Chapter 2
Radiation

In this chapter, we discuss the basic principles of electromagnetic radiation due to simple and distributed sources.

2.1 General Considerations

An antenna is a device used to propagate or to capture electromagnetic waves. When an antenna is used for transmission (propagation) of radio waves, electric currents are made to oscillate over the antenna. Energy from this oscillating charge is emitted into space as electromagnetic radio waves. When an antenna is used for reception, these waves induce a weak electric current in the antenna. This current is amplified by the radio receiver. An antenna can generally be used for reception and transmission on the same wavelength.

Electric energy is fed to an antenna by means of a transmission line, or a coaxial cable. In reflector antennas, microwave energy is reflected from a metallic paraboloid that shapes it into a narrow beam.

The dimensions of an antenna usually depend on the wavelength, or frequency, of the radio wave for which the antenna is designed. The length of an antenna must be such that it resonates electrically at the desired wavelength. The basic antenna length must be at least half the wavelength of the radio waves it is designed to transmit or receive. It can also be an integral multiple of the one-half wavelength. Antennas with such dimensions are called resonant antennas. A resonant antenna is an efficient propagator and receptor of electromagnetic energy at its design wavelength.

Let a source distribution ($\mathbf{J}$, ρ) be confined in a region V in free space. We have seen that the Hertz vector potential at $\mathbf{r}$ due to $\mathbf{J}$ satisfies

$$\nabla^2 \pi + k_0^2 \pi = -\frac{\mathbf{J}}{j\omega\epsilon_0} \tag{2.1}$$

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so that

$$\pi(\mathbf{r}) = -j\frac{Z_0}{k_0}\int\limits_V \mathbf{J}(\mathbf{r}')\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}dv' \tag{2.2}$$

where k_0 is the free space wavenumber. The electromagnetic fields are then given by

$$\mathbf{E} = \nabla\nabla\cdot\pi + k_0^2\pi \tag{2.3}$$

$$\mathbf{H} = j\omega\epsilon_0\nabla\times\pi \tag{2.4}$$

We begin our discussion of the radiating systems by elementary sources, and then extend the concepts to distributed sources.

2.2 Elementary Sources

In this section, we are concerned with simple sources which are *small* in extent. For these sources, the maximum dimension of the source is much less than free space wavelength. We shall assume that the point of observation or the field point P is located at a large distance from the source. This implies that

$$k_0 r' \ll \lambda \tag{2.5}$$

$$r' \ll r \tag{2.6}$$

Note that no restriction is placed on the order of magnitude of r in comparison with the wavelength. Later on, we define various field regions based on the magnitude of r with respect to the wavelength.

We now modify (2.2) on the basis of the approximations represented by (2.5) and (2.6). Referring to Fig. 2.1, we have

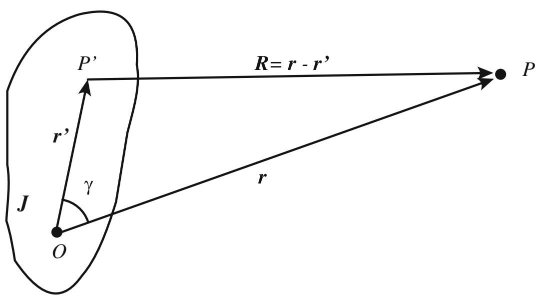

Fig. 2.1 An elementary current source radiating in free space

$$|\mathbf{r}-\mathbf{r}'| = R \simeq r - r'\cos\gamma \tag{2.7}$$

where γ is the included angle between $\mathbf{r}$ and $\mathbf{r}'$. That is

$$\cos\gamma = \widehat{r}\cdot\widehat{r'} \tag{2.8}$$

Also, since $k_0 r'\cos\gamma \ll 1$, we have

$$e^{-jk_0|\mathbf{r}-\mathbf{r}'|} \simeq e^{-jk_0 r}(1 + jk_0 r'\cos\gamma) \tag{2.9}$$

and

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \simeq \frac{1}{r}(1 + \frac{r'}{r}\cos\gamma) \tag{2.10}$$

Using the above, (2.2) can now be written as

$$\pi(\mathbf{r}) \simeq -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}\int_V \mathbf{J}(\mathbf{r}')[1 + (jk_0 + \frac{1}{r})r'\cos\gamma]dv' \tag{2.11}$$

Note that although the second term within the bracket is small compared with unity, we have retained it in (2.11), because in some cases the integral $\int_V \mathbf{J}(\mathbf{r}')dv'$ is zero in which case $\int_V \mathbf{J}(\mathbf{r}')(jk_0 + \frac{1}{r}r'\cos\gamma dv')$ would be the leading term. If $\int_V \mathbf{J}(\mathbf{r}')dv'$ is not zero, then we generally neglect the term involving r' in (2.11).

2.2.1 *The Short Electric Dipole*

Consider a small linear current element of length ℓ carrying a constant current I_0 (actually $I_0 e^{j\omega t}$), and oriented along the $\widehat{z}$ direction (Fig. 2.2). For such a current element

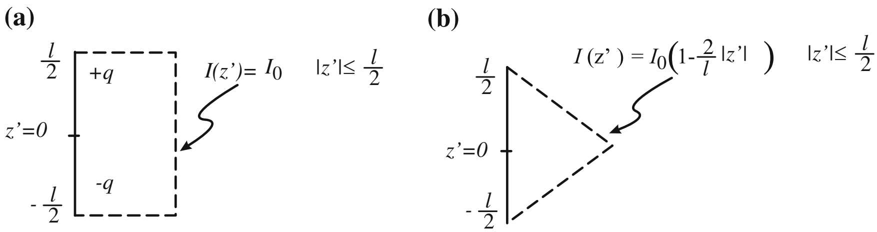

Fig. 2.2 $\widehat{z}$-directed dipoles located at the origin **a** a short Hertzian dipole, **b** an Abraham dipole

$$I(z) = I_0, \quad |z| \leq \ell/2 \tag{2.12}$$

We define the total current moment as

$$\mathbf{p}_i = \int_v \mathbf{J}(\mathbf{r}')dv' = \widehat{z} \int_{-\ell/2}^{\ell/2} I(z')dz' = I_0 \ell \widehat{z} \tag{2.13}$$

