z gX[\varepsilon] \\ &= T^z p^x[\varepsilon] gX[\varepsilon] \\ &= T^z gY[\varepsilon]. \end{aligned}$$

For the last part of this lemma, we invoke Lemma 2.3,

$$(T^z g[\varepsilon]) \cap X[\varepsilon] = (T^z g[\varepsilon]) \cap (T^z X[\varepsilon]) = T^z(gX[\varepsilon] \cap X[\varepsilon]),$$

since $g(\eta_1) = \eta_2 = T^{-z} T^z \eta_2$ for some η_1 and η_2 if and only if $T^z \cdot (g(\eta_1)) = T^z \cdot (\eta_2)$. □

Now we have all the necessary ingredients to prove a key theorem simplifying the algorithm for the calculation of contrapuntal symmetries; it says the intervals with respect to the tonic are the essential ones.

Theorem 2.2. *If $\xi = x + \varepsilon.k \in X[\varepsilon]$ is a consonant interval and g is one of its counterpoint symmetries, then there is a symmetry $h \in H$ such that we can verify it is contrapuntal using the set $hX[\varepsilon] = gX[\varepsilon]$. Furthermore, to check the required properties, we can restrict ourselves to the interval $\varepsilon.k$, the symmetry $h^{(x)} \in H$, and the polarity $p^0[\varepsilon]$. Last, but not least, the set of admitted successors $(hX[\varepsilon]) \cap X[\varepsilon]$ coincides with the set*

$$T^x \cdot (h^{(x)}X[\varepsilon]) \cap X[\varepsilon]).$$

In sum: to calculate the admitted successors of $x + \varepsilon.k$, we may first calculate those of $0 + \varepsilon.k$ with the smaller group of candidates H and translate the results accordingly once finished.

Proof. The first replacement of g is justified by Lemma 2.1. For the second replacement, by Lemma 2.4 we have

$$T^{-x}hX[\varepsilon] = h^{(x)}X[\varepsilon]$$

and using Lemma 2.5 with $z = -x$, we may verify that h is contrapuntal examining $h^{(x)}$ with the interval $T^{-x}\xi = \varepsilon.k$ and the polarity $p^{-x+x}[\varepsilon] = p^0[\varepsilon]$. From Lemma 2.5 we also have

$$h^{(x)}X[\varepsilon] \cap X[\varepsilon] = T^{-x}hX[\varepsilon] \cap X[\varepsilon] = T^{-x}(hX[\varepsilon] \cap X[\varepsilon])$$

and thus

$$hX[\varepsilon] \cap X[\varepsilon] = T^x \cdot ((h^{(x)}X[\varepsilon]) \cap X[\varepsilon])$$

as claimed. □

The next lemma from additive combinatorics is useful to prove a weak version of the so-called *counterpoint theorem*.

Lemma 2.6. *Let S be a finite subset of a cyclic group $\mathbb{Z}_n$ and $u \in \mathbb{Z}_n{}^{\times}$. Then*

$$\sum_{x \in \mathbb{Z}_n} \left| T^x \cdot u(S) \cap S \right| = \left| S \right|^2.$$

Proof. This is a straightforward consequence of the associativity and commutativity of the sum:

$$\begin{aligned}
\sum_{x\in\mathbb{Z}_n}\left|T^x\cdot u(S)\cap S\right| &= \sum_{x\in\mathbb{Z}_n}\sum_{y\in S}\sum_{z\in S}[uz+x=y]\\
&= \sum_{y\in S}\sum_{z\in S}\sum_{x\in\mathbb{Z}_n}[uz+x=y]\\
&= \sum_{y\in S}\sum_{z\in S}\sum_{x'\in\mathbb{Z}_n}[x'=y]\\
&= \sum_{y\in S}\sum_{z\in S}1\\
&= \sum_{y\in S}\left|S\right|\\
&= \left|S\right|^2.
\end{aligned}$$

The third equality follows from the fact that $\mathbb{Z}_n$ being a cyclic group with respect to addition, it is generated by any of its elements. □

