Surfaces of Revolution 2

A surface of revolution is generated by rotation of a plane curve $z = f(x)$ about an axis Oz called *the axis of the surface of revolution*. The resulting surface therefore always has *azimuthal symmetry*. Hence, an explicit equation of a surface of revolution can be presented in the following form:

$$z = f(r) = f(\sqrt{x^2 + y^2}),$$

where $r = \sqrt{x^2 + y^2}$ is the distance a point of the surface from the axis of rotation. Right cylindrical and conical surfaces are examples of surfaces generated by a straight line when the line is coplanar with the axis, as well as hyperboloids of one sheet when the line is skew to the axis. *A sphere* is a surface of revolution of a circle around an axis which runs through the center of the circle. If the circle is rotated about *a coplanar axis*, not crossing the circle, then it generates *a torus*.

Meridians are the lines of intersections of a surface of revolution with planes passing through an axis of rotation. All meridians of one surface of revolution are congruent to the rotated curve. A plane passing through the axis of the surface of revolution is called *the meridian plane*. It is *the plane of symmetry* of the surface. Any surface of revolution has the infinite number of planes of symmetry. *Parallels* are the lines of intersection of the surface with planes orthogonal to an axis of rotation. Meridians and parallels of a surface of revolution are the lines of principal curvatures. Any normal of surfaces of revolution intersects its axis of rotation. A surface of revolution having more than one axis of rotation is *a sphere* or *a plane*.

Tangents to all meridians in the points located on one parallel circle are lines on *the tangent conical* (or *cylindrical*) *surface of revolution*, which is created by the revolution of the tangent about the axis of the rotation. A vertex of the tangent conical surface is located on the axis of revolution.

A parallel is called *the neck circle*, if tangent planes to the surface of revolution in the points on this circle are parallel to the axis of revolution and the tangent cylindrical surface is located inside the surface of revolution. A parallel is called *the equator circle*, if tangent planes to the surface of revolution in the points on this circle are parallel to the axis of revolution and the tangent cylindrical surface is located outside the surface of revolution. A parallel is called *the crater circle*, if tangent plane to the surface of revolution in the points on this circle is perpendicular to the axis of revolution and normal to the surface of revolution in the points of this parallel are parallel to the axis of revolution and form the normal cylindrical surface.

Umbilical points of a surface of revolution are placed on those latitudes on which *a center of curvature* of a meridian is located on the axis of rotation. *Sphere* is umbilical surface. Under Alexis-Claude Clairaut theorem, the product of a radius of a parallel into cosines of an angle of intersection of the geodesic line with the parallel is constant along the geodesic line.

A surface of revolution admits *bending* into another surface of revolution and a net of lines of principal curvatures is remained.

Parametrical equations of arbitrary surface of revolution are

$$\mathbf{r} = \mathbf{r}(r, \beta) = r\cos\beta\mathbf{i} + r\sin\beta\mathbf{j} + f(r)\mathbf{k}.$$

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_2, © Springer International Publishing Switzerland 2015

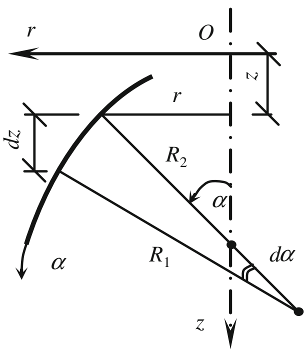


Fig. 1

Assume an equation of a meridian in the form $r = r(\alpha)$ where α is the angle of the normal to the surface passing through a given point with the axis of rotation (Fig. 1) then $r = R_2 \sin\alpha$. Coefficients of the fundamental forms of the surface of revolution can be obtained with the help of formulas:

$$A = A(\alpha) \ = R_1\alpha,\ B = B(\alpha) \ = r = R_2 \sin\alpha,\ F = \ 0;$$
$$L = R_1(\alpha),\ M = \ 0,\ N = R_2 \sin\alpha,$$

where R_1 is the principal radius of curvature of the meridian that is the coordinate line of α, R_2 is the principal radius of curvature of the parallel. The lines α = const are parallels and the lines β = const are meridians.

If an equation of a meridian is given in the form $r = r(z)$ (Fig. 1) then an equation of a surface of revolution can be written with the help of three scalar equations:

$$x = r\ \sin\beta, \quad y = r\cos\beta, \quad z = z$$

where $r = r(z)$ is a function that determines the shape of the meridian (*a profile curve*); β is the angle of rotation of the plane of the meridian and then

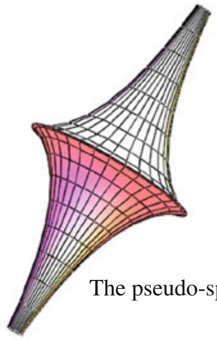
$$A = \sqrt{1 + r'^2}, \quad F = 0, \quad B = r(z),$$
$$k_1 = \frac{1}{R_1} = -\frac{r''}{\left(1 + r'^2\right)^{3/2}}, \quad k_2 = \frac{1}{R_2} = \frac{1}{r\sqrt{1 + r'^2}},$$

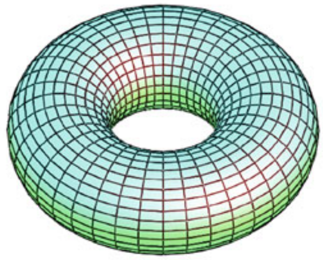
where the derivatives with respect to z are denoted by primes; k_1, k_2 are principal curvatures of the surface. A normal curvature of a surface in the direction of the meridian is equal to a curvature of the meridian, i.e., k_1. Meridians of surface of revolution are geodesic lines.

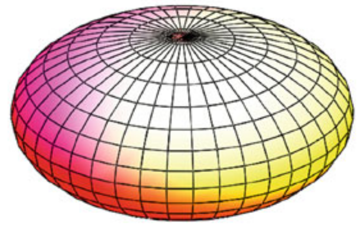
Catenoid is the only one *minimal surface of revolution*. *One-sheet hyperboloid of revolution, right circular cylinder* and *right circular cone* are *the only ruled surfaces*. The last two surfaces are the only developable surface of revolution. If a beginning and an end of unclosed rotated line are placed on an axis of rotation then the surface of revolution will be the closed one.

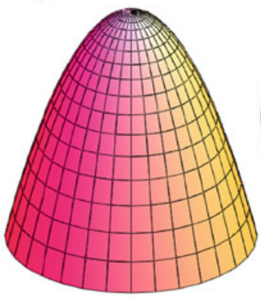
A great deal of surfaces of revolution exists and is studied in different scientific publications. Tens of surfaces of revolution are presented in this encyclopedia and shown on pages 101–104. Such surfaces of revolution as “Lochdiskus”, “Jet Surface”, “Apple Surface”, “Kidney Surface”, “Fish Surface”, “Limpet Torus”, Darwin-de Sitter spheroid, and others are known but used less and may be found in other original sources.

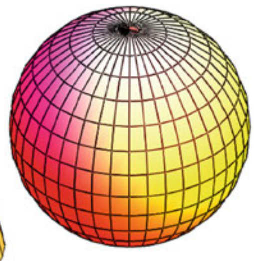
■ Surfaces of Revolution Presented in the Encyclopedia

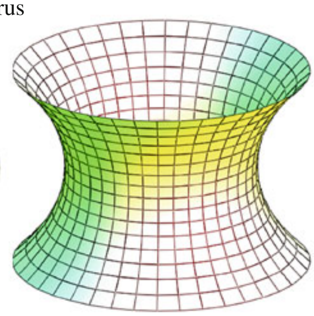

The pseudo-sphere

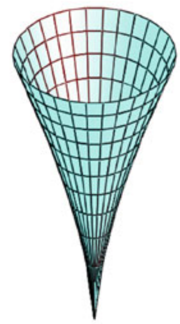

The circular torus

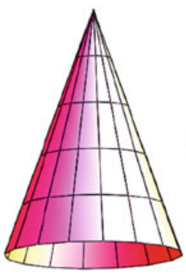

The ellipsoid of revolution

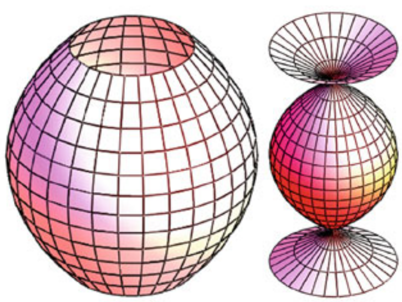

Paraboloid of revolution

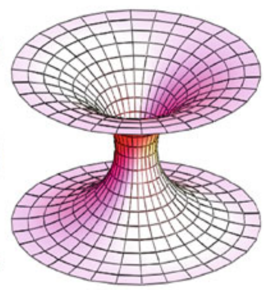

The spherical surface (sphere)

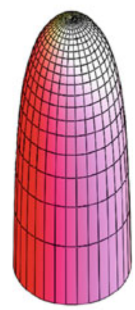

The catenoid

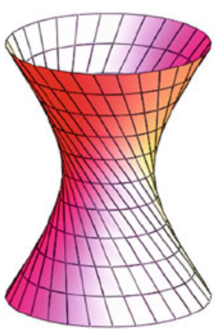

The surface of rotation of a Neil's parabola

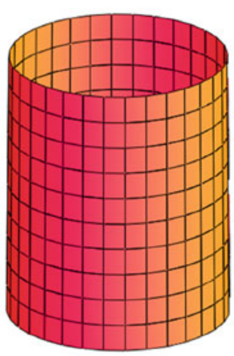

Conical surface of revolution

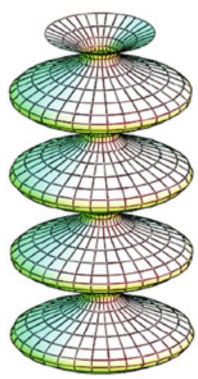

Surface of revolution of a parabola

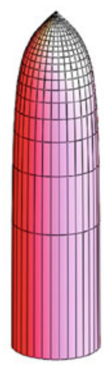

Globoid (toroid)

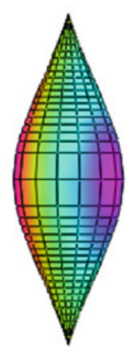

The Kappa surface

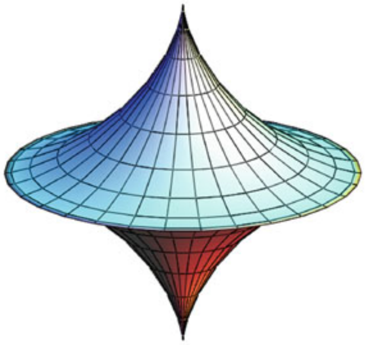

The one-sheet hyperboloid of revolution

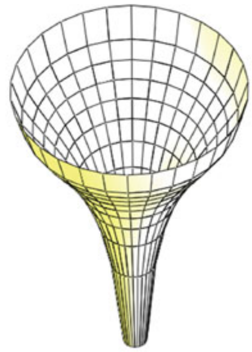

The cylindrical surface of revolution

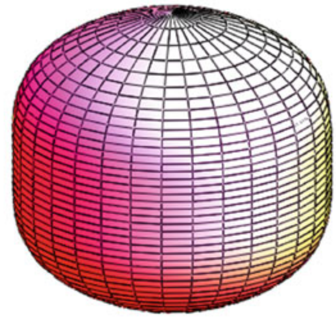

Corrugated surface of revolution of a common sinusoid

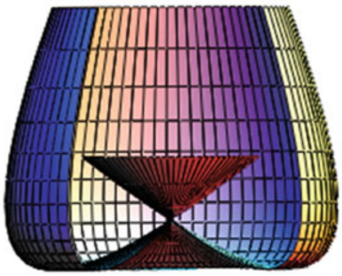

The bullet nose

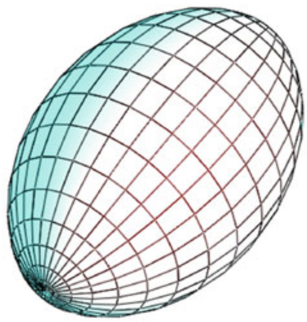

The surface of revolution of a Agnesi curl

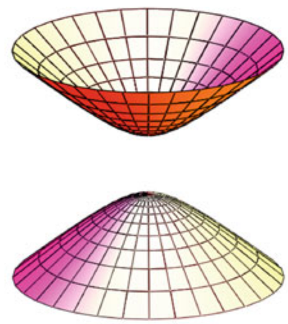

The surface of revolution of a astroid

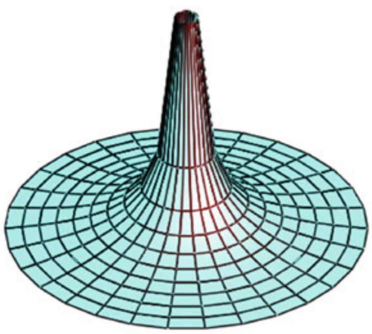

The surface of revolution of the Agnesi curl

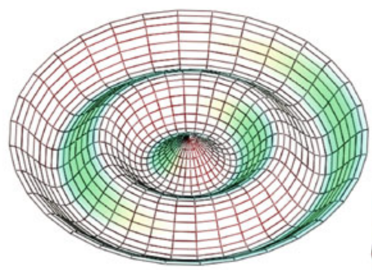

Surface of revolution of the biquadrate parabola

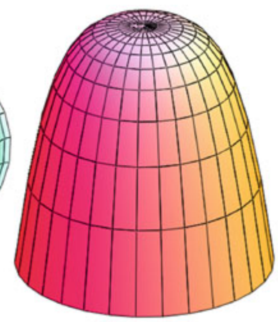

Surface of revolution of the parabola of arbitrary position

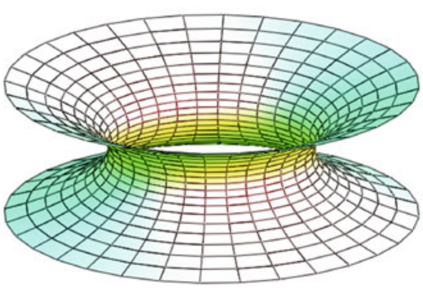

The surface of revolution of a cycloid


Two-sheeted hyperboloid of revolution

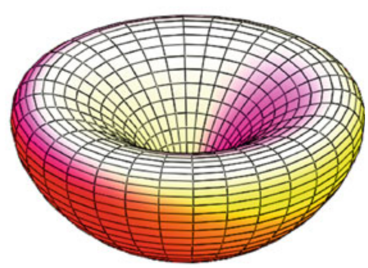

The surface of revolution of a hyperbola $z = b/x$ around the Oz axis

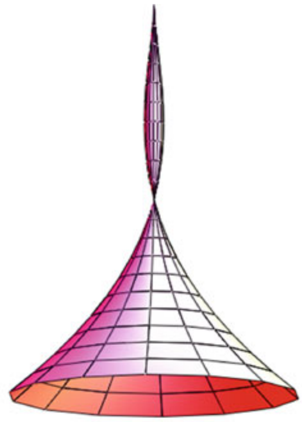

The surface of revolution of a sinusoid


The fourth order paraboloid of revolution

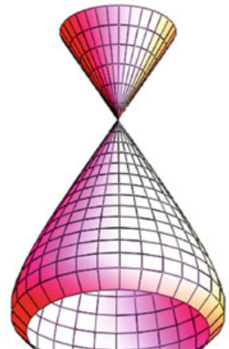

The pseudo-catenoid


“Penka”

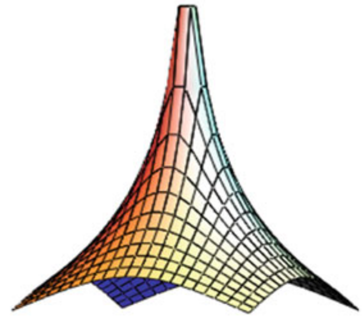

The elliptic torus



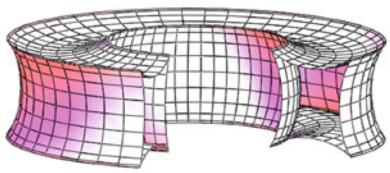
The hyperbolic-and-logarithmic surface of revolution



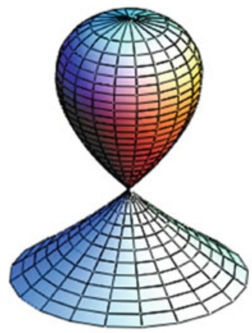
The parabolic-and-logarithmic surface of revolution



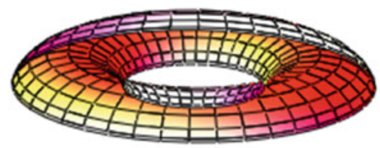
Surface of revolution given by a harmonic function $z = \ln[x^2 + y^2]^{1/2}$



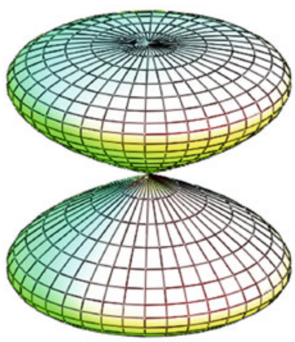
The astroidal torus



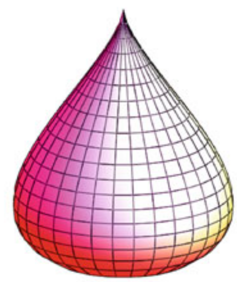
The Ding–Dong surface



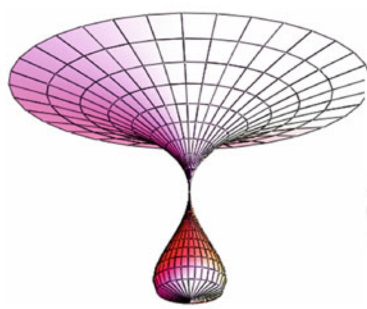
The cycloidal torus



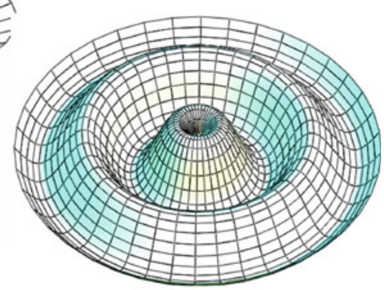
“Eight surface”



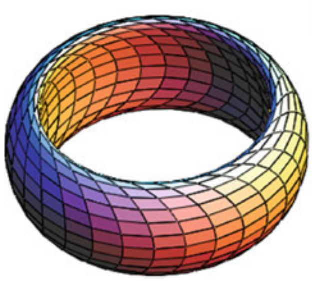
Surface of revolution «Pear»



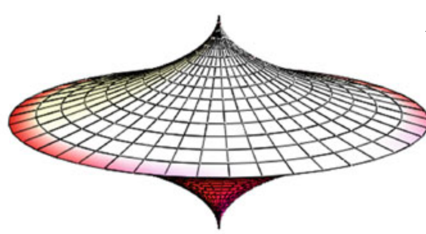
“Kiss surface”



The surface of revolution with damping circle waves

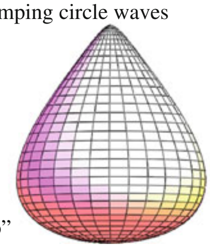


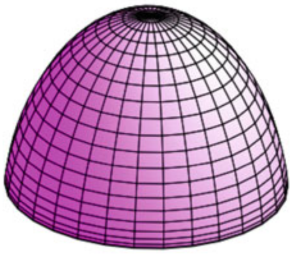
The cyclic surface of revolution “Wedding-ring”



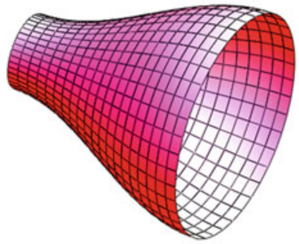
The parabolic humming-top

“Drop”

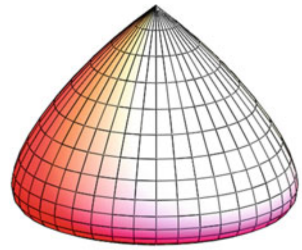




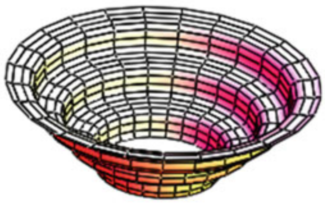
Fairing of cycloidal type



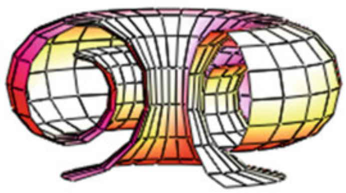
The surface of conjugation of two coaxial cylinders of different diameters



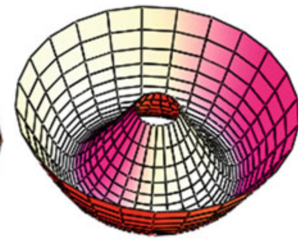
The deformed sphere



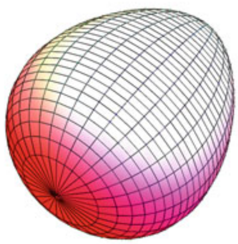
Surface of revolution of the inclined sinusoid



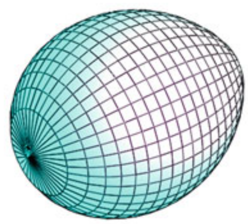
Surface of revolution of the evolvent of the circle



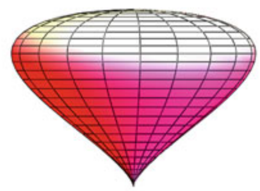
Surface of revolution of the hyperbola of arbitrary position



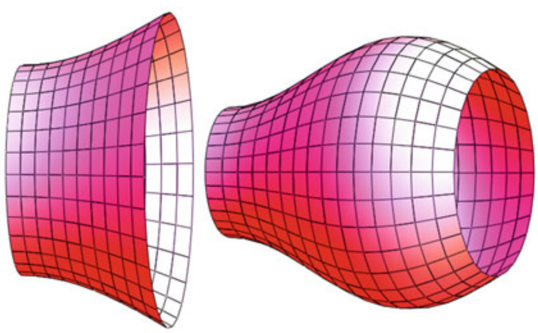
Surface of revolution "Egg" of the third order



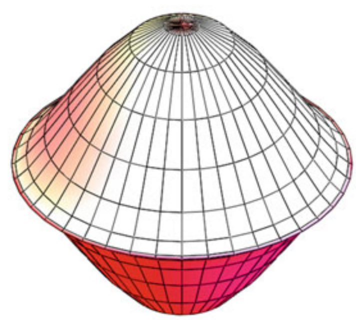
Surface of revolution "Egg" of the fourth order



The Piriform Surface



The surface of conjugation of the coaxial cylinder and the cone



Soucoupoid

Additional Sources

Parametrische Flächen und Körper. http://www.3d-meier.de/tut3/

http://www.wolframalpha.com/input/?i=surface+of+revolution (2014).

■ One-Sheet Hyperboloid of Revolution

One-sheet hyperboloid of revolution is generated by the rotation of a hyperbola

$$x^2/a^2 - z^2/c^2 = 1$$

about the Oz axis (Fig. 1). These are *twice ruled surfaces*. Through every point of the surface, two straight lines, lying on the hyperboloid, pass (Fig. 2). A hyperboloid can be constructed by rotation of a generatrix straight line about the Oz axis but the straight generatrix and the axis are skew lines (Figs. 3 and 4). The surface is the only one *ruled surface of revolution of negative Gaussian curvature*. The parallel lying in a plane $z = 0$ has a radius $r = a$ and is called *a waist circumference* that represents *a geodesic line*. All of the rest of the geodesic lines besides the equator go from infinity coming nearer to the equator. One of them intersects the equator and goes to other half of the surface but others do not reach the equator and touching the some parallel, turn back; the third geodesic lines come nearer asymptotically to the equator.

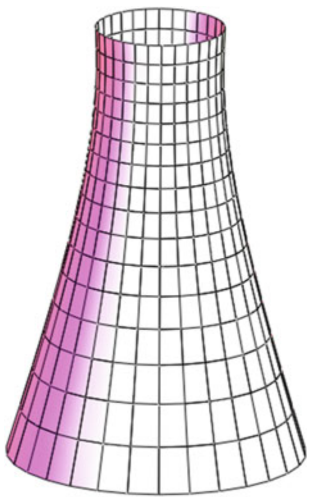


Fig. 1

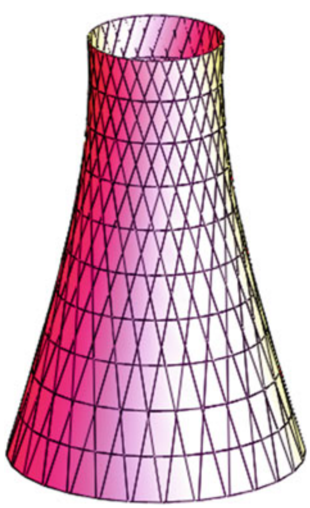


Fig. 2

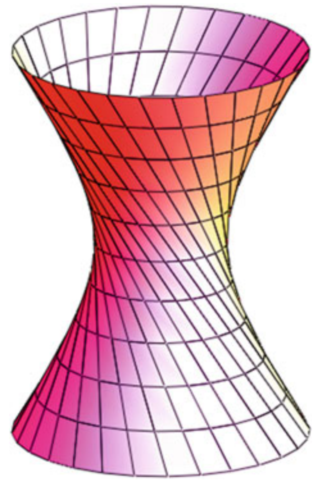


Fig. 3

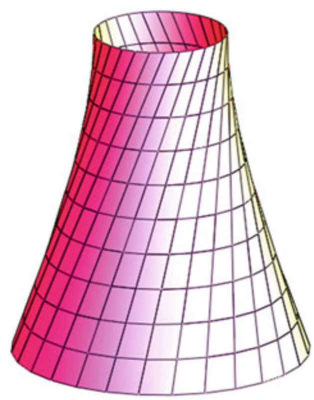


Fig. 4

Forms of definition of one-sheet hyperboloid of revolution

(1) Implicit equation (canonical equation):

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1.$$

If $a = c$, then a hyperboloid is called *a right hyperboloid*.

