Chapter 2
Verification Tests

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This chapter presents a set of closed form solutions that may serve as THMC test examples. The material has been arranged in sections of simulation exercises. All examples have been checked by OGS, FE-meshes and time steps are designed to reproduce the closed form solutions. The observed deviations are always less than one percent and are smaller by several orders of magnitude in many cases.

Throughout this chapter we will be concerned with the formal aspects of each exercise and present the closed form solution of the underlying initial or boundary value problem. The first four sections focus on single processes. Within each section we start from 1D problems and move to more advanced levels covering steady-state and transient problems up to 2D or 3D. Most of the material has been adopted from standard references. The series representations involved proved to converge rapidly and to serve well for numerical evaluation.

From section five onwards we will be concerned with coupled processes. Various transient problems will be solved with the aid of operational calculus; the Laplace transform solution method turns out to be an appropriate tool. Once that the Laplace transform is known numerical inversion will be employed to obtain the required values of the inverse transform.

The numerical method returns values of the function $f(t)$ for that the Laplace transform

$$\bar{f}(s) = \int\limits_0^\infty \mathrm{e}^{-st} f(t)\, dt \tag{2.1}$$

is known. The method selected subsequently is based on a theorem by Crump [1]: Let $a = max\{Re(P); P$ is a singularity of $\bar{f}\}$ and $T > 0$. Except from the relative error E for every t in $(0, T)$ the value of the invers Laplace transform is given by the

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O. Kolditz et al. (eds.), *Thermo-Hydro-Mechanical-Chemical Processes in Fractured Porous Media: Modelling and Benchmarking*, Terrestrial Environmental Sciences,
DOI 10.1007/978-3-319-11894-9_2

trigonometric series

$$f(t) = \frac{2}{T} e^{At} \left[\frac{1}{2} \bar{f}(A) + \sum_{k=1}^{\infty} c(k) \right], \tag{2.2}$$

where $A = a - ln(E)/T$ and for $k = 1, 2, \ldots$

$$c(k) = Re[\bar{f}(A + \frac{2k\pi i}{T})] \cos \frac{2k\pi t}{T} - Im[\bar{f}(A + \frac{2k\pi i}{T})] \sin \frac{2k\pi t}{T}. \tag{2.3}$$

The algorithm outlined above may easily be applied to the various transforms cited below. The convergence of the trigonometric series has been accelerated as described in [2].

2.1 Heat Conduction

The examples presented here have either been adopted from Carslaw and Jaeger [3] or they are based on ideas outlined there. Throughout this section we are concerned with the evaluation of temperature distributions.

2.1.1 A 1D Steady-State Temperature Distribution, Boundary Conditions of 1st Kind

The domain is a rectangular beam extending along the positive x-axis. It is composed of $10 \times 1 \times 1$ cubic elements of 10 m edge size each, the material has been assigned a thermal conductivity of $1\,\mathrm{W/(m \cdot K)}$. Specified temperatures prevail at the beam ends with prescribed values of $T_0 = 1\,°\mathrm{C}$ at $x = L = 100\,\mathrm{m}$ and zero temperature at $x = 0\,\mathrm{m}$. The simulation comprises one time step to establish the steady-state temperature distribution $T(x)$.
The Laplace equation is the governing equation describing the steady-state temperature distribution. It reads

$$\frac{d^2T}{dx^2} = 0 \tag{2.4}$$

for 1D heat flow along the x-axis, hence, the temperature is given by

$$T(x) = ax + b. \tag{2.5}$$

The free constants, a and b, have to be determined from the specified boundary conditions at $x = L$ and $x = 0\,\mathrm{m}$, therefore,

$$T(x) = T_0 \frac{x}{L}. \tag{2.6}$$

2.1.2 A 1D Steady-State Temperature Distribution, Boundary Conditions of 1st and 2nd Kind

The domain is a rectangular beam of length $L = 100\,\text{m}$ extending along the positive x-axis and composed of $10 \times 1 \times 1$ cubic elements. The domain is composed of two groups of materials, thermal conductivities $\lambda_1 = 100\,\text{W/(m} \cdot \text{K)}$ and $\lambda_2 = 300\,\text{W/(m} \cdot \text{K)}$ have been assigned for $x < 2L/5$ and $x > 2L/5$, respectively. A specified temperature $T_0 = 1\,°\text{C}$ prevails at $x = 0\,\text{m}$, the specific heat flow $q_{th} = -1.5\,\text{W/m}^2$ is prescribed at $x = L$, which acts as heat source to the domain. The simulation comprises one time step to establish the steady-state temperature distribution $T(x)$ (Fig. 2.1).

The Laplace equation is the governing equation describing the steady-state temperature distribution. It reads

$$\frac{d^2T}{dx^2} = 0 \tag{2.7}$$

for 1D heat flow along the x-axis, hence, the temperature is given by

$$T(x) = \begin{cases} a_1 x + b_1 & \text{for } x \le 2L/5, \\ a_2 x + b_2 & \text{for } x > 2L/5. \end{cases} \tag{2.8}$$

The constants a_1, b_1, a_2, and b_2, have to be determined from the specified boundary conditions and continuity of temperature and energy flow at the material boundary. Temperature T_0 prevails at $x = 0$, hence,

$$b_1 = T_0. \tag{2.9}$$

Specific heat flow q_{th} has been assigned at $x = L$. Then, by Fourier's law,

$$q_{th} = -\lambda_2 \frac{dT}{dx}|_{x=L} = -\lambda_2 a_2. \tag{2.10}$$

Continuity of the heat flow at the material boundary (i.e. $x = 2L/5$) yields

$$a_1 \lambda_1 = \lambda_2 a_2, \tag{2.11}$$

and via continuity of temperature at the material boundary

$$b_2 + a_2 2L/5 = a_1 2L/5 + b_1. \tag{2.12}$$

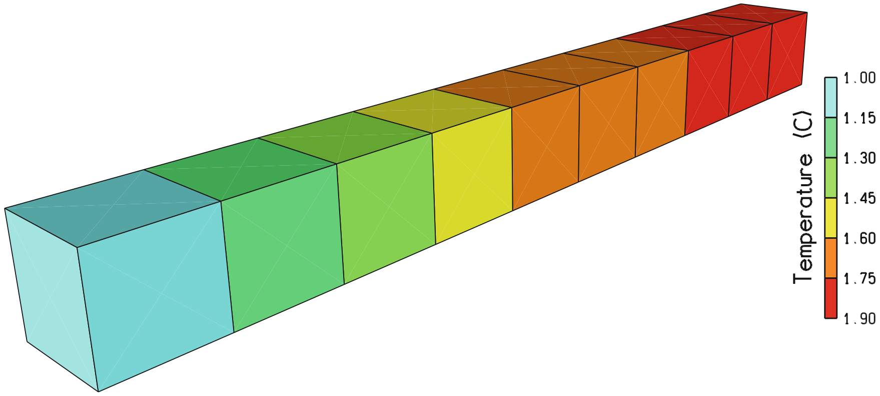

Fig. 2.1 Temperature distribution

The temperature distribution $T(x)$ thus becomes

$$T(x) = \begin{cases} -\dfrac{q_{th}}{\lambda_1}x + T_0 & \text{for } x \leq 2L/5, \\ -\dfrac{q_{th}}{\lambda_2}x + T_0 + q_{th}\dfrac{2L}{5}\left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right) & \text{for } x > 2L/5. \end{cases} \tag{2.13}$$

2.1.3 A 2D Steady-State Temperature Distribution, Boundary Conditions of 1st Kind

Given length $L = 1\,\text{m}$ the domain represents the square $[0, L] \times [0, L]$ in the x-y-plane. It is discretized by $50 \times 50 \times 1$ equally sized hexahedral elements, the material has been assigned a thermal conductivity of $1\,\text{W/(m}\cdot\text{K)}$. Prescribed temperatures prevail at the lateral boundaries of the domain as specified below. The simulation comprises one time step to establish the steady-state temperature distribution $T(x, y)$. The Laplace equation is the governing equation describing the steady-state temperature distribution. It reads

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.14}$$

for 2D heat flow in the x-y-plane. With the aid of temperature $T_0 = 1\,°\text{C}$ the applied boundary conditions read

$$\begin{aligned} T(x,0) &= 0 && \text{for } 0 \le x \le L, \\ T(0,y) &= 0 && \text{for } 0 \le y \le L, \\ T(x,L) &= T_0 \frac{x}{L} && \text{for } 0 \le x \le L, \\ T(L,y) &= T_0 \frac{y}{L} && \text{for } 0 \le y \le L. \end{aligned} \tag{2.15}$$

The temperature distribution

$$T(x, y) = T_0 \frac{x}{L} \frac{y}{L} \tag{2.16}$$

satisfies the Laplace equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

2.1.4 A 2D Steady-State Temperature Distribution, Boundary Conditions of 1st and 2nd Kind

Given length $L = 1\,\mathrm{m}$ the domain represents the rectangle $[0, 2L] \times [0, L]$ in the x-y-plane. It is discretized by an irregular mesh of hexahedral elements, the material has been assigned the thermal conductivity $\lambda = 1\,\mathrm{W/(m \cdot K)}$. Prescribed conditions prevail at the lateral boundaries of the domain as specified below. The simulation comprises one time step to establish the steady-state temperature distribution $T(x, y)$.