Consider now two time-varying charges $\pm q$ separated by a distance ℓ and placed along the z-axis. For small ℓ, this represents a time varying dipole with moment $\mathbf{p}$ defined as

$$\mathbf{p} = q\ell\widehat{z} \tag{2.14}$$

Substituting $I_0 = j\omega q$, we have

$$\mathbf{p} = \frac{I_0 \ell}{j\omega}\widehat{z} \tag{2.15}$$

which indicates that the current moment is related to the dipole moment by

$$\mathbf{p}_i = j\omega \mathbf{p} \tag{2.16}$$

We previously defined the polarization vector $\mathbf{P}$ in connection with the Hertz potential as

$$\mathbf{P} = \frac{\mathbf{J}}{j\omega} \tag{2.17}$$

Thus

$$\int_v \mathbf{P}(\mathbf{r}')dv' = \frac{1}{j\omega}\int \mathbf{J}(\mathbf{r}')dv' = \frac{I_0\ell}{j\omega}\widehat{z} = \mathbf{p} \tag{2.18}$$

Hence, $\mathbf{P}$ is indeed related to the dipole concept. The time varying dipole is referred to as the *Hertzian* dipole. For an elementary current source having the distribution

$$I(z) = I_0(1 - \frac{2}{\ell}|z|), \quad |z| \leq \ell/2 \tag{2.19}$$

the current moment is

$$\mathbf{p}_i = \frac{I_0 \ell}{2}\widehat{z} \tag{2.20}$$

which is half the current moment of a Hertzian dipole of the same length. This is called the *Abraham* dipole.

Using the approximation similar to (2.11), it can be shown that the potential for a Hertzian dipole of moment $\mathbf{p}_i = j\omega\mathbf{p}$ is

$$\pi(\mathbf{r}) \simeq \frac{e^{-jk_0 r}}{j\omega 4\pi\epsilon_0 r}\mathbf{p}_i = \mathbf{p}\frac{e^{-jk_0 r}}{4\pi\epsilon_0 r} \tag{2.21}$$

The magnetic field is given by

$$\begin{aligned}\mathbf{H} &= j\omega\epsilon_0 \nabla \times \pi \\ &= \frac{j\omega}{4\pi}[\frac{1}{r^2} + \frac{jk_0}{r}]e^{-jk_0 r}(\mathbf{p} \times \widehat{r})\end{aligned} \tag{2.22}$$

where $\widehat{r}$ is the unit vector in the direction of the field point. The electric field $\mathbf{E}$ can be obtained from

$$\begin{aligned}\mathbf{E} &= \nabla\nabla \cdot \pi + k_0^2 \pi \\ &= \frac{e^{-jk_0 r}}{4\pi\epsilon_0}[(\frac{1}{r^3} + \frac{jk_0}{r^2})\{3(\mathbf{p} \cdot \widehat{r})\widehat{r} - \mathbf{p}\} - \frac{k_0^2}{r}\{\widehat{r} \times (\widehat{r} \times \mathbf{p})\}]\end{aligned} \tag{2.23}$$

In the far zone ($k_0 r \gg 1$), only the terms depending on $1/r$ dominate, and we have

$$\mathbf{E} = -\frac{k_0^2}{4\pi\epsilon_0}\frac{e^{-jk_0 r}}{r}[\widehat{r} \times (\widehat{r} \times \mathbf{p})] \tag{2.24}$$

$$\mathbf{H} = \frac{\omega k_0}{4\pi}\frac{e^{-jk_0 r}}{r}(\widehat{r} \times \mathbf{p}) \tag{2.25}$$

For a current element $I_0\mathbf{d}\ell$, the above expressions can be used provided that $\mathbf{p}$ is replaced by $I_0\mathbf{d}\ell/j\omega$.

The above expressions for the electromagnetic fields are valid for any orientation of the dipole. For a $\widehat{z}$-oriented dipole, we have

$$\mathbf{p} = p\cos\theta\,\widehat{r} - p\sin\theta\,\widehat{\theta} \tag{2.26}$$

The spherical components of the electromagnetic fields are, therefore, given by

$$\begin{aligned}E_r &= p\frac{e^{-jk_0 r}}{2\pi\epsilon_0}[\frac{1}{r^3} + \frac{jk_0}{r^2}]\cos\theta \\ E_\theta &= p\frac{e^{-jk_0 r}}{4\pi\epsilon_0}[\frac{1}{r^3} + \frac{jk_0}{r^2} - \frac{k_0^2}{r}]\sin\theta \\ H_\phi &= j\omega p\frac{e^{-jk_0 r}}{2\pi}[\frac{1}{r^2} + \frac{jk_0}{r}]\end{aligned} \tag{2.27}$$

while

$$E_\phi = H_r = H_\theta = 0 \tag{2.28}$$

The radiation fields of a $\widehat{z}$-directed Hertzian dipole located at the origin are given in component form as

$$\begin{aligned} E_\theta &= j\frac{k_0^2}{4\pi\omega\epsilon_0} I_0 dz \frac{e^{-jk_0 r}}{r}\sin\theta \\ H_\phi &= j\frac{k_0}{4\pi} I_0 dz \frac{e^{-jk_0 r}}{r}\sin\theta \end{aligned} \tag{2.29}$$

with $E_r = E_\phi = H_r = H_\theta = 0$. Note that in the far field, the ratio of the transverse field components is

$$\frac{|E_\theta|}{|H_\phi|} = \frac{k_0}{\omega\epsilon_0} = Z_0 \tag{2.30}$$

where Z_0 is the free space intrinsic impedance. The radiation fields of an electric dipole behave as a transverse electromagnetic (TEM) wave with their amplitude decreasing as $1/r$. Also, the Poynting vector is given by

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^\star = \frac{p^2\omega k_0^3}{32\pi^2\epsilon_0 r^2}\sin^2\theta\,\widehat{r} \tag{2.31}$$

representing a real power flow density in the $\widehat{r}$-direction.