(2) Parametrical equations (Figs. 3 and 4):

$$x = x(u, v) = -a \sin u \pm av \cos u,$$
$$y = y(u, v) = a \cos u \pm av \sin u,$$
$$z = z(v) = \pm cv.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2(1+v^2),\ B^2 = a^2 + c^2,\ F = \mp a^2,$$
$$L = \mp ca^2(1+v^2)/(A^2B^2 - F^2)^{1/2},$$
$$M = a^2c/(A^2B^2 - F^2)^{1/2},\ N = 0.$$

Coordinate lines v (u = const) coincide with one system of straight lines but the lines u are the parallels of the hyperboloid of one sheet. In Fig. 3, the hyperboloid is shown with taking into consideration the upper signs in the parametrical equations of the surface. The lower signs are taken into account in Fig. 4.

(3) Parametrical equations (Fig. 1):

$$x = x(r,\beta) = r\cos\beta,\quad y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = c\sqrt{r^2 - a^2}/a.$$

Coordinate lines r and β (parallels and meridians) are the lines of principal curvatures.

(4) Parametrical equations (Fig. 1):

$$x = x(z,\beta) = \frac{a}{c}\sqrt{c^2 + z^2}\sin\beta,$$
$$y = y(z,\beta) = \frac{a}{c}\sqrt{c^2 + z^2}\cos\beta,$$
$$z = z.$$

Coordinate lines z and β (meridians and parallels) are the lines of principal curvatures.

(5) Parametrical equations (Fig. 1):

$$x = x(\beta,\alpha) = a\mathrm{ch}\alpha\cos\beta,\quad y = y(\beta,\alpha) = a\mathrm{ch}\alpha\ \sin\beta,$$
$$\mathrm{z} = \mathrm{z}(v) = c\mathrm{sh}\alpha.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = a\mathrm{ch}\alpha,\quad F = 0,\quad B^2 = a^2\mathrm{sh}^2\alpha + c^2\mathrm{ch}^2\alpha,$$
$$L = -ac\mathrm{ch}^2\alpha/B,\quad M = 0,\ N = ac/B,$$
$$k_1 = -c/(aB),\quad k_2 = ac/B^3.$$

Fig. 5 The planetarium in Saint Louis, USA

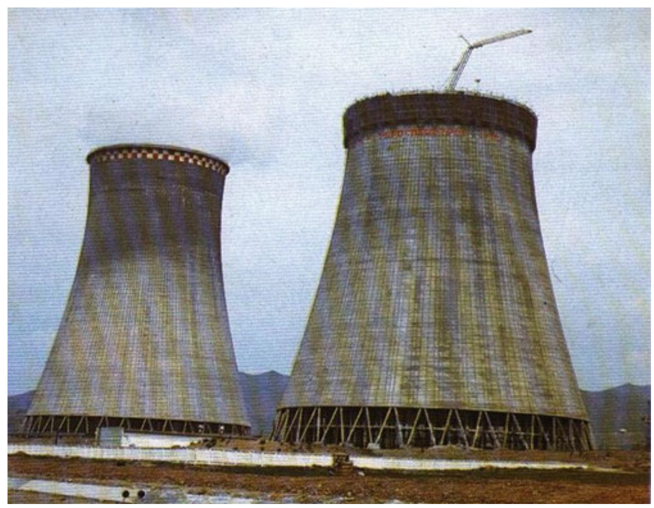


Fig. 6 The Cooling Towers, Uzbekistan

The surface is widely used in civil (Fig. 5) and industrial (Fig. 6) engineering.

Additional Literature

Krivoshapko SN. Static, vibration, and buckling analyses and applications to one-sheet hyperboloidal shells of revolution. Applied Mechanics Reviews. 2002; Vol. 55, No. 3, p. 241-270 (261ref.).

■ Fairing of Cycloidal Type

A surface of a fairing of cycloidal type is formed by the rotation of *a cycloidal curve*

$$x = x(t) = a(t + \sin t), \quad z = z(t) = c(1 + \cos t)$$

about an axis Oz (Fig. 1). If $a = c$, then a generatrix curve becomes a typical cycloid. The form of fairing is defined by a form of meridian that is given with the help of splines. Assume a curve generated by the trajectories of the points of an axis of symmetry of *a limaçon of Pascal* in the process of its rolling along a cycloid as a generatrix curve of a surface of revolution.

Forms of definition of the surface

(1) Parametrical equations (Figs. 1, 2 and 3):

$$\begin{aligned} x &= x(z, \beta) = r(z)\sin\beta, \\ y &= y(z, \beta) = r(z)\cos\beta, \\ z &= z, \end{aligned}$$

where

$$r = r(z) = a\left[\frac{\sqrt{z(2c - z)}}{c} + \arccos\left(\frac{z}{c} - 1\right)\right],$$

β is the angle counted off from the coordinate axis Oy in the direction of the axis Ox; $0 \le \beta \le 2\pi$; $0 \le z \le 2c$. In Fig. 1, it is assumed that $c = 2a$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned} A^2 &= 1 + \frac{a^2 z}{c^2(2c - z)}, \quad F = 0,\ B = r(z), \\ k_1 &= k_z = -\frac{r''(z)}{A^3} = \frac{az}{A^3(2cz - z^2)^{3/2}}, \\ M &= 0,\ k_2 = k_\beta = \frac{1}{AB}. \end{aligned}$$

The contour parallel $z = 0$ is the only geodesic parallel on the surface because the tangent lines to the meridians in its points are parallel to the axis of rotation. Choosing the parameters a и c, it is possible to seek necessary characteristics for a fairing. The ratio of maximum height H of the surface to the diameter ($2r_{\max} = 2a\pi$) of the geodesic parallel and a radius of curvature of the meridian in the frontal point ($z = 2c$) are the main characteristics of the fairing.

A radius of curvature R of the meridians in the frontal point of the surface is defined by a formula:

$$R = \frac{4a^2}{c}.$$

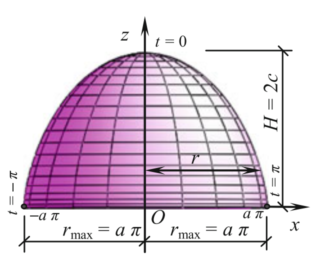


Fig. 1

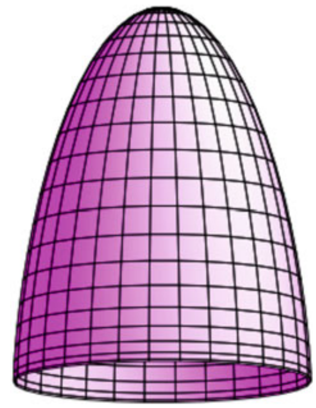


Fig. 2 $c = 4a$

(2) Parametrical equations (Figs. 1, 2 and 3):

$$\begin{aligned} x &= x(t, \gamma) = a(t + \sin t)\cos\gamma, \\ y &= y(t, \gamma) = a(t + \sin t)\sin\gamma, \\ z &= z(t) = c(1 + \cos t), \end{aligned}$$

where γ is the angle counted off from the coordinate axis Ox in the direction of the axis Oy; $0 \le \gamma \le 2\pi$; $0 \le t \le \pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned} A^2 &= a^2(1 + \cos t)^2 + c^2\sin^2 t, \quad F = 0, \quad B = a(t + \sin t), \\ L &= -\frac{ac(1 + \cos t)}{A}, \ M = 0,\ N = -\frac{cB}{A}\sin t, \\ k_1 &= k_t = -\frac{ac(1 + \cos t)}{A^3}, \ k_2 = k_\gamma = -\frac{c\sin t}{AB}. \end{aligned}$$

(3) A particular case of parametrical equations (Fig. 3).

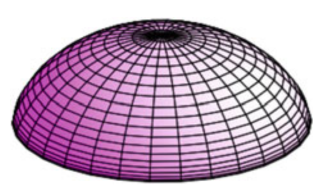


Fig. 3 $c = a$

If one takes $c = a$, then a surface of rotation of a typical cycloid about an axis of Oz will be:

$$x = x(t,\gamma) = a(t + \sin t)\cos\gamma,$$
$$y = y(t,\gamma) = a(t + \sin t)\sin\gamma,$$
$$z = z(t) = a(1 + \cos t).$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 2a^2(1 + \cos t), \quad F = 0, \quad B = a(t + \sin t),$$

$$L = -\frac{A}{2}, \quad M = 0, \ N = -\frac{aB}{A}\sin t,$$

$$k_1 = k_t = -\frac{1}{2A}, k_2 = k_\gamma = -\frac{a\sin t}{AB}, K = \frac{a\sin t}{2A^2B} > 0.$$

References

Krutov AV. On movement defined by centroid-and- trajectory pairs. Izv. vuzov. Mashinostroenie. 2001; No. 2-3, p. 3-6 (11 ref.).

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■ Pseudo-Sphere

Gaussian curvature ($K = k_1k_2$) is equal to a constant negative number, i.e.

$$K = -1/a^2,$$

in all points of *a pseudo-spherical surface* (Figs. 1 and 2). *A pseudo-sphere* or *Beltrami surface* is formed by rotation of a *tractrix* that is *trahere* in Latin, about an axis Oz. A tractrix is an evolvent of the catenary:

$$r = a\mathrm{ch}\frac{z}{a}.$$

Parametrical equations of a tractrix are written as

$$x = a\sin u,$$
$$z = a\left[\cos u + \ln\tan\frac{u}{2}\right],$$

where $0 < u < \pi$, u is the angle of the axis Oz with the tangent to the tractrix.

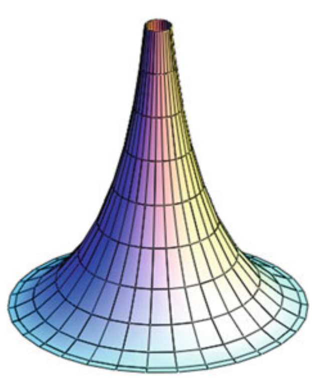


Fig. 1

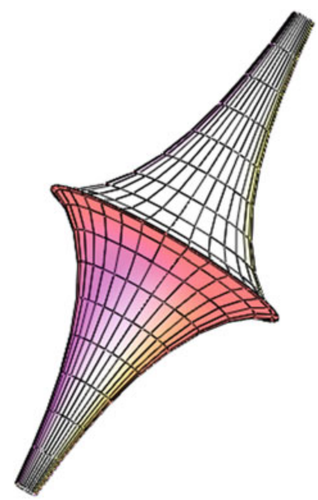


Fig. 2

A tractrix can be defined by an explicit equation:

$$z = a\ln\frac{a \pm \sqrt{a^2 - r^2}}{r} \mp \sqrt{a^2 - r^2},$$

where the upper signs concern the positive branch $z > 0$, lower signs concern the negative branch $z < 0$ (Fig. 2). A length of fragment of the tangent line to the tractrix from the point of tangency till the point of intersection with the Oz axis is constant and equal to $a > 0$. The line of the cross section of a pseudo-sphere by a plane xOy (*an edge of a pseudo-sphere*) is the circle with a radius a, all of the rest of parallels have a less radius r, that is $r < a$.

A volume of one part of a pseudo-sphere is

$$V = \frac{\pi a^3}{3}.$$

The inner geometry of pseudo-sphere coincides locally with the Lobachevski geometry.

Forms of definition of the surface

(1) Parametrical form of definition:

$$x = x(u, v) = a \sin u \cos v,$$
$$y = y(u, v) = a \sin u \sin v,$$
$$z = z(u) = a\left[\cos u + \ln \tan \frac{u}{2}\right],$$

where u is the angle of the axis Oz with the tangent to the meridian. An edge of a pseudo-sphere has $u = \pi/2$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = a \operatorname{co}\tan u, \quad F = 0, \quad B = a \sin u,$$
$$L = -ac \tan u, \quad M = 0, \quad N = a \sin u \cos u,$$
$$k_1 = -\tan u / a, \quad k_2 = \operatorname{co}\tan u / a.$$

Meridians u and parallels v except the edge of the pseudo-sphere ($u = \pi/2$) are the lines of principal curvatures.

(2) Parametrical equations:

$$x = x(r, \beta) = r \cos \beta, \quad y = y(r, \beta) = r \sin \beta,$$
$$z = z(r) = a \ln\left[\left(a + \sqrt{a^2 - r^2}\right)/r\right] - \sqrt{a^2 - r^2},$$

where r is the distance an axis of rotation from a corresponding point of the pseudo-sphere ($r < a$), the circumference $r = a$ is the edge of the pseudo-sphere.

An area of the fragment of a pseudo-sphere between the parallels $r = a$ and $r = r_o$ is

$$S = 2\pi a(a - r_o).$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{a}{r}, \; F = 0, \; B = r,$$
$$L = \frac{a}{r\sqrt{a^2 - r^2}}, \; M = 0, \; N = -\frac{r\sqrt{a^2 - r^2}}{a},$$
$$k_1 = \frac{r}{a\sqrt{a^2 - r^2}}, \; k_2 = -\frac{\sqrt{a^2 - r^2}}{ar}.$$

(3) Parametrical equations:

$$x = x(\gamma, t) = \frac{1}{\gamma} \cos at, \quad y = y(\gamma, t) = \frac{1}{\gamma} \sin at,$$
$$z = z(\gamma) = a \ln\left(a\gamma + \sqrt{a^2\gamma^2 - 1}\right) - \sqrt{a^2 - 1/\gamma^2}.$$

Coefficients of the fundamental forms of the surface:

$$A = B = \frac{a}{\gamma}, \; F = 0,$$
$$L = -\frac{a}{\gamma^2\sqrt{a^2\gamma^2 - 1}}, \; M = 0,$$
$$N = \frac{a\sqrt{a^2\gamma^2 - 1}}{\gamma^2},$$
$$K = -1/a^2 = \text{const}.$$

Here, using the substitution $\gamma = 1/r$ and $t = \beta/a$, we reduced a linear element of the surface to *isothermal form* that is when $A = B$.

Additional Literature

Popov AG. Pseudo-spherical surfaces and some problems of mathematical physics. Fundamental and Applied Mathematics. 2005; Vol. 11, No. 1, p. 227-239.

■ Paraboloid of Revolution

A paraboloid of revolution is created by the rotation of a parabola

$$x^2 = 2pz$$

about an axis z (Fig. 1). The parabolic surface can be generated also by translation of a movable parabola $y^2 = 2pz$ along the fixed parabola $x^2 = 2pz$ (Fig. 2).

The peak of the movable parabola must slide along the fixed parabola but the plane and the axis of the moving parabola must remain parallel. The concavities of the both parabolas must be directed in one side.

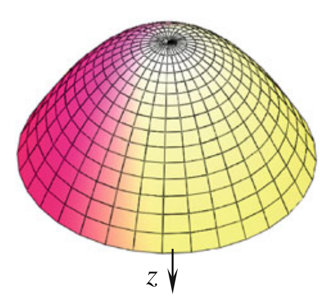


Fig. 1

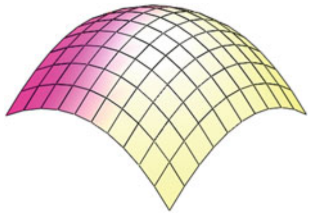


Fig. 2

Paraboloid of revolution possesses the interesting optical property. The light rays coming from the focus after the reflection of them from the surface of the paraboloid will go parallel to the axis of paraboloid of revolution.

Forms of definition of the surface

(1) Explicit form of definition (Fig. 2):

$$2z = (x^2 + y^2)/p.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + \frac{x^2}{p^2}, \quad F = \frac{xy}{p^2}, \quad B^2 = 1 + \frac{y^2}{p^2},$$
$$L = \frac{1}{\sqrt{p^2 + x^2 + y^2}} = N, \quad M = 0, \quad k_{>} = \frac{L}{A^2},$$
$$k_{-} = \frac{L}{B^2}, \; k_1 = L, \quad k_2 = p^2 L^3.$$

On the surface of a paraboloid of revolution, coordinate lines x, y generate *Tchebychef's net*, i.e., every quadrangle formed by the lines of curvilinear coordinate net has equal opposite sides. The coordinate net is non-orthogonal ($F \neq 0$) but conjugate ($M = 0$).

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are much less than one in strength analyses of real shallow shell objects and that is why it is possible to neglect squares of the derivatives in comparison with 1. So, the formulas obtained will take the simplified form for shallow middle surfaces of shells:

$$A = B = 1, \quad F = 0, \quad L = 1/p = N, \quad M = 0,$$
$$k_x = k_y = 1/p.$$

(2) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = r^2/(2p).$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + r^2/p^2, \; F = 0, \; B = r,$$
$$L = 1/(pA), \; M = 0, \; N = r^2/(pA),$$
$$k_1 = 1/(pA^3), \; k_2 = L.$$

(3) Parametrical equations (Fig. 1):

$$x = x(u, v) = a\sqrt{u/h}\cos v,$$
$$y = y(u, v) = a\sqrt{u/h}\sin v,$$
$$z = z(u) = u \;\; \text{where } u \geq 0; \; 0 \leq v \leq 2\pi.$$

The paraboloid has a radius $r = a$ at the height of $z = h$. An area of the lateral surface of a paraboloid of revolution is

$$S = \pi a\left[\left(a^2 + 4h^2\right)^{3/2} - a^3\right]/\left(6h^2\right).$$

A volume of a paraboloid of revolution is $V = \pi a^2 h/2$ if $0 \leq v \leq 2\pi, \; 0 \leq u \leq h$.

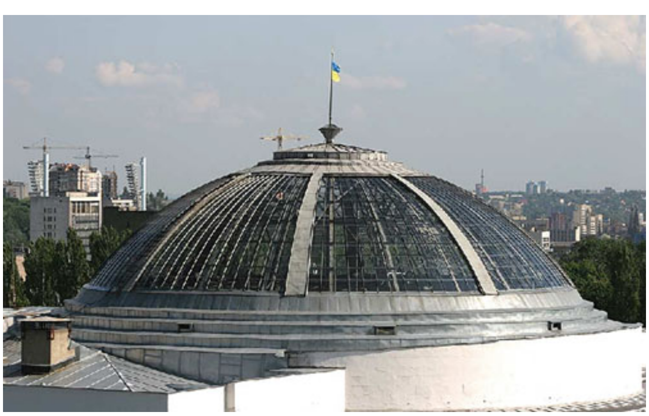


Fig. 3 The glass dome of museum, Kiev, Ukraine

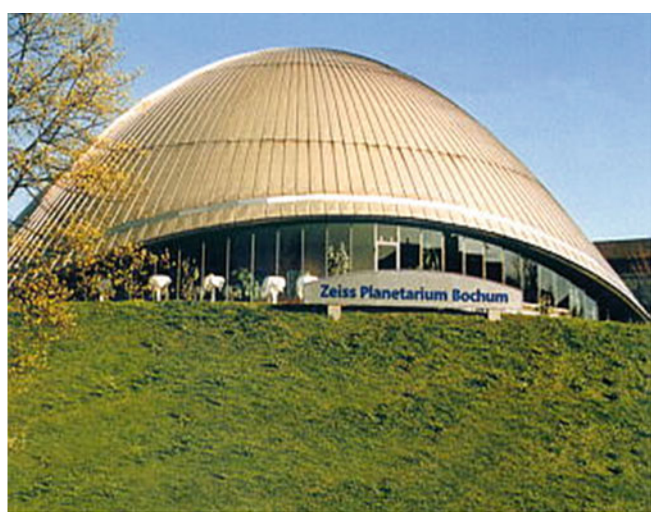


Fig. 4 A planetarium in Bochum, Germany

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2}{4uh},\ F = 0,\ B^2 = \frac{a^2 u}{h},$$
$$L = \frac{a}{2u\sqrt{a^2 + 4uh}},\ M = 0,\ N = \frac{2au}{\sqrt{a^2 + 4uh}},$$
$$k_1 = \frac{L}{A^2},\ k_2 = \frac{N}{B^2}.$$

The surface is widely used in civil (Fig. 3) and industrial (Fig. 4) engineering.

Additional Literature

Krivoshapko SN. Parabolic shells of revolution. Montazhn. i spetz. raboty v stroitelstve. 1999; No. 12, p. 5-12 (63 ref.).

■ Circular Torus

A circular torus or *torus* in Latin is formed by rotation of a circumference

$$(x - a)^2 + z^2 = b^2$$

about an axis Oz. *An open torus* is a torus (Fig. 1) generated by rotation of a circumference about an axis lying outside limit of this circle ($a > b$). *A closed torus* (*Horn Torus*) is a torus generated by rotation of a circumference about an axis touching ($a = b$, Fig. 2) or intersecting ($a < b$, Figs. 3 and 4) the circle. The inner part of surface of an open torus is a surface of negative Gaussian curvature but the outer surface is a surface of positive Gaussian curvature (Figs. 1, 2 and 3).

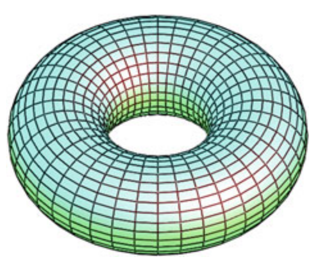


Fig. 1 The torus with $a > b$ (the open torus)

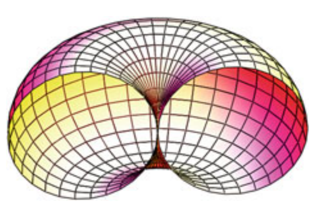


Fig. 2 The torus with $a = b$ (the closed torus)

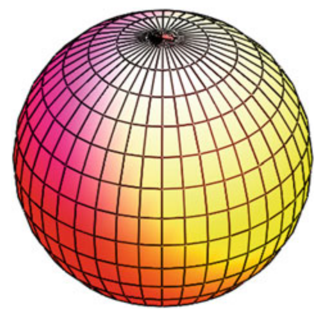


Fig. 3 The torus with $a = 0$ (a sphere)

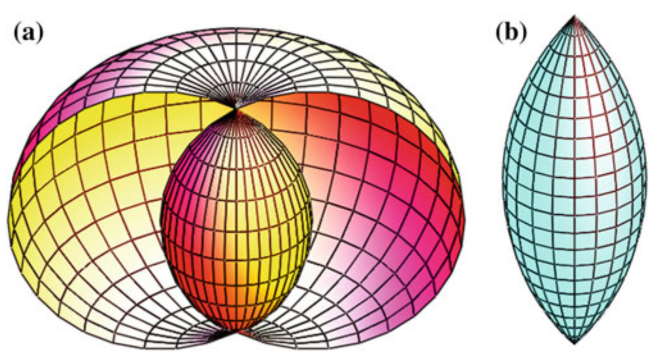


Fig. 4 The torus with $a < b$ (the closed torus)

Forms of definition of the surface

(1) Implicit equations:

$$\left(x^2 + y^2 + z^2 + a^2 - b^2\right)^2 = 4a^2\left(x^2 + y^2\right).$$

(2) Parametrical equations:

$$x = x(u, v) = (a + b\cos v)\cos u,$$
$$y = y(u, v) = (a + b\cos v)\sin u,$$
$$z = z(v) = b\sin v,$$

where a is a radius of the centers of generatrix circles, b is a radius of a generatrix circle, an angle u is called *an inner latitude* of a point of the torus; $0 \le u \le 2\pi$, $0 \le v \le 2\pi$; a ratio b/a is *an eccentricity of torus*. On a circular torus besides parallels and meridians, two families of plane circles, called *Villarceau circles*, exist. They can be seen in the cross sections of a torus by a plane touching the torus at two points. A radius of Villarceau circles is equal to a.

An area of the whole surface of a torus is $4\pi^2 ab$, its volume is $2\pi^2 ab^2$.

Coefficients of the fundamental forms of the surface:

$$A = a + b\cos v, \quad F = 0, \quad B = b,$$
$$L = -(a + b\cos v)\cos v, \quad M = 0, \quad N = -b,$$
$$K = \cos v/(bA).$$

Assume $a < b$ (Fig. 4), then the angle v changes in the limit of

$$-\arccos(-a/b) \le v \le \arccos(-a/b),$$

but if we want to have the torus (*the lemon*) shown in Fig. 4b then we must take

$$\arccos(-a/b) \le v \le 2\pi + \arccos(-a/b).$$

(3) Parametrical equations:

$$x = x(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\cos u,$$

$$y = y(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\sin u,$$

$$z = \frac{b\beta}{\sqrt{a^2 + \beta^2}}, \quad \beta = a\tan\alpha,$$

where α is the angle of the straight line, connecting the center of the generatrix circle of the radius b with arbitrary point of the torus, with a plane $z = 0$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{a(\sqrt{a^2 + \beta^2} - b)}{\sqrt{a^2 + \beta^2}}, \quad F = 0, \quad B = \frac{ab}{a^2 + \beta^2},$$

$$L = -\frac{a^2(\sqrt{a^2 + \beta^2} - b)}{a^2 + \beta^2}, \quad M = 0, \quad N = \frac{a^2 b}{(a^2 + \beta^2)^2},$$

$$k_1 = k_u = -1/(\sqrt{a^2 + \beta^2} - b), \quad k_2 = k_v = 1/b.$$

(4) Parametrical equations of a circular torus if $a = b$ (Fig. 2):

$$x = x(\gamma, u) = \frac{a(\mathrm{ch}\gamma \pm 1)}{\mathrm{ch}\gamma}\cos u,$$
$$y = y(\gamma, u) = \frac{a(\mathrm{ch}\gamma \pm 1)}{\mathrm{ch}\gamma}\sin u,$$
$$z = a\,\mathrm{th}\gamma.$$

Additional Literature

Gulyaev VI, Bazhenov VA, Gotzulyak EA, Gaydaychuk VV. An Analysis of Shells of Complex Form. 1990; Kiev: Budivelnik, 192 p.

Kutzenko GV. Axis-symmetrical deformation of a circular torus. PM. 1979; Vol. 15, No. 11, p. 46-51.

■ Elliptic Torus

An elliptic torus is generated by the rotation of an ellipse of arbitrary position (Fig. 1):

$$x = x(v) = a + r\cos v, \quad z = z(v) = r\sin v,$$

where $r = r(v) = \frac{cb}{\sqrt{b^2\sin^2\beta + c^2\cos^2\beta}}$, $\beta = v - \theta$, about an axis Oz; θ = const is the slope angle of the semi-axis of the ellipse ξ with the plane xOy.

