Chapter 2
Preliminaries on Species and Set Operads

In discrete mathematics it is usual to work with classes or families of finite labeled structures, for example, relational structures like simple undirected graphs, networks, lists, partially ordered sets, and general k-ary relations, and set systems like a set of parts of a set, hypergraphs, finite topological spaces, and systems of winner coalitions in game theory.

We usually hope these classes will be closed by relabeling. That is to say, for example, if we arbitrarily change the labels of the vertices of a graph, we expect the resulting structure to be a graph; it does not matter if the labels are in a set of cows, flowers, distinguishable fleas, or blocks of a set partition. Let us begin by saying that, informally, *a combinatorial species is a class of finite labeled structures that is closed by relabeling*. For example, a graph $G = (V, \mathcal{E})$ comes with a set of labels V for its vertices, and a set of unordered pairs of vertices $\mathcal{E}$ (edges) that defines the structure over V. If we change the labels, that is, if we define a bijection f between V and other finite set W, G is automatically refurnished by f in a new graph whose labels are in W and whose edges are transported accordingly $\{a, b\} \mapsto \{f(a), f(b)\}$, for $\{a, b\}$ in $\mathcal{E}$. Let us now put together all the graphs sharing the same set of labels V, and denote this set by $\mathscr{G}[V]$. This is a finite set and any bijection $f : V \to W$ induces another one, that we denote by $\mathscr{G}[f] : \mathscr{G}[V] \to \mathscr{G}[W]$, and that transports by f, each graph of $\mathscr{G}[V]$ into a graph of $\mathscr{G}[W]$ (see Fig. 2.1). This example leads us to the formal definition of a combinatorial species.

We denote the category (groupoid) of finite sets and bijections by $\mathbb{B}$, and the category of finite sets and arbitrary functions by $\mathbb{F}$.

Definition 2.1 A *combinatorial species* is a covariant functor from the category $\mathbb{B}$ to the category $\mathbb{F}$.

This short definition involves a lot of information. For a given species R, we have

- For each finite set V, a finite set $R[V]$, the structures in the class R (R-structures) with labels in V. We set $R[n] := R[\{1, 2, \ldots, n\}]$.

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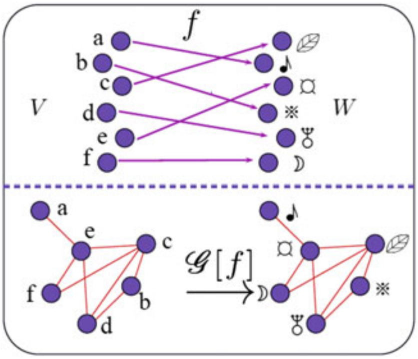

Fig. 2.1 Relabeling of a graph by a function f from $V = \{a, b, c, d, e, f\}$ to the exotic set of labels $W = \{$🍃, ♪, ¤, ※, ☿, ☽$\}$

- For a bijection $f : V \to W$, a function $R[f] : R[V] \to R[W]$, such that

$$R[1_V] = 1_{R[V]} \tag{2.1}$$

$$R[g \circ f] = R[g] \circ R[f]. \tag{2.2}$$

1_A being the identity of a set A, f, and g arbitrary composable bijections $V \xrightarrow{f} W \xrightarrow{g} U$. Note that $R[f]$ has to be a bijection, since $R[f] \circ R[f^{-1}] = R[f \circ f^{-1}] = R[1_V] = 1_{R[V]}$. We shall represent a generic R-structure as in Fig. 2.2.