If a $\widehat{z}$-directed dipole of current $I(z)$ is located at a position $\mathbf{r}'$ as shown in Fig. 2.3, then the far fields are given by

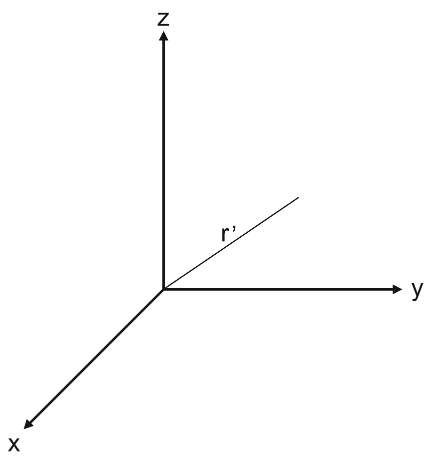

Fig. 2.3 A short $\widehat{z}$-directed Hertzian dipole positioned at $\mathbf{r}'$

$$E_\theta = j\frac{k_0^2}{4\pi\omega\epsilon_0}\sin\theta\frac{e^{-jk_0(r-\mathbf{r}'\cdot\widehat{r})}}{r}I(z)dz \tag{2.32}$$

$$H_\phi = j\frac{k_0}{4\pi}\sin\theta\frac{e^{-jk_0(r-\mathbf{r}'\cdot\widehat{r})}}{r}I(z)dz \tag{2.33}$$

2.2.2 The Small Magnetic Dipole

The half-wavelength dimensional rule applies to all antennas except wire loop antennas. Small loop antennas used in transistor radios are resonant at the long, 300 m wavelengths of the broadcast (AM) band because they contain a core of magnetic material called ferrite. Ferrite loop antennas are used in ultracompact transistor radios.

Consider a small loop of area Δ carrying a current I_0 as shown in Fig. 2.4. Such a current loop is called a magnetic dipole. For this loop, we note that

$$\int_V \mathbf{J}(\mathbf{r}')dv' = \oint_C I_0\mathbf{d}\ell = I_0\oint_C \mathbf{d}\ell = 0 \tag{2.34}$$

However, this does not imply that the current moment is zero. We define the magnetic dipole moment as

$$\mathbf{m} = \frac{1}{2}\int_V \widehat{r'}\times\mathbf{J}dv' \tag{2.35}$$

Applying this definition to the present case, we have

$$\mathbf{m} = \frac{1}{2}\oint_C \widehat{r'}\times I_0\mathbf{d}\ell' = I_0\oint_C \frac{1}{2}(\widehat{r'}\times\mathbf{d}\ell') \tag{2.36}$$

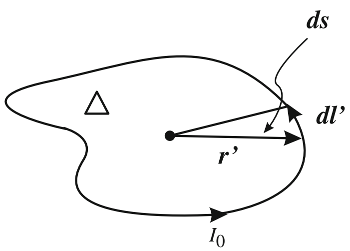

Fig. 2.4 A small magnetic dipole

Therefore,

$$\mathbf{m} = \widehat{n} I_0 \int_S ds' = \widehat{n} I_0 \Delta \tag{2.37}$$

which is clearly independent of the shape of the loop. In order to find the Hertz potential, we invoke (2.11). Thus

$$\begin{aligned}\pi(\mathbf{r}) &\simeq -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}\int_V \mathbf{J}(\mathbf{r}')(jk_0 + \frac{1}{r}) r' \cos\gamma\, dv' \\ &= -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}(jk_0 + \frac{1}{r})\int_V \mathbf{J}(\mathbf{r}') r' \widehat{r'} \cdot \widehat{r}\, dv' \\ &= -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}(jk_0 + \frac{1}{r}) I_0 \oint_C \frac{\mathbf{r}' \cdot \mathbf{r}}{r}\mathbf{d}\ell' \end{aligned} \tag{2.38}$$

Employing the vector theorem

$$\oint_C \psi\, \mathbf{d}\ell' = -\int_S \nabla'\psi \times \widehat{n} ds' \tag{2.39}$$

we obtain

$$\begin{aligned}\pi(\mathbf{r}) &= j\frac{Z_0 I_0}{k}\frac{e^{-jk_0 r}}{4\pi r}(jk_0 + \frac{1}{r})\int_S \frac{\nabla'(\mathbf{r}' \cdot \mathbf{r}) \times \widehat{n} ds'}{r} \\ &= j\frac{Z_0 I_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}(jk_0 + \frac{1}{r})\int_S \widehat{r} \times \widehat{n} ds' \end{aligned} \tag{2.40}$$

Substituting for the magnetic current moment **m**, we have

$$\pi(\mathbf{r}) = -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}(\mathbf{m} \times \widehat{r})(jk_0 + \frac{1}{r}) \tag{2.41}$$

and if the loop is oriented so that $\widehat{n} = \widehat{z}$, we find

$$\pi(\mathbf{r}) = -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r} m (jk_0 + \frac{1}{r}) \sin\theta \widehat{\phi} \tag{2.42}$$

It is noted that the Hertz potential is directed entirely in the $\widehat{\phi}$ direction, regardless of the size of the loop. Using (2.22) and (2.23), it can be shown that the complete electromagnetic fields for a $\widehat{z}$ oriented magnetic dipole are given by

$$\mathbf{E} = m\frac{e^{-jk_0 r}}{4\pi}k_0^2 Z_0\left(-\frac{jk_0}{r^2}+\frac{1}{r}\right)\sin\theta\widehat{\phi} \tag{2.43}$$

$$\mathbf{H} = m\frac{e^{-jk_0 r}}{4\pi}[2\left(\frac{1}{r^3}+\frac{jk_0}{r^2}\right)\cos\theta\widehat{r}+\left(\frac{1}{r^3}+\frac{jk_0}{r^2}-\frac{k_0^2}{r}\right)\sin\theta\widehat{\theta}] \tag{2.44}$$

For such a magnetic dipole, $E_r = E_\theta = h_\phi \equiv 0$.

In the far zone, as r goes to infinity, we obtain from the above

$$\begin{aligned}\mathbf{E} &\simeq m\frac{e^{-jk_0 r}}{4\pi r}k_0^2 Z_0 \sin\theta\widehat{\phi} \\ \mathbf{H} &\simeq -m\frac{e^{-jk_0 r}}{4\pi r}k_0^2 \sin\theta\widehat{\theta}\end{aligned} \tag{2.45}$$

Also, the Poynting vector is given by

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^\star = \frac{m^2 k_0^4 Z_0}{32\pi^2 r^2}\sin^2\theta\,\widehat{r} \tag{2.46}$$

again representing real power flow in the radial direction.