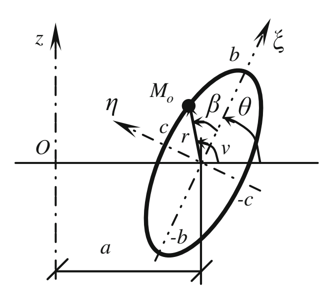


Fig. 1

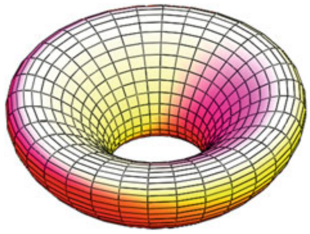


Fig. 2

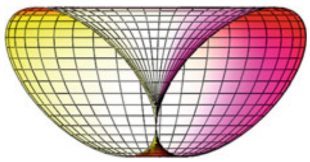


Fig. 3

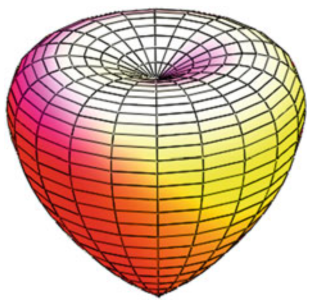


Fig. 4

An open elliptic torus is a torus formed by the rotation of an ellipse about an axis Oz lying outside of the limit of this ellipse (Figs. 1 and 2).

A closed torus is a torus generated by rotation of an ellipse about an axis Oz touching (Fig. 3) or intersecting (Fig. 4) the ellipse.

An ellipse touches an axis of rotation if the condition $\partial x/\partial v = 0$ carries out or

$$r^2\left(c^2 - b^2\right)\ \sin[2(v-\theta)] = 2c^2b^2\tan v.$$

Parametrical equations of the surface have the following form:

$$x = x(u, v) = (a + r\cos v)\cos u,$$
$$y = y(u, v) = (a + r\cos v)\sin u,$$
$$z = z(v) = r\sin v,$$

where a is the radius of the circle generated by the point of the intersection of the axes ξ and η of the generatrix ellipse (Fig. 1); r is the distance the point of the intersection of the ellipse's axes from an arbitrary point M_o belonging to the ellipse; b, c are the semi-axes of the ellipse; $0 \le u \le 2\pi$, $0 \le v \le 2\pi$; u is the angle of the axis Ox with the axis Oy.

If one of the axes of the generatrix ellipse, for example, the ξ axis, is parallel to the axis of rotation Oz, then it is necessary to assume $\theta = \pi/2$. If we take $b = c$, then we shall have $r = b$, $v = \beta$, but an elliptical torus will degenerate into *a circular torus* where a will be a radius of the centers of generatrix circles with the radius of b.

Coefficients of the fundamental forms of the surface:

$$A = a + r\cos v,\ F = 0,$$
$$B^2 = \frac{(b^4\sin^2\beta + c^4\cos^2\beta)r^6}{c^4b^4},$$
$$L = -\frac{A}{B}r\left[\frac{c^2 - b^2}{2c^2b^2}r^2\sin 2\beta\sin v + \cos v\right],$$
$$M = 0,\ N = -\frac{r^6}{c^2b^2B}.$$

Having assumed $a = 0$, we can design *an oblique ellipsoid of revolution* (Fig. 5a, b and c).

Additional Literature

Clark RA, Girloy TI. and Reissner E. Stresses and deformation of toroidal shells of elliptical cross section. J. Appl. Mech. 1953; Vol. 20, No. 4.

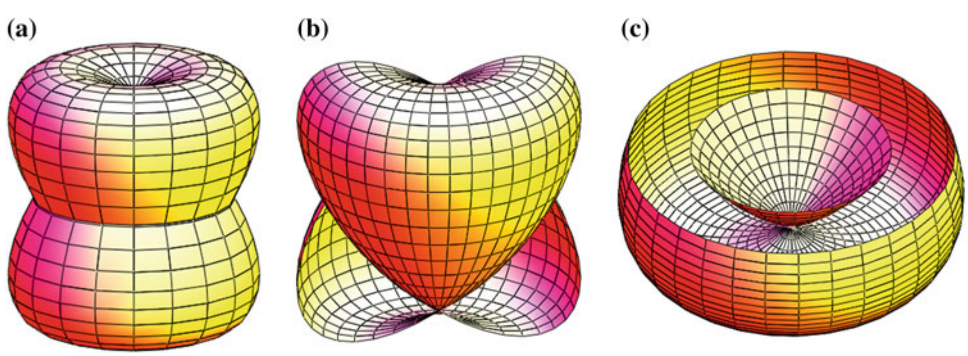


Fig. 5

■ Surface of Revolution of a Curve $z = b\ \exp(-a^2x^2)$ Around the Z Axis

The surface is formed by rotation of a curve $z = be^{-a^2x^2}$ about a coordinate axis z.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u) = u, \quad y = y(v) = v, \quad z = b\ \exp[-a^2(u^2 + v^2)].$$

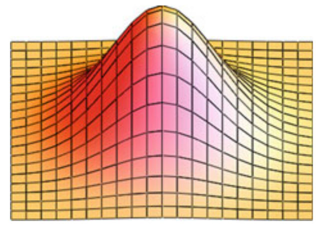
The surface is called «Die Glocke» in German.

(2) Parametrical equations (Fig. 2):

$$x = x(r, \beta) = r\cos\beta; \quad y = y(r, \beta) = r\sin\beta;$$
$$z = z(r) = be^{-a^2r^2},$$

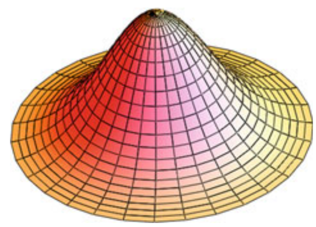
where $0 \le r < \infty$; $0 \le \beta \le 2\pi$; $z \le b$.

(3) An explicit equation (Fig. 1): $z = be^{-a^2(x^2+y^2)}$



$a = b = 1$;
$-2 \le u, v \le 2$m

Fig. 1



$a = b = 1$;
$0 \le r \le 2$m

Fig. 2

■ Two-Sheeted Hyperboloid of Revolution

Two-sheeted hyperboloid of revolution is formed by rotation of a hyperbola

$$-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,$$

about its *focal axis* (an axis Oz). The surface has two separate sheets when the axis of revolution is *the transverse axis.*

A section of a hyperboloid by a plane $z = h > c =$ const gives a circle with a radius $r = a\sqrt{h^2 - c^2}/c$ (Fig. 1). If we cut a hyperboloid by a plane $y = t =$ const, then hyperbolas $z = \pm c\sqrt{a^2 + t^2 + x^2}/a$ will be in the cross section (Fig. 2), but having intersected a hyperboloid by a plane $x = p =$ const, we can have hyperbolas $z = \pm c\sqrt{a^2 + p^2 + y^2}/a$ (Fig. 2).

The peaks of two sheets of hyperboloid are placed at the points with coordinates (0, 0, $\pm c$). The signs correspond two sheets of hyperboloid. Two-sheeted hyperboloid of revolution belongs to a class of *not closed central surfaces of the second order*. It is a particular case of *hyperboloid of two sheets* which is presented in Chap. "35. *Surfaces of the second order.*"

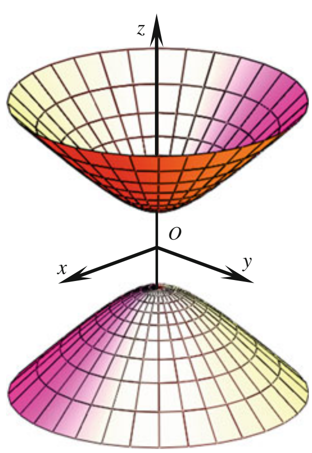


Fig. 1

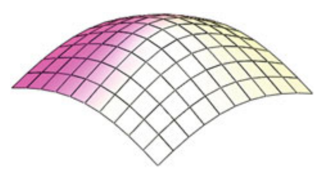


Fig. 2

Forms of definition of the surface

(1) Implicit equation:

$$\frac{-x^2 - y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

where a and c are the semi-axes of a hyperboloid of revolution, $|z| \geq c$; $a^2/c = p$ is a focal parameter of meridian. A hyperboloid is called *a right hyperboloid of revolution* if $a = c$. It is formed by rotation of *an equilateral hyperbola*. *An asymptotical cone* of two-sheeted hyperboloid of revolution is defined by an implicit equation:

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 0.$$

A hyperboloid of revolution is *a quadric surface*.

(2) Explicit equation (Fig. 2):

$$z = \pm \frac{c}{a} \sqrt{a^2 + x^2 + y^2}$$

(3) Parametrical equations (Fig. 1):

$$x = x(u, v) = a\mathrm{sh}u \cos v, \quad y = y(u, v) = a\mathrm{sh}u \sin v,$$
$$z = \pm c\mathrm{ch}u.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = a^2\mathrm{ch}^2 u + c^2\mathrm{sh}^2 u,$$
$$F = 0, \quad B = a\mathrm{sh}u,$$
$$L = \pm \frac{ac}{A}, \quad M = 0,$$
$$N = \pm \frac{ac}{A}\mathrm{sh}^2 u,$$
$$k_1 = \pm \frac{ac}{A^3}, \quad k_2 = \pm \frac{c}{aA}.$$

Coordinate lines u, v are the lines of principal curvatures.

(4) Parametrical equations (Fig. 1):

$$x = x(z, \beta) = r \sin \beta,$$
$$y = y(z, \beta) = r \cos \beta,$$
$$z = z, \quad \text{where} \quad r = \frac{a}{c}\sqrt{z^2 - c^2}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \sqrt{1 + r'^2}, \quad F = 0, \quad B = r(z),$$
$$k_1 = \frac{1}{R_1} = -\frac{r''}{\left(1 + r'^2\right)^{3/2}}, \quad k_2 = \frac{1}{R_2} = \frac{1}{r\sqrt{1 + r'^2}},$$

where the first and second derivatives of r with respect to parameter z are denoted by primes.

(5) A parametrical form of definition with the help of polar coordinates of the meridians (Fig. 1):

$$x = x(\varphi, \beta) = \rho \sin \varphi \sin \beta,$$
$$y = y(\varphi, \beta) = \rho \sin \varphi \cos \beta,$$
$$z = z(\varphi) = \rho \cos \varphi,$$

where

$$\rho = \frac{p}{1 - e \cos \varphi}, \quad p = \frac{a^2}{c}, \quad e = \sqrt{1 + \frac{a^2}{c^2}},$$
$$\theta \leq \varphi \leq \pi + \theta, \quad \cos \theta = \frac{1}{e}.$$

Additional Literature

Vasil'ev AN. Stability of anisotropic two-sheeted hyperboloid of revolution with filling material. Kazan: KFEI, 1991; 14 p., 6 ref., Dep. v VINITI 08.07.91, No. 2887-B91.

Gritskevich OV, *Meshcheryakov NA*, *Pod'yapol'skii YuV*, Precision laser processing of curved surfaces of revolution, QUANTUM ELECTRON. 1996; 26 (7), p. 644-646.

■ Surface of Conjugation of Two Coaxial Cylinders of Different Diameters

A surface of conjugation of two coaxial cylinders of different diameters may be included as a component of the two classes of surfaces. These are a class of cyclic surfaces and a class of surfaces of revolution.

The surface is formed by rotation of the sinusoid about a common axis of two conjugated cylinders (Fig. 1).

Parametrical equations of the surface of conjugation are (Figs. 1 and 2).

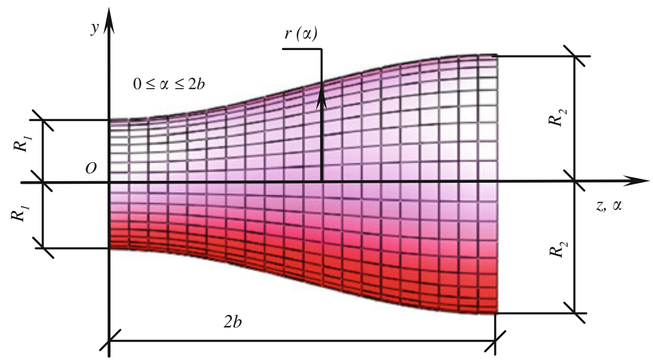


Fig. 1

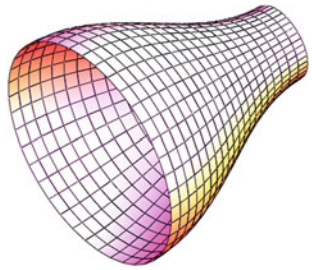


Fig. 2

$$x = x(\alpha, \beta) = r(\alpha)\cos\beta, \quad y = y(\alpha, \beta) = r(\alpha)\sin\beta,$$
$$z = \alpha,$$

where

$$r = r(\alpha) = \frac{R_2 - R_1}{2}\left(1 - \cos\frac{\pi\alpha}{2b}\right) + R_1$$
$$= (R_2 - R_1)\sin^2\frac{\pi\alpha}{4b} + R_1$$

is a law of change of a radius of the studied surface of conjugation along an axis Oz (an axis of rotation); $R_2 \geq R_1$; $0 \leq \alpha \leq 2b$; $2b$ is a length of a segment between two cylinders of different diameters; β is the angle in the planes of parallels taken from the axis Ox in the direction of the axis Oy; $0 \leq \beta \leq 2\pi$.

Two parallels placed in the cross sections $z = 0$ and $z = 2b$ are *geodesic lines*, because the tangent to the meridians at the points of these parallels are parallel to the axis of rotation.

All meridians of the surface of revolution are geodesic lines too.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{\pi^2}{16b^2}(R_2 - R_1)^2\sin^2\frac{\pi\alpha}{2b}, \quad F = 0, \quad B = r(\alpha),$$
$$L = -\frac{\pi^2(R_2 - R_1)}{8b^2A}\cos\frac{\pi\alpha}{2b}, \quad M = 0, \quad N = \frac{B}{A},$$
$$k_\alpha = k_1 = -\frac{\pi^2(R_2 - R_1)}{8b^2A^3}\cos\frac{\pi\alpha}{2b}, \quad k_\beta = k_2 = \frac{1}{AB},$$
$$K = -\frac{\pi^2(R_2 - R_1)}{8b^2A^4B}\cos\frac{\pi\alpha}{2b},$$
$$H = \frac{\pi^2(R_2 - R_1)\{R_2 - R_1 - (R_2 + R_1)\cos[\pi\alpha/(2b)]\} + 16b^2}{32b^2A^3B}.$$

A curvilinear coordinate net is given in lines of principal curvatures α, β. If $R_2 > R_1$, then the surface has a segment of negative Gaussian curvature if $0 \leq \alpha \leq b$ and of positive Gaussian curvature if $b \leq \alpha \leq 2b$. In Fig. 2, the surface of conjugation is shown with

$$R_2 = 3R_1; \quad b = 3R_1; \quad 0 \leq \alpha \leq 2b; \quad 0 \leq \beta \leq 2\pi.$$

The surface in issue is a component of subclass "*Corrugated surface of revolution of a common sinusoid*" contained also in a class "Surface of revolution." A surface of conjugation degenerates into a cylindrical surface of revolution if $R_1 = R_2$.

Additional Literature

Gulyaev VI, Bazhenov VA, Gotzulyak EA, Gaydaychuk VV. An Analysis of Shells of Complex Form. 1990; Kiev: Budivelnik, 192 p.

■ Surface of Revolution "Wellenkugel"

Information about a surface of revolution "Wellenkugel" is presented in sites given in References. This surface has parametrical equations:

$$x = u\cos(\cos u)\cos v;$$
$$y = u\cos(\cos u)\sin v;$$
$$z = u\sin(\cos u).$$

In Fig. 1, the surface with $0 \leq u \leq 14,5$ m; $0 \leq v \leq 1,5\pi$ is shown.

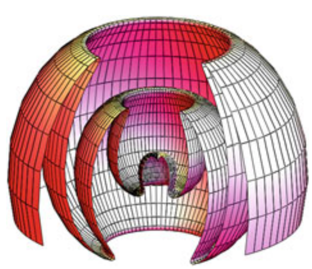


Fig. 1

References

1. Mathematics Museum (Japan). Introduction to Geometry, Ibaraki University, 2002, http://mathmuse.sci.ibaraki.ac.jp/MuseumE.html
2. Parametrische Flächen und Körper.—http://www.3d-meier.de/tut3/Seite63.html

■ Surface of Conjugation of Coaxial Cylinder and Cone

A surface of conjugation of coaxial cylinder and cone is a fragment of *a corrugated surface of revolution of a common sinusoid.* It is formed by rotation of a curve

$$y = a[1 - \cos(2\pi z/c)] + R_1$$

about an axis Oz. Having assumed two necessary conditions

$$\frac{2\pi a}{c}\sin\left(\frac{2\pi b}{c}\right) = \tan\varphi \quad \text{and}$$
$$a\left[1 - \cos\left(\frac{2\pi b}{c}\right)\right] + R_1 = R_2,$$

we may design a surface of conjugation of coaxial cylinder with a radius R_1 and circular cone with the angle φ at the vertex and with a base having a radius R_2 (Fig. 1). So, having six constants R_1, R_2, a, b, c, and φ, one may take four constants as desired but two remaining geometrical constants are derived from the system of two presented equations. Moreover, it is necessary to take $a < 0$ when $R_1 > R_2$.

For example, let us consider that R_1, R_2, c, and φ are given, then the rest two parameters a and b can be obtained with the help of formulas:

$$a = \frac{-1}{R_1 - R_2}\left[\frac{(R_1 - R_2)^2}{2} + \frac{c^2\tan^2\varphi}{8\pi^2}\right];$$
$$b = \frac{c}{2\pi}\arcsin\frac{c\tan\varphi}{2\pi a} \quad \text{if} \quad \varphi > 0,$$

$R_2 > R_1$ (Fig. 1) or $\varphi < 0$, $R_2 < R_1$ and

$$b = \frac{c}{2} - \frac{c}{2\pi}\arcsin\frac{c\tan\varphi}{2\pi a} \quad \text{if} \quad \varphi < 0, \quad R_2 > R_1 \quad \text{or}$$
$$\varphi > 0, \quad R_2 < R_1.$$

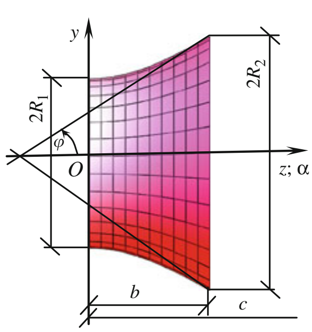


Fig. 1

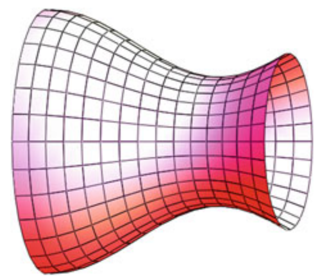


Fig. 2

Forms of definition of the surface

(1) Parametrical equations:

$$x = x(z, \beta) = r\cos\beta,$$
$$y = y(z, \beta) = r\sin\beta,$$
$$z = z$$

where

$$r = r(z) = a[1 - \cos(2\pi z/c)] + R_1;$$

$0 \le z \le b$; $b < c$; $0 \le \beta \le 2\pi$ (Figs. 1 and 2).

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{4\pi^2 a^2}{c^2}\sin^2\frac{2\pi z}{c}, \; F = 0, \; B = r(z),$$
$$L = -\frac{4a\pi^2}{c^2 A}\cos\frac{2\pi z}{c}, \; M = 0, \; N = \frac{r}{A}.$$
$$k_1 = k_z = -\frac{4a\pi^2}{c^2 A^3}\cos\frac{2\pi z}{c}, \; k_2 = k_\beta = \frac{1}{rA},$$
$$K = -\frac{4a\pi^2}{c^2 r A^4}\cos\frac{2\pi z}{c}.$$

All meridians and also the parallels $z = 0$, $z = c/2$, and $z = c$ on surface of a coaxial cylinder and a cone are geodesic lines. The surface of conjugation contains fragments of positive Gaussian curvature in the limits of $c/4 < z < 3c/4$ if $a > 0$ and fragments of negative Gaussian curvature in the limits of $0 < z < c/4$ and $3c/4 < z < c$ if $a > 0$.

The surface of conjugation shown in Fig. 1 has the following geometrical parameters: $R_2 = 1.5R_1$, $c = 4R_2$, and $\varphi = \pi/6$.

The surface of conjugation with $R_1 = 1.5R_2$, $c = 4R_2$, and $\varphi = \pi/6$ is presented in Fig. 2.

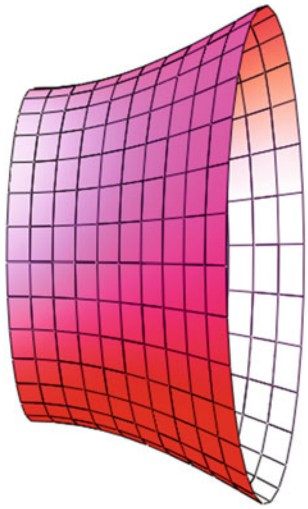


Fig. 3

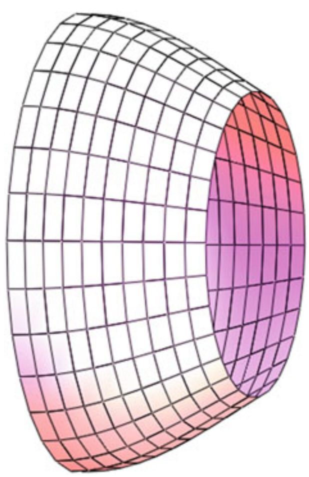


Fig. 4

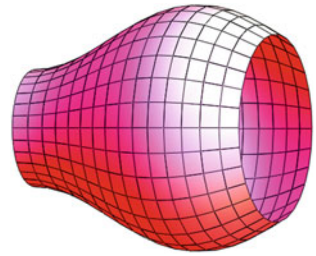


Fig. 5

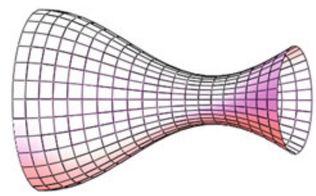


Fig. 6

(2) Parametrical equations:

$$\begin{aligned} x &= x(z, \beta) = r\cos\beta, \\ y &= y(z, \beta) = r\sin\beta, \\ z &= z, \\ r &= r(z) = a[1 - \cos(2\pi z/c)] + R_1; \\ a &= R_2 - R_1, \end{aligned}$$

where $b = c/4$; $c = 2\pi a/\tan\varphi$ if $\varphi > 0$, $a > 0$ (Fig. 3) or $\varphi < 0$, $a < 0$ (Fig. 4) and $b = 3c/4$; $c = -2\pi a/\tan\varphi$ if $\varphi < 0$, $a > 0$ (Fig. 5) or $\varphi > 0$, $a < 0$ (Fig. 6).

Coefficients of the fundamental forms of the surface are defined by the formulas given for the first variant.

The surfaces shown in Figs 1, 2, 3, 4, 5 and 6 are constructed when $|\varphi| = \pi/6$.

Reference

Krivoshapko SN. Model surfaces of connecting fragments of two pipe lines. Montazhn. i spetz. raboty v stroitelstve. 2005; No.10, p. 25-29.

■ Surface Formed by Rotation of a Meridian in the Form of Semicubical Parabola

A surface is generated by rotation of *a semicubical parabola* $z = bx^{2/3}$ (*Neil's parabola*) about an axis Oz. This surface of revolution has *a singular point* with coordinates (0, 0, 0).

Forms of definition of the surface

(1) Explicit equation:

$$z = b\sqrt[3]{x^2 + y^2}.$$

(2) Parametrical equations:

$$x = u^3, \quad y = v^3, \quad z = b\left(u^6 + v^6\right)^{1/3}.$$

(3) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta,$$
$$y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = br^{\frac{2}{3}}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{4b^2}{9}r^{-\frac{2}{3}},\ F = 0,\ B = r,$$
$$L = -\frac{2b}{9A}r^{-\frac{4}{3}},\ M = 0,\ N = \frac{2b}{3A}r^{\frac{2}{3}},$$

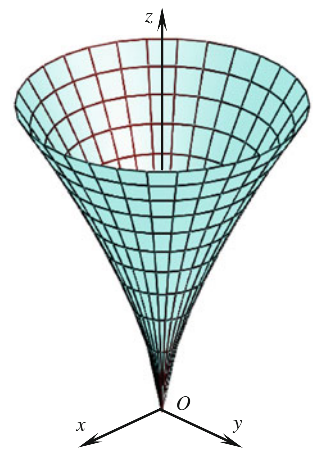


Fig. 1

$$k_1 = -\frac{2b}{9A^3}r^{-\frac{4}{3}},\ k_2 = \frac{2b}{3A}r^{-\frac{4}{3}}.$$

This is a surface of negative total curvature, i.e., $K < 0$.

■ Surface of Revolution of a Hyperbola $z = b/x$ About the Oz Axis

Forms of Definition of the Surface

(1) Explicit equation:

$$z = \frac{b}{\sqrt{x^2 + y^2}}.$$

A surface of rotation of a hyperbola $z = b/x$ about the axis Oz can be reckoned also in *Tzitzéica's surface with central affine invariant equal to* $I = -4/(27b^2)$.

(2) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = b/r,$$

where $x > 0$, $y > 0$, $r = b/r$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{b^2}{r^4}, \quad F = 0, \quad B = r,$$
$$L = \frac{2b}{Ar^3}, \quad M = 0, \quad N = -\frac{b}{Ar},$$
$$k_1 = \frac{2b}{r^3A^3}, \quad k_2 = -\frac{b}{r^3A}.$$

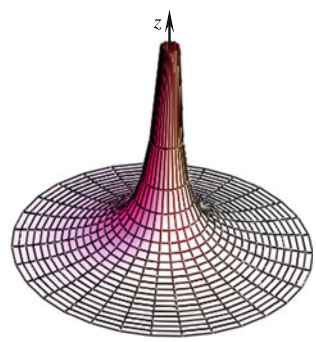


Fig. 1

The surface of rotation of a hyperbola is a surface of strictly negative Gaussian curvature. Not a single parallel will be a geodesic line.

