Chapter 1
Introduction

> *"So, naturalists observe, a flea*
> *Has smaller fleas that on him prey;*
> *And these have smaller still to bite 'em,*
> *And so proceed ad infinitum."*
> *Jonathan Swift*

1.1 What Is a Set Operad?

Let us start with an example. Consider a set of (injective) parenthesized words.

$$A = (a(bhd)),\ B = ((ec)g),\ C = (((nim)j)k).$$

Suppose, we have another parenthesized word $F = ((AB)C)$ for the symbols that represent the previous set of words. Then we define a "product" between the set of internal words and the external one as follows,

$$\eta(\{A, B, C\}, F) = (((a(bhd))((ec)g))(((nim)j)k)))$$

which is simply the substitution of the internal words in the places assigned by the external one. Now, assume that we have assigned other values to the letters A, B, and C. Let $A = (a)$, $B = (b)$, $C = (c)$, and we again "multiply" $\{A, B, C\}$ by F. In this case we obtain the word $(((a)(b))(c)) = ((ab)c)$ that can be identified with F. On the other hand, assume $F = (A)$ is a singleton word, with A being $(a(bhd))$, for example . The product is then equal to $((a(bhd)))$, that can be identified with A. The singleton words are like the "identity" for our operation (product) η. We can also add a third, fourth, or even more levels in a hierarchy of nested parenthesized words. If we perform the operation of substitution on each of the levels, we shall get the same result, independently of the order on the levels that we may choose. Due to this, we say that the substitution product η is associative.

The family of parenthesized words is an operad. We shall see in Chap. 4 that this operad was "counted" by Hipparchus as early as around 160 BC. Now, we can give an intuitive and informal description of a what a set operad is. A set operad consists of the following data:

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- A family of labeled combinatorial structures
- An "associative" mechanism η that creates larger structures from smaller, using as assembler an external structure in the same family
- Identity structures over the singleton sets

Set operads are the simplest kind of operads. This monograph is aimed at being an introduction of this concept as a first approach to the general subject.

1.2 Some Historical Remarks

Operads first appeared in the context of algebraic topology, in the work of Boardman and Vogt [BV68] under the name of *operators in normal form* and of May [May72] where the name *operad* was introduced. The interested reader is referred to the introduction of [MSS07] for an extended historical account on the origins of operads.

Set operads on the other hand, have had an independent and informal life. From decomposition theory (which we discuss below) initiated in the 1960's, [BE65, Sha61, Sha63, Sha67], to the combinatorial interpretation of umbral calculus by Reiner [Rei78], many examples of set operads have been considered. Although general techniques were developed, no acknowledgment of the implicit general subjacent structure was given until 1981. Joyal in [Joy81], gave a general brief definition of set operads in the context of the theory of species and the monoidal structures that implicit its operations. Using Joyal's approach, a general way of constructing posets from cancellative operads was introduced in [Mén89, MY91].

After the breakthrough brought along by the introduction of the Koszul duality for operads by Ginzburg and Kapranov [GK94], a *renaissance* took place in the study of operads. From the point of view of a combinatorialist, Koszul duality (either for operads or for algebras), can be seen as a sophisticated way of inverting a formal power series, with respect to the operation of substitution for operads, and with respect to the operation of product, for algebras. Of course, it is much more than that, but this pedagogical exaggeration can be used as a motivating starting point. That *renaissance* gave rise to research in many areas, from algebraic topology to theoretical physics [MSS07], which have continued to yield important results to our days. Let us mention only a few in the case that concerns us, set operads that can be constructed by combinatorial methods and that lead to interesting algebraic structures. The introduction of the diassociative Dias [Lod01], Trias [LR04], Quad [AL04] operads. The permutative Perm operad studied in [Cha01], the nonassociative permutative operad NAP [Liv06a], the operad ^{2}Com of two compatible associative commutative algebras [DK07]. More recently, an interesting construction from ordinary monoids to set operads with many applications to combinatorics was given in [Gir14].

Regarding Koszulness, in [Val07], Vallette introduced a method for proving this property for a cancellative operad by studying the Cohen–Macaulay properties of the associated posets. This method was successfully used by Chapoton and Vallette to prove the Koszulness of the Dias and Trias, and their commutative versions Perm and

Comtrias[CV06]. In [CL07] Chapoton and Livernet used the same approach to prove the Koszulness of the NAP operad. In [Str08], Strohmayer proved the Koszulness of the ^{2}Com operad. See also [Val08] where complete families of operads were proved to be Koszul by the poset method.

