Chapter 2
Partial Asymptotic Stability

Abstract A class of abstract dynamical systems with multivalued flows of solutions in a metric space is introduced in this chapter. For this class of systems, the property of partial asymptotic stability with respect to a continuous functional is studied. In order to characterize the limit set of a trajectory of a multivalued system, a modification of the invariance principle is proposed. This result is applied to derive sufficient conditions for partial asymptotic stability of an equilibrium by using a continuous Lyapunov functional. Such conditions are also formulated for particular classes of systems governed by differential inclusions, ordinary differential equations, and nonlinear semigroups in a Banach space. For further applications of these results to the partial stability analysis of nonlinear abstract differential equations, conditions for the relative compactness of trajectories are derived by considering nonlinear perturbations of dissipative operators. The partial stabilization problem is studied by using differentiable Lyapunov functions for control affine systems in a finite-dimensional space. This treatment is illustrated by examples of the attitude stabilization of a satellite controlled by thrust jets or flywheels.

2.1 Partial Stability of Multivalued Dynamical Systems

Let X be a metric space endowed with the distance $\boldsymbol{\rho} : X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, +\infty)$. The evolution of abstract dynamical processes on X will be described by functions $x(t) \in X$ defined for $t \in \mathbb{R}^+$. We denote by κ the set of all functions

$$x : \mathbb{R}^+ \to X,$$

the set of all subsets of κ is denoted by 2^κ. To introduce the notion of a multivalued dynamical system, we associate a set $\boldsymbol{\pi}(x^0) \subset \kappa$ with each $x^0 \in X$.

Definition 2.1 [1] A map $\boldsymbol{\pi} : X \to 2^\kappa$ is called a *multivalued D-system on X* if:

(A_1) $\boldsymbol{\pi}(x^0) \neq \emptyset$ for all $x^0 \in X$;
(A_2) $x(0) = x^0$ for each $x(\cdot) \in \boldsymbol{\pi}(x^0)$;

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(A_3) for any $x^0 \in X, s \in \mathbb{R}^+, x(\cdot) \in \boldsymbol{\pi}(x^0)$, and $z(\cdot) \in \boldsymbol{\pi}(x(s))$, the following conditions are satisfied: $u(\cdot) \in \boldsymbol{\pi}(x(s))$ and $v(\cdot) \in \boldsymbol{\pi}(x^0)$, where $u(t) = x(t+s)$ and

$$v(t) = \begin{cases} x(t), & t \le s, \\ z(t-s), & t > s; \end{cases}$$

(A_4) given $x^0 \in X$, $\boldsymbol{\varepsilon} > 0$, and $T > 0$, there is a $\boldsymbol{\delta}(x^0, \boldsymbol{\varepsilon}, T) > 0$ such that

$$\boldsymbol{\rho}(\tilde{x}^0, x^0) < \boldsymbol{\delta},\ \tilde{x}(\cdot) \in \boldsymbol{\pi}(\tilde{x}^0) \Rightarrow \inf_{x(\cdot)\in\boldsymbol{\pi}(x^0)} \left(\sup_{t\in[0,T]} \boldsymbol{\rho}(\tilde{x}(t), x(t)) \right) < \boldsymbol{\varepsilon};$$

(A_5) for any $x^0 \in X$, $T > 0$, and a sequence $\{x_n(\cdot)\}_{n=1}^{\infty} \subset \boldsymbol{\pi}(x^0)$, there exists an $x(\cdot) \in \boldsymbol{\pi}(x^0)$ such that

$$\liminf_{n\to\infty} \left(\sup_{t\in[0,T]} \boldsymbol{\rho}(x_n(t), x(t)) \right) = 0.$$

We will refer to an element $x(t)$ of $\boldsymbol{\pi}(x^0)$ as to *a solution of the initial value problem* $x(0) = x^0$ for $\boldsymbol{\pi}$. When considering an autonomous system of differential equations, the above $\boldsymbol{\pi}(x^0)$ represents the set of all solutions for the Cauchy problem on $\mathbb{R}^+$. Assumptions A_1 and A_2 state the global existence property, while A_3 means that the translation of a solution is a solution. Conditions in A_4 and A_5 provide extra regularity properties without assuming the uniqueness of solutions.

Definition 2.2 Let $x(\cdot) \in \boldsymbol{\pi}(x^0)$. An element $q \in X$ is said to be a (*positive*) *limit point of* x if there is a sequence $t_n \to +\infty$ such that $x(t_n) \to q$ as $n \to \infty$. The set of all such limit points is denoted by $\Omega(x)$ and called the (*positive*) *limit set of* x.

Definition 2.3 A set $F \subset X$ is said to be *semi-invariant for* $\boldsymbol{\pi}$ if, for every $x^0 \in F$, there exists at least one $x(\cdot) \in \boldsymbol{\pi}(x^0)$ such that $x(t) \in F$ for all $t \in \mathbb{R}^+$.

Definition 2.4 We say that $x(\cdot) \in \boldsymbol{\pi}(x^0)$ is *precompact* if $\bigcup_{t\ge 0}\{x(t)\}$ is contained in a (sequentially) compact subset of X.

An important property of the limit sets of the autonomous differential equations is that they are invariant (cf. [2, App. III]). The following lemma extends this well-known result for the class of multivalued D-systems on a metric space.

Lemma 2.1 *Let $\boldsymbol{\pi}$ be a multivalued D-system and let $x(\cdot) \in \boldsymbol{\pi}(x^0)$. If the trajectory x is precompact then $\Omega(x)$ is nonempty and semi-invariant.*

Proof The precompactness of $\{x(t)\,|\,t \ge 0\}$ implies that, for any sequence $t_n \to +\infty$, the sequence $\{x(t_n)\}_{n=1}^{\infty}$ has a limit point, therefore, $\Omega(x) \ne \emptyset$.

Let us show that, for each $T > 0$ and $x_0^* \in \Omega(x)$, there is a function $\xi(\cdot) \in \boldsymbol{\pi}(x_0^*)$ satisfying the condition $\xi(t) \in \Omega(x)$ for all $t \in [0, T]$. Since $x_0^* \in \Omega(x)$, there exists a sequence $t_n \to +\infty$ such that $x(t_n) \to x_0^*$ as $n \to \infty$. Define the sequence

$\{\phi_n(\cdot)\}_{n=1}^{\infty} \subset \kappa$: $\phi_n(t) = x(t_n + t)$, $t \in \mathbb{R}^+$. Then $\phi_n(\cdot) \in \boldsymbol{\pi}(x(t_n))$ because of A_3. Let $\{\phi_{n(k)}(\cdot)\}_{k=1}^{\infty}$ be a subsequence of $\{\phi_n(\cdot)\}_{n=1}^{\infty}$ satisfying the condition $\boldsymbol{\rho}(\phi_{n(k)}(0), x_0^*) < \boldsymbol{\delta}_k$, where the numbers $\boldsymbol{\delta}_k = \boldsymbol{\delta}(x_0^*, 1/k, T) > 0$ are chosen as in A_4. By assumption A_4, there is a sequence $\{\boldsymbol{\psi}_k(\cdot)\}_{k=1}^{\infty} \subset \boldsymbol{\pi}(x_0^*)$ such that

$$\sup_{t\in[0,T]} \boldsymbol{\rho}(\phi_{n(k)}(t), \boldsymbol{\psi}_k(t)) < \frac{1}{k}, \quad k = 1, 2, \ldots \tag{2.1}$$

Then A_5 implies that there exist $\xi(\cdot) \in \boldsymbol{\pi}(x_0^*)$ and a subsequence $\{\boldsymbol{\psi}_{k(m)}(\cdot)\}_{m=1}^{\infty}$:

$$\lim_{m\to\infty} \left(\sup_{t\in[0,T]} \boldsymbol{\rho}(\boldsymbol{\psi}_{k(m)}, \xi(t)) \right) = 0.$$

The above formula together with (2.1) imply

$$\lim_{m\to\infty} \left(\sup_{t\in[0,T]} \boldsymbol{\rho}(\phi_{n(k(m))}, \xi(t)) \right) = 0.$$

Since each $\phi_n(\cdot)$ is the translation of $x(\cdot)$, each value of $\xi(t)$ $(0 \le t \le T)$ belongs to $\Omega(x)$.

To conclude the proof, we apply the above construction infinitely many times at the points $x_i^* = \xi_{i-1}(T)$, where $\xi_0(\cdot) = \xi(\cdot)$. As a result, we get the system of functions $\xi_i(\cdot) \in \boldsymbol{\pi}(\xi_{i-1}(T))$ such that $\xi_i(t) \in \Omega(x)$ for all $t \in [0, T]$, $i = 1, 2, \ldots$ Then A_3 implies that the function $x^*(t) = \xi_{[t/T]}(\{t/T\}T)$ is an element of $\boldsymbol{\pi}(x_0^*)$, where $[t/T]$ and $\{t/T\}$ denote the integer and the fractional parts of t/T, respectively. Moreover, $x^*(t) \in \Omega(x)$ for all $t \in \mathbb{R}^+$. □

The limit sets of a dynamical system can be characterized in the terms of a Lyapunov function. A powerful machinery in this area is given by the invariance principle that is valid for the abstract systems on a Fréchet space [3]. We prove here a similar proposition for the D-systems in the sense of Definition 2.1.