The Laplace equation is the governing equation describing the steady-state temperature distribution. It reads

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.17}$$

for 2D heat flow in the x-y-plane. With the aid of temperature $T_0 = 1\,^\circ\mathrm{C}$ the applied boundary conditions read

$$\begin{aligned} T(x,0) &= \frac{T_0}{L} x && \text{for } 0 \le x \le 2L, \\ T(x,L) &= \frac{T_0}{L}(x + 2L) && \text{for } 0 \le x \le 2L, \\ T(2L,y) &= \frac{T_0}{L}(2L + 2y) && \text{for } 0 \le y \le L, \\ \frac{\partial T}{\partial x}(0,y) &= \frac{T_0}{L} && \text{for } 0 \le y \le L. \end{aligned} \tag{2.18}$$

The temperature distribution

$$T(x, y) = \frac{T_0}{L}(x + 2y) \tag{2.19}$$

satisfies the Laplace equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

Data input represents the second kind boundary condition with the aid of a specific heat flow q_{th} assigned at the face $x = 0\,\text{m}$. By Fourier's law,

$$q_{th} = -\lambda \frac{\partial T}{\partial x}|_{x=0}. \tag{2.20}$$

Hence, for the present example

$$q_{th} = -\lambda \frac{T_0}{L} = -1\ \text{W/m}^2, \tag{2.21}$$

specified at the face $x = 0\,\text{m}$.

2.1.5 A 3D Steady-State Temperature Distribution

Given length $L = 1\,\text{m}$ the domain represents the cube $[0, L] \times [0, L] \times [0, L]$ discretized by $5 \times 5 \times 6$ equally sized hexahedral elements, the material has been assigned a thermal conductivity of $1\,\text{W/(m}\cdot\text{K)}$. Prescribed temperatures prevail at the surface of the domain as specified below. The simulation comprises one time step to establish the steady-state temperature distribution $T(x, y, z)$.

The Laplace equation is the governing equation describing the steady-state temperature distribution. It reads

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{2.22}$$

With the aid of temperature $T_0 = 1\,°\text{C}$ the applied boundary conditions read

$$\begin{aligned}
T(0, y, z) &= T_0 \left(0 + \frac{y}{L} + \frac{z}{L}\right) \text{ on the face } x = 0,\\
T(x, 0, z) &= T_0 \left(\frac{x}{L} + 0 + \frac{z}{L}\right) \text{ on the face } y = 0,\\
T(x, y, 0) &= T_0 \left(\frac{x}{L} + \frac{y}{L} + 0\right) \text{ on the face } z = 0,\\
T(L, y, z) &= T_0 \left(1 + \frac{y}{L} + \frac{z}{L}\right) \text{ on the face } x = L,\\
T(x, L, z) &= T_0 \left(\frac{x}{L} + 1 + \frac{z}{L}\right) \text{ on the face } y = L,\\
T(x, y, L) &= T_0 \left(\frac{x}{L} + \frac{y}{L} + 1\right) \text{ on the face } z = L.
\end{aligned} \tag{2.23}$$

The temperature distribution

$$T(x, y, z) = T_0 \left(\frac{x}{L} + \frac{y}{L} + \frac{z}{L} \right) \tag{2.24}$$

satisfies the Laplace equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

2.1.6 A Transient 1D Temperature Distribution, Time-Dependent Boundary Conditions of 1st Kind

Given length $L = 50\,\mathrm{m}$ the domain is a beam extending from $-L$ to L along the x-axis, it is subdivided into $200 \times 1 \times 1$ equally sized hexahedral elements. Explicitly assigned properties of the material are thermal conductivity $\lambda = 0.5787037\,\mathrm{W/(m \cdot K)}$, heat capacity $c = 0.01\,\mathrm{J/(kg \cdot K)}$, and density $\rho = 2{,}500\,\mathrm{kg/m^3}$. The temperature $T_1 \cdot t$ ($T_1 = 2\,°\mathrm{C/d}$) increases linearly with time t, it is applied at the beam ends for times $t > 0$. Starting from zero initial temperature the simulation evaluates the transient temperature distribution $T(x, t)$ with output after 0.25 days and 0.5 days.

The heat conduction equation is the governing equation describing the transient temperature distribution. It reads

$$\rho c \frac{\partial T}{\partial t} = \lambda \nabla \cdot \nabla T. \tag{2.25}$$

Introducing the notation

$$\chi = \frac{\lambda}{\rho c} \tag{2.26}$$

the present 1D problem is governed by the parabolic equation

$$\frac{1}{\chi} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \tag{2.27}$$

the initial condition

$$T(x, 0) = 0 \quad \text{for } -L \le x \le L, \tag{2.28}$$

and linearly increasing temperatures imposed at the beam ends

$$\begin{aligned} T(-L, t) &= T_1 \cdot t \quad \text{for } t > 0, \\ T(L, t) &= T_1 \cdot t \quad \text{for } t > 0. \end{aligned} \tag{2.29}$$

The closed form solution of the above problem is given by Carslaw and Jaeger [3], who arrive at the series representation

$$T(x,t) = T_1 \cdot t + \frac{T_1 \cdot (x^2 - L^2)}{2\chi} + \frac{16 \cdot T_1 \cdot L^2}{\chi \pi^3} \times \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos\left(\frac{(2n+1)\pi x}{2L}\right) \exp\left(-\chi (2n+1)^2 \pi^2 \frac{t}{4L^2}\right). \tag{2.30}$$

2.1.7 Transient 1D Temperature Distributions, Time-Dependent Boundary Conditions of 2nd Kind

The domain is composed of two beams in parallel (Beam1 and Beam2) extending along the positive x-axis, each $L = 25\,\mathrm{m}$ long and subdivided into $25 \times 1 \times 1$ cubic elements. Explicitly assigned properties of the material are density $\rho = 2000\,\mathrm{kg/m^3}$, thermal conductivity $\lambda = 1.1574074\,\mathrm{W/(m \cdot K)}$, and heat capacities $c_1 = 0.01\,\mathrm{J/(kg \cdot K)}$ and $c_2 = 0.02\,\mathrm{J/(kg \cdot K)}$ assigned to Beam1 and Beam2, respectively. No-flow boundary conditions prevail at the $x = 0\,\mathrm{m}$ faces. A specific heat flow is prescribed at $x = L$ for times $t > 0$. It acts as heat source to the domain and increases linearly with time via $q_{th1} \cdot t$, where $q_{th1} = 0.385802\,\mathrm{W/(d \cdot m^2)}$ has been assumed. Starting from zero initial temperature the simulation evaluates the transient temperature distributions with output after 0.045 and 0.09 days (Fig. 2.2).

Let λ denote any of λ_1 or λ_2. The heat conduction equation is the governing equation describing the transient temperature distribution. It reads

$$\rho c \frac{\partial T}{\partial t} = \lambda \nabla \cdot \nabla T. \tag{2.31}$$

Introducing the notation

$$\chi = \frac{\lambda}{\rho c} \tag{2.32}$$

the present 1D problems are governed by the parabolic equation

$$\frac{1}{\chi} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \tag{2.33}$$

the initial condition

$$T(x, 0) = 0 \quad \text{for } 0 \le x \le L, \tag{2.34}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial T}{\partial x}(0,t) &= 0 && \text{for } t > 0, \\ \lambda \frac{\partial T}{\partial x}(L,t) &= q_{th1} \cdot t && \text{for } t > 0. \end{aligned} \tag{2.35}$$

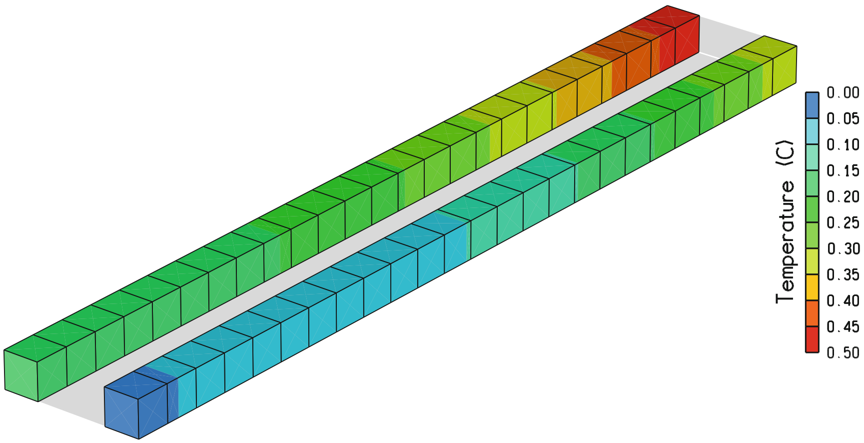

Fig. 2.2 Temperature distributions after 0.09 days

The closed form solution of the above problem is given by Carslaw and Jaeger [3], who arrive at the series representation

$$T(x,t) = \frac{8q_{th1}\sqrt{\chi t^3}}{\lambda}\sum_{n=0}^{\infty}\left[i^3\mathrm{erfc}\frac{(2n+1)L-x}{2\sqrt{\chi t}} + i^3\mathrm{erfc}\frac{(2n+1)L+x}{2\sqrt{\chi t}}\right], \quad (2.36)$$

where i^3erfc denotes the third repeated integral of the complementary error function. See [4] for its numerical evaluation.

2.1.8 Transient 1D Temperature Distributions, Non-Zero Initial Temperature, Boundary Conditions of 1st and 2nd Kind

The domain is composed of two beams in parallel (T1-beam and T2-beam) extending along the positive x-axis, each $L = 100$ m long and subdivided into $100 \times 1 \times 1$ cubic elements. Explicitly assigned properties of the material are thermal conductivity $\lambda = 0.5787037$ W/(m $\cdot$ K), heat capacity $c = 0.01$ J/(kg $\cdot$ K), and density $\rho = 2{,}000$ kg/m^3. Prescribed conditions prevail at the beams ends as specified below. Given temperature $T_0 = 1\,^\circ$C and

$$f(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \dfrac{L}{10}, \\ \dfrac{10}{3L}x - \dfrac{1}{3} & \text{for } \dfrac{L}{10} \leq x \leq \dfrac{4L}{10}, \\ 1 & \text{for } \dfrac{4L}{10} \leq x \leq \dfrac{6L}{10}, \\ 3 - \dfrac{10}{3L}x & \text{for } \dfrac{6L}{10} \leq x \leq \dfrac{9L}{10}, \\ 0 & \text{for } \dfrac{9L}{10} \leq x \leq L, \end{cases} \tag{2.37}$$

the simulation starts from the initial temperature distribution $T(x, 0) = T_0 \cdot f(x)$ and evaluates the transient temperature distribution $T(x, t)$ with output after 0.05 days and 0.1 days.