If we assume $b = 1$, i.e., $z = 1/x$ on $[1, \infty]$, then we have *Gabriel's Horn*, or *Gabriel's Trumpet*, due to a highly unusual and paradoxical trait. The volume of Gabriel's Horn is equal to π on $[1, \infty]$ and the area of lateral surface is equal to infinity, i.e., $A = \infty$, on $[1, \infty]$. So, we have a surface with infinitive surface area enclosing a finite volume.

Additional Literature

Tzitzéica G. Sur une nouvelle classe de surface. Comptes Rendus, Acad. Sci. Paris. 1907; 144, p. 1257-1259.

■ Parabolic Humming-Top

A surface "*Parabolic humming-top*" has a parabola, as a meridian, the axis of which is perpendicular to the axis of rotation but a peak of the parabola is lying at the axis of rotation, i.e., on an axis z (Fig. 1).

This surface called also "Der Kreisel" can be given by parametrical equations (Fig. 2):

$$x = \frac{(|z| - h)^2}{2p}\cos\beta,$$
$$y = \frac{(|z| - h)^2}{2p}\sin\beta, \quad z = z,$$

where h is a height of one sheet of the surface; $h^2/(2p)$ is a radius of the equator of the surface of revolution (Fig. 1); $-h \le z \le h$; $0 \le \beta \le 2\pi$. The peaks of two generatrix parabolas are placed in the points with coordinates (0; 0; $\pm$ h). This surface contains two segments of *a surface of rotation of a parabola* (Page 123).

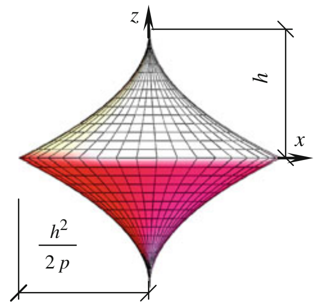


Fig. 1

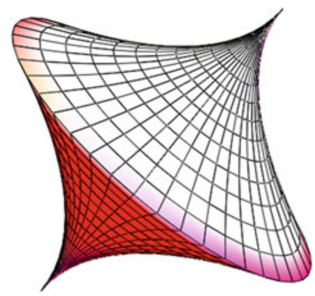


Fig. 2

■ Surface of Revolution of an Astroid

A surface of revolution of an astroid can be generated by the rotation of a astroid $x^{2/3} + z^{2/3} = a^{2/3}$ about its axis Ox or Oz (Fig. 1).

Forms of definition of the surface

(1) Explicit equation:

$$z = \pm\left[a^{2/3} - \left(x^2 + y^2\right)^{1/3}\right]^{\frac{3}{2}}.$$

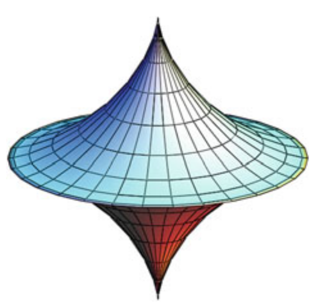


Fig. 1

The surface has two singular points in the poles of the surface with the coordinates $x = y = 0$, $z = \pm à$ and an edge of regression that is the parallel $r = a$ when $z = 0$.

(2) Parametrical equations:

$$x = x(r, \beta) = r\cos\beta,$$
$$y = y(r, \beta) = r\sin\beta,$$
$$z = \pm\left(a^{2/3} - r^{2/3}\right)^{3/2},$$

where $0 \le r \le a$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \left(\frac{a}{r}\right)^{1/3}, \quad F = 0, \quad B = r,$$
$$L = \frac{a^{1/3}}{3r\sqrt{a^{2/3} - r^{2/3}}}, \quad M = 0, \quad N = -\frac{r\sqrt{a^{2/3} - r^{2/3}}}{a^{1/3}},$$
$$k_1 = \frac{1}{3(ar)^{1/3}\sqrt{a^{2/3} - r^{2/3}}}, \quad k_2 = -\frac{\sqrt{a^{2/3} - r^{2/3}}}{ra^{1/3}},$$
$$K = -\frac{1}{3r^{4/3}a^{2/3}} < 0.$$

(3) Parametrical equations:

$$x = x(t, \beta) = a\sin^3 t\cos\beta, \quad y = y(t, \beta) = a\sin^3 \sin\beta,$$
$$z = z(t) = a\cos^3 t.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 3a\sin t\cos t, \quad F = 0, \quad B = a\sin^3 t,$$
$$L = 3a\sin t\cos t, \quad M = 0, \quad N = -a\sin^3 t\cos t,$$
$$k_1 = \frac{2}{3a\sin 2t}, \quad k_2 = -\frac{\cos t}{a\sin^3 t}, \quad K < 0.$$

■ Astroidal Torus

A surface of the rotation of an astroid is formed by an astroid

$$x^{2/3} + z^{2/3} = a^{2/3}$$

rotating about any of two its axes Ox or Oz. If an astroid

$$x = x(u) = a\cos^3 u, \quad z = z(u) = a\sin^3 u$$

is placed at the r distant from the axis of rotation, then we will have *an astroidal torus*. An inner area bounded by an astroid is

$$A = \frac{3}{8}\pi a^2.$$

A length of full astroid is $6a$. It can be noted that an astroid is *an evolute of the ellipse*. *The evolute of an astroid* is another astroid.

An astroidal torus can be defined by parametrical equations:

$$X = X(u, v) = [r + x(u)\cos\theta - z(u)\sin\theta]\cos v;$$
$$Y = Y(u, v) = [r + x(u)\cos\theta - z(u)\sin\theta]\sin v;$$
$$Z = Z(u) = x(u)\sin\theta + z(u)\cos\theta,$$

where θ is the angle of rotation of local axes x, z of the generatrix astroid in the vertical plane containing the axis The local coordinate system is rotated counter-clockwise if the θ angle has positive value.

An astroidal torus degenerates into *an astroidal surface of revolution* when $r = 0$, $\theta = 0$ (Fig. 1).

In Fig. 1, the astroidal torus is given when $a = 1$ m, $r = 2$ m, $\theta = 0$, $0 \le v \le 2\pi$; $-\pi \le u \le \pi$.

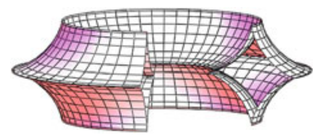


Fig. 1

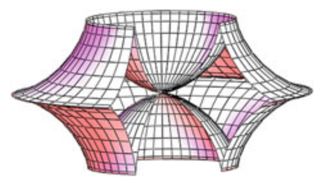


Fig. 2

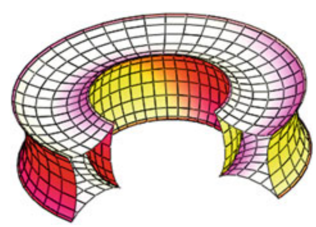


Fig. 3

The astroidal torus with $\theta = 0$, $0 \le v \le 2\pi$, $-\pi \le u \le \pi$, $a = r = 1$ m is given in Fig. 2.

The right astroidal torus is represented in Fig. 3 when $a = 1$ m, $r = 2$ m, $\theta = 0.25\pi$; $0 \le v \le 2\pi$; $-\pi \le u \le \pi$.

Additional Literature

Weisstein EW. Astroid from MathWorld.

■ Surface of Revolution of the Agnesi Curl

The meridians of *a surface of revolution of the Agnesi curl* about its asymptote intersect the plane $z = 0$, perpendicular to the rotation axis, at angle of 90° (Fig. 1). An implicit equation of an Agnesi curl is

$$z^2 y = 4a^2(2a - y).$$

The circle with a radius $2a$ lies in the cross section of this surface of revolution by the plane $z = 0$. This parallel is a geodesic line.

Forms of definition of the surface

(1) Implicit equation:

$$z^2 = 4a^2\left(\frac{2a}{\sqrt{x^2+y^2}} - 1\right).$$

(2) Parametrical equations (Fig. 1):

$$x = x(r,\beta) = r\cos\beta, \quad y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = 2a(2a/r{-}1)^{1/2}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

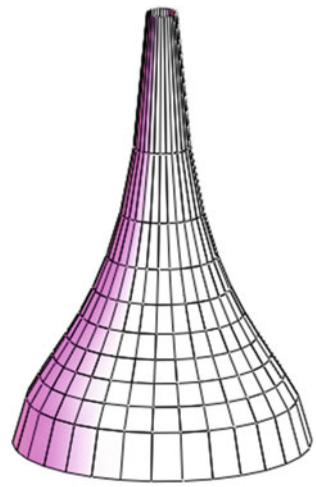


Fig. 1

$$A^2 = 1 + \frac{4a^4}{r^4(2a/r-1)}, \quad F = 0, \quad B = r,$$
$$L = \frac{2a^2(3a-2r)}{Ar^4(2a/r-1)^{3/2}}, \quad M = 0, \quad N = -\frac{2a^2}{Ar\sqrt{2a-1}},$$
$$k_1 = k_r = \frac{L}{A^2}, \quad k_2 = k_\beta = \frac{N}{B^2} < 0.$$

So, $K > 0$ if $r > 1.5a$; $K < 0$ if $r < 1.5a$ and $K = 0$ on the parallel $r = 1.5a$.

■ Deformed Sphere

Surface of revolution "*Deformed Sphere*" is a closed surface consisting of two parts one of which is a surface of positive Gaussian curvature but another one is of negative Gaussian curvature. These parts of the surface are jointed along the plane circle with parabolic points.

"Deformed Sphere" has the following parametrical equations (Fig. 1):

$$x = x(u,v) = \cos u\cos v,$$
$$y = y(u,v) = \cos u\sin v,$$
$$z = z(u) = \sin(u-a)$$

where a is a constant parameter, $-\pi/2 \le u \le \pi/2$, $0 \le v \le 2\pi$.

A "Deformed Sphere" is degenerated into a sphere when $a = 0$ and $a = \pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \sin^2 u + \cos^2(u-a), \quad F = 0,\ B = \cos u,$$
$$L = \frac{\cos a}{A}, \quad M = 0, \quad N = \frac{\cos u\cos(u-a)}{A},$$
$$k_1 = \frac{\cos a}{A^3}, \quad k_2 = \frac{\cos(u-a)}{A\cos u}.$$

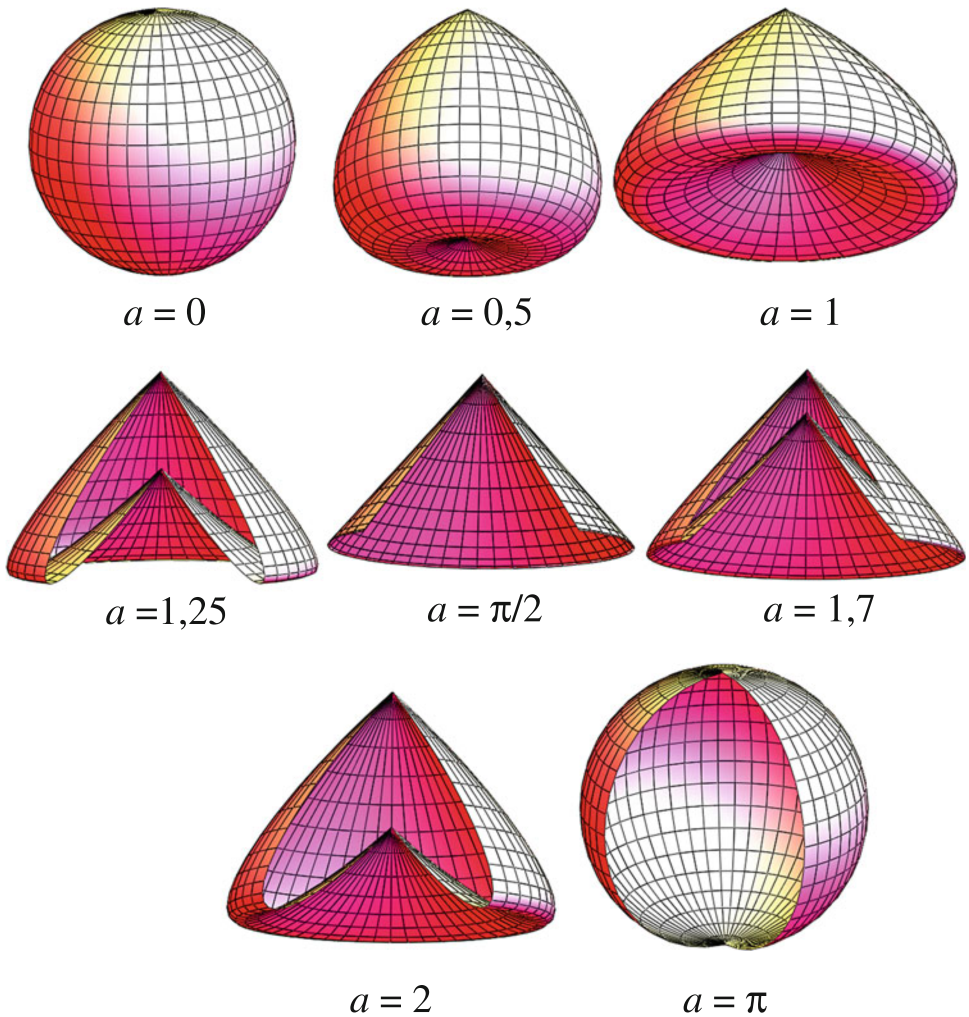
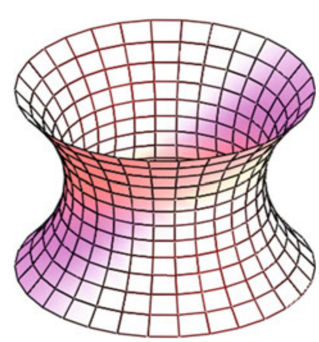


Fig. 1

■ Surface of Revolution of a Parabola

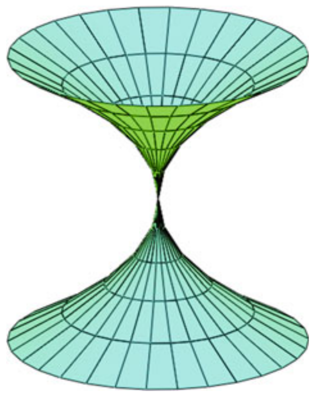
A paraboloid of revolution is formed by rotation of a parabola about its axis of symmetry, i.e., about the axis of the parabola. *A surface of revolution of a parabola* is generated by rotation of a parabola about a straight line that is perpendicular to the axis of the parabola, i.e., is parallel to the directrix of the parabola. A parabola has the only one directrix which is p away from its focus.

The general surface of revolution of a parabola is obtained when a parabolic arc is rotated about an arbitrary axis. In the encyclopedia, this surface is called *a surface of revolution of a parabola of arbitrary position*.



(a>0)

Fig. 1



$(a = b = 0)$

Fig. 2

Forms of the definition of the surface

(1) Parametrical equations (Fig. 1):

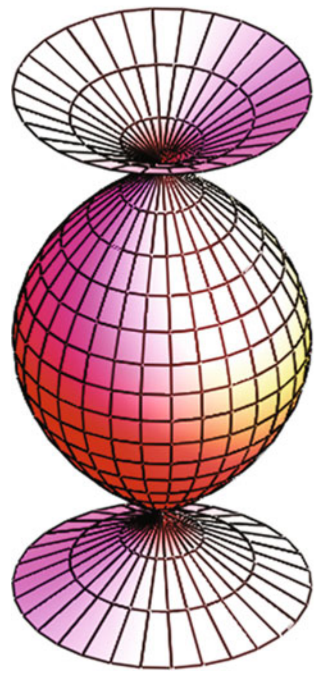
$$x = x(r, \beta) = r \cos \beta,$$
$$y = y(r, \beta) = r \sin \beta,$$
$$z = z(r) = \sqrt{2p(r - a)},$$

where $r = a$ is the radius of *the waist circle*, p is a distance the focus from the directrix of the parabolic meridian, $|x| \geq a,\ |y| \geq a,\ 0 \leq \beta \leq 2\pi$. The surface of revolution is formed by the rotation of a parabola $z^2 = 2p(x - a)$ about the z axis. The surface of revolution with $a > 0$ is shown in Fig. 1. If one takes $a = 0$, then he will design the surface represented in Fig. 2.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{p}{2(r - a)},\ F = 0,\ B = r,$$
$$L = -\frac{p^2}{A[2p(r - a)]^{3/2}},$$
$$M = 0,\ N = \frac{pr}{A\sqrt{2p(r - a)}},$$
$$k_1 = k_r = -\frac{p^2}{A^3[2p(r - a)]^{3/2}},$$
$$k_2 = k_\beta = \frac{p}{Ar\sqrt{2p(r - a)}},\quad K < 0.$$

A surface of revolution of a parabola belongs to surfaces of negative Gaussian curvature if $a \geq 0$. A directrix of the family of meridians becomes the axis of rotation when $a = p$.



$(b > 0)$

Fig. 3

(2) Parametrical equations (Figs. 1 and 2):

$$x = x(z, \beta) = \left[a + z^2/(2p)\right] \cos \beta,$$
$$y = y(z, \beta) = \left[a + z^2/(2p)\right] \sin \beta,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{z^2}{p^2},\quad F = 0,\quad B = r = a + \frac{z^2}{2p},$$
$$L = \frac{1}{pA},\quad M = 0\quad N = -\frac{B}{A},$$
$$k_1 = k_z = \frac{1}{pA^3},\quad k_2 = k_\beta = -\frac{1}{AB},$$
$$K = -\frac{1}{pA^4B} < 0.$$

(3) Parametrical equations (Figs. 3 and 4):

$$x = x(z, \beta) = \left[\frac{z^2}{2p} - b\right] \cos \beta,$$
$$y = y(z, \beta) = \left[\frac{z^2}{2p} - b\right] \sin \beta,$$
$$z = z,$$

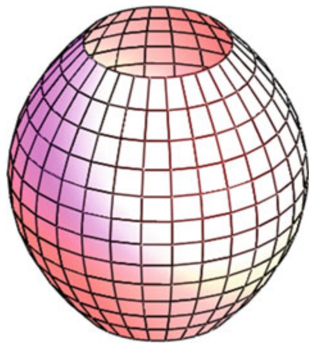


Fig. 4

where $b \geq 0$ is the distance the peak of the parabola from the axis of rotation. The surface shown in Fig. 2 is formed when $b = 0$. In Fig. 3, the surface with $b > 0$ is presented.

Having assumed $b > 0$ and $-\sqrt{2pb} < z < \sqrt{2pb}$, we can design *a barrel-shaped surface* of revolution (Fig. 4).

In several works, the surfaces shown in Figs 1, 2, 3 and 4 were called *a parabolic torus*.

Additional Literature

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Nedeshev YuB, Popov AYu. A method of determination of particular dimensions of shells of revolution. Izv. AN SSSR, MTT. 1991; No. 3, p. 118-126.

■ Parabolic-and-Logarithmic Surface of Revolution

A parabolic-and-logarithmic surface of revolution of positive Gaussian curvature is formed by rotation of a plane curve

$$r = r(z) = a\sqrt{cz + b}\ln(cz + b)$$

about the z axis.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(z, \beta) = r(z)\sin\beta,$$
$$y = y(z, \beta) = r(z)\cos\beta,$$
$$z = z.$$

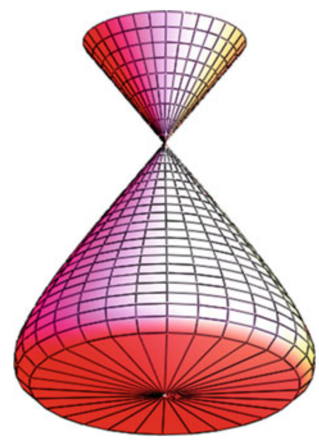

Fig. 1

The indeterminacy in the form of $0 \cdot \infty$ existing at the point z_o ($cz_o + b = 0$) is disclosed and leads to an equality $r(z_o) = 0$. The parallel, lying in the plane $z = 0$, has a radius $r_o = ab^{1/2}\ln b$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2c^2}{cz + b}\left[\frac{\ln(cz + b)}{2} + 1\right]^2, \quad F = 0,$$
$$B = r(z) = a\sqrt{cz + b}\ln(cz + b),$$

$$L = \frac{ac^2\ln(cz + b)}{4A(cz + b)^{3/2}}, \quad M = 0, \quad N = \frac{r(z)}{a},$$
$$k_1 = \frac{ac^2\ln(cz + b)}{4A^3(cz + b)^{3/2}},$$
$$k_2 = \frac{1}{r(z)A}, \quad K = \frac{c^2}{4A^4(cz + b)^2} > 0.$$

Additional Literature

Nazarov GI, Puchkov AA. An equilibrium of a parabolic-and-logarithmic surface of revolution. Prikl. Mat. i Mehanika (Moscow). 1991; 55, No. 5, p. 867-869.

■ Hyperbolic-and- Logarithmic Surface of Revolution

A hyperbolic-and-logarithmic surface of revolution of negative Gaussian curvature has meridians:

$$r = r(z) = a(z+b)^2 \ln(z+b),$$

where $a > 0$ is a constant characterizing the form of the surface (Fig. 1). A constant b does not influence on the form of the surface but the position of the beginning of coordinates depends on the parameter b. The beginning of a system of Cartesian coordinates is placed at the peak of the surface of revolution when $b = 0$. The axis Oz is an axis of rotation. The indeterminacy in the form of $0 \cdot \infty$ existing at the peak when $z = -b$ is disclosed due to de l'Hopitale rule. So, one will obtain:

$$r = r(z = -b) = 0.$$

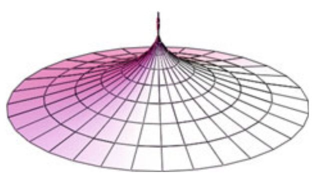

Fig. 1

Parametrical equations of the studied surface of revolution can be written as (Fig. 1):

$$x = x(z, \beta) = r(z)\sin\beta,$$
$$y = y(z, \beta) = r(z)\cos\beta,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + a^2(z+b)^2[1 + 2\ln(z+b)]^2,$$
$$F = 0,\ B = r(z) = a(z+b)^2\ln(z+b),$$
$$L = -\frac{a[2\ln(z+b)+3]}{A},\ M = 0,\ N = \frac{r(z)}{A},$$
$$k_1 = -\frac{a[2\ln(z+b)+3]}{A^3},\ k_2 = \frac{1}{r(z)A},$$
$$K = -\frac{3 + 2\ln(z+b)}{(z+b)^2\ln(z+b)A^4} < 0.$$

In Fig. 1, the hyperbolic-and-logarithmic surface of revolution is shown when $a = 0.5$; $b = 0$; $0.1 \le z \le 4$ m; $r_{\max} = 11.09$ m if $z = 4$ m.

Additional Literature

Nazarov GI, *Puchkov AA*. An inverse problem for a shell of revolution of negative Gaussian curvature. Izv. Vuzov: Stroit. i Arhitectura. 1990; No. 12, p. 22-24.

■ Bullet Nose

"*Bullet Nose*" is formed by rotation of a curve: $x = \pm az/\sqrt{b^2 + z^2}$ (Figs. 1 and 2) about a coordinate axis z.

Forms of definition of the surface

(1) Parametrical equations (рис. 3):

$$x = x(u, v) = a\cos v\cos u,$$
$$y = y(u, v) = a\cos v\sin u,$$
$$z = z(v) = -b/\tan v,$$
$$x < a;\ y < a;\ 0 \le u \le 2\pi;\ 0 < v \le \pi/2.$$

(2) Implicit equation

$$(b^2 + z^2)(x^2 + y^2) = a^2 z^2$$

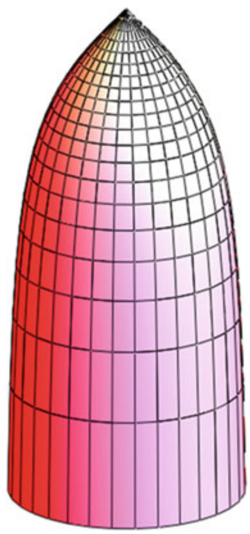

Fig. 1

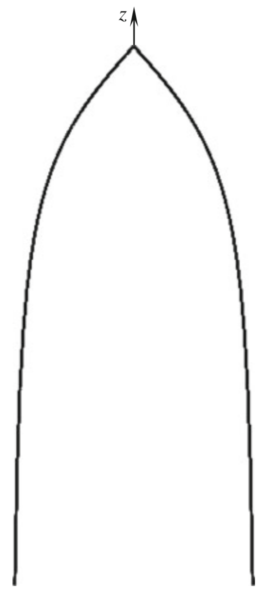

Fig. 2

■ The Fourth-Order Paraboloid of Revolution

The fourth-order paraboloid of revolution is formed by rotation of biquadratic parabola $x^4 = cz$ about an axis z (Fig. 1). This surface is also called *a quartoid*.

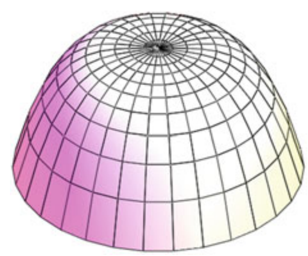

Fig. 1

Forms of definition of the surface

(1) Explicit equation:

$$cz = \left(x^2 + y^2\right)^2.$$

Having assumed $c = a^3$, we can get *a poweroid* (Jackway and Deriche).

In the cross section of the surface of revolution by the planes $z = h = \text{const}$, circles with radii

$$r = \sqrt[4]{hc}$$

are placed; $h > 0$.