2.3 Wire Antennas

Consider a straight wire antenna driven by a current distribution $\Re e[I(z)e^{j\omega t}]$ similar to half of what is shown in Fig. 2.5. Using expressions (2.32) and (2.33), we may find the fields due to a wire antenna by the superposition integral. Thus, the electric field is given by

$$E_\theta \simeq \frac{jk_0^2}{4\pi\omega\epsilon_0}\sin\theta\frac{e^{-jk_0 r}}{r}\int I(z')e^{jk_0\mathbf{r}'\cdot\widehat{r}}dz' \tag{2.47}$$

We may write the above expression as

$$E_\theta \simeq \frac{jk_0^2\ell}{4\pi\omega\epsilon_0}\frac{e^{-jk_0 r}}{r}I_0 e^{j\phi_0}\psi(\theta) \tag{2.48}$$

where ψ is defined as

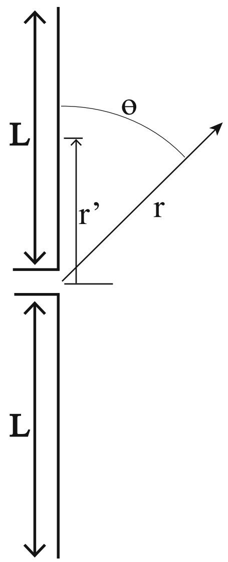

Fig. 2.5 A center-fed wire antenna of length $2L$

$$\psi(\theta) \equiv \frac{\sin\theta}{\ell} \int_{-\ell/2}^{\ell/2} I(z') e^{jk_0 \mathbf{r}' \cdot \hat{r}} dz' \tag{2.49}$$

and ℓ is the length of the antenna. The quantity ψ is dimensionless and is independent of I_0 and ϕ_0. It displays the angular dependence of the radiated field and is sometimes referred to as the *field radiation pattern*. In order to evaluate ψ, the current distribution should be known.

Example 2.1 Let the current distribution on a center-fed wire antenna of length $2L$ shown in Fig. 2.5 be given by

$$I(z) = I_m \frac{\sin k_0(L - |z|)}{\sin(k_0 L)}$$

Using (2.49), we obtain

$$\psi(\theta) = \frac{\sin\theta}{2L} \int_{-L}^{L} \frac{\sin k_0(L - |z'|)}{\sin(k_0 L)} e^{jk_0 z' \cos\theta} dz'$$

Evaluating the integral, we find that

$$\psi(\theta) = \frac{1}{k_0 L \sin(k_0 L)} \left[\frac{\cos(k_0 L) - \cos(k_0 L \cos\theta)}{\sin\theta} \right]$$

The radiated electric field is given by (2.48). □

2.4 Field Regions

The vector potential at a point P due to a source distribution $\mathbf{J}$ in free space is given by (2.2). Depending on the location of the observation point, the maximum linear dimension of the source and the wavelength, various approximations are made to evaluate (2.2), and hence the fields.

We consider point sources and extended sources separately.

2.4.1 Point Sources

If the maximum linear dimension of the source is small compared to the wavelength, that is, $r' \ll \lambda$ or $kr' \ll 1$, then the exponential term in the integrand of (2.2) may be approximated by the first two terms of its Taylor series expansion. This will yield (2.11) applicable for point or small sources. For such cases, the nature of the fields produced are different for $kr \ll 1$ and $kr \gg 1$, with the source assumed to be located at the origin. The two regions, so defined, are called the (reactive) near field and (radiating) far field regions of the source (Fig. 2.6). The common boundary of these two regions are arbitrarily chosen to be at $kr = 1$ or, equivalently, $r = \lambda/2\pi$.

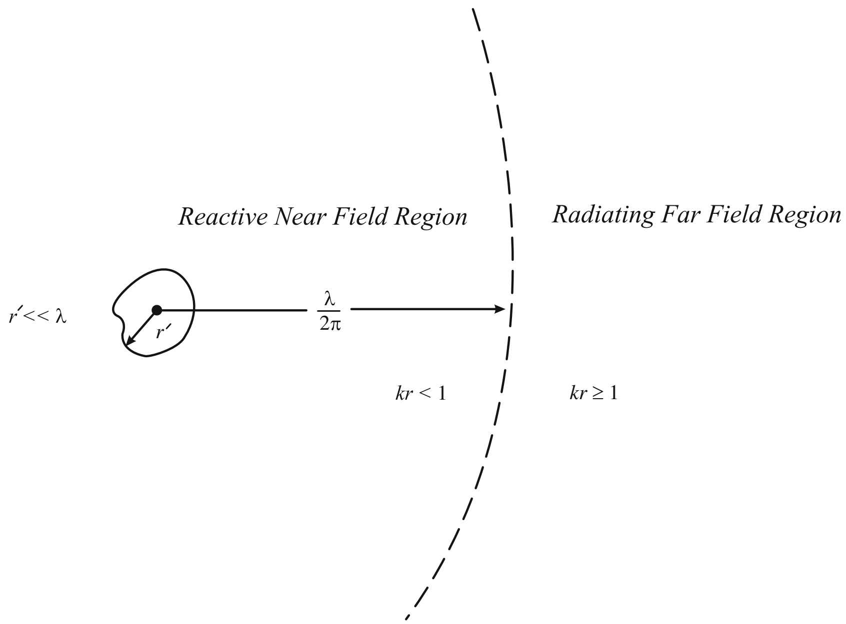

Fig. 2.6 Field regions for point sources in the near field region, the reactive energy is dominant, while in the far field, the radiating energy is dominant

2.4.2 Extended Sources

In many cases, the maximum dimension of the source may be much larger than λ. Due to the extended nature of the source, the far field region can no longer be assumed to start at $\frac{\lambda}{2\pi}$.