The heat conduction equation is the governing equation describing the transient temperature distribution. It reads

$$\rho c \frac{\partial T}{\partial t} = \lambda \nabla \cdot \nabla T. \tag{2.38}$$

Introducing the notation

$$\chi = \frac{\lambda}{\rho c} \tag{2.39}$$

the present 1D problems are governed by the parabolic equation

$$\frac{1}{\chi} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \tag{2.40}$$

the initial condition

$$T(x, 0) = T_0 \cdot f(x) \quad \text{for } 0 \leq x \leq L, \tag{2.41}$$

and the boundary conditions imposed at the beams ends. These boundary conditions are specified zero temperatures for the T1-beam,

$$\begin{aligned} T(0, t) &= 0 \quad \text{for } t > 0, \\ T(L, t) &= 0 \quad \text{for } t > 0, \end{aligned} \tag{2.42}$$

and no-flow boundary conditions for the T2-beam,

$$\begin{aligned} \frac{\partial T}{\partial x}(0, t) &= 0 \quad \text{for } t > 0, \\ \frac{\partial T}{\partial x}(L, t) &= 0 \quad \text{for } t > 0. \end{aligned} \tag{2.43}$$

Closed form solutions of the above problems are given by Carslaw and Jaeger [3], who arrive at series representations that read for the T1-beam

$$\begin{aligned} T(x,t)/T_0 &= T1(x,t) \\ &= \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right)\int_0^L f(x')\sin\frac{n\pi x'}{L}\,dx' \end{aligned} \tag{2.44}$$

and for the T2-beam

$$\begin{aligned} T(x,t)/T_0 &= T2(x,t) \\ &= \frac{1}{L}\int_0^L f(x')\,dx' + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\, \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right)\int_0^L f(x')\cos\frac{n\pi x'}{L}\,dx'. \end{aligned} \tag{2.45}$$

Now, with $f(x)$ as defined above, the integrals involved may be evaluated by elementary analytical methods. The series representations take the form

$$\begin{aligned} T1(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\, \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right) \\ \times \frac{80}{3(n\pi)^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{4}\sin\frac{3n\pi}{20}, \end{aligned} \tag{2.46}$$

$$\begin{aligned} T2(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\, \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right) \\ \times \frac{80}{3(n\pi)^2}\cos\frac{n\pi}{2}\sin\frac{n\pi}{4}\sin\frac{3n\pi}{20}. \end{aligned} \tag{2.47}$$

2.1.9 A Transient 2D Temperature Distribution, Non-Zero Initial Temperature, Boundary Conditions of 1st and 2nd Kind

The domain represents the square $[0, L] \times [0, L]$ with an edge size of $L = 100\,\mathrm{m}$, located in the x-y-plane and subdivided into $100 \times 100 \times 1$ cubic elements. Explicitly assigned properties of the material are thermal conductivity $\lambda = 0.5787037\,\mathrm{W/(m \cdot K)}$, heat capacity $c = 0.01\,\mathrm{J/(kg \cdot K)}$, and density $\rho = 2000\,\mathrm{kg/m^3}$. Prescribed conditions prevail at the lateral boundaries of the domain as specified below. Given temperature $T_0 = 1\,^\circ\mathrm{C}$ and

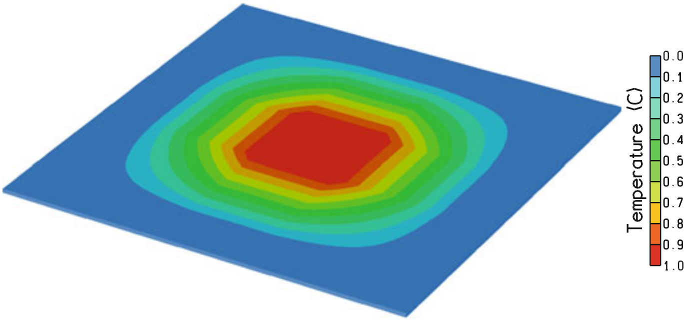

Fig. 2.3 Initial temperature distribution

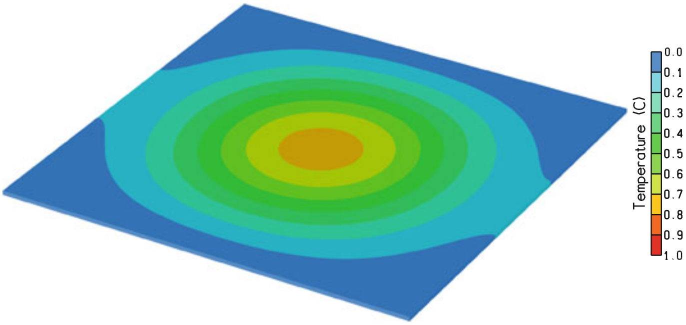

Fig. 2.4 Temperature distribution after 0.04 days

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{L}{10}, \\ \frac{10}{3L}x - \frac{1}{3} & \text{for } \frac{L}{10} \le x \le \frac{4L}{10}, \\ 1 & \text{for } \frac{4L}{10} \le x \le \frac{6L}{10}, \\ 3 - \frac{10}{3L}x & \text{for } \frac{6L}{10} \le x \le \frac{9L}{10}, \\ 0 & \text{for } \frac{9L}{10} \le x \le L, \end{cases} \tag{2.48}$$

the simulation starts from the initial temperature distribution $T(x, y, 0) = T_0 \cdot f(x) \cdot f(y)$ (Fig. 2.3) and evaluates the transient temperature distribution $T(x, y, t)$ with output after 0.02 and 0.04 days (Fig. 2.4).

The heat conduction equation is governing equation describing the transient temperature distribution. It reads

$$\rho c \frac{\partial T}{\partial t} = \lambda \nabla \cdot \nabla T. \tag{2.49}$$

Introducing the notation

$$\chi = \frac{\lambda}{\rho c} \tag{2.50}$$

the present 2D problem is governed by the parabolic equation

$$\frac{1}{\chi}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}, \tag{2.51}$$

the initial condition

$$T(x, y, 0) = T_0 \cdot f(x) \cdot f(y) \quad \text{for} \quad 0 \le x, y \le L, \tag{2.52}$$

and the applied boundary conditions. These are specified zero temperatures at $x = 0$ and $x = L$

$$T(0, y, t) = 0 \quad \text{for } 0 \le y \le L, \tag{2.53}$$

$$T(L, y, t) = 0 \quad \text{for } 0 \le y \le L, \tag{2.54}$$

and no-flow boundary conditions at $y = 0$ and $y = L$

$$\frac{\partial T}{\partial y}(x, 0, t) = 0 \quad \text{for } 0 \le x \le L, \tag{2.55}$$

$$\frac{\partial T}{\partial y}(x, L, t) = 0 \quad \text{for } 0 \le x \le L. \tag{2.56}$$

The closed form solution of the above problem is obtained in terms of the 1D T1-beam and T2-beam solutions given in the context of the previous example

$$\begin{aligned} T1(x, t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \exp\left(-\chi n^2 \pi^2 \frac{t}{L^2}\right) \\ \times \frac{80}{3(n\pi)^2} \sin\frac{n\pi}{2} \sin\frac{n\pi}{4} \sin\frac{3n\pi}{20}, \end{aligned} \tag{2.57}$$

$$\begin{aligned} T2(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{L} \exp\left(-\chi n^2 \pi^2 \frac{t}{L^2}\right) \\ \times \frac{80}{3(n\pi)^2} \cos\frac{n\pi}{2} \sin\frac{n\pi}{4} \sin\frac{3n\pi}{20}. \end{aligned} \tag{2.58}$$

We will next verify that the closed form solution of the above problem is given by

$$T(x, y, t) = T_0 \cdot T1(x, t) \cdot T2(y, t). \tag{2.59}$$

Both, $T1(x,t)$ and $T2(x,t)$, satisfy the initial condition

$$T1(x,0) = T2(x,0) = f(x). \tag{2.60}$$

Then

$$T(x,y,0) = T_0 \cdot T1(x,0) \cdot T2(y,0) = T_0 \cdot f(x) \cdot f(y), \tag{2.61}$$

hence, $T(x,y,t)$ satisfies the initial condition.

Both, $T1(x,t)$ and $T2(y,t)$, satisfy the 1D heat conduction equation. Then

$$\begin{aligned}\frac{1}{\chi}\frac{\partial T}{\partial t} &= T_0\left[\frac{1}{\chi}T2\frac{\partial T1}{\partial t} + \frac{1}{\chi}T1\frac{\partial T2}{\partial t}\right]\\ &= T_0\left[T2\frac{\partial^2 T1}{\partial x^2} + T1\frac{\partial^2 T2}{\partial y^2}\right]\\ &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},\end{aligned} \tag{2.62}$$

hence, $T(x,y,t)$ satisfies the differential equation.

Zero boundary temperatures are satisfied by $T1(x,t)$

$$T1(0,t) = 0 \quad \text{for } t > 0, \tag{2.63}$$

$$T1(L,t) = 0 \quad \text{for } t > 0, \tag{2.64}$$

no-flow boundary conditions are satisfied by $T2(y,t)$

$$\frac{\partial T2}{\partial y}(0,t) = 0 \quad \text{for } t > 0, \tag{2.65}$$

$$\frac{\partial T2}{\partial y}(L,t) = 0 \quad \text{for } t > 0. \tag{2.66}$$

Then

$$\begin{aligned}T(0,y,t) &= T_0 \cdot T1(0,t) \cdot T2(y,t) &&= 0 \cdot T2(y,t) &&= 0,\\ T(L,y,t) &= T_0 \cdot T1(L,t) \cdot T2(y,t) &&= 0 \cdot T2(y,t) &&= 0,\\ \frac{\partial T}{\partial y}(x,0,t) &= T_0 \cdot T1(x,t) \cdot \frac{\partial T2}{\partial y}(0,t) &&= T_0 \cdot T1(x,t) \cdot 0 = 0,\\ \frac{\partial T}{\partial y}(x,L,t) &= T_0 \cdot T1(x,t) \cdot \frac{\partial T2}{\partial y}(L,t) &&= T_0 \cdot T1(x,t) \cdot 0 = 0,\end{aligned} \tag{2.67}$$

hence, $T(x,y,t)$ satisfies the boundary conditions.