Lemma 2.2 *Let $\boldsymbol{\pi}$ be a multivalued D-system, $x(\cdot) \in \boldsymbol{\pi}(x^0)$. Suppose that there exists a continuous map $V : X \to \mathbb{R}^+$ such that $\xi_0 \in X$ and $\xi(\cdot) \in \boldsymbol{\pi}(\xi_0)$ imply $V(\xi(t))$ is non-increasing on $\mathbb{R}^+$. If $x(\cdot)$ is precompact, then*

$$\Omega(x) \subset \{p \in X |\, V(\xi(t)) = c \;\text{ for some }\; \xi(\cdot) \in \boldsymbol{\pi}(p),\; t \in \mathbb{R}^+\} \tag{2.2}$$

for some constant c.

Proof Precompactness of $\bigcup_{t\ge0}\{x(t)\}$ and continuity of V imply that $\Omega(x) \ne \emptyset$ and $V(x(t))$ is bounded on $\mathbb{R}^+$. Since $V(x(t))$ is non-increasing, there exists the limit

$$\lim_{t\to+\infty} V(x(t)) = c \ne -\infty.$$

As V is continuous, $V(x^*) = c$ for any $x^* \in \Omega(x)$. It means that $\Omega(x)$ is a subset of $\{p \in X|\, V(p) = c\}$. But as $\Omega(x)$ is semi-invariant (Lemma 2.1), for any $p \in \Omega(x)$ there should be a $\xi(\cdot) \in \boldsymbol{\pi}(p)$ such that $\xi(t) \in \Omega(x)$. It implies $V(\xi(t)) = c$ for all $t \in \mathbb{R}^+$. $\square$

Our goal is to apply the invariance principle for the analysis of partial asymptotic stability in abstract spaces. To introduce this notion, let us call $x^0 \in X$ an *equilibrium of* $\boldsymbol{\pi}$ if the function $x(t) \equiv x^0$ belongs to $\boldsymbol{\pi}(x^0)$.

Definition 2.5 Let $\boldsymbol{\pi}$ be a multivalued D-system on X, and let $\mu : X \to \mathbb{R}^+$. The equilibrium x^0 of $\boldsymbol{\pi}$ is said to be *asymptotically stable* with respect to μ if

(i) Given $\boldsymbol{\varepsilon} > 0$ there exists $\boldsymbol{\delta}(\boldsymbol{\varepsilon}) > 0$ such that $\boldsymbol{\rho}(\tilde{x}^0, x^0) < \boldsymbol{\delta}$ implies $\mu(\tilde{x}(t)) < \boldsymbol{\varepsilon}$ for all $\tilde{x}(\cdot) \in \boldsymbol{\pi}(\tilde{x}^0)$ and all $t \in \mathbb{R}^+$.
(ii) There exists a $\Delta > 0$ such that $\boldsymbol{\rho}(\tilde{x}^0, x^0) < \Delta$ implies

$$\lim_{t\to\infty} \mu(\tilde{x}(t)) = 0. \tag{2.3}$$

Remark 2.1 The notion of stability with respect to two metrics was introduced by Movčan in the paper [4]. Our approach differs from Movčan's work as we consider the partial stability for multivalued processes here.

To formulate stability results, we introduce the standard class $\mathscr{K}$ of comparison functions that consists of all continuous strictly increasing functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\alpha(0) = 0$. Then we prove the following theorem on sufficient conditions of partial asymptotic stability in terms of a continuous Lyapunov functional V on X.

Theorem 2.1 *Let $\boldsymbol{\pi}$ be a multivalued D-system on a metric space X, and let x^0 be its equilibrium. Assume that there is a pair of continuous functionals $\mu,\ V : X \to \mathbb{R}^+$ satisfying the following conditions.*

C_1. *There exist $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathscr{K}$ such that*

$$\alpha_1(\mu(x)) \le V(x) \le \alpha_2(\boldsymbol{\rho}(x^0, x)) \quad \text{for all} \quad x \in X. \tag{2.4}$$

C_2. *For any $\tilde{x}^0 \in X$, $\tilde{x}(\cdot) \in \boldsymbol{\pi}(\tilde{x}^0)$, the function $V(\tilde{x}(t))$ is non-increasing on $\mathbb{R}^+$.*
C_3. *There exists a $\Delta > 0$ such that $\boldsymbol{\rho}(\tilde{x}^0, x^0) < \Delta$ and $\tilde{x}(\cdot) \in \boldsymbol{\pi}(\tilde{x}^0)$ imply precompactness of $\tilde{x}(\cdot)$.*
C_4. *The set*

$$M_1 = \{p \in X|\ V(\tilde{x}(t)) \text{ is constant on } \mathbb{R}^+$$

$$\text{for some } \ \tilde{x}(\cdot) \in \boldsymbol{\pi}(p)\} \tag{2.5}$$

is contained in

$$\operatorname{Ker} \mu = \{p \in X|\ \mu(p) = 0\}.$$

Then the equilibrium x^0 is asymptotically stable with respect to μ.

Proof First we prove the property (i) from Definition 2.5 by generalizing Rumyantsev's theorem [5, Theorem 5.1], [6] on partial stability with respect to a part of the variables. Then we apply Lemma 2.2 to show the property (ii).

Condition C_2 implies $V(\tilde{x}(t)) \leq V(\tilde{x}^0)$ for all $\tilde{x}^0 \in X, \tilde{x}(\cdot) \in \boldsymbol{\pi}(\tilde{x}^0)$, and $t \in \mathbb{R}^+$. By combining this inequality with (2.4), we get

$$\mu(\tilde{x}(t)) \leq {\alpha_1}^{-1}\left(\alpha_2\left(\boldsymbol{\rho}\left(\tilde{x}^0, x^0\right)\right)\right), \tag{2.6}$$

where the function ${\alpha_1}^{-1}(\tau)$ exists and increases at least for small enough $\tau > 0$, since $\alpha(\cdot) \in \mathscr{K}$. Therefore, the function

$$\gamma(\boldsymbol{\delta}) = {\alpha_1}^{-1}\left(\alpha_2(\boldsymbol{\delta})\right)$$

is continuous, nonnegative, and strictly increasing on some interval $[0, \boldsymbol{\delta}^*)$, $0 < \boldsymbol{\delta}^* \leq +\infty$. It means that for arbitrary $\boldsymbol{\varepsilon} > 0$ there exists $\boldsymbol{\delta} \in (0, \boldsymbol{\delta}^*)$ such that $\gamma(\boldsymbol{\delta}) \leq \boldsymbol{\varepsilon}$. Hence, if $\boldsymbol{\rho}(\tilde{x}^0, x^0) < \boldsymbol{\varepsilon}$ then (2.6) implies

$$\mu(\tilde{x}(t)) \leq \gamma(\boldsymbol{\rho}(\tilde{x}^0, x^0))$$

for all $\tilde{x}(\cdot) \in \boldsymbol{\pi}(\tilde{x}^0)$ and all $t \in \mathbb{R}^+$.

To conclude the proof, it suffices to establish the limit existence for (2.3). Let Δ be chosen as in C_3, and let $\boldsymbol{\rho}(\tilde{x}^0, x^0) < \Delta$. Therefore, for any $\tilde{x}(\cdot) \in \tilde{x}^0$, the set $\Omega(\tilde{x}) \neq \emptyset$ is included in (2.5) because of Lemma 2.2. Condition C_4 implies

$$\Omega(\tilde{x}) \subset \operatorname{Ker} \mu. \tag{2.7}$$

To show (2.3), let us assume the contrary: there are some $\boldsymbol{\varepsilon} > 0$ and $t_n \to +\infty$ as $n \to \infty$ such that

$$\mu(\tilde{x}(t_n)) > \boldsymbol{\varepsilon}, \quad n = 1, 2, \ldots \tag{2.8}$$

Since $\tilde{x}(\cdot)$ is precompact, there exists a subsequence $\{t_{n(k)}\}_{k=1}^{\infty}$ such that $\tilde{x}(t_{n(k)}) \to x^* \in \Omega(\tilde{x})$ as $k \to \infty$. By (2.7), $\mu(x^*) = 0$. Continuity of μ implies

$$|\mu(\tilde{x}(t_{n(k)})) - \mu(x^*)| = \mu(\tilde{x}(t_{n(k)})) \to 0, \quad \text{as } k \to \infty.$$

But the above contradicts to (2.8). Therefore,

$$\lim_{t \to +\infty} \mu(\tilde{x}(t)) = 0$$

for all $\tilde{x}(\cdot) \in \boldsymbol{\pi}(\tilde{x}^0)$ provided that $\boldsymbol{\rho}(\tilde{x}^0, x^0) < \Delta$. $\square$

2.2 Application to Differential Inclusions and Ordinary Differential Equations

An important application of the concept of partial stability for systems with multi-valued flow comes from differential inclusions and Filippov's approach to ordinary differential equations with discontinuous right-hand sides.

2.2.1 Differential Equations with Discontinuous Right-Hand Sides

Consider a system of ordinary differential equations

$$\dot{x}(t) = f(x(t)), \quad x(t) \in X \subseteq \mathbb{R}^n, \tag{2.9}$$

where the function $f : X \to \mathbb{R}^n$ is assumed to be bounded on each compact subset D of domain X. In the sequel, we assume that $0 \in X$ and $f(0) = 0$, so that system (2.9) admits the trivial solution $x(t) \equiv 0$.