As a consequence of the definition, the symmetric group of permutations of V, $\mathbb{S}_V$, acts on $R[V]$. This yields two things. In the first place, we get a class of actions of the symmetric groups

$$\mathbb{S}_V \times R[V] \to R[V], V \in \mathbb{B}.$$

On the other hand, since any bijection $f : V \to W$ connects the groups $\mathbb{S}_V$ and $\mathbb{S}_W$, a permutation σ in $\mathbb{S}_V$ is transported by conjugation with f into a unique permutation τ in $\mathbb{S}_W$, $\tau = f \circ \sigma \circ f^{-1}$. Hence, in the second place, we get that the actions for two sets of the same cardinal are connected by R,

$$R[\sigma] = R[f^{-1}] \circ R[\tau] \circ R[f] = R[f]^{-1} \circ R[\tau] \circ R[f],$$

and what we obtain is a sequence of actions of the symmetric groups $\mathbb{S}_n$

$$\mathbb{S}_n \times R[n] \to R[n], n \geq 0.$$

We will call such sequence of actions as an $\mathbb{S}$-set. Considering $\mathbb{S}$-sets instead of species means that we impose a limitation over our structures: we forbid them to wear labels different from those in the sets $[n]$, $n \geq 0$. We shall see that keeping this awkward restriction would bring us unnecessary complications when defining combinatorial operations and operads.

But this situation can be reversed by the following procedure. Denote by $\mathrm{Bij}[n, V]$ the set of bijections from $[n]$ to V, n being the cardinal of V. The set $R[n] \times \mathrm{Bij}[n, V]$

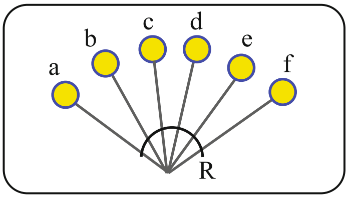

Fig. 2.2 Representation of a generic R-structure with labels in the set $V = \{a, b, c, d, e, f\}$

consists of pairs (r, f), with r as an element of $R[n]$, an R-structure with labels in $[n]$, and f a relabeling giving r the right to have another set of labels. But now r has labels in two sets. We erase the first set of labels by taking the equivalence relation $\sim_n$ on $R[n] \times \mathrm{Bij}[n, V]$ defined by $(r, f) \sim_n (r', g)$ if there exists σ in $\mathbb{S}_n$ such that $(r', g) = (R[\sigma]r, f \circ \sigma^{-1})$. If we now define,

$$R[V] := R[n] \times \mathrm{Bij}[n, V]/ \sim_n, \tag{2.3}$$

we get a species back. We leave to the reader to define the transport of structures by a function and to check the functorial properties of R.

The exponential generating function of R

$$R(x) = \sum_{n=0}^{\infty} |R[n]| \frac{x^n}{n!} \tag{2.4}$$

contains all information regarding the cardinal of the set of structures in each $R[V]$ of the class R. Operations on species (sum, product, substitution, and derivative) were designed by mimicking the analogous operations on generating functions. We will deal with these operations in the next chapter, but in order to talk a bit about set operads, we will have a glimpse at the operation of substitution.

The substitution of species formalizes the notion, present in many combinatorial constructions, of finite structures that are "inside" other finite structures. For example, a complex computer program consists of a number of subroutines doing partial jobs inside a master program that integrates all of them. A complex electric circuit is better understood if we "factor" it as a "circuit of circuits," a simple external circuit connecting smaller circuits inside it.

Analogously, given two species H and R and a finite set V, the structures in the set $H(R)[V]$ are constructed as follows. Take a set partition of the set V, place an R-structure on each of the blocks of the partition (internal structures), and then, place an H-structure over the set of blocks (the external structure). Remember that blocks of a set partition, as the elements of any other set, can be used as labels. The operation of substitution can be performed as many times as you like and, so doing, we get a formal definition for hierarchically nested structures. For example, circuits that are elements of a bigger circuit, that are elements of a bigger circuit, and so on.

It is useful to think of a set operad $\mathcal{O}$ as a species together with a self-reproducing mechanism $\mathcal{O}(\mathcal{O}) \xrightarrow{\eta} \mathcal{O}$, that assembles a set of $\mathcal{O}$-structures (pieces) using an external $\mathcal{O}$-structure (assembler or pattern), obtaining a bigger one in the same class.

This mechanism η satisfies axioms of associativity and existence of the identity, the same structural properties satisfied by an ordinary monoid. This fact will be extensively used in the following chapters.

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