Under the assumption

$$r \gg r', \quad k_0 r \gg 1 \tag{2.50}$$

we may write

$$|\mathbf{r} - \mathbf{r}'| \simeq r - (\widehat{r} \cdot \mathbf{r}') + \frac{1}{2r}[r'^2 - (\widehat{r} \cdot \mathbf{r}')^2] + \mathcal{O}\left(\frac{r'}{r}\right)^3 \tag{2.51}$$

when making phase calculations and

$$|\mathbf{r} - \mathbf{r}'| \simeq r \tag{2.52}$$

for amplitude considerations. Under these assumptions, (2.2) reduces to

$$\pi(\mathbf{r}) \simeq -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}\int_V \mathbf{J}(\mathbf{r}')e^{jk_0[(\widehat{r} \cdot \mathbf{r}') + \frac{(\widehat{r} \cdot \mathbf{r}')^2}{2r} - \frac{r'^2}{2r}]}dv' \tag{2.53}$$

According to the IEEE Standard, the far field region, also known as the *Fraunhofer* region starts at a distance r where

$$\frac{r'^2_{max}}{2r} = \lambda/16 \tag{2.54}$$

that is

$$r = 8r'^2_{max}/\lambda \tag{2.55}$$

Assuming D to be the maximum linear dimension of the source, $r'_{max} = D/2$, and we obtain the far field region definition as

$$r \geq 2D^2/\lambda \tag{2.56}$$

Under this condition, only the $(\widehat{r} \cdot \mathbf{r}')$ term in the exponential integrand of (2.53) is retained.

$$\pi(\mathbf{r}) \simeq -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}\int_V \mathbf{J}(\mathbf{r}')e^{jk_0\mathbf{r}' \cdot \widehat{r}}dv' \tag{2.57}$$

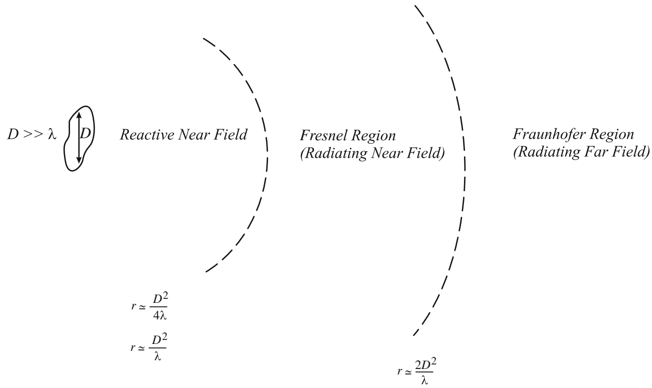

Fig. 2.7 Field regions for extended sources. In the Fresnel region, the angular field distribution depends on the distance from the antenna, while in the Franuhofer region, it is independent of the distance

This is the usual approximation used when determining the radiation fields of antennas. The neglect of lower order terms introduces a maximum phase error of $k_0 r'^2/2r$ implying an error of $\pi/8$ at the far field boundary $r = 2D^2/\lambda$. For larger distances, the error will be less.

The far field region is dominated by radiating energy and is very important in antenna analysis and design.

The region in which the second order term $k_0 r'^2/2r$ must be retained for field calculations is referred to as the *Fresnel region*. This is also known as the quasi-far field or the radiating near field region. There is no clearly marked boundary for the specification of the Fresnel region. However, for electrically large sources ($D \gg \lambda$), the region may be defined as

$$D^2/4\lambda \leq r < 2D^2/\lambda \tag{2.58}$$

as shown in Fig. 2.7.

The near field region extends from the source up to the lower boundary of the Fresnel region. For this region in which the reactive energy dominates, no general approximation is made in the evaluation of the potential and the fields.

2.5 Far Field Calculation for General Antennas

The far field expression (2.57) for the Hertz potential may be written as

$$\pi(\mathbf{r}) = -j\frac{Z_0}{k_0}\frac{e^{-jk_0 r}}{4\pi r}\mathbf{N}(\theta, \phi) \tag{2.59}$$

where

$$\mathbf{N} = \int\limits_V \mathbf{J}(\mathbf{r}')e^{jk_0\mathbf{r}'\cdot\widehat{r}}dv' \tag{2.60}$$

The magnetic field is given by

$$\mathbf{H}(\mathbf{r}) = j\omega\epsilon_0\nabla\times\pi(\mathbf{r}) \tag{2.61}$$

Since we are interested in the far fields, we wish to express the result only retaining those terms behaving like e^{-jk_0r}/r as $r\to\infty$. Carrying out the differentiation, we have

$$\begin{aligned}\nabla\times\{\frac{e^{-jk_0r}}{r}\mathbf{N}(\theta,\phi)\} &= \nabla(\frac{e^{-jk_0r}}{r})\times\mathbf{N} + \frac{e^{-jk_0r}}{r}\nabla\times\mathbf{N}\\ &= -(\frac{jk_0}{r}+\frac{1}{r^2})e^{-jk_0r}(\widehat{r}\times\mathbf{N}) + \frac{e^{-jk_0r}}{r}\nabla\times\mathbf{N}\\ &= (1+\frac{j}{k_0r})\frac{e^{-jk_0r}}{r}(-jk_0)\widehat{r}\times\mathbf{N} + \frac{e^{-jk_0r}}{r}\nabla\times\mathbf{N}\end{aligned} \tag{2.62}$$

Note that the curl of the vector $\mathbf{N}$ in the above equation is

$$\nabla\times\mathbf{N} = \mathcal{O}(1/r) \tag{2.63}$$

Therefore

$$\nabla\times\{\frac{e^{-jk_0r}}{r}\mathbf{N}(\theta,\phi)\} \simeq \frac{e^{-jk_0r}}{r}(-jk_0)(\widehat{r}\times\mathbf{N}) + \mathcal{O}(\frac{1}{r^2}) \tag{2.64}$$

Hence, the operation $(\nabla\times)$ can be replaced by $(-jk_0\widehat{r}\times)$ in the far field calculations. We may, therefore, write

$$\mathbf{H}(\mathbf{r}) = -jk_0\frac{e^{-jk_0r}}{4\pi r}(\widehat{r}\times\mathbf{N}) + \mathcal{O}(\frac{1}{r^2}) \tag{2.65}$$

Noting that $\widehat{r}\times\mathbf{N} = \widehat{r}\times\mathbf{N}_t$, we have

$$\mathbf{H}(\mathbf{r}) = -jk_0\frac{e^{-jk_0r}}{4\pi r}(\widehat{r}\times\mathbf{N}_t) \tag{2.66}$$

where

$$\mathbf{N}_t = \mathbf{N} - \widehat{r}N_r = -\widehat{r}\times\widehat{r}\times\mathbf{N} \tag{2.67}$$

is referred to as the radiation vector of the current distribution[1] and the terms neglected are of the order $1/k_0 r^2$. The radiation vector has the dimension of $[A.m]$. We may write

$$\mathbf{N}_t = I_i \widehat{h}(\theta, \phi) \tag{2.68}$$

where I_i is a reference current, usually taken as the input current to the antenna. Then $\widehat{h}(\theta, \phi)$ is called the *vector effective height function*.