It is a well-known fact [7] that classical solutions of system (2.9) may not exist if f is a discontinuous function. To ensure the existence and extendability of solutions to system (2.9), one should use a generalized notion of solutions which is applicable for differential equations with discontinuous right-hand sides. We use here the following definition of solutions due to Filippov.

Definition 2.6 *A solution of system* (2.9) is an absolutely continuous function $x(t) \in X$, defined for $-\infty \le T^- < t < T^+ \le +\infty$, that satisfies the following differential inclusion

$$\dot{x}(t) \in \mathrm{co}\, H(x(t)), \tag{2.10}$$

almost everywhere on $t \in (T^-, T^+)$. For each $x \in X$, the set $H(x) \subset \mathbb{R}^n$ contains $f(x)$ and the set of all limit points of $f(y)$ as $y \to x$. Here $\mathrm{co}\, H(x)$ is the convex hull of $H(x)$.

Definition 2.6 corresponds to the simplest convex definition according to [7, Sect. 4] (see also Proposition 1 in [8, Chap. 2]). It is obvious that each classical solution of (2.9) is also a solution in the sense of Filippov. Definition 2.6 is equivalent to the classical definition of solutions if the function $f : X \to \mathbb{R}^n$ is continuous.

For a given $x^0 \in X$, a solution $x(t)$ to the Cauchy problem for differential inclusion (2.10) with initial data

$$x(0) = x^0 \in X \tag{2.11}$$

may not be unique. For further analysis, we will make an extra assumption.

Assumption 2.1 If $x(t)$ is a solution of differential inclusion (2.10) on $t \in I$ then either $I \supset [0, +\infty)$ or $x(t)$ may be extended to some interval $t \in \tilde{I} \supset [0, +\infty)$.

Under this assumption, we introduce the following set-valued map on X:

$$x^0 \mapsto \boldsymbol{\pi}(x^0) = \{x(\cdot) \mid x(t) \text{ is a solution to the Cauchy problem (2.10), (2.11) on } t \geq 0\}. \tag{2.12}$$

By exploiting regularity results from [7], we show that the above defined map $\boldsymbol{\pi}$ is a multivalued D-system.

Lemma 2.3 *Let the Assumption 2.1 be satisfied. Then the set-valued map $x^0 \mapsto \boldsymbol{\pi}(x^0)$, given by (2.12), is a multivalued D-system on X in the sense of Definition 2.1.*

Proof As the function $f(x)$ is bounded on each compact subset of X, the set-valued function $F(x) = \operatorname{co} H(x)$, introduced in Definition 2.6, is upper semicontinuous by Lemma 1 of [7, Sect. 6]. Hence, $F(x)$ satisfies the *basic conditions* of [7, Sect. 7], i.e. the set $F(x)$ is nonempty, bounded, and closed for all $x \in X$, and $F(x)$ is upper semicontinuous in x. Then, for each $x^0 \in X$, there is a solution $x(t)$, $t \in [0, T^+)$ of the Cauchy problem (2.10), (2.11) according to Theorem 1 of [7, Sect. 7], and $T^+ = +\infty$ by Assumption 2.1. This implies that the set-valued map $\boldsymbol{\pi}(x^0)$, defined by (2.12), satisfies conditions (A_1) and (A_2) of Definition 2.1. Condition (A_3) also holds as $f(x)$ and $F(x)$ do not depend on t. Condition (A_4) is a consequence of Theorem 1 from [7, Sect. 8] on the dependence of solutions on the initial data. Condition (A_5) follows from the fact that the limit of a uniformly convergent sequence of solutions $\{x_n(t)\}$ of differential inclusion (2.10) is a solution of (2.10) (see Corollary 1 of [7, Sect. 7]). Thus, all the conditions (A_1)–(A_5) of Definition 2.1 are satisfied, and the set-valued map $\boldsymbol{\pi}(x^0)$, introduced in (2.12), is a multivalued D-system on X in the sense of Definition 2.1.

To present a finite-dimensional version of Theorem 2.1, we write the state vector x of system (2.9) as

$$x = (y_1, .., y_{n_1}, z_1, \dots, z_{n_2}),\ \ y = (y_1, \dots, y_{n_1}) \in \mathbb{R}^{n_1},\ \ z = (z_1, \dots, z_{n_2}) \in \mathbb{R}^{n_2},$$
$$n_1 + n_2 = n. \tag{2.13}$$

We also assume that X is a domain of form

$$X = \{x \in \mathbb{R}^n \mid z \in \mathbb{R}^{n_2},\ \|y\| < N\} \tag{2.14}$$

where $\|\cdot\|$ is the standard Euclidean norm of a vector and N is a positive constant. For a Filippov solution $x(t)$ of system (2.9), we will refer to its y- and z-components as $y(t)$ and $z(t)$, respectively.

Let us recall the following definition of asymptotic stability with respect to a part of variables in the sense of Lyapunov [9] and Rumyantsev [5].

Definition 2.7 [5, 6] The solution $x = 0$ of system (2.9) is *asymptotically y-stable* if, for any $\boldsymbol{\varepsilon} > 0$, there exists a $\boldsymbol{\delta} > 0$ such that any solution $x(t)$ of (2.9) with $\|x(0)\| < \boldsymbol{\delta}$ is defined on $\mathbb{R}^+$, $\|y(t)\| < \boldsymbol{\varepsilon}$ for all $t \geq 0$, and $\|y(t)\| \to 0$ as $t \to +\infty$.

If we treat solutions of system (2.9) in the sense of A.F. Filippov, then it is easy to see that Definition 2.7 is equivalent to Definition 2.5 with

$$\mu(x) = \|y\|,$$

provided that the set-valued map $x^0 \in X \mapsto \boldsymbol{\pi}(x^0)$ is defined by (2.12).

Remark 2.2 The concepts of weak and strong stability of solutions are usually addressed in the theory of differential inclusions [10]. Definition 2.7 is related to the strong partial stability, so that $\|y(t)\| < \boldsymbol{\varepsilon}$ and $y(t) \to 0$ for *each* solution $x(t)$ whenever $\|x(0)\| < \boldsymbol{\delta}$.

For a differentiable function $V(x)$ in X, its upper time derivative along the trajectories of differential inclusion (2.10) is

$$\dot{V}^*(x) = \sup_{p \in \operatorname{co} H(x)} \langle \nabla V(x), p \rangle,$$

where $\nabla V(x)$ is the gradient of $V(x)$ and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

As a corollary of Theorem 2.1, we obtain the following result.

Theorem 2.2 *Assume that, for some $\Delta > 0$, each Filippov solution $x(t)$ of system (2.9) is bounded for $t \geq 0$ provided that $\|x(0)\| < \Delta$. Let $V \in C^1(\overline{X})$ be a function such that $V(0) = 0$ and the following conditions hold:*

(1) $V(x) \geq \alpha(\|y\|)$ *for some* $\alpha \in \mathscr{K}$*;*
(2) $\dot{V}^*(x) \leq 0$ *for all* $x \in X$*;*
(3) *the set* $M = \{x \mid y = 0\}$ *is invariant for* (2.10) *with* $t \geq 0$*;*
(4) *the set* $\{x \mid \dot{V}^*(x) = 0\} \backslash M$ *does not contain any weakly invariant subset for* (2.10) *with* $t \geq 0$.

Then the solution $x = 0$ of system (2.9) is asymptotically y-stable.

Proof Let $x(t)$, $t \in I$ be a Filippov solution of (2.9) with $\|x(0)\| < \Delta$. Without loss of generality we assume that $I \supset [0, +\infty)$, otherwise, as each solution is bounded, $x(t)$ may be extended to some interval $\tilde{I} \supset [0, +\infty)$ by Theorem 2 of [7, Sect. 7]. Thus, the Filippov solutions of system (2.9) correspond to a multivalued D-system $x^0 \mapsto \boldsymbol{\pi}(x^0)$ on X by Lemma 2.3. The boundedness of the solutions implies that condition C_3 of Theorem 2.1 holds as each bounded subset of a finite dimensional space is precompact. Let us show that condition C_1 of Theorem 2.1 holds with $\alpha_1 = \alpha \in \mathscr{K}$ given in condition (1) and

$$\alpha_2(\boldsymbol{\rho}) = \sup_{\|x\| \leq \boldsymbol{\rho},\, x \in X} V(x).$$

The above defined α_2 is a function of class $\mathscr{K}$ because $V(0) = 0$ and $V(x)$ is continuous. If $x(t)$ is a solution of differential inclusion (2.10) then

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= \lim_{h\to 0}\frac{V(x(t+h)) - V(t)}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\{\langle\nabla V(x(t)), x(t+h) - x(t)\rangle + o(\|x(t+h) - x(t)\|)\}\\ &= \lim_{h\to 0}\left\{\langle\nabla V(x(t)),\, p\rangle_{p\in \mathrm{co}H(x(t))} + o(1)\right\} \le \dot V^*(x(t)) \le 0\end{aligned}$$

almost everywhere on $t \in [0, +\infty)$. This implies that $V(x(t))$ is a non-increasing function of $t \ge 0$, so that condition C_2 of Theorem 2.1 holds.