2.2 Liquid Flow

The same tools that work for the solution of heat conduction problems may also be applied to liquid flow problems. Here we are concerned with the evaluation of pressure distributions, for the underlying theory see Freeze and Cherry [5].

2.2.1 A 1D Steady-State Pressure Distribution, Boundary Conditions of 1st Kind

The domain is a rectangular beam extending along the positive x-axis and composed of $10 \times 1 \times 1$ cubic elements of 10 m edge size each. An isotropic permeability of $10^{-15}\,\mathrm{m}^2$ is assumed for the material. Liquid viscosity is $1\,\mathrm{mPa\cdot s}$ and gravity is neglected via zero liquid density. Specified pressures prevail at the beam ends with prescribed values of $p_0 = 1\,\mathrm{MPa}$ at $x = L = 100\,\mathrm{m}$ and zero pressure at $x = 0\,\mathrm{m}$. The simulation comprises one time step to establish the steady-state pressure distribution $p(x)$.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{d^2p}{dx^2} = 0 \tag{2.68}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = ax + b. \tag{2.69}$$

The free constants, a and b, have to be determined from the specified boundary conditions at $x = L$ and $x = 0\,\mathrm{m}$, therefore,

$$p(x) = p_0 \frac{x}{L}. \tag{2.70}$$

2.2.2 A 1D Steady-State Pressure Distribution, Boundary Conditions of 1st and 2nd Kind

The domain is a rectangular beam of length $L = 100\,\mathrm{m}$ extending along the positive x-axis and composed of $10 \times 1 \times 1$ cubic elements. The domain is composed of two groups of permeable materials with isotropic permeabilities $k_1 = 10^{-12}\,\mathrm{m}^2$ and $k_2 = 3 \times 10^{-12}\,\mathrm{m}^2$ for $x < 2L/5$ and $x > 2L/5$, respectively. Liquid viscosity is $\mu = 1\,\mathrm{mPa\cdot s}$ and gravity is neglected via zero liquid density. The specified pressure $p_0 = 1\,\mathrm{MPa}$ prevails at $x = 0\,\mathrm{m}$, the specific discharge $q = -1.5 \times 10^{-5}\,\mathrm{m/s}$ is prescribed at $x = L$, which acts as a source to the domain. The simulation comprises one time step to establish the steady-state pressure distribution $p(x)$.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{d^2 p}{dx^2} = 0 \tag{2.71}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = \begin{cases} a_1 x + b_1 & \text{for } x \leq 2L/5, \\ a_2 x + b_2 & \text{for } x > 2L/5. \end{cases} \tag{2.72}$$

The constants a_1, b_1, a_2, and b_2, have to be determined from the specified boundary conditions and continuity of pressure and specific discharge at the material boundary.

Pressure p_0 prevails at $x = 0$, hence,

$$b_1 = p_0. \tag{2.73}$$

Specific discharge q has been assigned at $x = L$. Then, by Darcy's law,

$$q = -\frac{k_2}{\mu}\frac{dp}{dx}|_{x=L} = -\frac{k_2}{\mu}a_2. \tag{2.74}$$

Continuity of the specific discharge at the material boundary (i.e. $x = 2L/5$) yields

$$a_1\frac{k_1}{\mu} = \frac{k_2}{\mu}a_2, \tag{2.75}$$

and via continuity of pressure at the material boundary

$$b_2 + a_2 2L/5 = a_1 2L/5 + b_1. \tag{2.76}$$

The pressure distribution $p(x)$ thus becomes

$$p(x) = \begin{cases} -\dfrac{q\mu}{k_1}x + p_0 & \text{for } x \leq 2L/5, \\ -\dfrac{q\mu}{k_2}x + p_0 + q\mu\dfrac{2L}{5}\left(\dfrac{1}{k_2} - \dfrac{1}{k_1}\right) & \text{for } x > 2L/5. \end{cases} \tag{2.77}$$

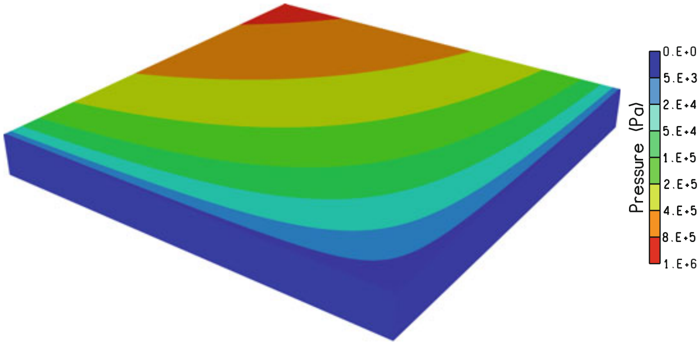

Fig. 2.5 Pressure distribution

2.2.3 A 2D Steady-State Pressure Distribution, Boundary Conditions of 1st Kind

Given length $L = 1\,\text{m}$ the domain represents the square $[0, L] \times [0, L]$ in the x-y-plane, discretized by $50 \times 50 \times 1$ equally sized hexahedral elements. An isotropic permeability of $10^{-15}\,\text{m}^2$ is assumed for the material. Liquid viscosity is $1\,\text{mPa} \cdot \text{s}$ and gravity is neglected via zero liquid density. Prescribed pressures prevail at the lateral boundaries of the domain as specified below. The simulation comprises one time step to establish the steady-state pressure distribution $p(x, y)$ (Fig. 2.5).

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{2.78}$$

for 2D flow in the x-y-plane. With the aid of pressure $p_0 = 1\,\text{MPa}$ the applied boundary conditions read

$$\begin{aligned} p(x, 0) &= 0 && \text{for } 0 \leq x \leq L, \\ p(0, y) &= 0 && \text{for } 0 \leq y \leq L, \\ p(x, L) &= p_0 \frac{x}{L} && \text{for } 0 \leq x \leq L, \\ p(L, y) &= p_0 \frac{y}{L} && \text{for } 0 \leq y \leq L. \end{aligned} \tag{2.79}$$

The pressure distribution

$$p(x, y) = p_0 \frac{x}{L} \frac{y}{L} \tag{2.80}$$

satisfies the Laplace equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

2.2.4 A 2D Steady-State Pressure Distribution, Boundary Conditions of 1st and 2nd Kind

Given length $L = 1\,\mathrm{m}$ the domain represents the rectangle $[0, 2L] \times [0, L]$ in the x-y-plane, discretized by an irregular mesh of hexahedral elements. An isotropic permeability $k = 10^{-14}\,\mathrm{m}^2$ is assumed for the material. Liquid viscosity is $\mu = 1\,\mathrm{mPa \cdot s}$ and gravity is neglected via zero liquid density. Prescribed conditions prevail at the lateral boundaries of the domain as specified below. The simulation comprises one time step to establish the steady-state pressure distribution $p(x, y)$.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{2.81}$$

for 2D flow in the x-y-plane. With the aid of pressure $p_0 = 1\,\mathrm{MPa}$ the applied boundary conditions read

$$\begin{aligned}
p(x, 0) &= \frac{p_0}{L} x && \text{for } 0 \le x \le 2L, \\
p(x, L) &= \frac{p_0}{L} (x + 2L) && \text{for } 0 \le x \le 2L, \\
p(2L, y) &= \frac{p_0}{L} (2L + 2y) && \text{for } 0 \le y \le L, \\
\frac{\partial p}{\partial x}(0, y) &= \frac{p_0}{L} && \text{for } 0 \le y \le L.
\end{aligned} \tag{2.82}$$

The pressure distribution

$$p(x, y) = \frac{p_0}{L} (x + 2y) \tag{2.83}$$

satisfies the Laplace equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

Data input represents the second kind boundary condition with the aid of a specific discharge q assigned at the face $x = 0\,\mathrm{m}$. By Darcy's law,

$$q = -\frac{k}{\mu} \frac{\partial p}{\partial x} |_{x=0}. \tag{2.84}$$

Hence, for the present example

$$q = -\frac{k}{\mu} \frac{p_0}{L} = -10^{-5}\ \mathrm{m/s}, \tag{2.85}$$

specified at the face $x = 0\,\mathrm{m}$.

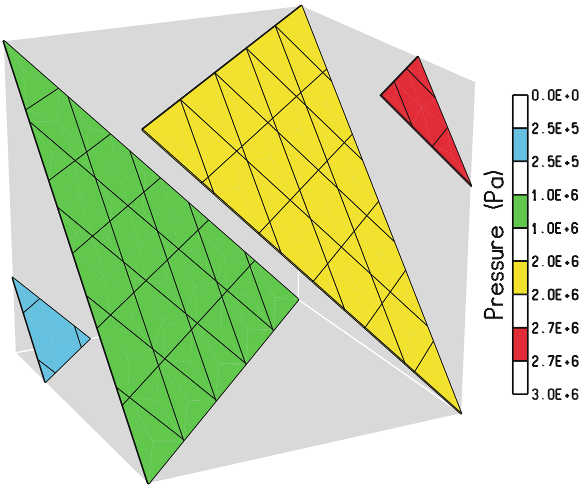

Fig. 2.6 Pressure contours

2.2.5 A 3D Steady-State Pressure Distribution

Given length $L = 1\,\text{m}$ the domain represents the cube $[0, L] \times [0, L] \times [0, L]$ discretized by $5 \times 5 \times 6$ equally sized hexahedral elements. An isotropic permeability of $10^{-10}\,\text{m}^2$ is assumed for the material. Liquid viscosity is $1\,\text{mPa}\cdot\text{s}$ and gravity is neglected via zero liquid density. Prescribed pressures prevail at the surface of the domain as specified below. The simulation comprises one time step to establish the steady-state pressure distribution $p(x, y, z)$ (Fig. 2.6).