Set Operads have permeated recently into other areas such as free probability and database theory in computer science. Male introduced a new notion of freeness [Mal13], by means of an algebra constructed from an operad. The structures of this operad are directed graphs with multiple edges and distinguished input and output vertices. His notion of freeness is nontrivial interbreeding between Voiculescu's [Voi85] and classical independence. Spivak has studied in [Spi13], the operad of wiring diagrams and the algebra of relations to model relational databases, plug-and-play devices, and recursion. The operadic point of view brings the advantage of formalizing effectively self-similarity and creation of larger structures from the smaller ones.

1.3 Objectives of this Monograph

This monograph has two main objectives. The first one is to give a self-contained exposition of the relevant facts about set operads, in the context of combinatorial species and their operations. This approach has various advantages; one of them, is that the definition of the basic combinatorial operations on species, sum, product, Hadamard product, substitution and derivative, is simple and natural. They were designed as the set theoretical counterparts of the operations with the same names on exponential generating functions, providing an immediate insight as to their combinatorial meaning. The definition of these operations using the alternative approach of $\mathbb{S}$-sets, that is, sequences of actions of the symmetric groups, $\mathbb{S}_n \times R[n] \to R[n]$, $n \geq 0$, requires the use of representation theory (induced representations) whose combinatorial meaning is not as clear. The same that can be said about operations is also true about operads, a concept whose combinatorial meaning relies on a notion that is intrinsic to the substitution of species, which are the structures placed inside other structures. Moreover, operads usually interact with combinatorial operations. For example, thanks to the chain rule for species, the derivative of an operad is a monoid with respect to the product of species, giving a nice link between operads and associative algebras. The pointing of an operad (distinguishing a vertex on each structure) gives rise naturally to another operad. The product of an operad with the uniform species (equivalent to taking partial structures) is also an operad. The language of species then provides a handy toolbox for a variety of combinatorial constructions. However, up to date there is neither elementary expository work addressing set operads, nor any kind of operads, from this point of view. Even though we use a categorical language, all the concepts are fully explained providing many examples and figures in Chaps. 2 and 3.

The second objective, relating set operads to decomposition theory, is more ambitious. Before formulating it, we present a brief historic account on the sources of this theory. For more than 40 years decompositions of discrete structures have been studied in different branches of discrete mathematics, for example, combinatorial optimization, network and graph theory, switching design or Boolean functions, simple multi-person games and clutters, etc. In 1965 Z. Birnbaum and J. D. Esary [BE65] proved a unique prime factorization theorem for monotonic Boolean functions. In 1967, the 2012 economy Nobel laureate L.T. Shapley proved a similar unique factorization for simple multiperson games [Sha67]. That same year T. Gallai [Gal67] gave a prime decomposition theorem for simple graphs. Gallai used his decomposition to prove a necessary and sufficient condition to recognize transitive orientable graphs, and exhibited an algorithm running in polynomial time to that end. Since then, a vast literature has flourished in what is now called "modular decomposition theory," or more concisely *decomposition theory*. Many efficient algorithms have been devised to solve particular decomposition problems in graph theory (see for example, [JSC72, HM79, CH94, MS94, TCHP08, DGM01, HPV99, BXHLdM09]). At the same time, decomposition theorems have been used to design a variety of divide and conquer algorithms, including one for drawing complex networks [PV06]. Modular decomposition of linear orders have been used in problems related to comparative genomics, in order to measure the evolutive distance between the genomes of two chromosomes (see for example, [BCdMR08]). To decompose or factorize a combinatorial structure means to always place the "factors," that are smaller structures, in the nodes of a tree, each factor being either "trivial" or "prime".

We can now formulate our second objective, which is to recast decomposition theory into the more general framework of set and algebraic operads. Within this framework, the terms "factor," "trivial," and "prime," have a general and precise meaning without specifying the family of structures we are dealing with. In Chap. 4 we survey many of the results of modular decomposition theory, integrated into the context of combinatorial operations with species and set operads, and interpret prime factorization in terms of decorated Schöder trees. By introducing the operation of amalgam between operads, we extend the classical notion of unique factorizable structures.

In Chap. 5 we study $\mathscr{L}$-species (classes of rigid structures), families of structures whose subjacent sets are totally ordered. In this case there are two kinds of product operations and two kinds of substitutions, ordinal (see [Joy81]) and shuffle (see [LV89, BLL98], and references therein). These two substitutions give rise, respectively, to two kinds of operads, nonsymmetric and shuffle. The decomposition of the nonsymmetric operad of permutations (see [AS02, AA05, BHV08]) and many others coming from the symmetric world follow the same general amalgam recipe of Chap. 4.

In Chap. 6 we study Koszulness of cancellative operads. In this context, by removing the hypothesis on homogeneity of generators, we present a generalization of Vallette's criterion relating Koszulness with Cohen–Macaulay posets from cancellative operads.

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