To prove condition C_4, let us assume the contrary: let $x(t)$, $t \ge 0$ be a solution of differential inclusion (2.10) such that $V(x(t)) \equiv 0$ and $y(\tau) \ne 0$ for some $\tau \ge 0$. The property $V(x(t)) \equiv 0$ together with condition $\dot V^*(x) \le 0$ imply that $x(t) \in M_0$ for all $t \ge 0$, where

$$M_0 = \{x \in X \mid \dot V^*(x) = 0\}.$$

As the set $M = \{x \mid \ y = 0\}$ is invariant [condition (3)], then $y(t) \ne 0$ for all $t \ge 0$ under our assumptions. This implies that $x(t) \in M_0\backslash M$ for all $t \ge 0$ which contradicts condition (4). This contradiction shows that the assumption C_4 is satisfied, so that the equilibrium $x = 0$ of system (2.9) is asymptotically stable with respect to $\mu(x) = \|y\|$ by Theorem 2.1.

Remark 2.3 In order to formulate sufficient conditions of partial stability, it is natural to assume that the solutions are *z-extendable* [5, 6], i.e. if $x(t)$ is a solution of (2.10) for $T^- < t < T^+ < +\infty$ and $z(t) \to \infty$ as $t \to T^+$ then $\|y(t)\| \to N$ as $t \to T^+$. In Theorem 2.2, such z-extendability assumption follows from the boundedness of the solutions.

Remark 2.4 If the right-hand side of system (2.9) is of class $C^1(X)$, then Theorem 2.2 is equivalent to the Risito–Rumyantsev theorem [11], [5, Theorems 19.1–19.2]. Further on, if $n_1 = n$ so that $y = x$ and $\mu(x) = \|x\|$, then Theorem 2.2 is reduced to the Barbashin–Krasovskii theorem [12] on asymptotic stability of the equilibrium $x = 0$.

For a given function $V(x)$ such that $\dot V(x) \le 0$ along the trajectories of system (2.9), the vector of variables y satisfying the property that the solution $x = 0$ of system (2.9) is y-asymptotically stable (in the sense of Definition 2.7) may be defined by using the method of the paper [13].

Remark 2.5 As Theorem 2.2 follows from Theorem 2.1, a part of its proof is actually based on the invariance principle (Lemma 2.2) with a differentiable Lyapunov function $V(x)$. A modification of the invariance principle with a non-smooth Lyapunov function was used for the stability analysis of differential inclusions in the paper [14].

2.3 Stabilization of Finite-Dimensional Systems with Respect to a Part of Variables

Let us consider a control system:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \quad x \in X \subset \mathbb{R}^n,\ u = (u_1, \ldots, u_m) \in \mathbb{R}^m, \tag{2.15}$$

where x is the state vector and u is the control. We assume that X is a domain of form (2.14), $f_i \in C(X)$ and $f_0(0) = 0$, so that system (2.15) admits the trivial solution $x(t) \equiv 0$ with $u = 0$.

Let us remark, that if system (2.15) has an integral then it is neither controllable nor stabilizable. In this case, only partial stabilization may be possible (see, e.g., [6]).

The goal of this section is to develop an effective strategy for partial stabilization of system (2.15) based on Theorem 2.2.

2.3.1 *Theorem on Partial Stabilization*

According to notations (2.13), system (2.15) may be written as

$$\dot{y} = f_{01}(y, z) + \sum_{i=1}^{m} u_i f_{i1}(y, z), \quad \dot{z} = f_{02}(y, z) + \sum_{i=1}^{m} u_i f_{i2}(y, z). \tag{2.16}$$

The functions $f_{ij}(y, z)$ are considered in the domain X. By an admissible feedback for system (2.16) we treat any function $k(x) \in C(X)$ such that $k(0) = 0$. We use the following definition.

Definition 2.8 Control system (2.16) is said to be *y-stabilizable* if there exists an admissible feedback law $u = k(x)$ such that the trivial solution of the corresponding closed-loop system is asymptotically y-stable.

The consideration of continuous feedback laws simplifies the stability analysis with Theorem 2.2 as the sets of classical and Filippov's solutions of the closed-loop system coincide. However, the multivalued framework of Sect. 2.2 is important for our study as systems with merely continuous right-hand sides may not exhibit the uniqueness of solutions.

Let $V(x)$ be a function of class $C^1(X)$. The time derivative of $V(x)$ along the trajectories of system (2.16) is:

$$\dot{V} = a(x) + \langle u, b(x) \rangle,$$

where

$$a(x) = \langle \nabla V(x),\ f_0(x)\rangle,\ \ b_i(x) = \langle \nabla V(x),\ f_i(x)\rangle,\ \ i = 1, 2, \ldots, m,$$
$$b(x) = (b_1(x), b_2(x), \ldots, b_m(x))\,. \tag{2.17}$$

The basic result we shall prove in this section is the following.

Theorem 2.3 *Let $V \in C^1(X)$ be a function such that $V(0) = 0$ and the following conditions hold:*

(1) $V(x) \geq \alpha(\|y\|)$ *for some* $\alpha \in \mathscr{K}$*;*
(2) *the equation* $a(x) + \langle u^0(x), b(x)\rangle = 0$ *has a solution* $u^0 \in C(X)$ *for which the set*

$$M_1 = \{x \mid b(x) = 0,\, y \neq 0\}$$

does not contain any trajectory $\{x(t) \mid t \geq 0\}$ *of system* (2.16) *with the control* $u = u^0(x)$*;*
(3) *there exist a positive number* $\Delta > 0$ *and a function* $h \in C(X)$, $h(x) > 0$ *such that each solution* $x(t)$ *of system* (2.16) *with the initial condition* $\|x(0)\| < \Delta$ *and the feedback control*

$$u = u^0(x) - h(x)b(x) \tag{2.18}$$

has bounded coordinates $z_j(t)$, $1 \leq j \leq n_2$, *for all* $t \geq 0$.

Then the solution $x = 0$ *of the closed-loop system* (2.16) *with* (2.18) *is asymptotically* y*-stable (i.e. system* (2.16) *is* y*-stabilizable).*

Proof By substituting (2.18) into (2.16) and computing $\dot{V}$, we get

$$\dot{V} = -h\,\|b(x)\|^2 \leq 0.$$

The time-derivative $\dot{V}$ vanishes on the following set:

$$M = \{x \mid b(x) = 0\}.$$

It is easy to see that $u(x) = u^0(x)$ for each $x \in M$ if the feedback is given by formula (2.18). Therefore, condition (2) implies that the set $M \setminus \{x : y = 0\}$ does not contain any positive semitrajectory for the closed-loop system (2.16) with (2.18).

Condition (3) together with the inequality $\dot{V} \leq 0$ guarantees the boundedness of the solutions starting from Δ-neighborhood of the origin.

Thus, all the conditions of Theorem 2.2 hold for the closed-loop system (2.16) with (2.18).

In the sequel, we apply Theorem 2.3 for studying a couple of examples of single-axis stabilization [15].

2.3.2 Partial Stabilization of a Rigid Body

Consider a model that describes the rotation of a satellite around its center of mass under the action of attitude control thrust jets (Fig. 2.1). We treat the satellite as a rigid body rotating around its center of mass (fixed point O). Let $Oe_1e_2e_3$ be a basis associated with the rigid body, and let v be a unit vector which is fixed in the inertial frame. The equations of motion we can be written in the Euler–Poisson form as follows [16]:

$$\dot{\omega}_1 = \frac{A_2 - A_3}{A_1}\omega_2\omega_3 + u_1,\ \ \dot{\omega}_2 = \frac{A_3 - A_1}{A_2}\omega_1\omega_3 + u_2,\ \ \dot{\omega}_3 = \frac{A_1 - A_2}{A_3}\omega_1\omega_2, \tag{2.19}$$

$$\dot{v}_1 = \omega_3 v_2 - \omega_2 v_3,\ \ \dot{v}_2 = \omega_1 v_3 - \omega_3 v_1,\ \ \dot{v}_3 = \omega_2 v_1 - \omega_1 v_2. \tag{2.20}$$

Here $\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$ is the angular velocity vector of the rigid body, $v = v_1 e_1 + v_2 e_2 + v_3 e_3$, and A_i is the moment inertia of the rigid body with respect to the axis defined by e_i, $i = 1, 2, 3$. We also assume that the directions of e_i are principal axes of inertia of the body. The action of jet torques is described by control parameters u_1 and u_2.

System (2.19) and (2.20) admits the following particular solution with $u_1 = 0$ and $u_2 = 0$:

$$\omega_1 = \omega_2 = \omega_3 = 0,\ v_1 = v_2 = 0,\ v_3 = 1. \tag{2.21}$$

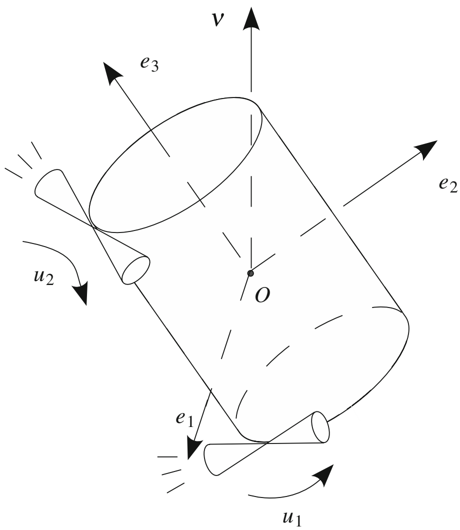

Fig. 2.1 Satellite with attitude control thrust jets

This solution corresponds to the equilibrium for which vectors e_3 and v coincide. Let us remark that solution (2.21) of system (2.19) and (2.20) cannot be made asymptotically stable (with respect all state variables) due to the geometric integral:

$$v_1^2 + v_2^2 + v_3^2 = \text{const}. \tag{2.22}$$

Stabilizability of the Euler equations of form (2.19) has been studied by many authors (see, e.g., [16–18] and references therein). Our investigation is based on the application of Theorem 2.3 in order to stabilize solution (2.21) with respect to the following variables:

$$(\omega_1, \omega_2, v_1, v_2). \tag{2.23}$$

This choice of variables correspond to the stabilization of the third principal axis of inertia (e_3) around the fixed direction v. So, v_1 and v_2 and their derivatives are required to be "small" and tending to zero as $t \to +\infty$, while the other ones are required to be merely bounded.