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0. \tag{2.86}$$

With the aid of pressure $p_0 = 1\,\text{MPa}$ the applied boundary conditions read

$$\begin{aligned}
p(0, y, z) &= p_0 \left(0 + \frac{y}{L} + \frac{z}{L}\right) \text{ on the face } x = 0,\\
p(x, 0, z) &= p_0 \left(\frac{x}{L} + 0 + \frac{z}{L}\right) \text{ on the face } y = 0,\\
p(x, y, 0) &= p_0 \left(\frac{x}{L} + \frac{y}{L} + 0\right) \text{ on the face } z = 0,\\
p(L, y, z) &= p_0 \left(1 + \frac{y}{L} + \frac{z}{L}\right) \text{ on the face } x = L,\\
p(x, L, z) &= p_0 \left(\frac{x}{L} + 1 + \frac{z}{L}\right) \text{ on the face } y = L,
\end{aligned} \tag{2.87}$$

$$p(x, y, L) = p_0 \left(\frac{x}{L} + \frac{y}{L} + 1 \right) \text{ on the face } z = L.$$

The pressure distribution

$$p(x, y, z) = p_0 \left(\frac{x}{L} + \frac{y}{L} + \frac{z}{L} \right) \tag{2.88}$$

satisfies the Laplace equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

2.2.6 A Hydrostatic Pressure Distribution

The domain is a cuboid of height $H = 30\,\text{m}$ and edges parallel to the x-y-z coordinate axes. It is discretized by an irregular mesh of hexahedral elements. The domain is composed of four groups of isotropic permeable materials, $\rho = 1019.368\,\text{kg/m}^3$ is the liquid density. The simulation setup employs a prescribed zero pressure at the top and explicitly specified no-flow conditions along the lateral boundaries and bottom (unless otherwise specified, no-flow boundary conditions will be assigned by default). The simulation comprises one time step to establish the hydrostatic pressure distribution $p(x, y, z)$.

The hydrostatic pressure distribution neither depends on the material properties nor on the coordinates x and y and is given by

$$p(x, y, z) = \rho g (H - z), \tag{2.89}$$

where $g = 9.81\,\text{m}^2/\text{s}$ is the magnitude of gravity, and z is the vertical coordinate extending from 0 to H.

2.2.7 A Transient 1D Pressure Distribution, Time-Dependent Boundary Conditions of 1st Kind

Given length $L = 50\,\text{m}$ the domain is a beam extending from $-L$ to L along the x-axis, it is subdivided into $200 \times 1 \times 1$ equally sized hexahedral elements. A permeable material represents the porous medium, which contains a liquid of small and constant compressibility. Gravity is neglected via zero liquid density. Explicitly assigned properties of matrix and liquid are an isotropic permeability $k = 10^{-14}\,\text{m}^2$ and liquid viscosity $\mu = 1.728\,\text{mPa} \cdot \text{s}$. Porosity ϕ and liquid compressibility κ have been incorporated in the storage $\phi\kappa = 2.5 \times 10^{-10}\,\text{1/Pa}$. The pressure $p_1 \cdot t$ ($p_1 = 2 \times 10^6\,\text{Pa/d}$) increases linearly with time t, it is applied at the beam ends for times $t > 0$. Starting from zero initial pressure the simulation evaluates the transient pressure distribution $p(x, t)$ with output after 0.25 and 0.5 days.

For liquids of small and constant compressibility Darcy's law and continuity equation yield the pressure conduction equation as the governing equation describing the transient pressure distribution. It reads

$$\phi\kappa \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla \cdot \nabla p. \tag{2.90}$$

Introducing the notation

$$\chi = \frac{k}{\phi\mu\kappa} \tag{2.91}$$

the present 1D problem is governed by the parabolic equation

$$\frac{1}{\chi}\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}, \tag{2.92}$$

the initial condition

$$p(x, 0) = 0 \quad \text{for } -L \leq x \leq L, \tag{2.93}$$

and linearly increasing pressures imposed at the beam ends

$$\begin{aligned} p(-L, t) &= p_1 \cdot t \quad \text{for } t > 0, \\ p(L, t) &= p_1 \cdot t \quad \text{for } t > 0. \end{aligned} \tag{2.94}$$

The closed form solution of the above problem is given by Carslaw and Jaeger [3], who arrive at the series representation

$$\begin{aligned} p(x, t) = p_1 \cdot t + \frac{p_1 \cdot (x^2 - L^2)}{2\chi} + \frac{16 \cdot p_1 \cdot L^2}{\chi\pi^3} \\ \times \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos\left(\frac{(2n+1)\pi x}{2L}\right) \exp\left(-\chi(2n+1)^2\pi^2 \frac{t}{4L^2}\right). \end{aligned} \tag{2.95}$$

2.2.8 Transient 1D Pressure Distributions, Time-Dependent Boundary Conditions of 2nd Kind

The domain is composed of two beams in parallel (Beam1 and Beam2) extending along the positive x-axis, each $L = 25\,\text{m}$ long and subdivided into $25 \times 1 \times 1$ cubic elements. A permeable material represents the porous medium, which contains a liquid of small and constant compressibility. Gravity is neglected via zero liquid density. Explicitly assigned properties of matrix and liquid are an isotropic permeability $k = 10^{-14}\,\text{m}^2$ and liquid viscosity $\mu = 0.864\,\text{mPa}\cdot\text{s}$. Porosity ϕ and liquid compressibility κ have been incorporated in the storage $\phi\kappa$ with values of

2×10^{-10} 1/Pa and 4×10^{-10} 1/Pa assigned to Beam1 and Beam2, respectively. No-flow boundary conditions prevail at the $x = 0$ m faces. A specific discharge is prescribed at $x = L$ for times $t > 0$. It acts as a source to the domain and increases linearly with time via $q_1 \cdot t$, where $q_1 = 3.85802 \times 10^{-6}$ m/(s · d) has been assumed. Starting from zero initial pressure the simulation evaluates the transient pressure distributions with output after 0.045 and 0.09 days.

For liquids of small and constant compressibility Darcy's law and continuity equation yield the pressure conduction equation as the governing equation describing the transient pressure distribution. It reads

$$\phi\kappa \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla \cdot \nabla p. \tag{2.96}$$

Introducing the notation

$$\chi = \frac{k}{\phi\mu\kappa} \tag{2.97}$$

the present 1D problems are governed by the parabolic equation

$$\frac{1}{\chi}\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}, \tag{2.98}$$

the initial condition

$$p(x, 0) = 0 \quad \text{for } 0 \le x \le L, \tag{2.99}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial p}{\partial x}(0, t) \quad &= 0 \quad \text{for } t > 0, \\ \frac{k}{\mu}\frac{\partial p}{\partial x}(L, t) &= q_1 \cdot t \quad \text{for } t > 0. \end{aligned} \tag{2.100}$$

The closed form solution of the above problem is given by Carslaw and Jaeger [3], who arrive at the series representation

$$p(x, t) = \frac{8q_1\sqrt{\chi t^3}}{k/\mu} \sum_{n=0}^{\infty} \left[i^3\text{erfc}\frac{(2n+1)L - x}{2\sqrt{\chi t}} + i^3\text{erfc}\frac{(2n+1)L + x}{2\sqrt{\chi t}} \right], \tag{2.101}$$

where i^3erfc denotes the third repeated integral of the complementary error function. See [4] for its numerical evaluation.

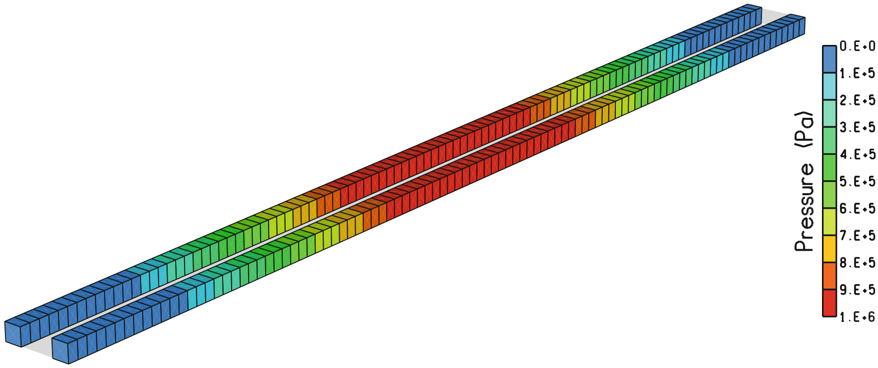

Fig. 2.7 Initial pressure distributions

2.2.9 Transient 1D Pressure Distributions, Non-Zero Initial Pressure, Boundary Conditions of 1st and 2nd Kind

The domain is composed of two beams in parallel (p1-beam and p2-beam) extending along the positive x-axis, each $L = 100\,\mathrm{m}$ long and subdivided into $100 \times 1 \times 1$ cubic elements. A permeable material represents the porous medium, which contains a liquid of small and constant compressibility. Gravity is neglected via zero liquid density. Explicitly assigned properties of matrix and liquid are an isotropic permeability $k = 10^{-14}\,\mathrm{m}^2$ and liquid viscosity $\mu = 1.728\,\mathrm{mPa \cdot s}$. Porosity ϕ and liquid compressibility κ have been incorporated in the storage $\phi\kappa = 2 \times 10^{-10}$ 1/Pa. Prescribed conditions prevail at the beams ends as specified below. Given pressure $p_0 = 1\,\mathrm{MPa}$ and

$$f(x) = \begin{cases} 0 & \text{for } \; 0 \leq x \leq \dfrac{L}{10}, \\ \dfrac{10}{3L}x - \dfrac{1}{3} & \text{for } \; \dfrac{L}{10} \leq x \leq \dfrac{4L}{10}, \\ 1 & \text{for } \; \dfrac{4L}{10} \leq x \leq \dfrac{6L}{10}, \\ 3 - \dfrac{10}{3L}x & \text{for } \; \dfrac{6L}{10} \leq x \leq \dfrac{9L}{10}, \\ 0 & \text{for } \; \dfrac{9L}{10} \leq x \leq L, \end{cases} \tag{2.102}$$

the simulation starts from the initial pressure distribution $p(x, 0) = p_0 \cdot f(x)$ (Fig. 2.7) and evaluates the transient pressure distribution $p(x, t)$ with output after 0.05 and 0.1 days (Fig. 2.8).