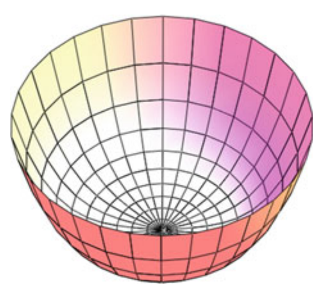

Fig. 2

(2) Parametrical equations (Figs. 1 and 2):

$$\begin{aligned} x &= x(r, \beta) = r \cos\beta, \\ y &= y(r, \beta) = r \sin\beta, \\ z &= z(r) = r^4/c. \end{aligned}$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + 16\frac{r^6}{c^2},\ F = 0,\ B = r,$$
$$L = \frac{12r^2}{cA},\ M = 0,\ N = \frac{4r^4}{cA},$$
$$k_r = k_1 = \frac{12r^2}{cA^3},\ k_\beta = k_2 = \frac{4r^2}{cA},$$
$$K = \frac{48r^4}{c^2A^4} > 0,\ H = \frac{2r^2}{cA}\left(1 + \frac{3}{A^2}\right).$$

The studied surface of revolution is given in the lines of principal curvatures r and β. A paraboloid of revolution of the fourth order is a surface of positive total curvature. The surface has zero Gaussian and mean curvatures ($K = H = 0$) only at one point $r = 0$. So, the peak of a paraboloid of revolution of the fourth order is *a plane point*.

(3) Parametrical equations (Figs. 1 and 2):

$$x = x(z, \beta) = \sqrt[4]{cz}\cos\beta,$$
$$y = y(z, \beta) = \sqrt[4]{cz}\sin\beta,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{c^2}{16(cz)^{3/2}},\quad F = 0,\quad B = \sqrt[4]{cz},$$
$$L = \frac{3c^2}{16AB^7},\quad M = 0,\quad N = \frac{B}{A},$$
$$k_z = k_1 = \frac{3c^2}{16A^3B^7},\quad k_\beta = k_2 = \frac{1}{AB},$$
$$K = \frac{3c^2}{16A^4B^8} = \frac{48z}{(\sqrt{c} + 16z^{3/2})^2} > 0.$$

The obtained values of the coefficients of the fundamental forms of surface show that the surface of rotation of a biquadratic parabola is given in lines of principal curvatures z and β but the fourth-order paraboloid of revolution is a surface of positive total curvature and only in one point $z = 0$, the surface has zero Gaussian and mean curvatures.

Additional Literature

Sun Bo-Hua, Zhang Wei, Yeh Kai-Yuan, Rimrott FPJ. Exact displacement solution of arbitrary degree paraboloidal shallow shell of revolution made of linear elastic materials. Int. J. Solids and Struct. 1996; 33, No. 16, p. 2299-2308 (14 ref.).
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Palm G. Robust segmentation of human cardiac contours from spatial magnetic resonance images. Diss. zur Erlangung des Doct. (Dr. rer.nat.), der Fakultät für Informatik der Universität Ulm.; 2004; 130 p.

■ Surface of Revolution with Damping Circular Waves

Having researched damped natural vibrations, one seeks the amplitude-time dependence in the form of a function

$$z = z(x) = ae^{-nx}\sin(\omega x + \varphi).$$

A surface of revolution with damping circular waves is traced by a curve $z = z(x)$ in the process of its rotation about an axis Oz.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(r, u) = r\cos u,$$
$$y = y(r, u) = r\sin u,$$
$$z = z(r) = ae^{-nr}\sin(\omega r + \varphi),$$

where $\omega = m\pi/b$, m is a number of integral half-waves, placed at the straight line segment with the b length; φ = const.

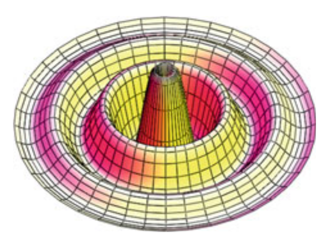


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + a^2e^{-2nr}[-n\sin(\omega r + \varphi) + \omega\cos(\omega r + \varphi)]^2,$$
$$F = 0,\quad B = r,$$
$$L = ae^{-nr}\left[(n^2 - \omega^2)\sin(\omega r + \varphi) - 2n\omega\cos(\omega r + \varphi)\right]/A,$$
$$M = 0, N = rae^{-nr}[-n\sin(\omega r + \varphi) + \omega\cos(\omega r + \varphi)]/A.$$

In Fig. 1, the surface of revolution with $m = 6$, $b = 6$ m; $a = 4$ m; $n = 0.5$; $0 \le r \le b$; $\varphi = 0$ is shown.

■ Kiss Surface

A "*Kiss Surface*" is *an algebraic surface of the fifth order* (Fig. 1). Sometimes this surface is called *a "Falling Drop."* It is traced by a curve $x = x(z) = z^2(1 - z)^{1/2}$ in the process of its rotation about an axis Oz.

Forms of definition of the surface

(1) Implicit form of the definition:

$$x^2 + y^2 = (1 - z)z^4, \text{ where } -\infty \le z \le 1.$$

(2) Explicit form of the definition:

$$x = \pm\sqrt{(1 - z)z^4 - y^2}.$$

(3) Parametrical equations (Fig. 1):

$$x = x(u, z) = z^2\sqrt{1 - z}\cos u, \quad y = y(u, z) = z^2\sqrt{1 - z}\sin u,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = z^4(1 - z), \quad F = 0,$$
$$B^2 = \frac{4(1 - z) + z^2(4 - 5z)^2}{4(1 - z)}, \quad M = 0,$$

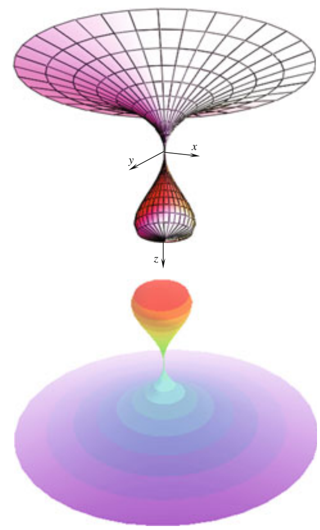


Fig. 1

$$L = \frac{2(1 - z)z^2}{\sqrt{4(1 - z) + z^2(4 - 5z)^2}},$$
$$N = \frac{15z^2 - 24z + 8}{2(1 - z)\sqrt{4(1 - z) + z^2(4 - 5z)^2}},$$
$$K = \frac{4(15z^2 - 24z + 8)}{z^2\left[4(1 - z) + z^2(4 - 5z)^2\right]^2}.$$

The surface contains the parts of positive and negative Gaussian curvatures. Parabolic points with $K = 0$ are placed at the cross section of the surface by a plane $z = 0.8 - 0.4(2/3)^{1/2} = 0.473$. In Fig. 2, the surface is shown when $-1 \le z \le 1$; $0 \le u \le 2\pi$.

■ Soucoupoid

Forms of Definition of the Surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = a\cos u\cos v, \quad y = y(u, v) = a\cos u\sin v,$$
$$z = z(u) = b\sin^3 u,$$

where coordinate lines *u, v* (meridians and parallels) are the lines of principal curvatures; *a, b* are constants; $-\pi/2 \le u \le \pi/2$, $0 \le v \le 2\pi$.

(2) Implicit equation: $z^2 = b^2\left(1 - \frac{x^2 + y^2}{a^2}\right)^3$.

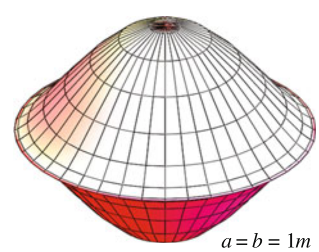


Fig. 1

Reference

Encyclopédie Des Formes Mathematiques Remarquables Surfaces.—http://mathcurve.com/surfaces/surfaces.shtml

■ Globoid (Toroid)

A globoid is a surface formed by rotation of an arc of the circle m about an axis z lying at the plane of this arc. A method of generation of a surface of a globoid shows that we have a segment of *the circular torus* which has a negative Gaussian curvature (Fig. 1). A line on the globoid generated by uniform motion of a point along the axis of the globoid with simultaneous steady rotation of the globoid about its axis is called *a globoidal helical line*.

A globoidal worm gearing is an example of application of globoid in the technique.

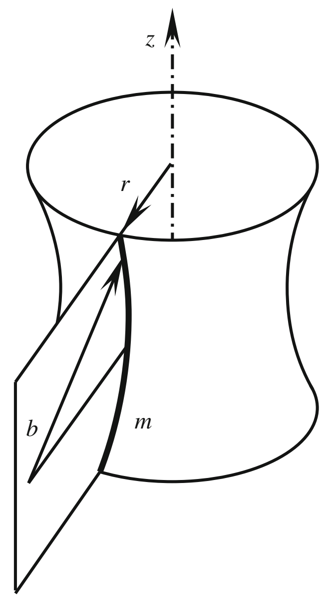


Fig. 1

Forms of definition of the surface

(1) Parametrical equations (Fig. 2):

$$x = x(u, v) = (a + b\cos v)\cos u,$$
$$y = y(u, v) = (a + b\cos v)\sin u,$$
$$z = z(v) = b\sin v,$$

where a is a radius of centers of generatrix circles; b is a radius of the generatrix circle, $0 \le u \le 2\pi$, $\pi/2 \le v \le (3/2)\pi$. In Fig. 3, a fragment of the surface bounded by the lines of principal curvatures is shown; $0 \le u \le \pi$ and $\pi \le v \le (3/2)\pi$.

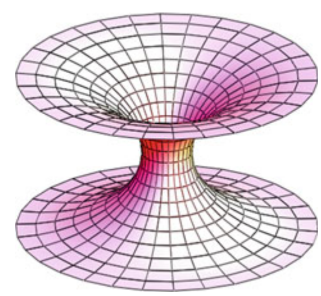


Fig. 2

Coefficients of the fundamental forms of the surface:

$$A = a + b\cos v, \quad F = 0, \quad B = b,$$
$$L = -(a + b\cos v)\cos v, \quad M = 0, \quad N = -b,$$
$$k_u = k_1 = -\frac{\cos v}{A}, \quad k_v = k_2 = -\frac{1}{b},$$
$$K = \frac{\cos v}{bA}.$$

(2) Parametrical equations:

$$x = x(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\cos u,$$
$$y = y(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\sin u,$$
$$z = \frac{b\beta}{\sqrt{a^2 + \beta^2}},$$

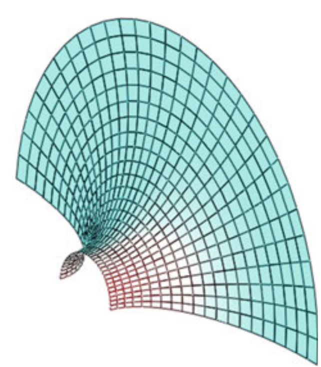


Fig. 3

where $\beta = a \tan \alpha$; α is the angle of a straight, connecting the center of generatrix circle with a radius b with an arbitrary point of the torus, with a plane $z = 0$. Positive direction is counted off anticlockwise; $-\pi/2 < \alpha < \pi/2$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{a(\sqrt{a^2 + \beta^2} - b)}{\sqrt{a^2 + \beta^2}}, \quad F = 0, \quad B = \frac{ab}{a^2 + \beta^2},$$
$$L = -\frac{a^2(\sqrt{a^2 + \beta^2} - b)}{a^2 + \beta^2}, \quad M = 0, \quad N = \frac{a^2 b}{(a^2 + \beta^2)^2},$$
$$k_1 = k_u = -\frac{1}{\sqrt{a^2 + \beta^2} - b}, \quad k_2 = k_v = \frac{1}{b}.$$

Coordinate lines u, v and u, β are the lines of principal curvatures. They coincide with the meridians and the parallels of surface of revolution.

(3) Parametrical equations:

$$x = x(\gamma, v) = \frac{a(\mathrm{ch}\gamma - 1)}{\mathrm{ch}\gamma}\cos v,$$
$$y = y(\gamma, v) = \frac{a(\mathrm{ch}\gamma - 1)}{\mathrm{ch}\gamma}\sin v,$$
$$z = a\,\mathrm{th}\gamma, \; -\infty < \gamma < +\infty,$$

The globoid has a degenerated point with coordinates (0, 0, 0) or when $\gamma = 0$; $a = b$.

Additional Literature

Blachut J and Jaiswal OR. Instabilities in torispheres and toroids under suddenly applied external pressure. Int. J. Impact. Eng. 1999; 22 (5), p. 511-530 (16 ref.).

■ Surface of Revolution of a Usual Cycloid

A surface of revolution of a usual cycloid is formed by the rotation of *an usual cycloid*

$$z_c = at - a\sin t, \; x_c = a - a\cos t$$

about the axis z_c, where t is *a real parameter*, corresponding to the angle through which the rolling circle has rotated, measured in radians. For given t, the circle's center lies at $z_c = at$, $x_c = a$.

A usual cycloid is generated by a point that is apart from a center of the circle with a radius a, rolling without sliding on the axis z_c, at the distance of a.

Let us study a general case when a cycloid is rotated about the axis z which is parallel to the axis z_c and is apart from it at the distance of c.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(t, \beta) = (a + c - a\cos t)\cos\beta,$$
$$y = y(t, \beta) = (a + c - a\cos t)\sin\beta,$$
$$z = z(t) = at - a\sin t.$$

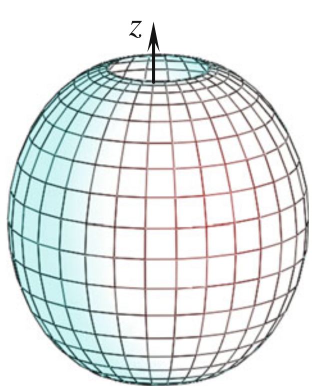


Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 2a\sin\frac{t}{2}, \quad F = 0, \quad B = c + 2a\sin^2\frac{t}{2},$$
$$L = \frac{A}{2}, \quad M = 0, \quad N = \frac{AB}{2a},$$
$$k_1 = k_t = \frac{1}{2A} = \frac{1}{4a\sin\frac{t}{2}},$$
$$k_2 = k_\beta = \frac{A}{2aB} = \frac{\sin\frac{t}{2}}{\left(c + 2a\sin^2\frac{t}{2}\right)},$$
$$K = \frac{1}{4aB} = \frac{1}{4a\left(c + 2a\sin^2\frac{t}{2}\right)} > 0.$$

Coordinate lines β and t (parallels and meridians) are the lines of principal curvatures.

A length of a meridian from a parallel $t = 0$ till a parallel $t = \mathrm{const}$ is calculated by a formula:

$$s = 4a\left(1 - \cos\frac{t}{2}\right).$$

In Fig. 2, the fragment of the surface bounded by the parallels $t = 0$, $t = 2\pi$ and by the meridians $\beta = 0$, $\beta = \pi$ is presented.

In Fig. 3, three sections of the surface of the rotation of a usual cycloid with $c = 0$ are given; but in Fig. 4, the surface with $c > 0$ is shown, $0 \le t \le 5\pi$.

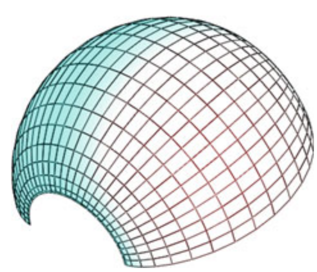


Fig. 2

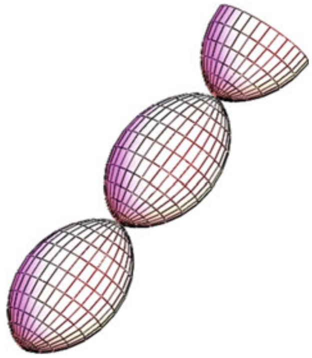


Fig. 3

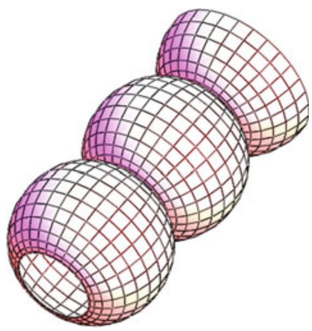


Fig. 4

Two sections of the surface presented in Fig. 3 belong to a category of *closed surfaces of revolution* because the beginning and the end of a not closed rotated usual cycloid is placed at the rotation axis.

An area of a surface of rotation of a segment of the meridian ($t_0 \le t \le t_1$) in the form of a usual cycloid can be defined by a formula:

$$A = 8a\pi\left[\frac{2a}{3}\cos^3\frac{t}{2}\bigg|_{t_0}^{t_1} - (c+2a)\cos\frac{t}{2}\bigg|_{t_0}^{t_1}\right], \quad 0 \le \beta \le 2\pi.$$

For example, an area of one closed section of the surface shown in Fig. 3 is

$$A_1 = \frac{64a^2\pi}{3}, \quad 0 \le t \le 2\pi, \ \ 0 \le \beta \le 2\pi.$$

Additional Literature

Barra Mario. The cycloid. Educ. Stud. Math. 1975; 6, No. 1, p. 93-98.

Churkin GM. The property of points of a cycloid. In-t him. Kinet. I goreniya SO AN SSSR, Novosibirsk, 1989; 10 p., 3 ref., Dep v VINITI 06.01.89, No. 156-B89.

Wells D. (1991). The Penguin Dictionary of Curious and Interesting Geometry. New York: Penguin Books. 1991; p. 445-47.

■ Pseudo-Catenoid

A catenoid is formed by the rotation of *a catenary*

$$x = a\cos\mathrm{h}(z/a)$$

about an Oz axis (Fig. 1). A catenoid is the only *minimal surface of revolution*, i.e., mean curvature of its surface is equal to zero at all points of the surface. It is the first minimal surface to be discovered.

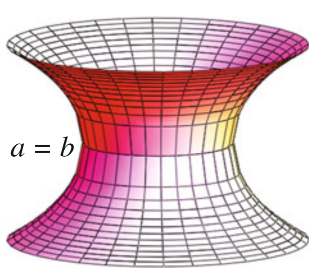


Fig. 1

A pseudo-catenoid is generated by the rotation of a curve

$$x = b\cos\mathrm{h}(z/a)$$

about an Oz axis. A pseudo-catenoid is a surface of rigorously negative Gaussian curvature but it is not a minimal surface.

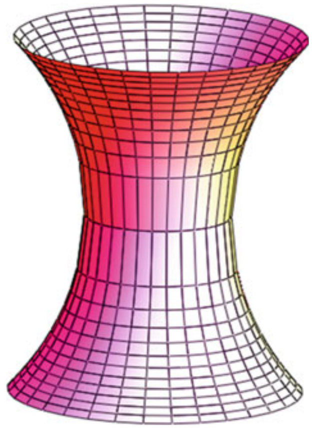
Forms of definition of the surface

(1) Explicit equation:

$$z = a\mathrm{Ar}\ \cos\mathrm{h}\sqrt{(x^2+y^2)/b^2}.$$

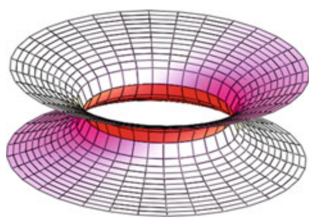
(2) Parametrical equations (Figs. 2 and 3):

$$x = x(r,\ \beta) = r\cos\beta,$$
$$y = y(r,\ \beta) = r\ \sin\beta,$$
$$z = z(r) = \pm a\mathrm{Ar}\cos\mathrm{h}(r/b),$$



$(a > b)$

Fig. 2



$(a < b)$

Fig. 3

where β is the angle taken from the axis Ox in the directions of the Oy axis.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \frac{r^2 - b^2 + a^2}{r^2 - b^2},\ F = 0,\ B = r,$$
$$L = \frac{-ar}{(r^2 - b^2)\sqrt{r^2 - b^2 + a^2}},$$
$$M = 0,\ N = \frac{ra}{\sqrt{r^2 - b^2 + a^2}},$$
$$k_1 = \frac{-ar}{(r^2 - b^2 + a^2)^{3/2}},\ k_2 = \frac{a}{r\sqrt{r^2 - b^2 + a^2}},$$
$$K = \frac{-a^2}{[r^2 - b^2 + a^2]^2} < 0,$$
$$H = \frac{a(a^2 - b^2)}{2r(r^2 - b^2 + a^2)^{3/2}} \neq 0.$$

Coordinate lines r and β (parallels and meridians) are the lines of principal curvatures (Figs. 1, 2 and 3). In Fig. 2, the pseudo-catenoid has $a > b$. The surface of revolution shown in Fig. 3 was created when $a < b$. And a pseudo-catenoid becomes a minimal surface if $a = b$ (Fig. 1) and this surface can be called a catenoid.

Substituting $a = b$ in the formulae for the determination of coefficients of the fundamental forms of surface, it is possible to obtain corresponding values of these coefficients for catenoid.

Additional Literature

Krivoshapko SN. On mistakes in the terminology on theory of surfaces and geometric modelling. Present Problems of Geometric Modelling: Proc. of Ukraine-Russian Scientific-and-Practical Conf. April 19-22, 2005. Kharkov, 2005; p. 82-87.

■ Surface of Revolution "Pear"

A surface of revolution called "*Pear*" is generated by rotating curve

$$b^2y^2 = z^3(a - z)$$

about its coordinate axis Oz.

Forms of definition of the surface

(1) Parametrical form of the definition (Fig. 1):

$$x = x(z, \beta) = r(z)\sin\beta;$$
$$y = y(z, \beta) = r(z)\cos\beta; \quad z = z,$$

where $r = r(z) = z\sqrt{z(a - z)}/b$; a and b are arbitrary constants; $0 \le z \le a$;

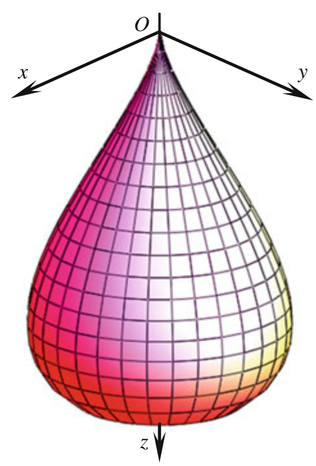


Fig. 1

$$0 \le r \le 3\sqrt{3}a^2/(16b).$$

A parallel $z = 3a/4$ with

$$r = r_{\max} = 3\sqrt{3}a^2/(16b)$$

is *a geodesic line*.

(2) Implicit equation:

$$z^3(a - z) - b^2(x^2 + y^2) = 0.$$

It means that the studied surface "Pear" is an algebraic surface of the fourth order.

Additional Literature

Gustavo Gordillo. A collection of famous plane curves. http://curvebank.calstatela.edu/famouscurves/famous.htm. August 14, 2001.

■ Surface of Revolution of a General Sinusoid

A surface of revolution of a general sinusoid

$$z = a\ \sin(n\pi x/R + \pi/2) = a\cos(n\pi x/R)$$

about an axis Oz is used in technics. *General sinusoid* in contrast to *usual sinusoid* ($z = \sin x$) is elongated $|a|$ times along the axis Oz and contracted $R/(n\pi)$ times along the axis Ox, where n is an integer, R is a dimension of an integer n of half-waves of the sinusoid, and is shifted to the left by a straight-line segment $R/(2n)$. A period of the function is $T = 2R/n$. The points of intersection of the sine function with the Ox axis have the coordinates $[(k + ½)R/n, 0]$. A surface of revolution of a general sinusoid has the parts of positive and negative Gaussian curvatures. This surface can be reckoned in a subclass of waving or corrugated surfaces.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = a\cos\frac{n\pi r}{R}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2n^2\pi^2}{R^2}\sin^2\frac{n\pi r}{R},$$
$$F = 0,\ B = r,$$
$$L = -\frac{an^2\pi^2}{AR^2}\cos\frac{n\pi r}{R},$$
$$M = 0,\ N = -\frac{an\pi}{AR}r\sin\frac{n\pi r}{R},$$
$$k_1 = k_r = -\frac{an^2\pi^2}{A^3R^2}\cos\frac{n\pi r}{R},$$
$$k_2 = k_\beta = -\frac{an\pi}{rAR}\sin\frac{n\pi r}{R},$$
$$K = \frac{a^2n^3\pi^3}{2rA^4R^3}\sin\frac{2n\pi r}{R}.$$

The curvilinear coordinate net is put down to lines of principal curvatures.

(2) Parametrical equations (Fig. 2):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = a\sin\frac{n\pi r}{R}.$$

The general generating sinusoid in contrast to *usual sinusoid* ($z = \sin x$) is elongated $|a|$ times along the axis Oz and contracted $R/(n\pi)$ times along the axis Ox, where n is an integer, R is a dimension of an integer n of half-waves of the sinusoid. A period of the function is $T = 2R/n$. The points of intersection of the sine function with the Ox axis have the coordinates $[kR/n, 0]$.

The presented surface of revolution can be given in an explicit form (Fig. 3):

$$z = a\sin\left(\frac{n\pi}{R}\sqrt{x^2 + y^2}\right).$$

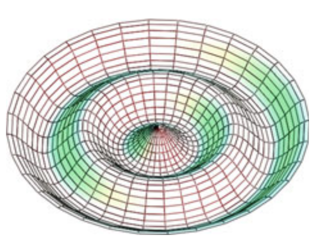


Fig. 1

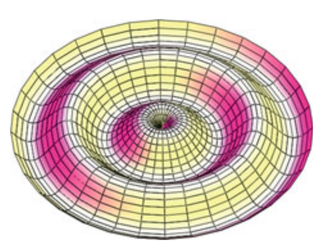


Fig. 2

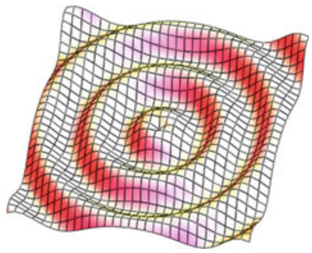


Fig. 3

The surface shown in Fig. 3 is called "Die Sinuswelle" in the German language scientific literature.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2n^2\pi^2}{R^2}\cos^2\frac{n\pi r}{R},$$
$$F = 0,\ B = r,$$
$$L = -\frac{an^2\pi^2}{AR^2}\sin\frac{n\pi r}{R},$$
$$M = 0, \quad N = \frac{an\pi}{AR}r\cos\frac{n\pi r}{R},$$
$$k_1 = k_r = -\frac{an^2\pi^2}{A^3R^2}\sin\frac{n\pi r}{R},$$
$$k_2 = k_\beta = \frac{an\pi}{rAR}\cos\frac{n\pi r}{R},$$
$$K = -\frac{a^2n^3\pi^3}{2rA^4R^3}\sin\frac{2n\pi r}{R}.$$

The parallels β and meridians r of the surface of revolution of a general sinusoid coincide with lines of principal curvatures.