We define a Lyapunov function candidate as follows:

$$2V = A_1\omega_1^2 + A_2\omega_2^2 + \varkappa(v_1^2 + v_2^2), \quad \varkappa > 0.$$

A straightforward application of formulas (2.17) yields

$$\begin{aligned} a(x) &= (A_2 - A_1)\omega_1\omega_2\omega_3 + \varkappa v_3(\omega_1 v_2 - \omega_2 v_1), \\ b_1(x) &= A_1\omega_1, \ b_2(x) = A_2\omega_2. \end{aligned} \tag{2.24}$$

A particular solution of the equation $a(x) + \langle u^0, b(x)\rangle = 0$ can be taken in the form

$$u_1^0(x) = \omega_2\omega_3 - \frac{\varkappa}{A_1} v_2 v_3, \ u_2^0(x) = -\omega_1\omega_3 + \frac{\varkappa}{A_2} v_1 v_3. \tag{2.25}$$

It is easy to check that all trajectories of system (2.19) and (2.20) with control (2.25) satisfy the condition $v_1 = v_2 = 0$ on the set

$$M_1 : \ A_1\omega_1 = A_2\omega_2 = 0,$$

provided that the initial value is taken in some neighborhood of solution (2.21). This proves that condition (2) of Theorem 2.3 holds.

All solutions of system (2.19) and (2.20) are bounded with respect to v_i because of integral (2.22). So, it suffices to ensure the boundedness of the solutions with respect to ω_3. In order to prove the boundedness, we apply Theorem 39.1 from the monograph [5] with the following function:

$$2W = A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2 + \varkappa(v_1^2 + v_2^2).$$

The time-derivative of $W(x)$ along the trajectories of the closed-loop system with the feedback of form (2.18) is

$$\dot{W}(x) = (A_1 - A_2)\omega_1\omega_2\omega_3 - h(x)\,(A_1^2\omega_1^2 + A_2^2\omega_2^2).$$

According to [5, Theorem 39.1], it is sufficient to show that

$$h(x)\,(A_1^2\omega_1^2 + A_2^2\omega_2^2) \geq (A_1 - A_2)\omega_1\omega_2\omega_3, \quad (h(x) > 0). \tag{2.26}$$

By using the inequality

$$2A_1A_2|\omega_1\omega_2| \leq A_1^2\omega_1^2 + A_2^2\omega_2^2,$$

we define $h(x)$ in the following manner:

$$h(x) = \left|\frac{A_1 - A_2}{2A_1A_2}\omega_3\right| + \boldsymbol{\varepsilon}, \tag{2.27}$$

where $\boldsymbol{\varepsilon}$ is an arbitrary positive constant. Then condition (2.26) holds.

Finally, by taking into account (2.24), (2.25), (2.27), expression (2.18) takes the form:

$$\begin{aligned} u_1 &= \omega_2\omega_3 - \frac{\varkappa}{A_1}v_2v_3 - \Big\{\frac{|A_1 - A_2|}{2A_2}|\omega_3| + \boldsymbol{\varepsilon}A_1\Big\}\omega_1, \\ u_2 &= -\omega_1\omega_3 + \frac{\varkappa}{A_2}v_1v_3 - \Big\{\frac{|A_1 - A_2|}{2A_1}|\omega_3| + \boldsymbol{\varepsilon}A_2\Big\}\omega_2, \quad (\varkappa > 0,\ \boldsymbol{\varepsilon} > 0). \end{aligned} \tag{2.28}$$

Let us remark that the feedback law (2.28) not only stabilizes the solution (2.21) of system (2.19) and (2.20) with respect to variables (2.23), but also ensures Lyapunov stability of the solution (2.21) due to the inequality $\dot{W} \leq 0$ (see Fig. 2.2).

Figure 2.2 illustrates the solution of the closed-loop system (2.19), (2.20), (2.28) for the following parameters[1]:

$$A_1 = 1,\ A_2 = 3/2,\ A_3 = 2,\ \varkappa = 1,\ \boldsymbol{\varepsilon} = 1/10,$$

and initial conditions

$$\omega(0) = 0,\ v_1(0) = 1/\sqrt{3},\ v_2(0) = v_3(0) = 0.$$

We see in Fig. 2.2 that the components $\omega_1(t), \omega_2(t), v_1(t)$, and $v_2(t)$ of this solution of the closed-loop system tend to zero for large t. We also note that the limit motion of the satellite corresponds to uniform rotations around v with constant angular velocity $\omega_3 \neq 0$.

[1] To simplify notations, we assume that all state variables and parameters are dimensionless in this section.

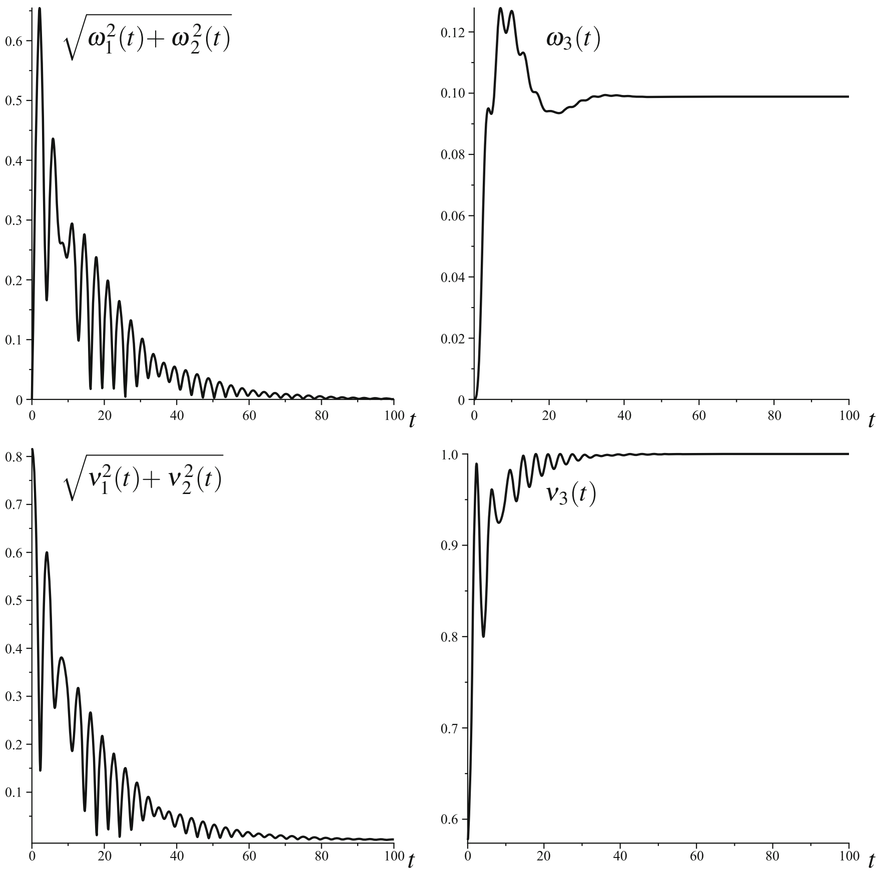

Fig. 2.2 Solution of the closed-loop system (2.19), (2.20) with control (2.28)

2.3.3 A Satellite with Moving Masses

Let us consider a system describing the rotation of a satellite with a pair of flywheels (see Fig. 2.3):

$$
\begin{aligned}
(A_1 - I_1)\dot{\omega}_1 &= (A_2 - A_3)\omega_2\omega_3 + I_2\Omega_2\omega_3 - u_1,\\
(A_2 - I_2)\dot{\omega}_2 &= (A_3 - A_1)\omega_1\omega_3 - I_1\Omega_1\omega_3 - u_2,\\
A_3\dot{\omega}_3 &= (A_1 - A_2)\omega_1\omega_2 + I_1\Omega_1\omega_2 - I_2\Omega_2\omega_1,\\
I_1(\dot{\Omega}_1 + \dot{\omega}_1) &= u_1, \quad I_2(\dot{\Omega}_2 + \dot{\omega}_2) = u_2,\\
\dot{v}_1 = \omega_3 v_2 - \omega_2 v_3,\ \dot{v}_2 &= \omega_1 v_3 - \omega_3 v_1,\ \dot{v}_3 = \omega_2 v_1 - \omega_1 v_2.
\end{aligned} \tag{2.29}
$$

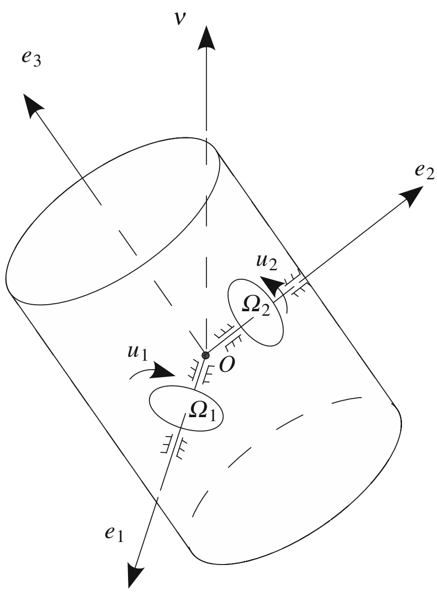

Fig. 2.3 Satellite controlled by a pair of flywheels

As in the previous example, the dynamics of the carrier body is characterized by the angular velocity vector $\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_2 e_2$ and coordinates of the fixed unit vector $v = v_1 e_1 + v_2 e_2 + v_3 e_3$ (see, e.g., [16]). We assume that the ith flywheel rotates around the direction of e_i with relative angular velocity Ω_i under the action of control torque u_i, and the moment of inertia of the ith flywheels is denoted by I_i, $i = 1, 2$ [15]. The control torques u_1 and u_2 are implemented by electric motors. We denote by A_i the moment of inertia of the whole system (i.e. the carrier body and flywheels) with respect to the e_i axis, $i = 1, 2, 3$. We assume that the motion of flywheels does not change the mass distribution in the system, and that A_i are principal moments of inertia.