For liquids of small and constant compressibility Darcy's law and continuity equation yield the pressure conduction equation as the governing equation describing

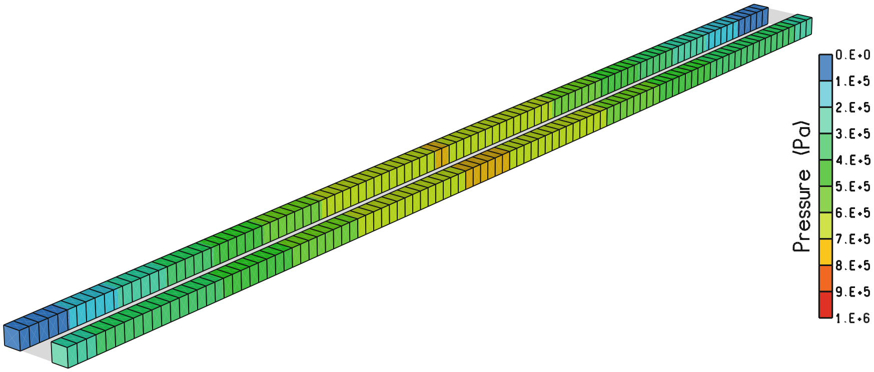

Fig. 2.8 Pressure distributions after 0.1 days

the transient pressure distribution. It reads

$$\phi\kappa \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla \cdot \nabla p. \tag{2.103}$$

Introducing the notation

$$\chi = \frac{k}{\phi\mu\kappa} \tag{2.104}$$

the present 1D problems are governed by the parabolic equation

$$\frac{1}{\chi} \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}, \tag{2.105}$$

the initial condition

$$p(x, 0) = p_0 \cdot f(x) \quad \text{for } 0 \leq x \leq L, \tag{2.106}$$

and the boundary conditions imposed at the beams ends. These boundary conditions are specified zero pressures for the p1-beam,

$$\begin{aligned} p(0, t) &= 0 \quad \text{for } t > 0, \\ p(L, t) &= 0 \quad \text{for } t > 0, \end{aligned} \tag{2.107}$$

and no-flow boundary conditions for the p2-beam,

$$\frac{\partial p}{\partial x}(0,t) = 0 \quad \text{for } t > 0,$$
$$\frac{\partial p}{\partial x}(L,t) = 0 \quad \text{for } t > 0. \tag{2.108}$$

Closed form solutions of the above problems are given by Carslaw and Jaeger [3], who arrive at series representations that read for the p1-beam

$$\begin{aligned} p(x,t)/p_0 &= p1(x,t) \\ &= \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\ \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right)\int_0^L f(x')\sin\frac{n\pi x'}{L}\,dx' \end{aligned} \tag{2.109}$$

and for the p2-beam

$$\begin{aligned} p(x,t)/p_0 &= p2(x,t) \\ &= \frac{1}{L}\int_0^L f(x')\,dx' + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\ \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right)\int_0^L f(x')\cos\frac{n\pi x'}{L}\,dx'. \end{aligned} \tag{2.110}$$

Now, with $f(x)$ as defined above, the integrals involved may be evaluated by elementary analytical methods. The series representations take the form

$$\begin{aligned} p1(x,t) = \sum_{n=1}^{\infty} &\sin\frac{n\pi x}{L}\ \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right) \\ &\times \frac{80}{3(n\pi)^2}\ \sin\frac{n\pi}{2}\ \sin\frac{n\pi}{4}\ \sin\frac{3n\pi}{20}, \end{aligned} \tag{2.111}$$

$$\begin{aligned} p2(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} &\cos\frac{n\pi x}{L}\ \exp\left(-\chi n^2\pi^2\frac{t}{L^2}\right) \\ &\times \frac{80}{3(n\pi)^2}\ \cos\frac{n\pi}{2}\ \sin\frac{n\pi}{4}\ \sin\frac{3n\pi}{20}. \end{aligned} \tag{2.112}$$

2.2.10 A Transient 2D Pressure Distribution, Non-Zero Initial Pressure, Boundary Conditions of 1st and 2nd Kind

The domain represents the square $[0, L] \times [0, L]$ with an edge size of $L = 100\,\text{m}$, located in the x-y-plane and subdivided into $100 \times 100 \times 1$ cubic elements. A permeable material represents the porous medium, which contains a liquid of small and constant compressibility. Gravity is neglected via zero liquid density. Explicitly assigned properties of matrix and liquid are an isotropic permeability $k = 10^{-14}\,\mathrm{m}^2$ and liquid viscosity $\mu = 1.728\,\mathrm{mPa \cdot s}$. Porosity ϕ and liquid compressibility κ have been incorporated in the storage $\phi\kappa = 2 \times 10^{-10}\,1/\mathrm{Pa}$. Prescribed conditions prevail at the lateral boundaries of the domain as specified below. Given pressure $p_0 = 1\,\mathrm{MPa}$ and

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{L}{10}, \\ \frac{10}{3L}x - \frac{1}{3} & \text{for } \frac{L}{10} \le x \le \frac{4L}{10}, \\ 1 & \text{for } \frac{4L}{10} \le x \le \frac{6L}{10}, \\ 3 - \frac{10}{3L}x & \text{for } \frac{6L}{10} \le x \le \frac{9L}{10}, \\ 0 & \text{for } \frac{9L}{10} \le x \le L, \end{cases} \tag{2.113}$$

the simulation starts from the initial pressure distribution $p(x, y, 0) = p_0 \cdot f(x) \cdot f(y)$ and evaluates the transient pressure distribution $p(x, y, t)$ with output after 0.02 and 0.04 days.

For liquids of small and constant compressibility Darcy's law and continuity equation yield the pressure conduction equation as the governing equation describing the transient pressure distribution. It reads

$$\phi\kappa \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla \cdot \nabla p. \tag{2.114}$$

Introducing the notation

$$\chi = \frac{k}{\phi\mu\kappa} \tag{2.115}$$

the present 2D problem is governed by the parabolic equation

$$\frac{1}{\chi}\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}, \tag{2.116}$$

the initial condition

$$p(x, y, 0) = p_0 \cdot f(x) \cdot f(y) \quad \text{for } 0 \le x, y \le L, \tag{2.117}$$

and the applied boundary conditions. These are specified zero pressures at $x = 0$ and $x = L$

$$p(0, y, t) = 0 \quad \text{for } 0 \le y \le L, \tag{2.118}$$

$$p(L, y, t) = 0 \quad \text{for } 0 \le y \le L, \tag{2.119}$$

and no-flow boundary conditions at $y = 0$ and $y = L$

$$\frac{\partial p}{\partial y}(x, 0, t) = 0 \quad \text{for } 0 \le x \le L, \tag{2.120}$$

$$\frac{\partial p}{\partial y}(x, L, t) = 0 \quad \text{for } 0 \le x \le L. \tag{2.121}$$

The closed form solution of the above problem is obtained in terms of the 1D p1-beam and p2-beam solutions given in the context of the previous example

$$\begin{aligned} p1(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, \exp\left(-\chi n^2\pi^2 \frac{t}{L^2}\right) \\ \times \frac{80}{3(n\pi)^2}\, \sin\frac{n\pi}{2}\, \sin\frac{n\pi}{4}\, \sin\frac{3n\pi}{20}, \end{aligned} \tag{2.122}$$

$$\begin{aligned} p2(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{L}\, \exp\left(-\chi n^2\pi^2 \frac{t}{L^2}\right) \\ \times \frac{80}{3(n\pi)^2}\, \cos\frac{n\pi}{2}\, \sin\frac{n\pi}{4}\, \sin\frac{3n\pi}{20}. \end{aligned} \tag{2.123}$$

We will next verify that the closed form solution of the above problem is given by

$$p(x, y, t) = p_0 \cdot p1(x,t) \cdot p2(y,t). \tag{2.124}$$

Both, $p1(x,t)$ and $p2(x,t)$, satisfy the initial condition

$$p1(x,0) = p2(x,0) = f(x). \tag{2.125}$$

Then

$$p(x,y,0) = p_0 \cdot p1(x,0) \cdot p2(y,0) = p_0 \cdot f(x) \cdot f(y), \tag{2.126}$$

hence, $p(x, y, t)$ satisfies the initial condition.

Both, $p1(x,t)$ and $p2(y,t)$, satisfy the 1D pressure conduction equation. Then

$$\begin{aligned} \frac{1}{\chi}\frac{\partial p}{\partial t} &= p_0\left[\frac{1}{\chi} p2 \frac{\partial p1}{\partial t} + \frac{1}{\chi} p1 \frac{\partial p2}{\partial t}\right] \\ &= p_0\left[p2\frac{\partial^2 p1}{\partial x^2} + p1 \frac{\partial^2 p2}{\partial y^2}\right] \\ &= \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}, \end{aligned} \tag{2.127}$$

hence, $p(x, y, t)$ satisfies the differential equation.