The electric field in the region away from the sources is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{j\omega\epsilon_0} \nabla \times \mathbf{H}(\mathbf{r}) \tag{2.69}$$

Thus, we have

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{1}{j\omega\epsilon_0}(-jk_0\widehat{r}) \times \mathbf{H}(\mathbf{r}) \\ &= jk_0 Z_0 \frac{e^{-jk_0 r}}{4\pi r}[\widehat{r} \times (\widehat{r} \times \mathbf{N}_t)]\end{aligned} \tag{2.70}$$

Therefore, the transverse components of the fields dominate in the far field.

Summarizing the above results, we use the following procedure to find the far fields of any antenna.

$$\begin{aligned}\mathbf{N} &= \int_V \mathbf{J}(\mathbf{r}') e^{jk_0 \mathbf{r}' \cdot \widehat{r}} dv' \\ \mathbf{N}_t &= N_\theta \widehat{\theta} + N_\phi \widehat{\phi} \\ &= -\widehat{r} \times \widehat{r} \times \mathbf{N}\end{aligned} \tag{2.71}$$

$$\begin{aligned}\mathbf{E} &= E_\theta \widehat{\theta} + E_\phi \widehat{\phi} \\ &= -jk_0 Z_0 \frac{e^{-jk_0 r}}{4\pi r} \mathbf{N}_t \\ \mathbf{H} &= \frac{1}{Z_0} \widehat{r} \times \mathbf{E}\end{aligned}$$

It is noted that the direction of the Poynting vector is that of $\mathbf{N}_t \times (\widehat{r} \times \mathbf{N}_t)$, that is $\widehat{r}$ and $\mathbf{E}$, $\mathbf{H}$ and $\widehat{r}$ form a right handed perpendicular system of vectors.

The above prescription is widely used to obtain the far fields of various antennas, provided the current distribution is known.

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[1] This vector is due to Schelkunoff.

2.6 Antenna Parameters

In this section, we discuss various antenna parameters. Some of these parameters such as the radiation intensity and the directive gain pertain to the far field behavior of the antenna, while the others like the antenna impedance are near field quantities.

2.6.1 Antenna Patterns and Radiation Intensity

A plot of $|\mathbf{E}|$ with constant r as a function of (θ, ϕ) is called the field radiation pattern of the antenna.

The power radiated from an antenna per unit solid angle is called the *radiation intensity* of the antenna. If $\mathbf{S}$ is the Poynting vector, then

$$U = r^2 S_r(\theta, \phi) \tag{2.72}$$

is the radiation intensity in Watts per unit solid angle. In the far field, the Poynting vector is given by $\mathbf{S} = \frac{1}{2Z_0}|\mathbf{E}|^2\widehat{r}$ and

$$U(\theta, \phi) = \frac{r^2}{2Z_0}|\mathbf{E}|^2 \tag{2.73}$$

A plot of U as function of (θ, ϕ) is called the antenna power pattern. These patterns are usually plotted in the far-field and are directly related to the magnitude of the vector effective height function. The normalized power pattern is defined as

$$U_n(\theta, \phi) = \frac{U(\theta, \phi)}{U(\theta, \phi)_{max}} \tag{2.74}$$

For a short electric dipole, The normalized field radiation and power patterns are shown in Fig. 2.8.

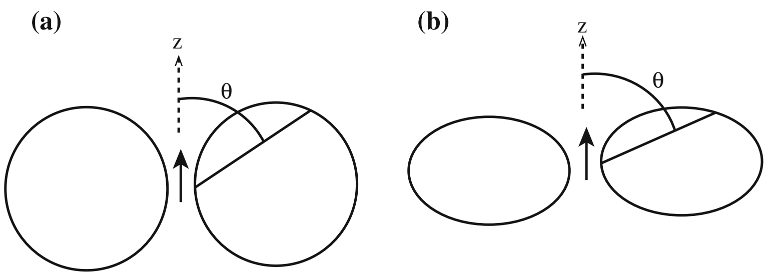

Fig. 2.8 The normalized **a** field radiation pattern, and **b** power pattern pf a short Hertzian dipole

2.6.2 Directive Gain

The *directive gain* of an antenna in a given direction is the ratio of the radiation intensity in that direction to the radiation intensity of an equivalent isotropic antenna radiating the same total average power. Thus

$$D_g = \frac{U}{U_0} = \frac{U}{P_{rad}/4\pi} \tag{2.75}$$

where P_{rad} can be written in terms of the radiated power density S_r

$$P_{rad} = \oint_S r^2 S_r d\Omega \tag{2.76}$$

where $d\Omega$ is the element of solid angle. Thus, using the above expression and (2.72), the directive gain (2.75) can also be expressed as

$$D_g(\theta, \phi) = \frac{4\pi S_r(\theta, \phi)}{\int_{4\pi} S_r(\theta, \phi) d\Omega} \tag{2.77}$$

If the direction is not specified, it is implied that D_g is specified in the direction of maximum gain. This is referred to as *directivity*. Directivity is denoted by D_0 and is given by

$$D_0 = \frac{U_{max}}{P_{rad}/4\pi} = \frac{4\pi}{\int_{4\pi} U_n(\theta, \phi) d\Omega} \tag{2.78}$$

The directive gain can be expressed in terms of the directivity and the normalized power pattern as

$$D(\theta, \phi) = D_0 U_n(\theta, \phi) \tag{2.79}$$

Example 2.2 A short Hertzian dipole transmits or receives most of its energy at right angles to the wire; little energy is transferred along the length of the wire. Such directivity is one of the most important electric qualities of an antenna. It allows transmission or reception to be beamed in a particular direction, to the exclusion of signals in other directions.