System (2.29) admits the following equilibria for $u_1 = u_2 = 0$:

$$\omega = 0,\ \Omega_1 = \text{const}, \Omega_2 = \text{const}, v_1 = v_2 = 0,\ v_3 = 1. \tag{2.30}$$

The equilibrium (2.30) cannot be stabilized with respect to all variables, since system (2.29) admits the following integrals:

$$\Phi_1 = (A_1\omega_1 + I_1\Omega_1)^2 + (A_2\omega_2 + I_2\Omega_2)^2 + (A_3\omega_3)^2 = \text{const};$$
$$\Phi_2 = (A_1\omega_1 + I_1\Omega_1)v_1 + (A_2\omega_2 + I_2\Omega_2)v_2 + A_3\omega_3 v_3 = \text{const};$$
$$\Phi_3 = v_1^2 + v_2^2 + v_3^2 = \text{const}.$$

In order to stabilize solution (2.30) of system (2.29) with respect to variables $y = (\omega_1, \omega_2, v_1, v_2)$, we apply Theorem 2.3 with the following Lyapunov function:

$$2V(x) = (A_1 - I_1)\omega_1^2 + (A_2 - I_2)\omega_2^2 + \varkappa(v_1^2 + v_2^2), \quad \varkappa > 0.$$

Then

$$a(x) = (A_2 - A_1)\omega_1\omega_2\omega_3 + (I_2\Omega_2\omega_1 - I_1\Omega_1\omega_2)\omega_3 + \varkappa v_3(\omega_1 v_2 - \omega_2 v_1),$$
$$b_1(x) = -\omega_1, \; b_2(x) = -\omega_2.$$

The function $u^0(x)$ from condition (2) of Theorem 2.3 is a solution of the following algebraic equation:

$$\big\{(A_2\omega_2 + I_2\Omega_2)\omega_3 + \varkappa v_2 v_3 - u_1^0\big\}\omega_1 - \big\{(A_1\omega_1 + I_1\Omega_1)\omega_3 + \varkappa v_1 v_3 + u_2^0\big\}\omega_2 = 0.$$

To satisfy this equation, we assume

$$u_1^0 = \varkappa v_2 v_3 + (A_2\omega_2 + I_2\Omega_2)\omega_3, \; u_2^0 = -\varkappa v_1 v_3 - (A_1\omega_1 + I_1\Omega_1)\omega_3. \quad (2.31)$$

It can be seen that the set

$$M_1 = \{(\omega_1, \omega_2, \omega_3, \Omega_1, \Omega_2, v_1, v_2, v_3) \mid \omega_1 = \omega_2 = 0, v_1^2 + v_2^2 \neq 0\}$$

does not contain any positive semi-trajectory of system (2.29) with the feedback law $u = u^0(x)$ in a neighborhood of (2.30).

Let us assume $h(x) = \boldsymbol{\varepsilon} = \text{const}$ and show that the feedback law

$$\begin{aligned} u_1 &= \varkappa v_2 v_3 + (A_2\omega_2 + I_2\Omega_2)\omega_3 + \boldsymbol{\varepsilon}\omega_1, \\ u_2 &= -\varkappa v_1 v_3 - (A_1\omega_1 + I_1\Omega_1)\omega_3 + \boldsymbol{\varepsilon}\omega_2. \end{aligned} \quad (2.32)$$

satisfied condition (3) of Theorem 2.3 for any $\varkappa > 0$ and $\boldsymbol{\varepsilon} > 0$. Indeed, the boundedness of the solutions with respect to variables $(\omega_3, \Omega_1, \Omega_2, v_3)$ follows from integrals Φ_1 and Φ_3 of system (2.29).

Thus, the solution (2.30) of system (2.29) is stabilizable with respect to variables (2.23) by means of the feedback law (2.32) by Theorem 2.3.

In order to illustrate the proposed stabilization scheme, we perform a numerical integration of the closed-loop system (2.29), (2.32) with the following parameters:

$$A_1 = 2, \; A_2 = 3, \; A_3 = 4, \; I_1 = I_2 = 1, \; \boldsymbol{\varepsilon} = 1/10, \; \varkappa = 1.$$

Time-plots of the solution of the closed loop-system are shown in Figs. 2.4 and 2.5 for the initial conditions

$$\omega(0) = 0, \; \Omega_1(0) = \Omega_2(0) = 0, \; v_1(0) = v_3(0) = 0, \; v_2(0) = -1.$$

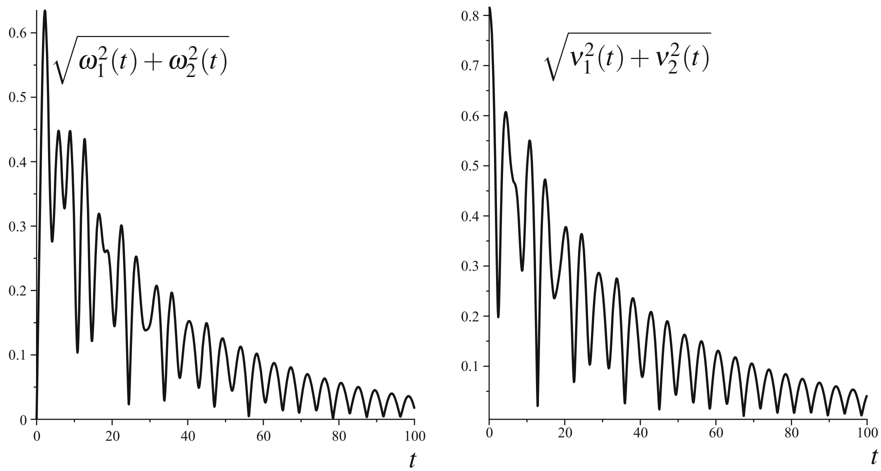

Fig. 2.4 Solution components of the closed-loop system (2.29) with control (2.32)

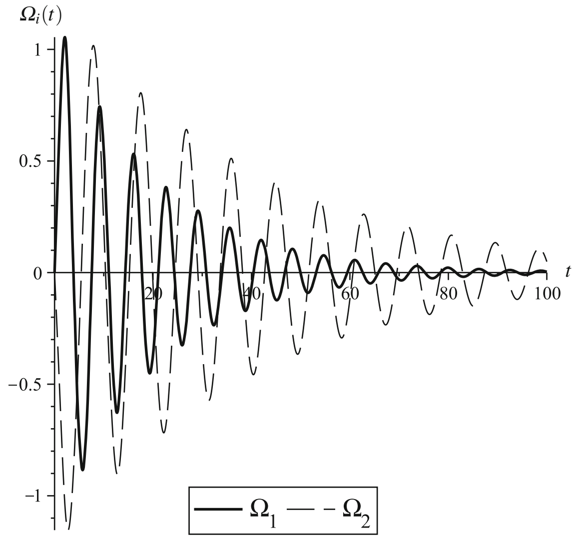

Fig. 2.5 Time plots of $\Omega_1(t)$ and $\Omega_2(2)$

Figures 2.4 and 2.5 confirm that the feedback law (2.32) stabilizes the system with respect to variables (2.23). We also observe that the limit position of the carrier body approximately corresponds to its equilibrium with $e_3 = \nu$, and the flywheels perform rotations with decaying angular velocities $\Omega_1(t)$ and $\Omega_2(t)$.

2.4 Partial Asymptotic Stability of Nonlinear Semigroups

In Sect. 2.1, we have obtained a general result on partial asymptotic stability without assuming the uniqueness of solutions as well as the differentiability of a Lyapunov functional. To derive more convenient stability conditions for the analysis of distributed parameter systems, let us consider a class of dynamical systems governed by differential equations in a Banach space.

Let E be a Banach space with the norm $\|\cdot\|$, and let X be its closed subset containing some ball $B_R = \{x \in E \mid \|x\| \leq R\}$ of radius $R > 0$. Then X is a metric space with respect to the distance $\boldsymbol{\rho}(a, b) = \|a - b\|$. Let F be a (nonlinear) operator from $D(F) \subset X$ into E, F is supposed to be closed and densely defined on X. Given $x^0 \in X$ consider the abstract Cauchy problem (cf. [19, Chap. 4], [20, Sect. 5.2]) for F with initial data x^0:

$$\frac{dx(t)}{dt} = Fx(t), \quad t \in \mathbb{R}^+, \; x(0) = x^0. \tag{2.33}$$

We assume that the operator F generates a *nonlinear continuous semigroup* on X in the sense of Definition 1.9.