(3) Explicit equation:

$$z = a\cos\left(\frac{n\pi}{R}\sqrt{x^2+y^2}\right).$$

Additional Literature

http://samoucka.ru/document22180.html

■ Corrugated Surface of Revolution of a General Sinusoid

A corrugated surface of revolution of a general sinusoid

$$x = a\sin\frac{n\pi z}{b} + c$$

about the axis Oz contains circular parts of both positive and negative curvatures.

General sinusoid in contrast to *usual sinusoid* ($x = \sin z$) is elongated $|a|$ times along the axis Ox and contracted $b/(n\pi)$ times along the axis Oz, where n is an integer, b is a dimension of an integer n of half-waves of the sinusoid.

A period of the function is $T = 2b/n$.

A volume of a body bounded by a surface of revolution of the half-wave of a usual sinusoid $x = \sin z$ is equal to $\pi^2/2$.

Forms of definition of the surface

(1) Explicit equation:

$$z = \frac{b}{n\pi}\arcsin\frac{\sqrt{x^2+y^2}-c}{a}.$$

(2) Implicit equation:

$$x^2 + y^2 - \left(a\sin\frac{n\pi z}{b} + c\right)^2 = 0.$$

(3) Parametrical equations (Fig. 1):

$$x = x(z,\beta) = r(z)\cos\beta,$$
$$y = y(z,\beta) = r(z)\sin\beta,$$
$$z(z) = z,$$

where $r = r(z) = a\sin\frac{n\pi z}{b} + c$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2n^2\pi^2}{b^2}\cos^2\frac{n\pi z}{b}, \quad F = 0, \quad B = r(z),$$
$$L = \frac{an^2\pi^2}{Ab^2}\sin\frac{n\pi z}{b}, \quad M = 0, \quad N = \frac{r(z)}{A},$$
$$k_1 = k_z = \frac{an^2\pi^2}{A^3b^2}\sin\frac{n\pi z}{b}, \quad k_2 = k_\beta = \frac{1}{r(z)A},$$
$$K = \frac{an^2\pi^2}{rA^4b^2}\sin\frac{n\pi z}{b}.$$

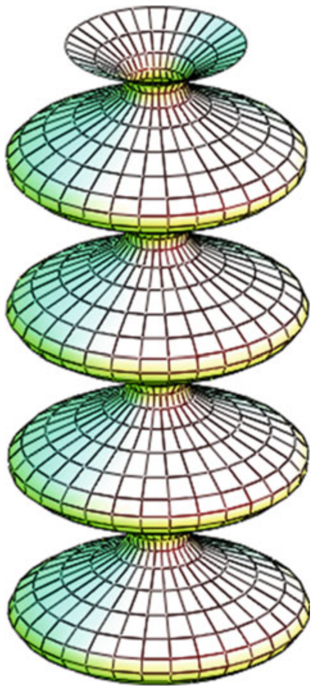


Fig. 1

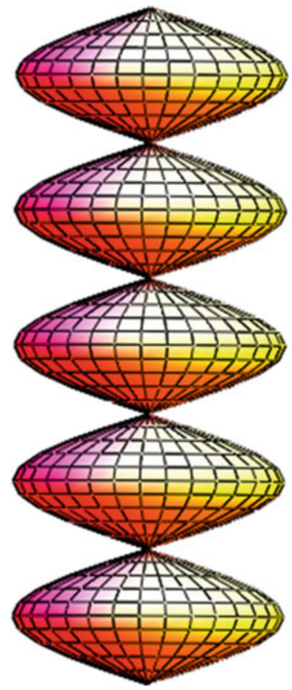


Fig. 3

Fig. 2

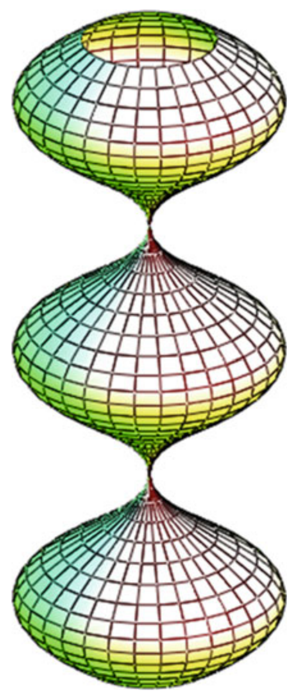


Fig. 4

The curvilinear coordinate net is put down to lines of principal curvatures β and z.

In Fig. 1, the corrugated surface of revolution of a general sinusoid is shown when $a < c$. Having assumed $c \gg a$, we can obtain *a corrugated cylinder* (Wolfram Demonstrations Project) or *a sinusoidal cylinder* (SpringerImages).

In Fig. 2, the surface of revolution has $a > c$; in Fig. 3, it is $c = 0$, and in Fig. 4, the surface of revolution has $a = c$. The surface of revolution represented in Fig. 1 is called "*Isolator*."

The surfaces of revolution shown in Figs. 1, 2, and 4 have the parts of both positive and negative Gaussian curvatures.

The surface of revolution represented in Fig. 3 is a surface of positive Gaussian curvature.

Additional Literature

Krivoshapko AN, *Halabi SM, Se Tsyan*. Analytical surfaces with a sine generatrix. Vestnik RUDN. "Engineering Researches". 2005; No. 1 (11), p. 115-120.

Zhulaev VP, *Sultanov BZ*. Screw pumping stations for recover of oil: Manual. Ufa: Izd-vo UShU, 1997; 43 p.

2014 Wolfram Demonstrations Project: http://demonstrations.wolfram.com/SinusoidalBellows/

SpringerImages: http://www.springerimages.com/Images/RSS/1-10.1007_s00348-005-0981-9-0

■ Surface of Revolution of a Parabola of Arbitrary Position

A surface of revolution of a parabola of an arbitrary position is formed by rotation of a parabola $Y(t) = ct^2$ with the axis Y, turned relatively to an axis of rotation Oz at the θ angle, about the axis Oz. A peak of the parabola lies at the distance a from the axis of rotation (Fig. 1).

Forms of definition of the studied surface

(1) Parametrical equations (Fig. 1):

$$x(u,t) = \left(a + t\cos\theta + ct^2\sin\theta\right)\cos u;$$
$$y(u,t) = \left(a + t\cos\theta + ct^2\sin\theta\right)\sin u;$$
$$z(u,t) = -t\sin\theta + ct^2\cos\theta.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \left(a + t\cos\theta + ct^2\sin\theta\right);$$
$$F = 0; \quad B^2 = 1 + 4c^2t^2;$$
$$L = \left(a + t\cos\theta + ct^2\sin\theta\right)\frac{2ct\cos\theta - \sin\theta}{B};$$
$$M = 0; N = \left(a + t\cos\theta + ct^2\sin\theta\right)\frac{2c}{B};$$
$$k_u = k_1 = \frac{2ct\cos\theta - \sin\theta}{AB},$$
$$k_t = k_2 = \frac{2c}{B^3}.$$

In Fig. 2, the surface of revolution of positive Gaussian curvature is shown when $a = 0.8$ m; $c = 2$ m^{-1}; $\theta = 0.2\pi$.

In Fig. 3, the studied surfaces of revolution of negative Gaussian curvature are presented. Here, the surface given in Fig. 3a has $\theta = \pi/2$, $a = 0$, $c = 1$ m^{-1}, but the surface in Fig. 3b has $\theta = -\pi/2$, $a = 0.8$ m; $c = 1$ m^{-1}. These surfaces are studied in the section "*Surface of revolution of a parabola*" of the Chap. "2. Surfaces of revolution".

In Fig. 4, two types of the studied surfaces of revolution are presented some more.

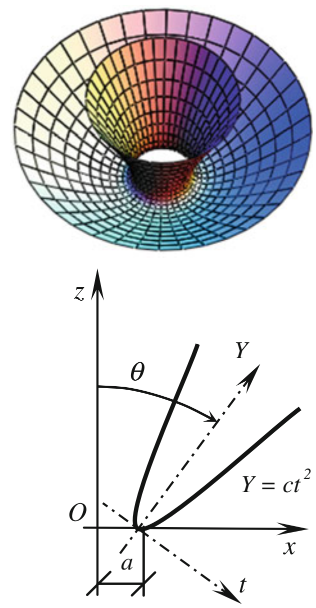
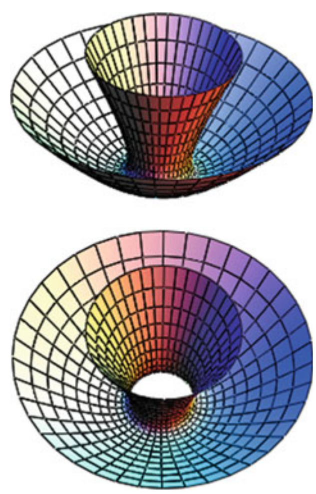
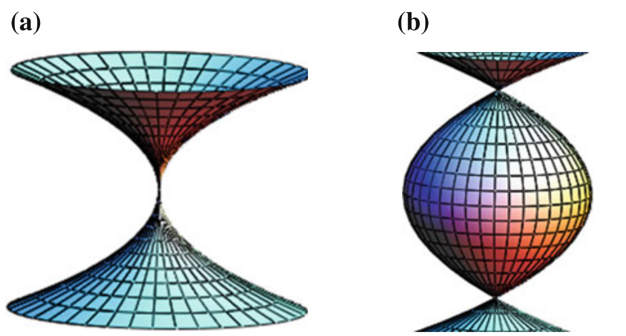


Fig. 1



$a = 0.8$ m; $c = 2$ m^{-1}; $\theta = 0.2\,\pi$

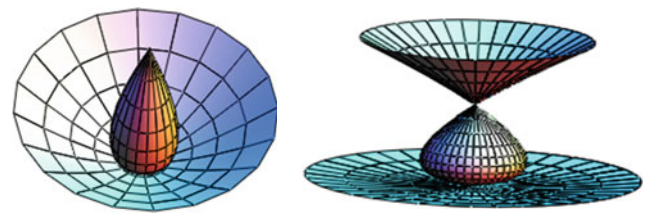
Fig. 2



(a) $a = 0$; $c = 1\ \mathrm{m}^{-1}$; $\theta = \pi/2$ (b) $a = 0.8$ m; $c = 1\ \mathrm{m}^{-1}$; $\theta = -\pi/2$

Fig. 3

Assume a slope angle of the axis of a parabola to an axis of rotation equal to zero (θ = 0) and the distance a peak of the parabola from the rotation axis equal to zero (a = 0) too, then the studied surface of revolution will degenerate into *a paraboloid of revolution* that is considered in section "Paraboloid of revolution".



$a = 0.5$ m; $c = 1\ \mathrm{m}^{-1}$; $\theta = -\pi/5$ $a = 0.8$ m; $c = 1\ \mathrm{m}^{-1}$; $\theta = -2\pi/5$

Fig. 4

Additional Literature

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■ Surface of Revolution of a Biquadrate Parabola

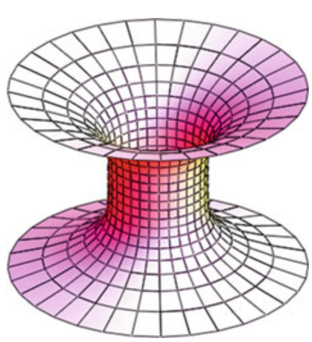
A paraboloid of revolution of the fourth order is generated by a rotating biquadrate parabola about its axis of symmetry, i.e., about the axis of the parabola.

A surface of revolution of a biquadrate parabola is formed in the process of rotation of a biquadrate parabola about a straight that is perpendicular to the parabola axis.

Forms of definition of the surface of revolution

(1) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = \sqrt[4]{c(r-a)},$$



$(a > 0)$

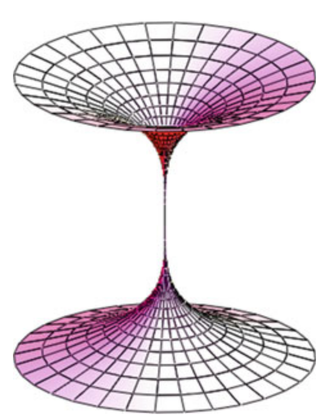
Fig. 1

where $r = a$ is a radius of *the waist circle*, $|x| \geq a$, $|y| \geq a$, $0 \leq \beta \leq 2\pi$. The surface is formed by rotation of *a parabola of the fourth order*

$$z^4 = c(x - a)$$

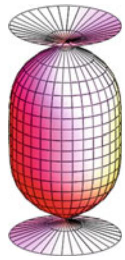
about the axis z. In Fig. 1, the surface of rotation of the biquadrate parabola is shown when $a > 0$.

Having assumed a = 0, we can design the surface of revolution presented in Fig. 2. If $a \geq 0$, then the surface of revolution of the biquadrate parabola belongs to a class of *surfaces of negative Gaussian curvature*.



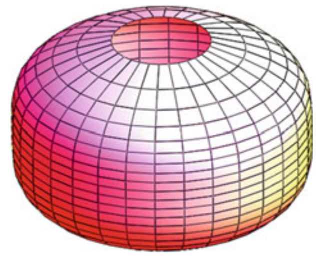
$(a = b = 0)$

Fig. 2



$(b > 0)$

Fig. 3



$(b > 0)$

Fig. 4

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{\sqrt{\ }}{16(r-a)^{3/2}}, \quad F = 0, \quad B = r,$$
$$L = -\frac{3-^{1/4}}{16A(r-a)^{7/4}}, \quad M = 0, \quad N = \frac{c^{1/4}r}{4A(r-a)^{3/4}},$$
$$k_1 = k_r = -\frac{3c^{1/4}}{16A^3(r-a)^{7/4}}, \quad k_2 = k_\beta = \frac{c^{1/4}}{4Ar(r-a)^{3/4}},$$
$$K = -\frac{3\sqrt{c}}{64rA^4(r-a)^{5/2}} < 0.$$

(2) Parametrical equations (Figs. 3 and 4):

$$x = x(z,\beta) = \left[\frac{z^4}{c} - b\right]\cos\beta,$$
$$y = y(z,\beta) = \left[\frac{z^4}{c} - b\right]\sin\beta,$$
$$z = z,$$

where $b \geq 0$ is a distance between a peak of the parabola and the axis of rotation.

If $b = 0$, then we can produce the surface shown in Fig. 2. In Fig. 3, the surface is shown when $b > 0$. Having assumed $b > 0$ and $-bc < z^4 < bc$, we can have *a barrel-shaped surface* of revolution of positive Gaussian curvature (Fig. 4). A surface of revolution of a biquadrate parabola has two conical points:

$$x = y = 0, \quad z = \pm(cb)^{1/4}.$$

If $z^4 > |bc|$, then a surface of revolution of a biquadrate parabola becomes a surface of negative Gaussian curvature.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + 16\frac{z^6}{c^2}, \quad F = 0, \quad B^2 = \left(\frac{z^4}{c} - b\right)^2,$$
$$L = -\frac{12z^2}{cA}, \quad M = 0, \quad N = \frac{B}{A},$$
$$k_1 = k_z = -12\frac{z^2}{cA^3}, \quad k_2 = k_\beta = \frac{1}{AB}, \; K = -\frac{12z^2}{cA^4B}.$$

■ Ellipsoid of Revolution

An ellipsoid of revolution is a surface formed by rotating of an ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$$

about its axis of symmetry Oz. An ellipsoid of revolution is *a closed quadric surface*. Older literature uses "*spheroid*" in place of "ellipsoid of revolution." *An oblate spheroid* (*oblate ellipsoid of revolution*) is formed by rotation of the ellipse about its minor axis (Fig. 1a). A special case arises when $a = b$, then the surface is *a sphere* and the intersection with any plane passing through it is a circle (Fig. 1b). *A prolate spheroid* (*prolate ellipsoid of revolution*) is formed by rotation of the ellipse about its major axis (Fig. 1c).

An ellipsoid of *revolution lies inside* the rectangular parallelepiped bounded by the sides $-a \leq x \leq a$; $-a \leq — \leq a$; $-b \leq z \leq b$. The geodesic line coincides with the equator parallel of an ellipsoid of revolution. The geodesic line passing through a pole point of an ellipsoid passes through an opposite pole point too. A volume contained inside the surface of ellipsoid of revolution is

$$V = \frac{4}{3}\pi a^2 b.$$

In cartography, the Earth is often approximated by an oblate spheroid instead of a sphere. The current World

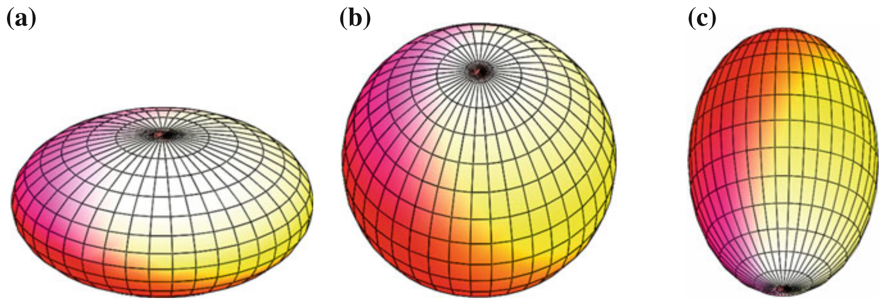


Fig. 1 **a** The oblate ellipsoid of revolution ($a > b$). **b** The sphere ($a = b$). **c** The prolate ellipsoid of revolution ($a < b$)

Geodetic System model uses a spheroid whose radius is 6,378.137 km at the equator and 6,356.752 km at the poles.

Forms of definition of the surface

(1) The standard equation of an ellipsoid of revolution centered at the origin of a Cartesian coordinate system and aligned with the axes is:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

(2) Parametrical equations (Fig. 1):

$$x = x(\alpha, \beta) = a \cos \alpha \cos \beta,$$
$$y = y(\alpha, \beta) = a \sin \alpha \cos \beta,$$
$$z = z(\beta) = b \sin \beta;$$
$$0 \le \alpha \le 2\pi; \; -\pi/2 \le \beta \le \pi/2.$$

Coefficients of the fundamental forms of the surface:

$$A = a \cos \beta, F = 0; B^2 = a^2 \sin^2 \beta + b^2 \cos^2 \beta;$$
$$L = ab \cos^2 \beta / B; M = 0; N = -ab/B.$$

Coordinate lines α and β (parallels and meridians) are lines of principal curvatures.

(3) Parametrical equations (Fig. 2):

$$x = x(u, v) = \rho \sin u \cos v,$$
$$y = y(u, v) = \rho \sin u \sin v,$$
$$z = z(u) = \rho \cos u,$$

where $\rho = \dfrac{b}{\sqrt{1 + \omega \sin^2 u \cos^2 v}}$; $\omega = \dfrac{b^2}{a^2} - 1.$

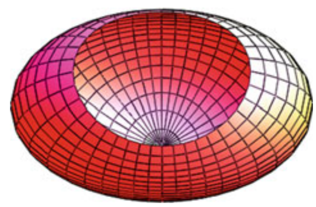


Fig. 2 The ellipsoid of revolution with the elliptical opening, $u_o \le u \le \pi$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \rho \sqrt{1 + \left(\frac{\omega}{2a^2} \rho^2 \sin 2u \cos^2 v\right)^2}; \; F = 0;$$

$$B = \rho \sin u \sqrt{1 + \left(\frac{\omega}{2a^2} \rho^2 \sin 2v \sin^2 u\right)^2};$$

$$k_1 = \frac{ab}{\left[b^2 + \omega(\rho \sin u \cos v)^2\right]^{3/2}};$$

$$k_2 = \frac{1}{\rho \sqrt{1 - (1 - a^4/b^4) \sin^2 u \cos^2 v}}.$$

Coordinate lines u, v form *the geographic system of coordinates* but they are not lines of principal curvatures.

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■ Ding–Dong Surface

A surface of revolution "*Ding–Dong Surface*" is like a surface of revolution "*Kiss surface.*"

Forms of definition of the surface

(1) Implicit equation: $x^2 + y^2 = (1 - z)z^2$

So, the studied surface of revolution is an algebraic surface of the third order. It is obtained by rotating curve

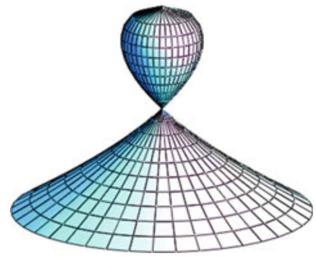
$$x = x(z) = z(1-z)^{1/2}$$

about an axis Oz.

(2) Parametrical equation (Fig. 1):

$$x = x(u, v) = r(v)\cos u,\ \ y = y(u, v) = r(v)\sin u,$$
$$z = z(v) = v,$$

where $r(v) = v\sqrt{1 - v}$; $-\infty \le v \le 1$; $0 \le u \le 2\pi$.



$(-1 \le v \le 1)$

Fig. 1

Additional Literature

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■ "Eight Surface"

A surface of revolution "*Eight Surface*" is generated by rotation of a curve

$$x = x(z) = 2z\left(1 - z^2\right)^{1/2}$$

about the axis Oz. The surface pictured in Fig. 1 is called an eight surface because it is a surface of revolution of a figure eight.

Forms of definition of the surface

(1) Implicit equation:

$$x^2 + y^2 = 4\left(1 - z^2\right)z^2.$$

Hence, the studied surface is *an algebraic surface of the fourth order*.

(2) Parametrical equations (Fig. 1):

$$x = x(u, v) = \cos u \sin 2v, \quad y = y(u, v) = \sin u \sin 2v,$$
$$z = z(v) = \sin v,$$

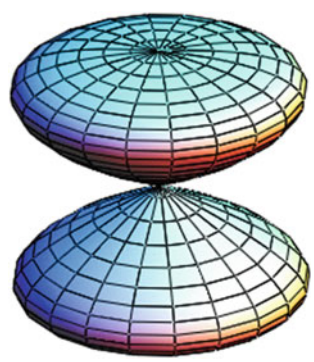


Fig. 1

where $-\pi/2 \le v \le \pi/2$; $0 \le u \le 2\pi$. The surface comes to a point at its very center.

Reference

The Eight Surface: http://www.math.hmc.edu/~gu/math142/mellon/curves_and_surfaces/surfaces/eightsurf.html

■ Surface of Revolution "Egg" of the Fourth Order

Eggshell is one of the perfect natural forms. Having researched *closed two-focus curves* of the fourth order, one can obtain an equation of mathematical model of the meridian cross section of an eggshell. G.V. Brandt considered that an egg form can be described by an implicit equation of the fourth order:

$$z^2 + y^2 = 3x(2a - x)\left[1 - c^2/(x + a)^2\right]/4,$$

where $2a$ is a length of major axis (an axis of rotation); c is the interfocus distance; $(a - c)/2$ is the distance the origin of a Cartesian coordinates from the first focus of meridional curve.

Parametrical equations of a surface of revolution "Egg" can be written in the form:

$$x = x, \quad y = y(x, \varphi) = r(x)\cos\varphi,$$
$$z = z(x, \varphi) = r(x)\sin\varphi,$$

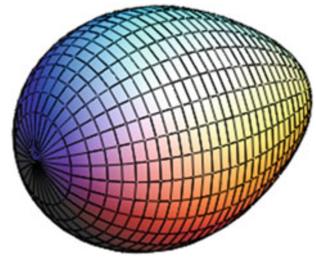


Fig. 1

where $r(x) = \sqrt{\frac{3}{4}x(2a - x)\left[1 - \frac{a^2\beta^2}{(x+a)^2}\right]}$, $\beta = c/a$ is a coefficient characterized a form of the meridian. A surface "*Quail Egg*" with $\beta = 0.75$ is presented in Fig. 1.

Reference

Brandt GV. The research of an equation of a shell formed by the two-focus curve. Sb. tr. VZPI: "Stroitelstvo i Arhitektura". Moscow: VZPI. 1973; p. 76-86.

■ Surface of Revolution "Egg" of the Third Order

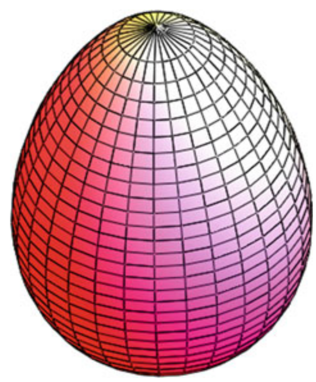
It is known also a surface of revolution "Egg" which is given by an implicit equation of the third order:

$$x^2 + y^2 = c^2 z(z - a)(z - b),$$

where a, b, c are constant parameters determining the form of a surface. Parametrical equations of the third-order surface of revolution "Egg" (Fig. 1) can be given as

$$x = x(u, v) = c\sqrt{u(u - a)(u - b)}\sin v,$$
$$y = y(u, v) = c\sqrt{u(u - a)(u - b)}\cos v,$$
$$z = z(u) = u,$$

where $a < = b$, then $0 \le v \le 2\pi$, $0 \le u \le a$.



$a = 1$ cm; $b = 1.5$ cm;
$c^2 = 0.85^2$ cm^{-1}

Fig. 1

■ Piriform Surface

This surface of revolution resembles a coming to the surface soft capacity with load. In English language literature, this surface is called "Piriform Surface".

Parametrical equations are

$$x = x(u, v) = b[\cos v(r + \sin v)]\cos u,$$
$$y = y(v) = a(r + \sin v),$$
$$z = z(u, v) = b[\cos v(r + \sin v)]\sin u,$$

where $0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$; a, b, and r are constant coefficients defining the form of the surface (Fig. 1).