2.4.2 Hichert's Algorithm

We are ready now to examine the final details regarding Hichert's algorithm for a more efficient calculation of contrapuntal symmetries. Let $\Delta = (X/Y)$ be a strong dichotomy, $\xi = \varepsilon.k$ with $k \in X$, and the symmetry $g = T^{\varepsilon.t}\cdot(u+\varepsilon.uv) \in H$; without loss of generality, we may choose uv in the linear part of the symmetry because u is invertible. Let us restate the conditions for contrapuntal character of these settings. To begin with, we have

$$\begin{aligned}
gX[\varepsilon] &= \bigcup_{x\in\mathbb{Z}_{2k}} g(x+\varepsilon.X)\\
&= \bigcup_{x\in\mathbb{Z}_{2k}} (ux+\varepsilon.(uvx+t)+\varepsilon.uX)\\
&= \bigcup_{y\in\mathbb{Z}_{2k}} (y+\varepsilon.(vy+t)+\varepsilon.uX) = \bigcup_{y\in\mathbb{Z}_{2k}} (y+\varepsilon.T^{vy+t}u(X)).
\end{aligned}$$

Letting $f(y) = T^{vy+t}u$, we rewrite the latter as

$$gX[\varepsilon] = \bigcup_{y\in\mathbb{Z}_{2k}} (y+\varepsilon.f(y)(X)) \tag{2.5}$$

and

$$(gX[\varepsilon])\cap X[\varepsilon] = \bigcup_{y\in\mathbb{Z}_{2k}} (y+\varepsilon.(f(y)(X)\cap X)). \tag{2.6}$$

Thus, from (2.5) it follows that the first condition for contrapuntality reduces to $k \notin f(0)X$. In other words, $k \in f(0)Y = f(0)p(X)$, which means that

$$k = f(0)p(s) = t + up(s)$$

for some $s \in X$. Hence

$$t \in \{k - up(s) : s \in X\}. \tag{2.7}$$

And thus there exists $s \in X$ such that

$$gX[\varepsilon] = \bigcup_{y \in \mathbb{Z}_{2k}} \left(y + \varepsilon.(T^{vy+k-up(s)} \cdot u(X))\right)$$

and therefore

$$\left|(gX[\varepsilon]) \cap X[\varepsilon]\right| = \sum_{y \in \mathbb{Z}_{2k}} \left|T^{vy+k-up(s)} \cdot u(X) \cap X\right|. \tag{2.8}$$

This expression for the cardinality of the intersection of deformed consonances and consonances is of great help for the estimation of its size.

Lemma 2.7. *Let $g = T^{\varepsilon.t} \cdot (u + \varepsilon.v) \in H$. Then the fact that $p^0[\varepsilon]$ is a polarity of $gX[\varepsilon]$*

$$p^0[\varepsilon]gX[\varepsilon] = gp^0[\varepsilon]X[\varepsilon] = gY[\varepsilon]$$

is equivalent to the commutativity condition

$$p^0[\varepsilon]g = gp^0[\varepsilon].$$

Proof. The sufficiency is obvious. For the necessity, let $\alpha + \varepsilon.\beta$ be any interval with $\beta \in X$. By hypothesis, there exists a $\gamma + \varepsilon.\delta$ with $\delta \in X$ such that

$$p^0[\varepsilon]g(\alpha + \varepsilon.\beta) = gp^0[\varepsilon](\gamma + \varepsilon.\delta).$$

Writing $p = T^r w$, we calculate

$$\begin{aligned} p^0[\varepsilon]g(\alpha + \varepsilon.\beta) &= T^{\varepsilon.r} \cdot w(u\alpha + \varepsilon.(v\alpha + u\beta + t)) \\ &= uw\alpha + \varepsilon.(vw\alpha + uw\beta + wt + r) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} gp^0[\varepsilon](\gamma + \varepsilon.\delta) &= T^{\varepsilon.t}(u + \varepsilon.v)(w\gamma + \varepsilon.(w\delta + r)) \\ &= uw\gamma + \varepsilon.(uw\delta + vw\gamma + ur + t). \end{aligned}$$

We are led to the system

$$\begin{aligned} uw\alpha &= uw\gamma, \\ vw\alpha + uw\beta + wt + r &= uw\delta + vw\gamma + ur + t. \end{aligned} \tag{2.10}$$

That allows us to deduce, using the invertibility of u and w, that

$$\gamma = \alpha,$$
$$\delta = \beta + ut(1-w) + wr(u-1) \in X.$$

Since Δ is strong, we must have $ut(1-w)+wr(u-1) = 0$, since the second equality holds for an arbitrary $\beta \in X$. Hence $\delta = \beta$ and $p^0[\varepsilon]g = gp^0[\varepsilon]$. □