As F is the infinitesimal generator of a continuous semigroup $\{S(t)\}$, the Cauchy problem (2.33) is well-posed, and any mild solution of (2.33) is given by

$$x(t) = S(t)x^0, \quad t \in \mathbb{R}^+, \; x^0 \in X.$$

In that case, at each x^0 we may associate the singleton $\boldsymbol{\pi}(x^0) = \{S(\cdot)x^0\}$. It is easy to check that the above defined $\boldsymbol{\pi} : X \to 2^{\mathcal{K}}$ is a multivalued D-system in the sense of Definition 2.1. (Assumption A_5 is satisfied by uniqueness of the solutions and A_4 is a consequence of continuity of the map $(t, x) \mapsto S(t)x$.)

Let $V : E \to \mathbb{R}$ be a differentiable functional, then $V(S(t)x^0)$ is differentiable along any classical solution of (2.33). The time-derivative of V along the trajectories of (2.33) at $x \in X$ is defined as

$$\dot{V}(x) = \lim_{t \to +0} \frac{V(S(t)x) - V(x)}{t}.$$

The above expression can also be written in the terms of vector fields for $x \in D(F)$:

$$\dot{V}(x) = (DV(x), Fx), \tag{2.34}$$

where $(\cdot, \cdot) : E^* \times E \to \mathbb{R}$ is the duality pairing of E and E^*, i.e. $(DV(x), \xi)$ is the value of a linear functional $DV(x) \in E^*$ at $\xi \in E$. When E is a Hilbert space, (2.34) takes the form

$$\dot{V}(x^0) = \langle \nabla V(x), Fx \rangle.$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product in E, ∇ denotes the gradient.

As a consequence of Theorem 2.1 and the regularity of V, we have

Theorem 2.4 *Let F be the infinitesimal generator of a nonlinear continuous semigroup $\{S(t)\}$ on X, $F(0) = 0$, and let $\mu : X \to \mathbb{R}^+$ be a continuous functional. Assume that there exists a differentiable functional $V : E \to \mathbb{R}$ satisfying the following conditions.*

(i) *There exist $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathscr{K}$ such that*

$$\alpha_1(\mu(x)) \le V(x) \le \alpha_2(\|x\|) \quad \text{for all} \quad x \in X.$$

(ii) $\dot{V}(x) \le 0$, *for all $x \in D(F)$.*
(iii) *There exists $\Delta > 0$ such that*

$$\bigcup_{t \ge 0} \{S(t)x^0\}$$

is precompact in X provided that $\|x^0\| < \Delta$.
(iv) $\operatorname{Ker} \mu = \{x \in X \mid \mu(x) = 0\}$ *is invariant for (2.33), i.e. $\mu(S(\tau)x^0) = 0$ and $\tau \ge 0$ imply $\mu(S(t)x^0) = 0$ for all $t \in \mathbb{R}^+$.*
(v) *The set*

$$M = \overline{\{x \in D(F) \mid \dot{V}(x) = 0\}} \setminus \operatorname{Ker} \mu$$

does not contain any semitrajectory of (2.33) defined for $t \in \mathbb{R}^+$.

Then the equilibrium $x^0 = 0$ of (2.33) is asymptotically stable with respect to μ.

Proof As $F(0) = 0$, the solution $x(t) \equiv 0$ is an equilibrium of the multivalued D-system defined by $x^0 \in X \mapsto \boldsymbol{\pi}(x^0) = \{S(\cdot)x^0\}$. It is easy to see that (i) and (iii) imply C_1 and C_3 in conditions of Theorem 2.1.

Let us prove that condition (ii) implies that $V(x(t))$ is non-increasing on any mild solution of (2.33) with $x^0 \in X$, $t \in \mathbb{R}^+$. If $x^0 \in D(F)$ then $x(t) = S(t)x^0$ is a classical solution, and $\dot{V}(x(t))$ given by formula (2.34) exists for all $t \ge 0$. As $V(x(t))$ is continuous and $\dot{V}(x(t)) \le 0$ on $\mathbb{R}^+$, the function $V(x(t))$ is non-increasing on $\mathbb{R}^+$. For arbitrary $x^0 \in X \setminus D(F)$ and $T > 0$, the mild solution $S(t)x^0$ $(0 \le t \le T)$ can be approximated by classical ones in the $L^\infty([0, T]; E)$ norm (it is a consequence of assumption A_4). Therefore, as V is non-increasing along any classical solution and V is continuous, $V(S(t)x^0)$ is non-increasing on $\mathbb{R}^+$ for each $x^0 \in X$.

To finish the proof, let us show that C_4 holds under our assumptions. If $p \in M_1$ in (2.5) then $\frac{d}{dt}V(S(t)p) = 0$ for all $t \in \mathbb{R}^+$. Therefore,

$$M_1 \subset M_0 = \overline{\{x \in D(F) \mid \dot{V}(x) = 0\}}.$$

(The closure in M_0 is taken because (2.34) defines $\dot{V}$ on $D(F)$ only, but F is densely defined.) On the other hand, as M_1 is semi-invariant,

$$M_1 \subset \{x \in M_0 \mid S(t)x \in M_0 \quad \text{for all} \quad t \in \mathbb{R}^+\}. \tag{2.35}$$

Suppose that x be an element of the right-hand side of (2.35). The kernel invariance (iv) implies either $S(t)x \in \operatorname{Ker}\mu$ or $S(t)x \notin \operatorname{Ker}\mu$ for all $t \in \mathbb{R}^+$. But the last case is impossible because of (v). Therefore, any $x \in M_1$ belongs to $\operatorname{Ker}\mu$ that proves C_4. □

Remark 2.6 The above theorem can be applied for studying strong asymptotic stability when $\mu(x) = \|x\|$. If F is linear then Definition 1.9 is equivalent to that of C_0-semigroups of linear bounded operators. Therefore, the assumption of Theorem 2.4 regarding the semigroup $\{S(t)\}$ can be checked via the Hille–Yosida or the Lumer–Phillips theorems for linear operators F (cf. [19, Chap. 1]). In more general case, there is a close relationship between quasicontractive semigroups (which are jointly continuous) and ω-accretive operators F [21]. If $\omega = 0$ then the generator of $\{S(t)\}$ is dissipative (cf. [22, Sect. 2.9]). The compactness assumption (iii) can be checked by the method of [23] for a class of monotone operators. We extend this approach for the case of bounded perturbations of C_0-semigroups in the next section.

2.5 Relative Compactness of Trajectories in a Banach Space

In order to apply Theorem 2.4, it is necessary to check the relative compactness of trajectories for a differential equation in a Banach space. One could observe that the accretivity condition (monotonicity), which is a crucial assumption of the paper [23], is violated for some important classes of flexible systems. In particular, the infinitesimal generator of the nonlinear system considered in [24] is not monotone. This fact stimulates the development of new tools for the analysis of the compactness for trajectories of distributed parameter systems. This section provides some compactness results based on a priori estimates of perturbations for a differential equation in a Banach space.

Let E be a real Banach space, and let $A : D(A) \to E$ be a closed linear operator with the domain of definition of $D(A) \subset E$. Consider the abstract Cauchy problem for $t \in [0, +\infty)$:

$$\dot{x}(t) = Ax(t),\ x(0) = x_0 \in E. \tag{2.36}$$

We assume that the domain $D(A)$ is dense in E, and that A is the infinitesimal generator of a C_0-semigroup of linear operators $\{e^{tA}\}_{t\ge 0}$ in E. So, the Cauchy problem (2.36) is well posed for $t \in [0, +\infty)$, and its mild solution can be represented in the form

$$x(t) = e^{tA}x_0,\ t \ge 0. \tag{2.37}$$

In order to study a wider class of equations (including the one with non-monotone operators), we introduce the perturbed Cauchy problem for $t \ge 0$ as follows:

$$\dot{x} = Ax + f(t)R(x, t),\ x(0) = x_0 \in E, \tag{2.38}$$

where $f : [0, +\infty) \to \mathbb{R}$ and $R : E \times [0, +\infty) \to E$ are continuous mappings.

We prove that the compactness property is preserved by passing from Eqs. (2.36)–(2.38) under some additional assumptions on the function f and the mapping R. This result will be applied to derive sufficient conditions for the compactness of trajectories of an autonomous differential equation in a Banach space.

2.5.1 Compactness Lemmas

Assume that the Banach space E has a basis $\{e_i\}$ $(i = 1, 2, \ldots)$. We denote by $\{f_j\} \subset E^*$ $(j = 1, 2, \ldots)$ the conjugate system of bounded linear functionals, i.e. $f_j(e_i) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Then, for each $x \in E$, $n \in \mathbb{N}$, we define the linear projection operators:

$$S_n(x) = \sum_{i=1}^{n} f_i(x)e_i, \;\; P_n(x) = x - S_n(x).$$

As $\{e_i\}$ is a basis then the operators $S_n : E \to E$ are uniformly bounded:

$$\|S_n\| \le M < \infty, \quad n = 1, 2, \ldots$$

To describe the compact subsets of E, we formulate two auxiliary results.