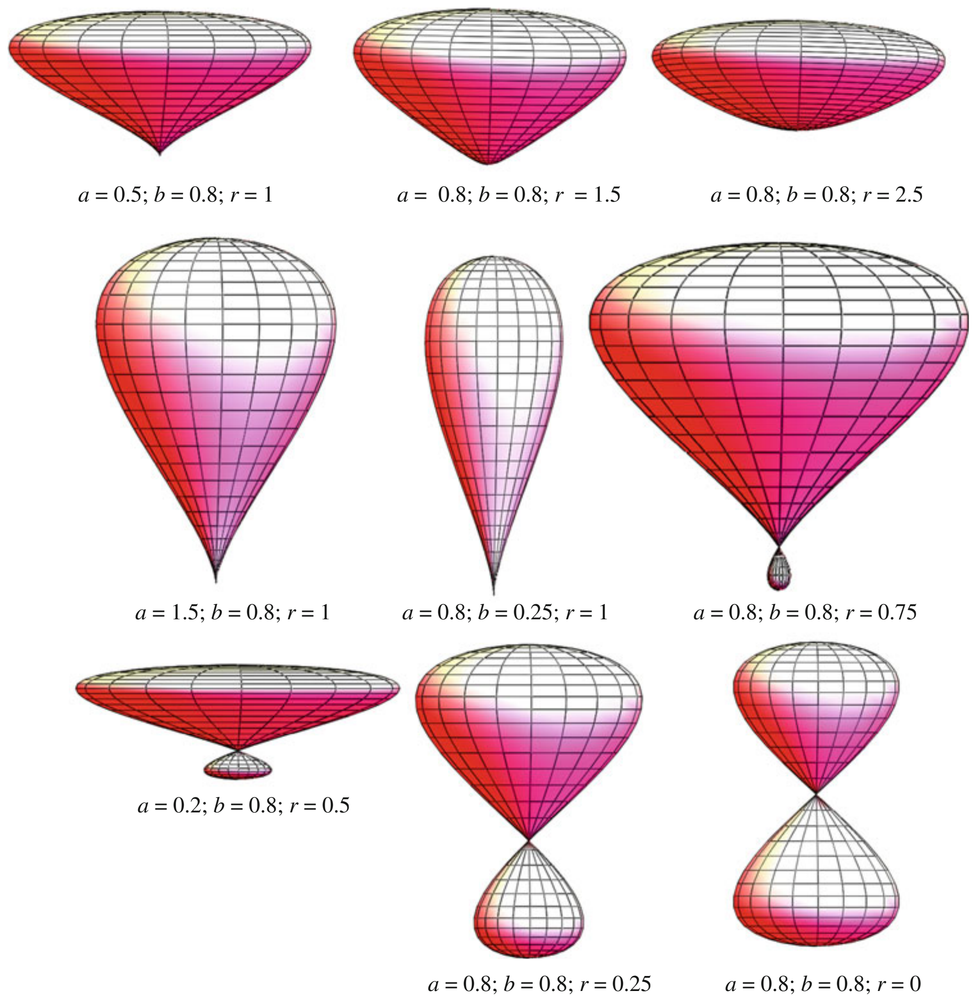


Fig. 1

■ "Drop"

Assuming certain values of constant parameters entering into parametrical equations of a surface of revolution "Drop," one can obtain the form of a drop in the process of falling.

Parametrical equations of the surface can be given as (Figs. 1 and 2):

$$x = x(u, v) = a(b - \cos u)\sin u \cos v,$$

$$y = y(u, v) = a(b - \cos u)\sin u \sin v,$$
$$z = z(u) = \cos u,$$

where $0 \le u \le \pi$, $0 \le v \le 2\pi$; a and b are constant coefficients defining the form of the surface.

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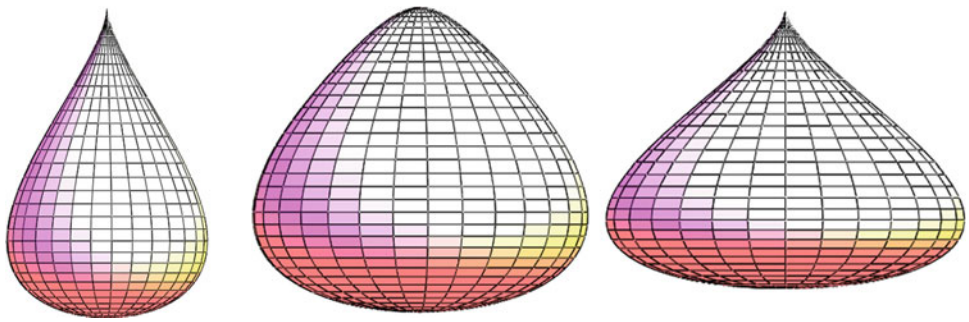
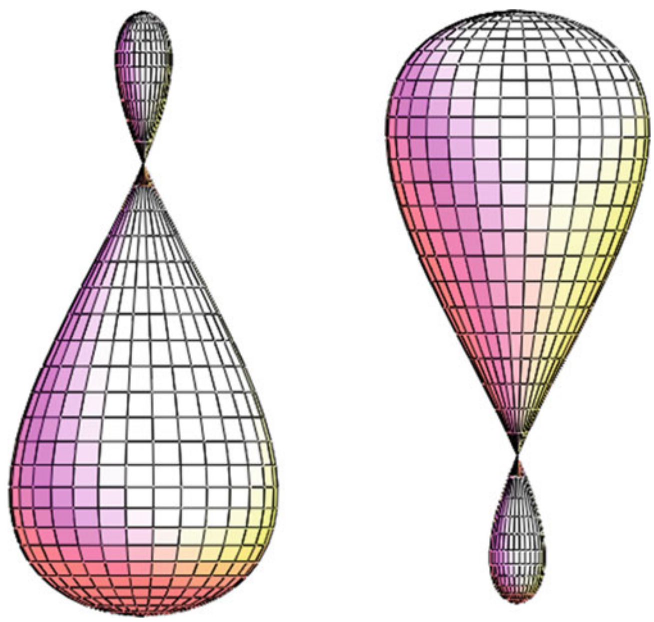
Parametrische Flächen und Körper. - http://www.3d-meir.de/tut3/Seite44.html



$a = 0.5; b = 1$ $a = 0.5; b = 2$ $a = 1; b = 1$

Fig. 1



$a = 0.5; b = 0.5$ $a = 0.5; b = -0.5$

Fig. 2

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Additional Literature

P.S.: Additional literature is given at the corresponding pages of the Chap. "2. Surfaces of Revolution".

2.1 Middle Surfaces of Bottoms of Shells of Revolution Made by Winding of One Family of Threads Along the Lines of Limit Deviation

Shells of revolution made by winding of one family of threads along the lines of limit deviation are used in pressure vessels from composite materials. They consist of a cylindrical fragment and two bottoms that are jointed smoothly just between themselves along the edges. The bottoms end by the pole openings with metal flange for the fixing of the cover. A pressure vessel from composed materials made by a method of winding of high-strength threads is more adaptable to streamlined production and gives a reduction of 30–50 % in weight in comparison with metal analogies.

Inner forces appearing in the bottom under inner pressure must be oriented along the threads in its every point.

An equation of *a middle surface of bottoms of shells of revolution made by winding of one family of threads along the lines of limit deviation* is derived from the decision of a nonlinear ordinary differential equation:

$$\frac{y''}{y'(1+y'^2)} = \frac{2r}{r^2 - t^2} - \frac{\mathrm{tg}^2\varphi}{r}$$

obtained on the base of a momentless theory of analysis of shells made of threads. The following conventions are used in the formula: $y = f_1(r)$ is an equation of a meridian of the middle surface of the bottom of revolution; r is a radial coordinate of a generatrix line of the bottom (meridian); the primes mean the differentiation with respect to a coordinate r; φ is an angle of the thread with a meridian of the surface of the bottom. In every point of the shell surface, a tread with an angle $+\varphi$ corresponds the thread with the angle $-\varphi$; a parameter t is equal to zero for the pole opening closed by the cover or to the radius r_p of the opening in the cover.

Trajectories of the threads of the shell must satisfy a condition of *technological realizably*, i.e., absolute value of tangent of the angle between the normal to the trajectory of a thread and the normal to the surface must not go over the coefficient of friction k of the thread on the surface in the process of winding. It can be written as

$$\left| \frac{r\varphi' \cos\varphi + \sin\varphi}{\frac{ry'' \cos^2\varphi}{1+y'^2} + y' \sin^2\varphi} \right| \le k.$$

For shell of revolution made by winding of one family of threads along the lines of limit deviation, an equation of generatrix surface $y = f_1(r)$ and an equation of the trajectories of the threads $\varphi = f_2(r)$ are calculated numerically from the solution of Augustin Louis Cauchy problem for a system of two differential equations that are the equation of generatrix curve of the surface of revolution and the equation of technological realizably with a sign of an equality in the right part and with a meaning $k_0 \le k$. An angle φ of a thread at the pole must be equal to 90° due to a condition of continuity of automatized winding.

The given differential equations give an opportunity to find a form of generatrixes of a surface of bottoms and the trajectory of threads of pressure vessels with maximally differing radiuses of pole openings.

References

Vasil'ev VV, Protasov VD, Bolotin VV et al. Composite Materials. Reference book. Moscow: "Mashinostroenie", 1990; 512 p.

Obraztzov IF, Vasil'ev VV, Bunakov VA. Optimal Design of Shells of Revolution from Composite Materials. Moscow: "Mashinostroenie", 1977; 144 p.

2.2 Middle Surfaces of Bottoms of Shells of Revolution Made by Plane Winding of Threads

Shells of revolution made by plane winding are used in pressure vessels from composite materials. They consist of a cylindrical fragment and two bottoms that are jointed smoothly just between themselves along the edges. The bottoms end by the pole openings with metal flange for the fixing of the cover. A pressure vessel from composed materials made by a method of winding of high-strength threads is more adaptable to streamlined production and gives a reduction of 30–50 % in weight in comparison with metal analogies.

Inner forces appearing in the bottom of the shell under action of inner pressure must be oriented along the threads in its every point. An equation of the generatrix of *the middle surface of bottoms of shells of revolution made by plane winding of threads* is derived from the decision of a nonlinear ordinary differential equation:

$$\frac{y''}{y'(1+y'^2)} = \frac{2r}{r^2 - t^2} - \frac{\mathrm{tg}^2\varphi}{r}$$

obtained on the base of a momentless theory of analysis of shells made of threads. The following conventions are used in the formula: $y = y(r)$ is an equation of a meridian of the middle surface of the bottom of revolution; r is a radial coordinate of a generatrix curve of the surface of revolution of bottom. The primes mean the differentiation with respect to a coordinate r; φ is an angle of the thread with a meridian of the surface of revolution of the bottom. In every point of the shell surface, a tread with an angle $+\varphi$ corresponds the thread with the angle $-\varphi$; a parameter t is equal to zero for the pole opening closed by the cover or to the radius r_p of the opening in the cover.

The threads of plane winding are placed on the surface of revolution in the planes tangent to the pole openings of the both bottoms in conformity with an equation

$$\mathrm{tg}\varphi = \frac{ry' - y}{\sqrt{1+y'^2}\sqrt{r^2\mathrm{ctg}^2\gamma - y^2}},$$

where γ is the angle of the plane with a thread with the axis of rotation of a surface of the bottom. An angle φ of a thread at the pole must be equal to 90° due to a condition of continuity of winding.

An equation of a meridian of the middle surface $y = y(r)$ for a shell of revolution made by plane winding is turn up from the solution of A.L. Cauchy problem for a nonlinear ordinary differential equation

$$\frac{y''}{y'(1+y'^2)} = \frac{2r}{r^2 - t^2} - \frac{(ry' - y)^2}{r(1+y'^2)(r^2\text{ctg}^2\gamma - y^2)},$$

which is obtained by equating corresponding parts of two given above differential equations. The given differential equations give an opportunity to find a form of generatrix curves of middle surfaces of bottoms and the trajectory of threads of pressure vessel both with equal and different radiuses of pole openings of two bottoms.

The calculated trajectory of laying of the thread in the process of winding must satisfy a condition of *technological realizably*, i.e., absolute value of tangent of the angle between the normal to the trajectory of a thread and the normal to the surface must not go over the coefficient of friction k of the thread on the surface in the process of winding. This condition is presented in the previous section.

Reference

Vasil'ev VV, *Protasov VD, Bolotin VV et al.* Composite Materials. Reference book. Moscow: "Mashinostroenie", 1990; 512 p.

2.3 Middle Surface of Bottoms of Shell of Revolution Made by Winding of Threads Along Geodesic Lines

Pressure vessels from composed materials made by a method of winding of high-strength threads along geodesic lines are more adaptable to streamlined production and give a reduction of 30–50 % in weight in comparison with metal analogies.

The laying of threads on a surface along geodesic lines maintains a stable position of threads in the process of their winding in conformity with A. Clairaut equation: $r \sin \varphi = r_{0,}$ where φ is the angle of the thread with the generatrix curve of a surface of revolution. In every point of the middle surface of a shell of revolution, a tread with an angle $+\varphi$ corresponds the thread with the angle $-\varphi$; r_0 is the radius of the pole opening. The form of a generatrix curve $y = y(r)$ of the middle surface of revolution of the bottom ensures the direction of inner forces, appearing in the shell of the bottom under action of inner pressure, along the threads. A generatrix of the surface of bottom with a flange is computed as a result of consistent solution of two differential equations:

$$\frac{\mathrm{d}y_1}{\mathrm{d}r} = -\frac{r(r^2 - t^2)\sqrt{a^2 - r^2}}{\sqrt{a^2(r^2 - r_0^2)(a^2 - t^2)^2 - r^2(a^2 - r_0^2)(r^2 - t^2)^2}}$$

where $b \leq r < a$,

$$\frac{\mathrm{d}y_2}{\mathrm{d}r} = -\frac{r(b^2 - r_0^2)\sqrt{(a^2 - r_0^2)(r - r_0^2)}}{\sqrt{a^2(b^2 - r_0^2)^2(a^2 - r_0^2)^2 - r^2(r^2 - r_0^2)(b^2 - r_0^2)^2(a - r_0^2)}}$$

$r_0 \leq r \leq b$, $y = y(r)$ is a axial coordinate of a generatrix curve of the bottom; a is the radius of the cylindrical segment of the shell of revolution; b is the maximal radius of the flange; a parameter t is equal to zero for the pole opening closed by the cover or to the radius r_p of the opening in the cover.

A.L. Cauchy problem for the first differential equation is solved with a initial condition that is $y_1 = 0$ if $r = a$. For the second differential equation, an initial condition is $y_2 = y_1$ if $r = b$. The first and the second equation can be solved in *elliptical integrals*. Maximal radius of the flange for the convex surface of the bottom must satisfy a condition:

$$b \geq \frac{\sqrt{3}}{2} r_0 \sqrt{1 + \sqrt{1 - \frac{8t^2}{9r_0^2}}}.$$

The form of the studied middle surface is shown in Fig. 1. An equation of the meridian $y = y(r)$ was derived numerically with the help of presented differential equations. A problem was solved for a surface of revolution with the following parameters: $a = 3$ m; $b = 1.3$ m; $r_0 = 1$ m, $t = 0$. The surface of revolution runs smoothly into the cylindrical segment of the pressure vessel.

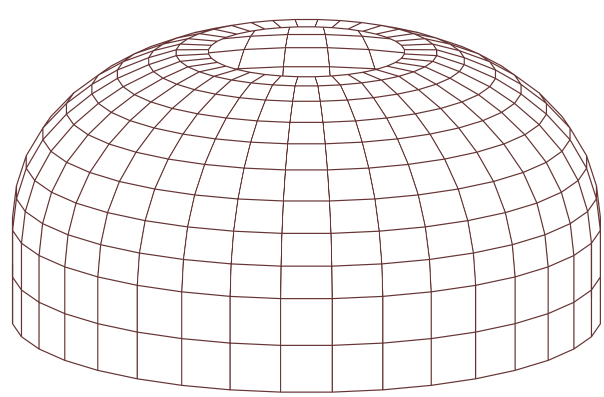

Fig. 1

References

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Obraztzov IF, Vasil'ev VV, Bunakov VA. Optimal Design of Shells of Revolution from Composite Materials. Moscow: "Mashinostroenie", 1977; 144 p.

2.4 Middle Surfaces of Shells of Revolution with Given Properties

Many scientific works devote to discovering form of a meridian of the middle surface of thin-walled shell of revolution with given properties in advance. It is known the following criterions of selection of optimal form of shell of revolution: a cost of a shell, minimal weight [1], the absence of bending moments and tensile normal forces [2], the given stress state for acting external load [3], the given bearing capacity for optimal slope [4], maximal external load; minimal weight under limitation for value of the natural frequency and maximal displacements [5]; the absence of bending moments with taking into account inner pressure, dead weight and centrifugal forces [6]; maximal critical load [7, 8] or the selection of a form with taking into consideration another set of presented demands.

A condition of equi-strength of thin-walled shell of reservoir is assumed as a basis of analysis of *drop-shaped reservoir* for the liquid products [9]. Geometry of the middle surface of a shell is chosen on condition that tensile meridional and circular forces will be equal to each other and constant ($N_1 = N_2 = N = \text{const}$) under an action of designed load. It means that a condition

$$1/R_1 + 1/R_2 = \gamma(h + y)/N = pN,$$

must be satisfied. This equation follows from the condition of equilibrium of a shell element (Laplace formula). Here R_1 and R_2 are radiuses of principle curvatures correspondingly in meridional and circular directions. The key designed load (inner pressure)

$$p = \gamma(h + y)$$

is a sum of hydrostatical pressure of liquid and uniform redundant pressure; y is the distance the peak from a considered point of the shell in the vertical direction; γ is a density of the product; h is a height of designed column of liquid.

In a paper [10], problems of existence of optimal forms of thin-walled shells possessing minimal mass and satisfying to corresponding geometrical limitations and satisfying to restrictions on acceptable number of cycles of external cyclical load were studied. In this paper, an equilibrium stress state of a membrane shell of revolution loaded by axisymmetric loads q_n, q_θ was described by the following equations:

$$\mathrm{d}(r_0 N_\alpha)/\mathrm{d}\alpha - N_\theta R_1 \cos\alpha + r_0 R_1 q_\alpha = 0,$$
$$N_\alpha/R_1 + N_\theta/R_2 = q_n,$$

$r_0 = R_2 \sin\alpha$. The symbolism is shown in Fig. 1 at Page 100.

E. Annaberdyev [11] offers a method of selection of the single surface of revolution passing through given parallels and having the given magnitudes of coefficients of the first fundamental form in the theory of surfaces

$$\mathrm{d}s^2 = E\mathrm{d}u^2 + G\mathrm{d}v^2.$$

We cannot design a surface of revolution when a finite number of its parallels is taken. A meridian of surface of revolution can be formed if we shall give the common tangents at the joints of the parallels for maintaining smoothness of the meridian.

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■ Surfaces of Revolution with Geometrically Optimal Rise

In applied geometry of surfaces, interest to methods of optimization of geometrical form of surfaces of revolution with given properties in advance arose time and again. It was considered that the most actual problem is the following: it is necessary to obtain a form of the surface with minimal area S covering the maximal volume V. It gives the lesser expenditure of materials and the lesser weight of the shell. The special criterion

$$n = V/S$$

was introduced into practice (Fig. 1).

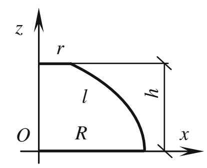

Fig. 1

An area S of the second-order surface and a volume covered by this surface can be defined with the help of the general formulas:

$$S = 2\pi \int_0^h x(z)\sqrt{1 + x(z)'^2}\mathrm{d}z, \quad V = \pi \int_0^h x(z)^2 \mathrm{d}z,$$

where $x = x(z)$ is an equation of a meridian; h is the rise of a surface, i.e., maximal rise of a surface over the plane xOy. A meridian is rotated about the axis Oz.

For concrete surfaces of revolution, these formulas give:

(1) **a truncated sphere**:

$$x = x(z) = \sqrt{a^2 - (z + \sqrt{a^2 - R^2})^2};$$
$$a = \sqrt{(R^2 - h^2 - r^2)^2/(4h^2) + R^2};$$
$$S = 2\pi a h; V = \frac{\pi h}{2}\left(R^2 + \frac{h^2}{3} + r^2\right);$$

$$n_{\text{sph.segm.}} = \frac{h(R^2 + h^2/3 + r^2)}{2\sqrt{(R^2 - h^2 - r^2)^2 + 4h^2R^2}};$$
$$n_{\text{sphere}} = \frac{R}{3};$$

(2) **a truncated cone**:

$$x = x(z) = R - (R - r)z/h;$$
$$S = \pi(R + r)\sqrt{(R - r)^2 + h^2};$$
$$V = \frac{\pi h}{3}\left(R^2 + r^2 + rR\right);$$
$$n_{\text{tr.c}} = \frac{h(R^2 + r^2 + rR)}{3(R + r)\sqrt{(R - r)^2 + h^2}},$$
$$n_{\text{c}} = \frac{hR}{3\sqrt{R^2 + h^2}};$$

(3) **a circular cylinder**:

$$S = 2\pi R h; \quad V = \pi R^2 h; \quad n_{\text{cyl.}} = R/2.$$

(4) **a truncated paraboloid of revolution**:

$$x = x(z) = \sqrt{R^2 - z(R^2 - r^2)/h};$$

$$S = \frac{4\pi h}{3(R^2 - r^2)}\left\{\left[R^2 + \frac{(R^2 - r^2)^2}{4h^2}\right]^{3/2} - \left[r^2 + \frac{(R^2 - r^2)^2}{4h^2}\right]^{3/2}\right\};$$

$$V = \frac{\pi h}{2}(R^2 + r^2);\ n_{\text{par.}} = \frac{3Rh^3}{\left[(4h^2 + R^2)^{3/2} - R^3\right]};$$

$$n_{\text{tr.par.}} = \frac{3(R^4 - r^4)}{8\left[R^2 + (R^2 - r^2)^2/(4h^2)\right]^{3/2} - 8\left[r^2 + (R^2 - r^2)^2/(4h^2)\right]^{3/2}};$$

(5) **a truncated ellipsoid of revolution**:

$$x = x(z) = \sqrt{a^2 - \left(z/k + \sqrt{a^2 - R^2}\right)^2};$$

$$a \geq R;\ m^2 = a^2 - R^2;$$

$$\frac{c}{a} = k = \frac{h}{\sqrt{a^2 - r^2} - \sqrt{a^2 - R^2}};$$

$$V = \pi\left[a^2 h - \frac{(h + km)^3}{3k^2} + \frac{km^3}{3}\right];$$

$$n_{tr.el.} = \frac{V}{S},$$

where for **an oblate ellipsoid** with semi-axes $a > c$ ($k < 1$); $b^2 = 1 - k^2$, one has

$$S = \pi b\left[\frac{h + km}{k^2}\sqrt{\frac{k^4 a^2}{b^2} + (h + km)^2} - m\sqrt{\frac{a^2}{b^2} - R^2} + \frac{k^2 a^2}{b^2}\ln\frac{h + km + \sqrt{k^4 a^2/b^2 + (h + km)^2}}{k(m + \sqrt{a^2/b^2 - R^2}}\right];$$

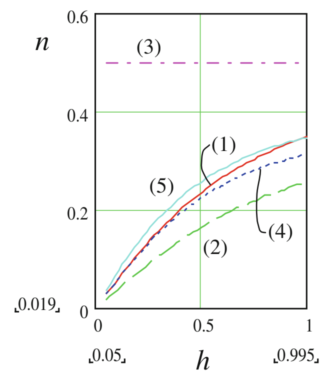

Fig. 2

for **a prolate ellipsoid** with semi-axes $a < c$ ($k > 1$); $t^2 = k^2 - 1 > 0$, one has

$$S = \pi t\left[\frac{h + km}{k^2}\sqrt{\frac{a^2 k^4}{t^2} - (h + km)^2} - m\sqrt{\frac{a^2 k^2}{t^2} - m^2} + \frac{a^2 k^2}{t^2}\left(\arcsin\frac{h + km}{ak^2}t - \arcsin\frac{mt}{ak}\right)\right].$$

Curves showing a change of the ratio $n = V/S$ with a change of a rise h give an opportunity to choose optimal parameters of the meridian for the given shell form (Fig. 2).

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■ Middle Surface of Non-Bending Shell of Revolution Under Uniform Pressure

Under action of uniform pressure with corresponding boundary conditions, not only spherical and circular cylindrical shells deform without bending but also endless two-parametrical family of shells of revolution which includes a sphere and a cylinder as a particular case. In the process of axisymmetrical deformation, all normals to a middle surface do not turn, i.e., their angle of turn in the meridional plane is equal to zero. Besides, the angles of shearing between the meridians and parallels are equal to zero too and the angles between them remain equal to $\pi/2$.

Having assumed these propositions and using the first condition of Peterson-Codazzi

$$\frac{dR_2}{d\theta} = (R_1 - R_2)\frac{\cos\theta}{\sin\theta},$$

V.I. Gurevich and V.S. Kalinin derived a condition of absence of bending in shells of revolution in forces in the form:

$$\frac{R_2}{R_1}\frac{\mathrm{d}(N_2-\nu N_1)}{\mathrm{d}\theta}+(1+\nu)(N_2-N_1)\frac{\cos\theta}{\sin\theta}=0$$

where R_1 and R_2 are the principal radiuses of curvatures of the meridian and the parallels accordingly; θ is the angle of a normal to the meridian with an axis of rotation; ν is Poisson's ratio in theory of elasticity; N_1 and N_2 are the normal tensile or compressive forces reckoned per unit of curvilinear coordinates' length acting in the tangent plane of middle surface of the shell of revolution,

$$N_1=\frac{pR_2}{2},\; N_2=0.5pR_2\left(2-\frac{R_2}{R_1}\right).$$

A condition of absence of bending is correctly for shells of revolution subjected to any axisymmetrical loading. Substituting the values of normal forces in this condition, we can obtain its new interpretation:

$$\left(3-\frac{R_2}{R_1}\right)\frac{\mathrm{d}R_2}{\mathrm{d}\theta}-R_2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_2}{R_1}\right)=0$$

defining radiuses of principal curvatures of shell of revolution deforming without bending under action of uniform pressure.