Zero boundary pressures are satisfied by $p1(x, t)$

$$p1(0, t) = 0 \quad \text{for } t > 0, \tag{2.128}$$
$$p1(L, t) = 0 \quad \text{for } t > 0, \tag{2.129}$$

no-flow boundary conditions are satisfied by $p2(y, t)$

$$\frac{\partial p2}{\partial y}(0, t) = 0 \quad \text{for } t > 0, \tag{2.130}$$
$$\frac{\partial p2}{\partial y}(L, t) = 0 \quad \text{for } t > 0. \tag{2.131}$$

Then

$$\begin{aligned}
p(0, y, t) &= p_0 \cdot p1(0, t) \cdot p2(y, t) &&= 0 \cdot p2(y, t) &&= 0,\\
p(L, y, t) &= p_0 \cdot p1(L, t) \cdot p2(y, t) &&= 0 \cdot p2(y, t) &&= 0,\\
\frac{\partial p}{\partial y}(x, 0, t) &= p_0 \cdot p1(x, t) \cdot \frac{\partial p2}{\partial y}(0, t) &&= p_0 \cdot p1(x, t) \cdot 0 &&= 0,\\
\frac{\partial p}{\partial y}(x, L, t) &= p_0 \cdot p1(x, t) \cdot \frac{\partial p2}{\partial y}(L, t) &&= p_0 \cdot p1(x, t) \cdot 0 &&= 0,
\end{aligned} \tag{2.132}$$

hence, $p(x, y, t)$ satisfies the boundary conditions.

2.3 Gas Flow

Because the gas density strongly depends on pressure, the governing equations become non-linear. We present a few steady-state solutions of isothermal flow problems, for the underlying theory see Freeze and Cherry [5].

2.3.1 A 1D Steady-State Gas Pressure Distribution, Boundary Conditions of 1st Kind

The domain is a rectangular beam extending along the positive x-axis and composed of $40 \times 1 \times 1$ equally sized hexahedral elements. An isotropic permeability of $10^{-15}\,\mathrm{m}^2$ is assumed for the material, gas viscosity has been assigned $10^{-5}\,\mathrm{Pa \cdot s}$, gravity is neglected by default. Specified pressures prevail at the beam ends with prescribed values of $p_1 = 10^5$ Pa at $x = L = 100\,\mathrm{m}$ and $p_0 = 2 \times 10^5$ Pa at $x = 0\,\mathrm{m}$. The simulation comprises one time step to establish the steady-state pressure distribution $p(x)$.

Darcy's law and continuity equation yield the Laplace equation governing the square of the steady-state gas pressure distribution. It reads

$$\frac{d^2 p^2}{dx^2} = 0 \tag{2.133}$$

for 1D gas flow along the x-axis, hence, the pressure is given by

$$p(x) = \sqrt{ax + b}. \tag{2.134}$$

The free constants, a and b, have to be determined from the specified boundary conditions at $x = L$ and $x = 0\,\text{m}$, therefore,

$$p(x) = \sqrt{(p_1^2 - p_0^2)\frac{x}{L} + p_0^2}. \tag{2.135}$$

2.3.2 A 1D Steady-State Gas Pressure Distribution, Boundary Conditions of 1st and 2nd Kind

The domain is a rectangular beam of length $L = 100\,\text{m}$ extending along the positive x-axis and composed of $40 \times 1 \times 1$ equally sized hexahedral elements. An isotropic permeability $k = 10^{-15}\,\text{m}^2$ is assumed for the material, gas viscosity $\mu = 10^{-5}\,\text{Pa}\cdot\text{s}$ has been assigned, gravity is neglected by default. Prescribed boundary conditions prevail at the beam ends as specified below. The simulation comprises one time step to establish the steady-state pressure distribution $p(x)$.

Darcy's law and continuity equation yield the Laplace equation governing the square of the steady-state gas pressure distribution. It reads

$$\frac{d^2 p^2}{dx^2} = 0 \tag{2.136}$$

for 1D gas flow along the x-axis, hence, the pressure is given by

$$p(x) = \sqrt{ax + b}. \tag{2.137}$$

The free constants, a and b, have to be determined from the specified boundary conditions at $x = L$ and $x = 0\,\text{m}$. Pressure $p_1 = 10^5\,\text{Pa}$ prevails at $x = L$, hence,

$$b = p_1^2 - aL. \tag{2.138}$$

Specific gas flow $Q = 0.17\,\text{Pa}\cdot\text{m/s}$ has been assigned at $x = 0\,\text{m}$. Then, by Darcy's law,

$$Q = -\frac{k}{\mu} p \frac{dp}{dx}|_{x=0} = -\frac{k}{2\mu}\frac{dp^2}{dx}|_{x=0} = -\frac{k}{2\mu}a. \tag{2.139}$$

The pressure distribution $p(x)$ thus becomes

$$p(x) = \sqrt{\frac{2Q\mu}{k}(L - x) + p_1^2}. \tag{2.140}$$

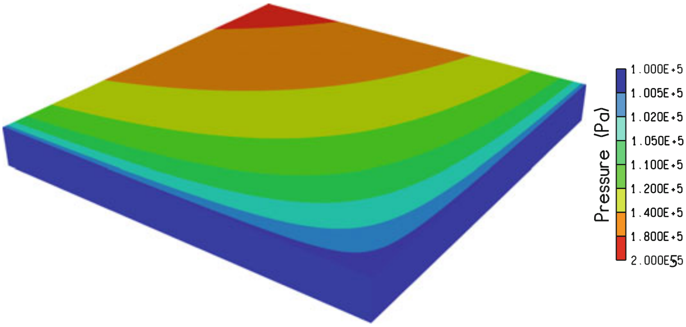

Fig. 2.9 Pressure distribution

2.3.3 A 2D Steady-State Gas Pressure Distribution

Given length $L = 1\,\mathrm{m}$ the domain represents the square $[0, L] \times [0, L]$ in the x-y-plane, discretized by an irregular mesh of hexahedral elements. An isotropic permeability $k = 10^{-15}\,\mathrm{m}^2$ is assumed for the material, gas viscosity $\mu = 10^{-5}\,\mathrm{Pa \cdot s}$ has been assigned, gravity is neglected by default. Prescribed conditions prevail at the lateral boundaries of the domain as specified below. The simulation comprises one time step to establish the steady-state pressure distribution $p(x, y)$ (Fig. 2.9).

Darcy's law and continuity equation yield the Laplace equation governing the square of the steady-state gas pressure distribution. It reads

$$\frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} = 0 \tag{2.141}$$

for 2D gas flow in the x-y-plane. With the aid of pressure $p_0 = 10^5\,\mathrm{Pa}$ the applied boundary conditions read

$$\begin{aligned}
p^2(0, y) &= p_0^2 && \text{for } 0 \le y \le L, \\
p^2(x, 0) &= p_0^2 && \text{for } 0 \le x \le L, \\
\frac{\partial p^2}{\partial x}(L, y) &= 3\frac{p_0^2}{L}\frac{y}{L} && \text{for } 0 \le y \le L, \\
\frac{\partial p^2}{\partial y}(x, L) &= 3\frac{p_0^2}{L}\frac{x}{L} && \text{for } 0 \le x \le L.
\end{aligned} \tag{2.142}$$

The pressure distribution

$$p(x, y) = p_0\sqrt{1 + 3\frac{xy}{L^2}} \tag{2.143}$$

satisfies the differential equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

Data input represents the second kind boundary conditions with the aid of a specific gas flow Q, which acts as a gas source to the domain at the faces $x = L$ and $y = L$. By Darcy's law,

$$Q = \frac{k}{\mu} p \frac{dp}{dx}|_{x=L} = \frac{k}{2\mu} \frac{dp^2}{dx}|_{x=L} = \frac{3k}{2\mu} \frac{p_0^2}{L} \frac{y}{L} \tag{2.144}$$

at the face $x = L$, and

$$Q = \frac{k}{\mu} p \frac{dp}{dy}|_{y=L} = \frac{k}{2\mu} \frac{dp^2}{dy}|_{y=L} = \frac{3k}{2\mu} \frac{p_0^2}{L} \frac{x}{L} \tag{2.145}$$

at the face $y = L$.

2.3.4 A 3D Steady-State Gas Pressure Distribution

Given length $L = 1\,\text{m}$ the domain represents the cube $[0, L] \times [0, L] \times [0, L]$ discretized by $21 \times 22 \times 23$ equally sized hexahedral elements. An isotropic permeability $k = 10^{-15}\,\text{m}^2$ is assumed for the material, gas viscosity $\mu = 10^{-5}\,\text{Pa} \cdot \text{s}$ has been assigned, gravity is neglected by default. Prescribed 2nd kind boundary conditions prevail at the entire surface of the domain as specified below. The simulation comprises one time step to establish the steady-state pressure distribution $p(x, y, z)$ (Fig. 2.10).