The complex Poynting vector for a short electric dipole is given by (2.31)

$$S_r = \frac{p^2 \omega k^3}{32\pi^2 \epsilon_0 r^2} \sin^2\theta$$

The directive gain is expressed as

$$D_g(\theta) = \frac{\sin^2\theta}{\frac{1}{2}\int_0^\pi \sin^3\theta d\theta} = \frac{3}{2}\sin^2\theta$$

and the directivity is $D_0 = 3/2$. □

The directive gain may also be written in terms of the vector effective height function $\widehat{h}(\theta, \phi)$ as

$$D_g = \frac{|\widehat{h}(\theta, \phi)|^2}{\frac{1}{4\pi}\int_\Omega |\widehat{h}(\theta, \phi)|^2 d\Omega} \tag{2.80}$$

2.6.3 Gain

The *gain* of an antenna in a specified direction is defined as the ratio of the power density radiated by the antenna, $S_r(\theta, \phi)$, to the power density radiated by a lossless isotropic antenna, S_{ri}, provided both antennas are supplied with the same amount of power, P_t

$$G(\theta, \phi) = \frac{S_r(\theta, \phi)}{S_{ri}} \tag{2.81}$$

The total power radiated by the antenna is given by

$$P_{rad} = \oint_S S_r(\theta, \phi) ds \tag{2.82}$$

while the total power radiated by the lossless isotropic antenna is given by

$$P_{rad}^i = 4\pi r^2 S_{ri} \tag{2.83}$$

This is equal to the total power delivered to the antenna P_t. However, due to the losses in the antenna system, part of the power is dissipated in the antenna structure. Designating this power loss as P_ℓ, the *radiation efficiency* is defined as

$$\eta_\ell = \frac{P_{rad}}{P_t} \tag{2.84}$$

Combining (2.82) to (2.84), we find that

$$S_{ri} = \frac{1}{4\pi\eta_\ell}\int_{4\pi} S_r(\theta, \phi) d\Omega \tag{2.85}$$

and substituting in (2.81), we obtain

$$G(\theta,\phi) = \frac{4\pi\eta_\ell S_r(\theta,\phi)}{\int_{4\pi} S_r(\theta,\phi)d\Omega} \tag{2.86}$$

In view of (2.77), the gain can be written in terms of the directive gain as

$$G(\theta,\phi) = \eta_\ell D(\theta,\phi) \tag{2.87}$$

The antenna gain accounts for ohmic losses in the antenna structure.

2.6.4 Effective Aperture

The ability of an antenna to capture energy from an incident wave and to convert it to an intercepted power for delivering to a matched load is characterized by the *effective aperture* A_e. If the incident power density at the position of the receiving antenna is S_i, then the intercepted power is given by

$$P_{int} = A_e S_i \tag{2.88}$$

The effective aperture is also known as *effective area* and *receiving cross section*.

The effective aperture can be written in terms of directivity of the antenna as

$$A_e = \frac{\lambda^2 D_0}{4\pi} \tag{2.89}$$

2.6.5 Antenna Impedance

Consider the antenna shown in Fig. 2.9. Enclose the antenna by a closed surface S consisting of three surfaces: Surface S_a in the far-field region, surface S_b enclosing the antenna and the generator, and surface S_c a tube-shaped surface connecting S_a and S_b. Thus, we may write $S = S_a + S_b + S_c$. We now write the Poynting theorem for the surface S enclosing the volume V

$$-\frac{1}{2}\oint_S \mathbf{S}\cdot\mathbf{dS} = j2\omega\int_V \frac{1}{4}[\mu_0|\mathbf{H}|^2 - \epsilon_0|\mathbf{E}|^2]dV + \int_V \frac{1}{2}\mathbf{E}\cdot\mathbf{J}_a dV \tag{2.90}$$

We may reduce the diameter of the tube S_c as far as we are pleased. Therefore, the contribution of the surface integral S_c is negligible. Hence, we have

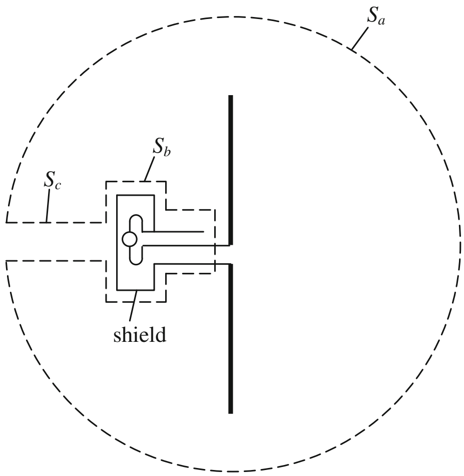

Fig. 2.9 The Poynting theorem for radiating antennas

$$-\oint_{S_a} \frac{1}{2}\frac{|\mathbf{E}|^2}{Z_0} dS - \oint_{S_b} \frac{1}{2}(\mathbf{E} \times \mathbf{H}^\star) \cdot \mathbf{dS} = j2\omega \int_V \frac{1}{4}[\mu_0|\mathbf{H}|^2 - \epsilon_0|\mathbf{E}|^2] dV \quad (2.91)$$

where we used the expression for **S** in the far-field region. We now give an interpretation for the second integral on the left. If, in accordance with the concepts of circuit theory, we ignore the displacement current and magnetic induction effects, we may write

$$\mathbf{E} = -\nabla\Phi \quad (2.92)$$

so that

$$-\oint_{S_b} \frac{1}{2}(\mathbf{E} \times \mathbf{H}^\star) \cdot \mathbf{dS} = -\oint_{S_b} \frac{1}{2}(-\nabla\Phi \times \mathbf{H}^\star) \cdot \mathbf{dS} \quad (2.93)$$

Using the vector identity

$$\nabla \times (\Phi\mathbf{H}^\star) \equiv \nabla\Phi \times \mathbf{H}^\star + \Phi\nabla \times \mathbf{H}^\star \quad (2.94)$$

we have

$$-\oint_{S_b} \frac{1}{2}(\mathbf{E} \times \mathbf{H}^\star) \cdot \mathbf{dS} = \frac{1}{2}\oint_{S_b} \nabla \times (\Phi\mathbf{H}^\star) \cdot \mathbf{dS} - \frac{1}{2}\oint_{S_b} (\Phi\nabla \times \mathbf{H}^\star) \cdot \mathbf{dS} \quad (2.95)$$

But since $\nabla \times \mathbf{H} = \mathbf{J}$,

$$-\oint_{S_b} \frac{1}{2}(\mathbf{E} \times \mathbf{H}^\star) \cdot \mathbf{dS} = -\frac{1}{2}\oint_{S_b} \Phi \mathbf{J}^\star \cdot \mathbf{dS} \tag{2.96}$$