It is obviously that not only radiuses of principal curvatures of sphere and cylinder satisfy this condition but shells with constant ratio $R_2/R_1=3$ too. In this case, $N_1=N_2$. Assume that $z=f(x)$ is an equation of unknown meridian, then

$$R_1=-\frac{(1+f'^2)^{3/2}}{f''},\quad R_2=\frac{x}{\sin\theta}=\frac{x\sqrt{1+f'^2}}{f'}.$$

After substituting of values $R_1=R_1(f)$ and $R_2=R_2(f)$ into the differential equation of absence of bending, we can derive an equation of left branch of the meridian in the form of an integral:

$$z=f(x)=-\int\limits_{-\eta}^{x}\frac{2C_1C_2x^3\mathrm{d}x}{\sqrt{(C_1-C_2x^2)^2-4C_1^2C_2^2x^6}},$$

which does not express itself in terms of elementary functions. Here, C_1 is constant.

In Fig. 1, taken from a paper of V.I. Gurevich and V.S. Kalinin, the meridians of non-bending shells of revolution having an angle $\theta=\pi/2$ when $x=\pm r_1$, i.e., $R_2=r_1$, where r_1 is the radius of the support circle, are presented.

The surfaces represented in Fig. 1 divide by a sphere into closed and unclosed at the peak. Unclosed surfaces divide by a circular cylinder into the surfaces of negative and positive Gaussian curvatures near the support part.

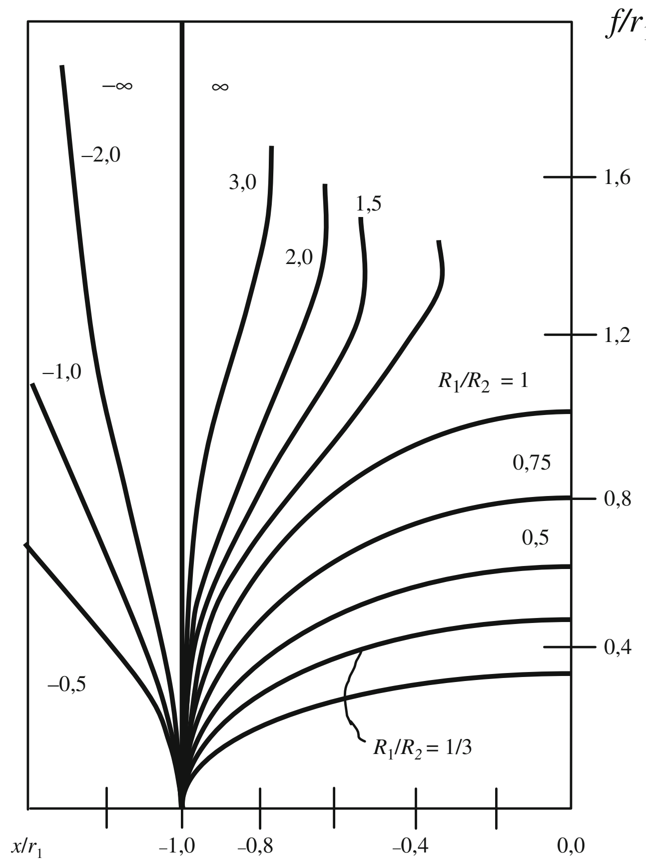

Fig. 1

Meridians were constructed under the condition that

$$C_1=\frac{R_1}{r_1(r_1-3R_1)},\quad C_2=\frac{R_1}{r_1^3(r_1-R_1)}.$$

Dissertation of N.V. Cherdyntzev is devoted to seeking of forms of shells of revolution and differential equations of stress-strain state of non-bending shell of revolution under uniform external pressure are presented. An integral defining a form of the shell was reduced to a sum of two elliptical integrals and was presented also in the form of power series.

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2.5 Surfaces of Revolution with Extreme Properties

Let a plane curve $r = r(z)$ (Fig. 1) passing through the given points has the given length L and revolving about an axis Oz, forms a surface of revolution of the given area S. Besides this, the volume V bounded by this surface and by two planes that are perpendicular to the axis of revolution must have the greatest value. This is a classical variational problem about conditional extremum: if a curve $r = r(z)$ gives an extremum to an integral

$$V = \int_D \pi \cdot r^2 \mathrm{d}r$$

under conditions

$$L = \int_D \sqrt{1 + r'^2}\, \mathrm{d}z \quad \text{and} \quad S = \int_D 2\pi r \sqrt{1 + r'^2}\, \mathrm{d}z$$

then the constants λ_0, λ_1, and λ_2 (*Lagrange multipliers*) exist and the curve $r = r(z)$ gives the extremum to an integral

$$Q = \int_D H \mathrm{d}z$$

where

$$H = \lambda_0 \pi r^2 + \lambda_1 \sqrt{1 + r'^2} + 2\lambda_2 \pi r \sqrt{1 + r'^2}.$$

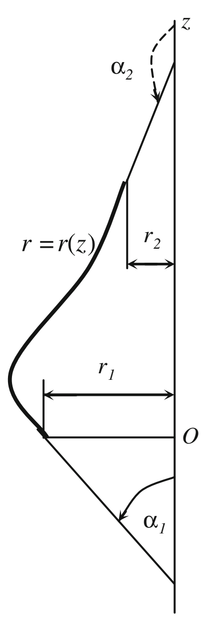

Fig. 1

Taking into consideration that this problem due to a reciprocity principle is equivalent to other two problems about conditional extremum:

(1) Obtain a plane curve $r = r(z)$ of a given length L which rotating about an axis Oz forms a surface of the minimal area bounding the given volume V.
(2) Obtain a plane curve $r = r(z)$ of the minimal length L which rotating about an axis Oz forms a surface of the given area S bounding the given volume V.

An Euler equation for the functional H is

$$H - \frac{\partial H}{\partial r'} = C,$$

because the function H does not depend explicitly on z, i.e., $H = H(r, r')$.

After transformation, we can derive an equation $z = z(r)$ in the integral form:

$$z = \int \frac{(C - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_1 + \lambda_2 r)^2 - (C - \lambda_0 r^2)^2}} + \gamma.$$

In general case, this integral can be expressed with the help of elliptical integrals. But having specific values of λ_0, λ_1, λ_2, and C, it is possible to integrate in the elementary functions. In this case, we shall obtain *a sphere* and *a torus* when $\lambda_1^2 - C\lambda_2^2 = 0$.

So, a sphere and a torus satisfy to all extremal conditions.

The expressions for Gaussian and mean curvatures of extreme surfaces have the following form:

$$K = \frac{(C - \lambda_0 r^2)(\lambda_2 C + \lambda_0 \lambda_2 r^2 + 2\lambda_0 \lambda_1 r)}{4r(\lambda_1 + \lambda_2 r)^3},$$

$$2H = \frac{\lambda_1 (C - 3\lambda_0 r^2) - 2\lambda_0 \lambda_2 r^3}{2r(\lambda_1 + \lambda_2 r)^2}$$

Giving different values to Lagrange constants, we can obtain different forms of surfaces possessing by extreme properties. There are well-known surfaces such as *cylindrical surface, sphere, torus*, *catenoid*, little known and insufficiently studied surfaces such as *nodoid* and *unduloid*, and recently presented surfaces such as “*Penka*” and *a surface of catenoidal type*, among them.

One paper is devoted to investigation of extremal surfaces of rotation for area-type functional. The solutions of differential Euler–Lagrange equation are obtained. Also, the symmetry property of this surface is proved; the examples of functionals are demonstrated and their corresponding solutions are given.

A theorem of existence for nonholonomic rotation surfaces of zero total curvature of the second kind was proved in a paper of O.V. Vasil'eva . An example of a nonholonomic surface of this class was constructed.

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■ Surface of Catenoidal Type

Substitute $\lambda_0^* = 0,\ \lambda_1^* \neq 0,\ \lambda_2^* \neq 0$ into general equation for generatrix curves

$$z = \int \frac{(C^* - \lambda_0^* r^2)\mathrm{d}r}{\sqrt{4(\lambda_1^* + \lambda_2^* r)^2 - (C^* - \lambda_0^* r^2)^2}} + \gamma$$

of surfaces of revolution possessing by extremal properties then we can formulate a problem in the following form: determine a surface formed by rotation of a curve $r = r(z)$ about an axis Oz limited by two planes, that are perpendicular to the axis of rotation, and having the least area of the surface with given length of a generatrix meridian $r = r(z)$.

Due to *reciprocity theorem*, such surface is equivalent to a surface of given area formed by rotation of a line $z = z(r)$ with the least length about an axis Oz. Then an expression for generatrix curves, represented before, will have the following form:

$$z = \int\limits_D \frac{C\mathrm{d}r}{\sqrt{4(\lambda_1 + r)^2 - C^2}},$$

where we introduced the following symbolisms:

$$\lambda_1 = \frac{\lambda_1^*}{\lambda_2^*},\ C = \frac{C^*}{2\lambda_2^*}.$$

Having fulfilled the specific manipulations, one can obtain an equation of the meridian $r = r(z)$ expressed in elementary functions:

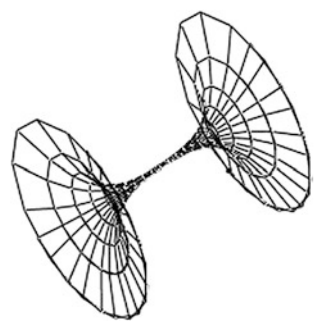

Fig. 1

$$r = C \cdot \cosh\frac{z-\gamma}{C} - \lambda_1.$$

The equation obtained is an equation of *a catenary* that is parallel transferred along an axis Oz at a distance of λ_1.

It should be noted that catenary is formed by *a focus of a parabola* in the process of rolling of this parabola along an axis Ox. The magnitude C is a parameter of the parabola. A value γ is defined by the initial position of the focus of the parabola.

A classical catenoid is formed by rotation of a catenary when this line is placed at the certain distance from the axis of rotation. A surface of revolution formed by rotation of a catenary displaced from this position will not be a minimal surface because the sum of principal curvatures of this surface is not equal to zero (Fig. 1).

Parametrical equations of *a surface of catenoidal type* can be written in the following form:

$$\begin{aligned} x &= x(z,\beta) &&= r(z)\cos\beta, \\ y &= y(z,\beta) &&= r(z)\sin\beta, \\ z &= z. \end{aligned}$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \operatorname{ch}\frac{z-\gamma}{C},\ F = 0,\ B = r,$$
$$L = -\frac{1}{C},\ M = 0,\ N = \frac{B}{A},$$
$$k_z = k_1 = \frac{-1}{CA^2},\ k_\beta = k_2 = \frac{1}{AB},$$
$$K = \frac{-1}{CBA^3} = \frac{-C^2}{r(r+\lambda_1)^3} < 0,$$
$$2H = \frac{C}{r(r+\lambda_1)^2} \neq 0.$$

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■ "Penka"

Assuming $\lambda_0 \neq 0;\ \ \lambda_1 \neq 0;\ \ \lambda_2 = 0$ in the equation for generatrix curves of surfaces of revolution possessing extreme properties

$$z = \int \frac{(C^* - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_1 + \lambda_2 r)^2 - (C^* - \lambda_0 r^2)^2}} + \gamma,$$

we can raise a problem in the following form: determine a curve

$$r = r(z)$$

of the given length in the process of rotation of which about an axis Oz, a surface of revolution is formed and together with two planes, that are perpendicular to the axis Oz, it envelops a maximal volume.

Assume $\lambda = \lambda_1/\lambda_0$, $C = C^*/\lambda_0$, then an integral expression for the generatrix meridian of a surface of revolution has the form:

$$z = \int \frac{(C - r^2)\mathrm{d}r}{\sqrt{4\lambda^2 - (C - r^2)^2}} + \gamma.$$

In this case, Gaussian and mean curvatures, radiuses of principal curvatures are

$$K = -\frac{(C - r^2)}{2\lambda^2};\quad 2H = \frac{C - 3r^2}{2\lambda r};$$
$$R_1 = \frac{\lambda}{r};\quad R_2 = \frac{2\lambda r}{(C - r^2)}.$$

Constants λ and C are determined due to the boundary conditions.

An equation of the generatrix meridian can be expressed with the help of elliptical integrals with taking into account the parameters λ, C and the conditions at the edges:

$$\left.\begin{aligned} z &= 2\sqrt{\pm\lambda}[\mathrm{E}(k,\varphi) - \mathrm{E}(k,\varphi_0)] + \sqrt{\pm\lambda}[\mathrm{F}(k,\varphi) - \mathrm{F}(k,\varphi_0)],\\ r &= \sqrt{|2\lambda \pm C|}\cos\varphi, \end{aligned}\right\}$$

or

$$\left.\begin{aligned} z &= -\frac{C}{\sqrt{|2\lambda \pm C|}}[\mathrm{F}(k,\varphi) - \mathrm{F}(k,\varphi_0)]\\ &\quad + \sqrt{|2\lambda \pm C|}[\mathrm{E}(k,\varphi) - \mathrm{E}(k,\varphi_0)];\\ r &= \sqrt{|2\lambda \pm C|}\sqrt{1 - k^2\sin^2\varphi}, \end{aligned}\right\}$$

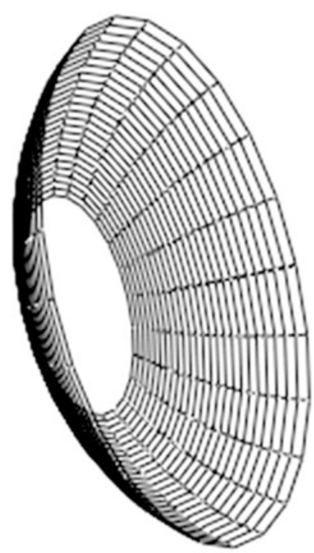

Fig. 1

where $F(k;\varphi)$ and $E(k;\varphi)$ are the elliptical integrals of the first and second orders, k is a module but φ is an amplitude of the elliptical integrals, φ_0 is an initial amplitude corresponding to $r = a$.

If $\lambda = \pm C/2$ and $C = 0$ then the integral expression for the generatrix curve is solved in quadrature: if $\lambda = \pm C/2$, then (Fig. 1).

$$z = \sqrt{\frac{C}{2}} \ln\left|\frac{\sqrt{2C} + \sqrt{2C - r_1^2}}{\sqrt{2C} + \sqrt{2C - r^2}} \cdot \frac{r}{r_1^2}\right| + \sqrt{2C - r^2} - \sqrt{2C - r_1^2};$$

if $C = 0$, then

$$z = \frac{1}{2}\left[r\sqrt{(2\lambda)^2 - r^2} - a\sqrt{(2\lambda)^2 - a^2}\right] - 2\lambda^2\left[\arcsin\frac{r}{2\lambda} - \arcsin\frac{a}{2\lambda}\right].$$

Having known the equation of a generatrix curve, it is easy to construct the surface of revolution with extremum properties with the help of parametrical equations:

$$x = x(r,\beta) = r\cos\beta,$$
$$y = y(r,\beta) = r\sin\beta,$$
$$z = z(r).$$

A surface with $\lambda = \pm C/2$ is called "PenKa" (Fig. 1).

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2.6 The Surfaces of Delaunay

In 1841, astronomer and mathematician C. Delaunay has picked out some surfaces of revolution described by him in his paper into an independent group.

In appendix of this paper, M. Sturm noted that the determination of equations of *Delaunay surfaces* is a variational problem on a conditional extremum.

For example, for *unduloid* and *nodoid*, the crux of the problem consists in the following: determine the functions $y(x)$, that are identified with meridians of surfaces of revolution, the volume of which can be calculated by a formula

$$V(y) = \pi \int_{x_0}^{x_1} y^2 \mathrm{d}x,$$

under condition of extremum of areas of their lateral surfaces

$$S(y) = 2\pi \int_{x_0}^{x_1} y \mathrm{d}s = 2\pi.$$

It is supposed that the edges of a surface of revolution are fixed.

This problem results in an equation of Euler–Lagrange:

$$y^2 + \frac{2ay}{\sqrt{1 + y'^2}} \mp b^2 = 0,$$

that is connected with an integral

$$F(y) = \pi \int_{x_0}^{x_1} (y^2 \mathrm{d}x + 2ay\mathrm{d}s) = \pi \int_{x_0}^{x_1} (y^2 + 2ay\sqrt{1 + y'^2})\mathrm{d}x.$$

Here, a is a corresponding real parameter; b is the second parameter.

It is recognized that the Delaunay surfaces are surfaces of revolution with *constant mean curvature*. With the exception of spheres, they are generated by *roulettes* in the process of their rotation about a curve along which the corresponding conics roll.

Roulettes are formed by focuses of parabola, ellipse, and hyperbola rolling without sliding along a straight line that is an axis of rotation.

Delaunay surfaces incorporate five surfaces of revolution that are *catenoids, unduloids, nodoids, spheres, and circular cylindrical surfaces*.

Let us present Euler–Lagrange equations for every type of surfaces of revolution:

$$\frac{y}{\sqrt{1 + y'^2}} - c = 0; \quad c > 0\,(\text{catenoid});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} + b^2 = 0, \quad \frac{1}{2H} > b > 0 \ (\text{unduloid});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} - b^2 = 0, \quad b > 0\,(\text{nodoid});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} = 0, \quad H > 0 \ (\text{sphere});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} + b^2 = 0, \quad H > 0, b > \frac{1}{2H}$$

(circular cylindrical surface).

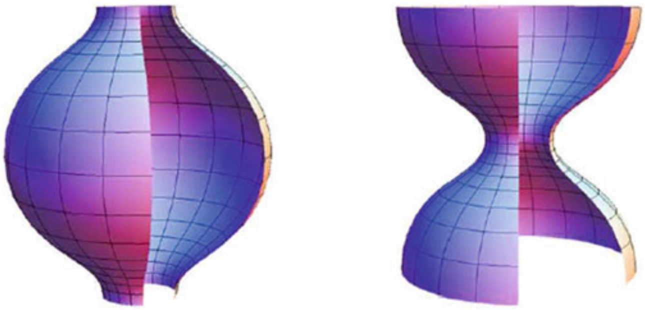

Fig. 1 Open parts of the bulb (*left*) and the neck (*right*) segments of the axially symmetric unduloid-like periodic surfaces of revolution obtained with the help of parametric equations by Djondjorov PA, et al

So, the Delaunay surfaces are included in a group of "Surfaces of Revolution with Extreme Properties" (p. 72). Axisymmetric surfaces of Delaunay's unduloids provide solutions of the shape equation in explicit parametric form. This class provides the analytical examples of surfaces with periodic curvatures studied by K. Kenmotsu and leads to some unexpected relationships among Jacobian elliptic functions and their integrals (Fig. 1).

Delaunay surfaces are used for description of processes in gas dynamics, for research of surfaces of soap films and bubbles.

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2.6.1 Nodoid and Unduloid Surfaces of Revolution

Substituting $\lambda_0 \neq 0,\ \lambda_1 = 0,\ \lambda_2 \neq 0$ into a general shape equation for generatrix curves of surfaces of revolution possessing extreme properties

$$z = \int \frac{(C - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_1 + \lambda_2 r)^2 - (C - \lambda_0 r^2)^2}} + \gamma$$

we can obtain an integral equation of the generatrix:

$$z = \int \frac{(C - \lambda_0 r^2)}{\sqrt{4(\lambda_2 r)^2 - (C - \lambda_0 r^2)^2}}\mathrm{d}r + \gamma.$$

This integral equation describes a family of *curves of Shturm*. that are lines generated by a focus of a parabola or hyperbola in the process of rolling of corresponding curves along a straight.

In that case, we can state a problem in the following form: find a plane curve $r = r(z)$ that forms a body of rotation of the given volume V. This curve rotates about an axis Oz but the body must cover a minimal area S.

Due to the principle of mutuality, this problem is equivalent to the following problem: determine a plane curve $r = r(z)$ rotating about an axis Oz that forms a body of minimal volume V limited by the surface of the given area S (Fig. 1).

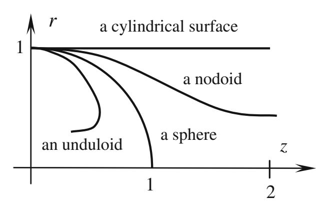

Fig. 1

Constant mean curvature is a remarkable property of nodoids and unduloids:

$$2H = -\frac{1}{\lambda_2} = \text{const},$$

but

$$K = \frac{(C^2 - r^4)}{4\lambda_2^2 r^4}.$$

So, an unduloid, or onduloid, is a surface with constant nonzero mean curvature obtained as a surface of revolution of an elliptic catenary: that is, by rolling an ellipse along a fixed line, tracing the focus, and revolving the resulting curve around the line. A nodoid is a surface of revolution with constant nonzero mean curvature obtained by rolling a hyperbola along a fixed line, tracing the focus, and revolving the resulting nodary curve about the line.

In 1828, Poisson has shown that a surface of separation of two mediums that are at balance is a surface of a constant mean curvature. But in this case, one neglects the dead weight. These surfaces can be modeled by soap films. A physical principle forming soap films, regulating their behavior, local and global properties is rather simple. A physical system keeps corresponding configuration only if the system cannot change easily the configuration having captured a position with less level of energy. An integral of general type is reduced into elliptical integrals of the first and second types:

$$x = -\frac{CF(k', \varphi)}{r} + rE(k', \varphi), \quad y = r\sqrt{1 - k'^2 \sin \varphi},$$

where

$$k = \frac{m}{r}; \quad k' = \sqrt{1 - k^2}$$

is an additional module of the integral. In ultimate cases, the integral for the studied surfaces can be reduced to an equation of sphere and circular cylindrical surface.

An analog of geometrical properties of shells of revolution under corresponding conditions is a condition of matching in strength (the same strength), i.e., an equality of circular and meridional forces in every cross section. A shell of revolution will be in equal strength state under action of inner pressure P and axial force P_0^z per unit length of the circular edge if

$$\frac{P_0^z}{2\pi r_1} = \lambda_2 P \sin \theta_0.$$

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■ Nodoid Surface Connecting Two Circular Cones

It is necessary to know Lagrange multipliers λ_0, λ_2; Euler constant C and a constant of integration γ for the unambiguous determination of a curve defined by an equation:

$$z = \int \frac{(C - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_2 r)^2 - (C - \lambda_0 r^2)^2}} + \gamma,$$

These values can be obtained without using of integral conditions for areas and volumes of the surface.

Let us construct a conjugation of two circular cones with known radiuses r_1 and r_2 and with slopes α_1, α_2 of rectilinear generatrixes of the cones (Fig. 1). For this case, we shall use a nodoidal surface. The length of the surface along an axis Oz turns automatically.

The integral equation becomes

$$z = \int \frac{(C^* - r^2)}{\sqrt{4 \cdot (\lambda r)^2 - (C^* - r^2)^2}} \cdot \mathrm{d}r + \gamma,$$

where $C^* = \frac{C}{\lambda_0}$, $\lambda = \frac{\lambda_2}{\lambda_0}$.

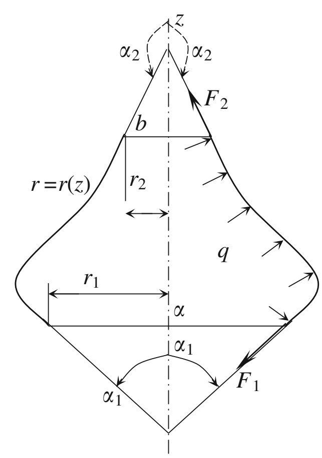

Fig. 1

The values r_1 and r_2, α_1 and α_2 must be connected between themselves.

Let us study a soap bubble subjected to inner pressure q. A contact of a soap film with the bases of the circular cones takes place in the sections a and b. In these sections, surface tension forces are directed along rectilinear generatrixes

These forces are

$$F_1 = \mu l_1 = 2\pi r_1 \mu \text{ and } F_2 = \mu l_2 = 2\pi r_2 \mu,$$

where μ is a coefficient of surface tension, l_i are the lengths of the contours of contact.

The conditions of equilibrium give

$$r_1 \sin\alpha_1 = r_2 \sin\alpha_2.$$

Using a Laplace formula for surface tension, we can get

$$\Delta p = 2H\mu.$$

So, we can design the surfaces both of positive and negative mean curvatures.

For the determination of coefficients λ and C, it is necessary to use an expression for derivative:

$$r'(z) = \frac{\sqrt{4\lambda^2 r^2 - (C^* - r^2)^2}}{(C^* - r^2)}$$

and boundary conditions: if $z = 0$ then $r = r_1$ and $r' = \tan\alpha_1$; but if $r = r_2$ then $r' = \tan\alpha_2$.

In addition, we have

$$\lambda = \frac{r_1^2 - r_2^2}{2(r_2\cos\alpha_1 - r_1\cos\alpha_2)},$$
$$C^* = \frac{r_1 r_2(r_1\cos\alpha_1 - r_2\cos\alpha_2)}{(r_2\cos\alpha_1 - r_1\cos\alpha_2)}.$$

In Fig. 2, copper nodoids are shown made by a method of galvanoplastics.

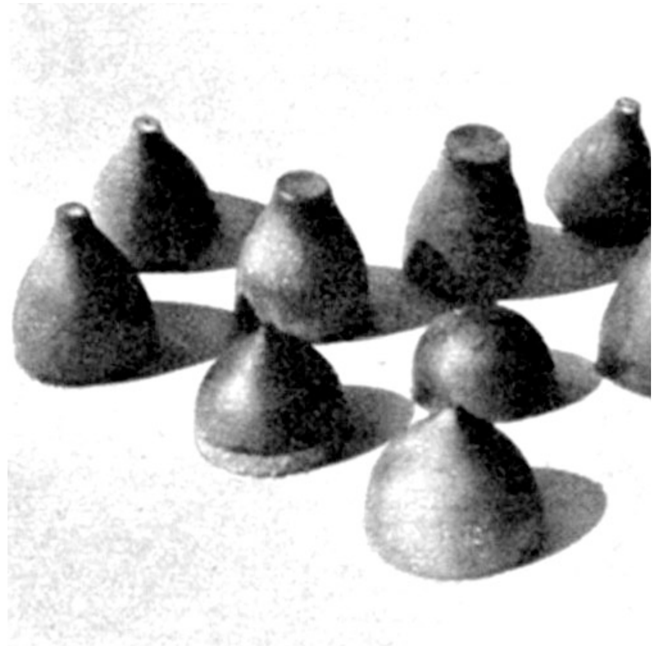

Fig. 2

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