Remark 2.2. *It is important to stress that the previous lemma is false if $g \notin H$. For example, take $g = T^1 \cdot 1$ and the induce polarity $p^0_{(K/D)}[\varepsilon] = T^{\varepsilon.2} \cdot 5$ for the dichotomy (K/D) from Example 2.2. It is clear that $p^0_{(K/D)}[\varepsilon]$ is a polarity of $(gK[\varepsilon]/gD[\varepsilon]) = (K[\varepsilon]/D[\varepsilon])$. Nevertheless*

$$\begin{aligned} g \circ p^0_{(K/D)}[\varepsilon] &= T^1 \cdot 1 \circ T^{\varepsilon.2} \cdot 5 \\ &= T^{1+\varepsilon.2} \cdot 5 \\ &\neq T^{5+\varepsilon.2} \cdot 5 \\ &= T^{\varepsilon.2} \cdot 5 \circ T^1 \cdot 1 = p^0_{(K/D)}[\varepsilon] \circ g. \end{aligned}$$

This lemma yields a quick criterion to check if $gX[\varepsilon]$ is polarized by $p^0[\varepsilon]$. Note that from (2.10) we know that the condition

$$wt + r = ur + t \tag{2.11}$$

must hold, because $\alpha = \gamma$ and $\beta = \delta$.

To finish the description of Hichert's algorithm, we need to analyze more carefully (2.8). If $\rho = \gcd(v, 2k)$, then

$$\left|gX[\varepsilon] \cap X[\varepsilon]\right| = \rho \sum_{j=0}^{\frac{2k}{\rho}-1} \left|(T^{j\rho+k-up(s)} \cdot u(X)) \cap X\right|.$$

When $\rho = 1$, this reduces to

$$\left|gX[\varepsilon] \cap X[\varepsilon]\right| = \sum_{y \in \mathbb{Z}_{2k}} \left|T^y \cdot u.X \cap X\right| = \left|X\right|^2 = k^2. \tag{2.12}$$

by Lemma 2.6.

Now observe that the cardinality of $T^{y-up(s)} \cdot uX \cap X$ cannot exceed $k-1$, because (X/Y) is strong and $y - p(s) \neq 0$, since p is the polarity. Therefore,

$$2k\left|T^{y-up(s)} \cdot u(X) \cap X\right| \leq 2k(k-1).$$

The equation $j\rho + y - p(s) = 0$ has at most one solution in the interval $0 \leq j < 2k/\rho$. This means that

$$\rho \sum_{j=0}^{2k/\rho-1} \left|T^{j\rho+y-up(s)} \cdot u(X) \cap X\right| \leq \rho\Big[\Big(\frac{2k}{\rho}-1\Big)(k-1)+k\Big]$$
$$= \rho + 2k(k-1)$$
$$\leq k + 2k(k-1) = 2k^2 - k.$$

Summarizing, we have the following result.

Theorem 2.3 (Kleiner Kontrapunktsatz, Mazzola [53], Agustín-Aquino [4]). *Let $\Delta = (X/Y)$ be a strong interval dichotomy, and let $\xi \in X[\varepsilon]$. The number N of admitted successors of ξ satisfies*

$$k^2 \leq N \leq 2k^2 - k.$$

Remark 2.3. *Both bounds are tight: The dichotomy*

$$W = (\{0,1,3\}/\{2,4,5\})$$

in $\mathbb{Z}_6$ is strong,[2] *and the number of admitted successors of $\varepsilon.1$ is exactly $15 = 2 \cdot 3^2 - 3$. Its unique contrapuntal symmetry is $g = T^{\varepsilon.3} \cdot (1+\varepsilon.3)$. On the other hand, the dichotomy*

$$X = (\{1,4,5,6,7,8,14,15\}/\{0,2,3,9,10,11,12,13\})$$

in $\mathbb{Z}_{16}$ is such that the number of admitted successors of $\varepsilon.6$ is $64 = 8^2$, and has the impressive number of 105 contrapuntal symmetries (considering that H in this case consist of 128 symmetries).