Clearly, the right hand side of the above equation is the power generated by the source

$$-\oint_{S_b} \frac{1}{2}(\mathbf{E} \times \mathbf{H}^\star) \cdot \mathbf{dS} = \frac{1}{2}\Phi_i I_i^\star \tag{2.97}$$

The complex voltage of the antenna is proportional to the complex terminal current

$$\Phi = Z_{ant} I \tag{2.98}$$

where Z_{ant} is the antenna impedance. The real part of the antenna can be defined as

$$\Re\mathrm{e} Z_{ant} \equiv R_{rad} = \frac{1}{Z_0}\oint_S \frac{|\mathbf{E}|^2}{|I_0|^2} dS \tag{2.99}$$

where R_{rad} is the *radiation resistance* and the imaginary part is related to the reactive power supplied to the antenna

$$\Im\mathrm{m} Z_{ant} = \frac{\omega \int_V (\mu_0|\mathbf{H}|^2 - \epsilon_0|\mathbf{E}|^2) dV}{|I_0|^2} \tag{2.100}$$

Example 2.3 The radiation resistance of a short electric dipole of length ℓ can be found by the total power radiated by the dipole in the far-field

$$P = \oint_S \mathbf{S} \cdot \mathbf{ds}$$

where $\mathbf{S}$ is the Poynting vector. Using (2.96), we have

$$\begin{aligned} P &= \int_0^{2\pi}\int_0^{\pi} (I_0\ell)^2 \frac{k_0^2 Z_0}{2(4\pi)^2} \sin^2\theta ds \\ &= (I_0\ell)^2 \frac{k_0^2 Z_0}{12\pi} = (\frac{I_0\ell}{\lambda_0})^2 \frac{\pi}{3} Z_0 \end{aligned}$$

The radiation resistance is given by

$$R_{rad} = 2P/I_0^2$$

Thus

$$R_{rad} = Z_0(\frac{2\pi}{3})(\ell/\lambda_0)^2$$

This expression is valid for short dipoles ($\ell \ll \lambda_0$), but it is a good approximation for dipoles of length $\ell \leq \lambda_0/4$. □

2.6.6 Friis Transmission Formula

Consider a transmitting and a receiving antenna positioned in the direction of their maximum gain in free space separated by a distance R, as shown in Fig. 2.10 . If the power transmitted by the transmitting antenna is P_t, the power density at the receiver is given by

$$S_r = G_t \frac{P_t}{4\pi R^2} \tag{2.101}$$

where G_t is the gain of the transmitting antenna. The intercepted power at the receiving antenna is expressed as

$$P_{int} = A_r S_r = A_r G_t \frac{P_t}{4\pi R^2} \tag{2.102}$$

where A_r is the effective aperture of the receiving antenna. The received power can be written in terms of the intercepted power as

$$P_{rec} = \eta_r P_{int} \tag{2.103}$$

where η_r is the receiving antenna efficiency. Substituting from (2.102), we get

$$P_{rec} = \eta_r A_r G_t \frac{P_t}{4\pi R^2} = G_t G_r P_t \left(\frac{\lambda}{4\pi R}\right)^2 \tag{2.104}$$

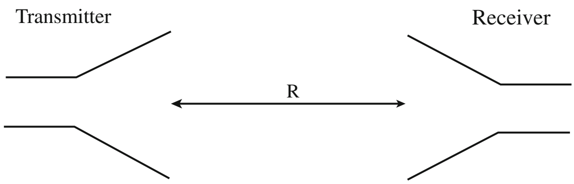

Fig. 2.10 The configuration for the derivation of the Friis transmission formula

where use has been made of (2.89). The power transfer ratio is given by

$$\frac{P_{rec}}{P_t} = G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2 \tag{2.105}$$

If the antennas are positioned arbitrarily, then this ratio is given by

$$\frac{P_{rec}}{P_t} = G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2 U_t(\theta_t, \phi_t) U_r(\theta_r, \phi_r) \tag{2.106}$$

which is known as *Friis transmission formula*.

Exercises

2.1: If $\mathbf{F} = \mathbf{A} f(r)$ where $\mathbf{A}$ is a constant vector, f is a function of r only, and $\mathbf{r} = r\widehat{r} = x\widehat{x} + y\widehat{y} + z\widehat{z}$ is the position vector,
(a) show

$$\nabla \times \mathbf{F} = \widehat{r} \times \mathbf{A} \frac{df}{dr}$$

(b) In particular, if $f(r) = \frac{e^{-jkr}}{r}$, show that for $kr \gg 1$ (corresponding to the far field)

$$\nabla \times \mathbf{F} \simeq -jk\widehat{r} \times \mathbf{F}$$

(c) More generally, if $\mathbf{F} = \mathbf{F}(r, \theta, \phi) = \mathbf{A}\frac{e^{-jkr}}{r}$ where $\mathbf{A}$ is any vector *independent of* r but not necessarily constant, show that in the far field

$$\nabla \times \mathbf{F} \simeq -jk\widehat{r} \times \mathbf{F}$$
$$\nabla \times \nabla \times \mathbf{F} \simeq -k^2\widehat{r} \times (\widehat{r}\mathbf{F})$$

In other words, in the far field the operator $\nabla(.)$ is equivalent to $-jk\widehat{r} \times (.)$. Thus, given a Hertz vector, the resulting $\mathbf{E}$ and $\mathbf{H}$ in the *far field region* can be obtained *without any differentiation*.

2.2: A straight wire of length L carrying the current $\widehat{z}I_0 e^{-j\beta z}$ lies on the z-axis ($0 \le z \le L$). With β a real constant, this represents a travelling wave antenna.
(a) State the Hertz vector(s) associated with this source.
(b) If the point of observation $\mathbf{r}$ is such that $r \gg L$, show that

$$|\mathbf{r} - \mathbf{r}'| \simeq r - z' \cos\theta$$

(c) Under the assumption that

$$\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \simeq \frac{e^{-jkr}}{r} e^{jkz'} \cos\theta$$

evaluate the Hertz vector(s) in the far field.
(d) Determine $\mathbf{E}$ in the far field, showing that $\mathbf{E} = \widehat{\theta} E_\theta$

2.3: Find the radiation resistance of a short Hertzian dipole.

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