Lemma 2.4 *Let $\{e_i\}$ be a basis of E. A bounded subset $C \subset E$ is relatively compact in E iff*

$$\lim_{n\to\infty} \sup_{x\in C} \|P_n x\| = 0. \tag{2.39}$$

Proof If C is precompact then the Hausdorff compactness criterion implies that, for any $\varepsilon > 0$, there exists a finite $\frac{\varepsilon}{2M}$-net $\{x^{(j)}\}$, $j = 1, 2, \ldots, m(\varepsilon)$ [25]. It means that, for each $x \in C$, there is a $j \le m(\varepsilon)$ such that $\|x - x^{(j)}\| < \frac{\varepsilon}{2M}$, i.e.

$$\|P_n x\| = \|P_n(x - x^{(j)}) + P_n x^{(j)}\| \le \|P_n\|\frac{\varepsilon}{2M} + \|P_n x^{(j)}\| < \frac{\varepsilon}{2} + \|P_n x^{(j)}\|. \tag{2.40}$$

We now show that, for sufficiently large n, the inequality $\|P_n x^{(j)}\| < \frac{\varepsilon}{2}$ holds for each $j = 1, 2, \ldots, m(\varepsilon)$. In fact, since $\{e_i\}$ is a basis, each element of the net is represented by a convergent series:

$$x^{(j)} = \sum_{i=1}^{\infty} c_i^{(j)} e_i. \tag{2.41}$$

According to the definition of P_n,

$$\|P_n x^{(j)}\| = \|\sum_{i=n+1}^{\infty} c_i^{(j)} e_i\|.$$

The last expression does not exceed $\frac{\varepsilon}{2}$, starting with some index $n = n(\varepsilon)$, since the series in (2.41) is convergent in the norm of E. Thus, (2.40) implies $\|P_n x\| < \varepsilon$ for $n \geq n(\varepsilon)$, which proves (2.39).

Conversely, if the set C is bounded, then all finite-dimensional projection

$$C_n = \{S_n x \mid x \in C\}, \quad n = 1, 2, \ldots$$

are precompact. Relation (2.39) implies that, for each $\varepsilon > 0$, any finite ε-net of the set C_n can be used to cover C for sufficiently large n.

We will call a C_0-semigroup of linear operators $\{e^{tA}\}_{t \geq 0}$ in E *uniformly bounded* if [19]:

$$\|e^{tA}\| \leq N, \quad \forall t \geq 0,$$

with some constant $N < \infty$.

Lemma 2.5 *Assume that $\{e_i\}$ is a basis in E, C is a compact subset of E, and $\{e^{tA}\}_{t \geq 0}$ is a uniformly bounded C_0-semigroup of linear operators in E, for which the trajectory $\gamma(x_0) = \{e^{tA} x_0 \mid t \geq 0\}$ is precompact for any $x_0 \in C$. Then*

$$\lim_{n \to \infty} \left(\sup_{t \geq 0, x \in C} \|P_n e^{tA} x\| \right) = 0. \tag{2.42}$$

Proof According to Lemma 2.4, to prove (2.42) is suffices to establish that the set

$$K = \{e^{tA} \mid x \in C, t \geq 0\}.$$

is precompact. Let $\{y_n\}$ be a sequence of elements of K, i.e. $y_n = e^{t_n A} x_n$ for some $\{t_n\} \subset [0, +\infty)$, $\{x_n\} \subset C$, $n = 1, 2, \ldots$ The compactness of C implies the existence of a convergent subsequence $x_{n(k)} \to x^* \in C$ as $k \to \infty$. As $\gamma(x^*)$ is precompact then there exists a convergent subsequence $e^{t_{n(k(m))} A} x^* \to y^* \in E$ as $m \to \infty$. By using the uniformly bounded semigroup $\{e^{tA}\}_{t \geq 0}$, we conclude that $e^{t_{n(k(m))} A} x_{n(k(m))} \to y^*$ as $m \to \infty$.

2.5.2 Trajectories of the Perturbed System

Let us recall that *a mild solution* of the inhomogeneous problem (2.38) on $0 \leq t < T \leq +\infty$ is a continuous function $x : [0, T) \to E$ that satisfies the integral equation [19]:

$$x(t) = e^{tA}x_0 + \int\limits_0^t e^{(t-s)A} f(s) R(x(s), s)\, ds. \tag{2.43}$$

The integral in formula (2.43) in treated the sense of Bochner. We formulate the following sufficient condition for the precompactness of trajectories to the perturbed differential equation.

Theorem 2.5 *Let E be a Banach space with a basis, A be the infinitesimal generator of a uniformly bounded C_0-semigroup of linear operators $\{e^{tA}\}_{t\geq 0}$ in E, $f \in L^1[0, +\infty)$, $R(x, t) \in K$ for all $x \in E$, $t \geq 0$, and K is a compact set. Assume that the set $\{e^{tA}y \mid t \geq 0\}$ is precompact for all $y \in K \cup \{x_0\}$.*

Then each mild trajectory $\{x(t) \mid t \geq 0\}$ of (2.38) is contained in a compact subset of E.

Proof Let $x(t)$ be a mild solution of (2.38) on the semi-interval $t \geq 0$. Then the integral equation (2.43) implies the compactness of $\{e^{tA}x_0 \mid t \geq 0\}$, and condition $f \in L^1[0, +\infty)$ together with $R \in K$ provides the boundedness of $x(t)$. According to Lemma 2.4, to prove the precompactness of $\{x(t) \mid t \geq 0\}$ is suffices to choose a basis $\{e_i\}$ in E and establish the existence of the limit

$$\lim_{n\to\infty} \sup_{t\geq 0} \|P_n x(t)\| = 0.$$

Applying the projection operator to (2.43), we get

$$\begin{aligned} \|P_n x(t)\| &\leq \|P_n e^{tA}x_0\| + \left\| \int\limits_0^t f(s) P_n \left(e^{(t-s)A} R(x(s), s)\right) \mathrm{d}s \right\| \\ &\leq \|P_n e^{tA}x_0\| + \|f\|_{L^1} \cdot \sup_{s\in[0,t],\, y\in K} \|P_n e^{sA} y\|. \end{aligned}$$

The proof is completed by applying Lemmas 2.4 and 2.5.

A certain class of autonomous differential equations with nonlinear infinitesimal generators can be transformed to the form (2.38) with an appropriate assumption on the function $f(t)$. We state the main result in this direction for the following abstract Cauchy problem:

$$\dot{x}(t) = Ax(t) + h(x(t))B(x(t)), \quad x(0) = x_0 \in E, \tag{2.44}$$

where $h : E \to \mathbb{R}$ and $B : E \to E$ are locally Lipschitz mappings. Recall that the mappings $h(x)$ and $B(x)$ are called *locally Lipschitz* if, for every $r \geq 0$, there exists a constant $L(r)$ such that

$$|h(x) - h(y)| \leq L(r)\|x - y\|, \; \|B(x) - B(y)\| \leq L(r)\|x - y\|$$

for all $\|x\| \leq r$, $\|y\| \leq r$. If $w : E \to \mathbb{R}$ is a Fréchet differentiable functional, then the function of time $w(x(t))$ is differentiable along each classical solution $x(t)$ to the problem (2.44). Then, for any $x \in D(A) \subset E$, the time-derivative of w along the trajectories of (2.44) can be written as

$$\dot{w}(x) = (Ax + B(x)h(x), \nabla_x w),$$

where $(\cdot, \cdot) : E \times E^* \to \mathbb{R}$ is the duality pairing of E and E^*, i.e. $(\xi, \nabla_x w)$ is the value of the linear functional $\nabla_x w \in E^*$ at $\xi \in E$.

Theorem 2.6 *Assume that E is a Banach space with a basis, A is the infinitesimal generator of a uniformly bounded C_0-semigroup of linear operators $\{e^{tA}\}_{t \geq 0}$ in E, the set $\{e^{tA} y \mid t \geq 0\}$ is precompact for all $y \in E$, and $B : E \to E$ is a compact operator. Assume, moreover, that $w : E \to \mathbb{R}$ is a Fréchet differentiable functional that satisfies the following conditions:*

(1) *the set $M_c = \{x \mid w(x) \leq c\}$ is bounded for each $c \in \mathbb{R}$;*
(2) $\inf_{\|x\| \leq r} w(x) > -\infty$ *for all $r > 0$;*
(3) *there exists a constant $k_1 > 0$ such that*

$$\dot{w}(x) \leq k_1 h(x) \leq 0, \quad \forall x \in D(A).$$

Then, for each $x_0 \in E$, the Cauchy problem (2.44) has the unique solution $x(t)$ on $[0, +\infty)$, and $\{x(t) \mid t \geq 0\}$ is precompact in E.

Proof According to Theorem 1.4 [19], for each $x_0 \in E$, there exists a unique maximal mild solution $x(t)$ of the problem (2.44), $t \in [0, t_{max})$. Conditions (1) and (3) imply that $x(t)$ is bounded, hence, $t_{max} = +\infty$. Let us consider equation (2.38) with $R(x, t) = B(x)$ and $f(t) = h(x(t))$. Then conditions (2) and (3) yield the property $f \in L^1[0, +\infty)$. Thus, the trajectory $\{x(t) \mid t \geq 0\}$ is precompact in E by Theorem 2.5.

Note that, since the set M_c is forward invariant under the condition $\dot{w}(x) \leq 0$, then Theorem 2.6 admits a local formulation on the subset of E located between level surfaces of the functional w.

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