Example 2.6. The case of $\mathbb{Z}_6$ is an interesting illustration of the generalization from 12 to $2k$, because it represents the whole-tone scale. On the one hand, the restriction of the (K/D) dichotomy to whole-tone intervals yields the sets

$$\overline{K} = \{0,4,8\}, \quad \overline{D} = \{2,6,10\}.$$

The set $\overline{K}$ contains the prime, the major third, and the minor sixth, while the major second, the tritone, and the minor seventh belong to $\overline{D}$. In the whole-tone scale, this corresponds to the dichotomy

$$(\{0,2,4\}/\{1,3,5\}),$$

which, unfortunately, is not strong ($T^3 \cdot 1$ is a non-trivial automorphism). On the other hand, the major second is "consonant" in the aforementioned dichotomy W, and the set of admitted successors for the interval $\varepsilon.1$ (ascending major second over C), is

[2] It is essentially the *only* strong dichotomy in $\mathbb{Z}_6$, except for affine isomorphic images.

$$\begin{aligned} S = gX[\varepsilon] \cap X[\varepsilon] = \{&\varepsilon.0, \varepsilon.3, 1+\varepsilon.0, 1+\varepsilon.1, 1+\varepsilon.3,\\ &2+\varepsilon.0, 2+\varepsilon.3, 3+\varepsilon.0, 3+\varepsilon.1, 3+\varepsilon.3,\\ &4+\varepsilon.0, 4+\varepsilon.3, 5+\varepsilon.0, 5+\varepsilon.1, 5+\varepsilon.3\}, \end{aligned}$$

with the claimed 15 elements. In fact, the only "forbidden" successors (apart from the interval itself) are $2+\varepsilon.1$ (the major second over E) and $4+\varepsilon.1$ (the major second over A♭). In other words, parallel major second progressions producing a "mi contra fa" cross relation are not allowed: C-D going to E-F♯ and C-D going to A♭-B♭.

All of the components of Hichert's algorithm are finally ready.

Algorithm 2.1 (J. Hichert, 1993, [38]). *We calculate the contrapuntal symmetries within H for the intervals $\varepsilon.k \in X[\varepsilon]$. Here $\chi(x,y)$ is a function that outputs the cardinality of the set $(T^x \cdot y(X)) \cap X$.*

Require: The strong dichotomy $\Delta = (X/Y)$ in $\mathbb{Z}_{2n}$ and its polarity $T^r \cdot w$.
Ensure: The set of counterpoint symmetries $\Sigma_k \subseteq H$ for each $\varepsilon.k \in X[\varepsilon]$.

```
1  foreach k ∈ X {
2      M ← 0, Σ_k ← ∅;
3      foreach u ∈ GL(Z_{2n}) {
4          foreach s ∈ X {
5              foreach v ∈ Z_{2n} {
6                  t ← k − u(ws + r);
7                  if wt + r = ur + t {
8                      if v = 0 {
9                          S ← 2nχ(t, u);
10                     }
11                     else if v ∈ GL(Z_{2n}) {
12                         S ← n^2;
13                     }
14                     else {
15                         ρ ← gcd(v, 2n);
16                         S ← ρ Σ_{j=0}^{2n/ρ − 1} χ(jρ + t, u);
17                     }
18                     if S > M {
19                         Σ_k ← {T^{ε.t} · (u + ε.uv)};
20                         M ← S;
21                     }
22                     else if S = M {
23                         Σ_k ← Σ_k ∪ {T^{ε.t} · (u + ε.uv)};
24                     }
25                 }
26             }
27         }
28     }
29  return (Σ_k);
30 }
```

Proof. The election of t in line 6 is justified by (2.7). In line 7 we check that the symmetry polarizes (gX/gY) using (2.11). The lines from 8 to 17 calculate the cardinality of $gX \cap X$ according to (2.12). Afterwards, from line 18 to 21, we update the set of contrapuntal symmetries S_k in a standard way. Since there are a finite number of symmetries to analyze, the algorithm terminates and Σ_k contains all the contrapuntal symmetries within H. □

http://www.springer.com/978-3-319-11235-0

Computational Counterpoint Worlds
Mathematical Theory, Software, and Experiments
Agustin-Aquino, O.A.; Junod, J.; Mazzola, G.
2015, X, 220 p. 57 illus., 16 illus. in color., Hardcover
ISBN: 978-3-319-11235-0