Darcy's law and continuity equation yield the Laplace equation governing the square of the steady-state gas pressure distribution. It reads

$$\frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} + \frac{\partial^2 p^2}{\partial z^2} = 0. \tag{2.146}$$

With the aid of pressure $p_0 = 10^5\,\text{Pa}$ the applied boundary conditions read

$$\begin{aligned} p^2(0,0,0) &= p_0^2, \\ \frac{\partial p^2}{\partial x}(0, y, z) &= \frac{3}{2} \frac{p_0^2}{L} \frac{y}{L} \quad \text{on the face } x = 0, \\ \frac{\partial p^2}{\partial x}(L, y, z) &= \frac{3}{2} \frac{p_0^2}{L} \frac{y}{L} \quad \text{on the face } x = L, \\ \frac{\partial p^2}{\partial y}(x, 0, z) &= \frac{3}{2} \frac{p_0^2}{L} \frac{x}{L} \quad \text{on the face } y = 0, \\ \frac{\partial p^2}{\partial y}(x, L, z) &= \frac{3}{2} \frac{p_0^2}{L} \frac{x}{L} \quad \text{on the face } y = L, \end{aligned} \tag{2.147}$$

$$\frac{\partial p^2}{\partial z}(x, y, 0) = \frac{3}{2}\frac{p_0^2}{L} \qquad \text{on the face } z = 0,$$
$$\frac{\partial p^2}{\partial z}(x, y, L) = \frac{3}{2}\frac{p_0^2}{L} \qquad \text{on the face } z = L.$$

The pressure distribution

$$p(x, y, z) = p_0\sqrt{1 + \frac{3}{2}\left(\frac{x}{L} \cdot \frac{y}{L} + \frac{z}{L}\right)} \tag{2.148}$$

satisfies the differential equation and the boundary conditions, hence, this is the closed form solution of the above boundary value problem.

Data input represents the second kind boundary conditions with the aid of a specific gas flow Q. It acts as a gas source to the domain on the faces $x = L$, $y = L$, and $z = L$ and acts as a sink on the remainder. On the face $x = L$ we have by Darcy's law

$$Q = \frac{k}{\mu} p \frac{\partial p}{\partial x}|_{x=L} = \frac{k}{2\mu}\frac{\partial p^2}{\partial x}|_{x=L} = \frac{3k}{4\mu}\frac{p_0^2}{L}\frac{y}{L}, \tag{2.149}$$

and similarly for the five other faces.

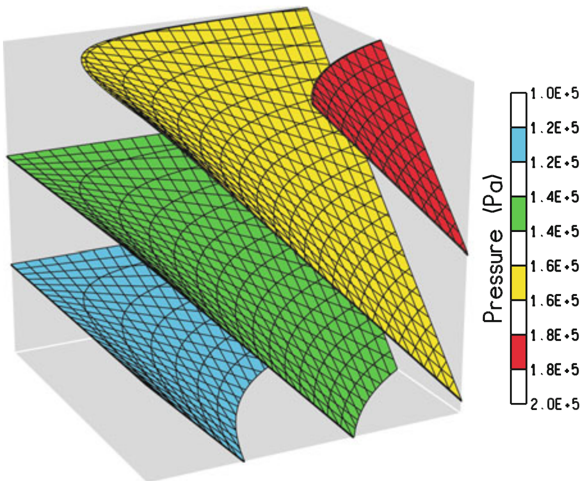

Fig. 2.10 Pressure contours

2.4 Deformation Processes

The linear elastic material is subject of the steady-state problems. Our transient problems focus on the Norton material. For the underlying theory see Jaeger and Cook [6].

2.4.1 An Elastic Beam Undergoes Axial Load

The domain is a rectangular beam extending along the positive x-axis. It has three faces located on the coordinate planes and is discretized by $20 \times 2 \times 2$ cubic elements of 0.05 m edge size each. The beam is represented by an elastic material. Poisson's ratio $\nu = 0.25$ and Young's modulus $E = 10{,}000\,\mathrm{MPa}$ have been assigned, gravity is neglected via zero material density. Faces on the coordinate planes are sliding planes, the top and the rear face of the beam are free, a tensile stress $\sigma_0 = 2\,\mathrm{MPa}$ is applied at the $x = 1$ m face. The simulation comprises one time step to establish the stresses, strains, and displacements.

Let $\boldsymbol{\sigma}$ denote the stress tensor. The equation of mechanical equilibrium

$$\nabla \cdot \boldsymbol{\sigma} = 0 \tag{2.150}$$

is satisfied by zero shear and constant stresses

$$\begin{aligned} \sigma_{11} &= \sigma_0, \\ \sigma_{22} &= \sigma_{33} = 0. \end{aligned} \tag{2.151}$$

Then, with principal axes equal to coordinate axes, Hooke's law gives for the strains

$$\begin{aligned} \epsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] = \frac{\sigma_0}{E}, \\ \epsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] = -\nu\frac{\sigma_0}{E}, \\ \epsilon_{33} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] = -\nu\frac{\sigma_0}{E}. \end{aligned} \tag{2.152}$$

Integrating the strains with respect to the fixities at the coordinate planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x(x) &= \frac{\sigma_0}{E}x, \\ u_y(y) &= -\nu\frac{\sigma_0}{E}y, \\ u_z(z) &= -\nu\frac{\sigma_0}{E}z. \end{aligned} \tag{2.153}$$

2.4.2 An Elastic Plate Undergoes Simple Shear

The domain is a rectangular plate located in the first octant. It has an extent of 10 m in x- and y-direction and is discretized by $8 \times 8 \times 2$ equally sized hexahedral elements. The plate is represented by an elastic material. Young's modulus $E = 10{,}000$ MPa and Poisson's ratio $\nu = 0.25$ have been assigned, gravity is neglected via zero material density. Load is applied with the aid of prescribed displacements which cover the entire surface. The simulation comprises two time steps with increasing deformation as specified below.

The equation of mechanical equilibrium and Hooke's law yield the Navier equations describing mechanical equilibrium in terms of the displacement vector (u_x, u_y, u_z). Employing the notation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = e \tag{2.154}$$

the Navier equations read

$$\begin{aligned}
\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial e}{\partial x} &= 0,\\
\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial e}{\partial y} &= 0,\\
\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} &= 0.
\end{aligned} \tag{2.155}$$

With the aid of slope m the specified boundary conditions are given by

$$\begin{aligned}
u_x(x, y, z) &= 0 && \text{on the entire surface,}\\
u_y(x, y, z) &= m\,x && \text{on the entire surface,}\\
u_z(x, y, z) &= 0 && \text{on the entire surface.}
\end{aligned} \tag{2.156}$$

The displacement vector

$$\begin{aligned}
u_x(x, y, z) &= 0,\\
u_y(x, y, z) &= m\,x,\\
u_z(x, y, z) &= 0
\end{aligned} \tag{2.157}$$

satisfies the Navier equations and the specified boundary conditions, hence, this is the required solution of the above boundary value problem. The only non-zero strain is

$$\epsilon_{12} = \frac{\partial u_y}{\partial x} = m, \tag{2.158}$$

and Hooke's law yields for the associated stress

$$\sigma_{12} = \frac{E}{2(1+\nu)}\, m. \tag{2.159}$$

The two time steps have m assigned the values -0.1 and -0.2, respectively.

2.4.3 An Elastic Cuboid Undergoes Load Due to Gravity

The domain is a cuboid of height $H = 30$ m and edges parallel to the x-y-z coordinate axes. It is discretized by an irregular mesh of hexahedral elements. The cuboid is represented by four groups of elastic materials, where each has been assigned density $\rho = 3058.104\,\mathrm{kg/m^3}$, Poisson's ratio $\nu = 0.25$ and Young's modulus $E = 10{,}000$ MPa. The bottom and the lateral faces are sliding planes, the top face is free. Gravity is the only load applied, $g = 9.81\,\mathrm{m/s^2}$ is the magnitude of gravity. The simulation comprises one time step to establish the stresses, strains, and displacements (Fig. 2.11).

Let $\boldsymbol{\sigma}$ denote the stress tensor. The equation of mechanical equilibrium

$$0 = \nabla \cdot \boldsymbol{\sigma} - (0, 0, \rho g) \tag{2.160}$$

is satisfied by zero shear, if the horizontal stresses σ_{11} and σ_{22} are functions of the vertical coordinate z only and the vertical stress σ_{33} satisfies

$$\frac{\partial \sigma_{33}}{\partial z} = \rho g. \tag{2.161}$$

The face $z = H$ is free, hence, integration gives

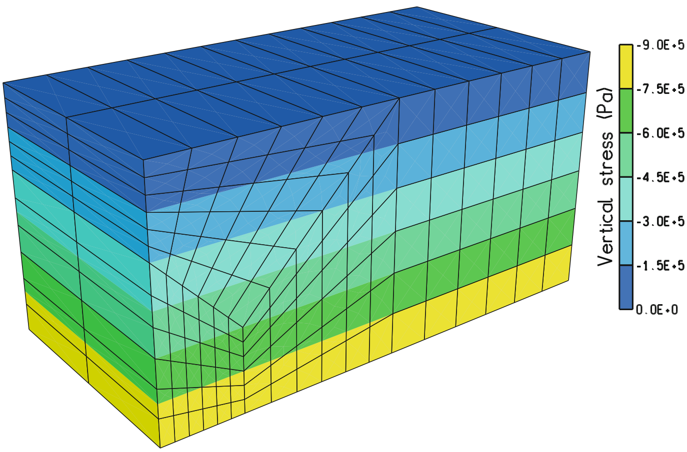

Fig. 2.11 Vertical stress

$$\sigma_{33} = \rho(-g)(H - z). \tag{2.162}$$

Assuming that there is no horizontal displacement anywhere we have for the horizontal strains

$$\epsilon_{11} = \epsilon_{22} = 0. \tag{2.163}$$

Then, with principal axes equal to coordinate axes, Hooke's law gives

$$\begin{aligned} 0 &= \sigma_{11} - \nu(\sigma_{22} + \sigma_{33}), \\ 0 &= \sigma_{22} - \nu(\sigma_{11} + \sigma_{33}), \\ E\epsilon_{33} &= \sigma_{33} - \nu(\sigma_{11} + \sigma_{22}). \end{aligned} \tag{2.164}$$

Solving for σ_{11}, σ_{22}, and the vertical strain ϵ_{33} yields

$$\begin{aligned} \sigma_{11} = \sigma_{22} &= \frac{\nu}{1-\nu}\sigma_{33} = \frac{\nu}{1-\nu}\,\rho(-g)(H - z), \\ \epsilon_{33} &= \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\rho(-g)(H - z) \end{aligned} \tag{2.165}$$

in terms of the vertical coordinate. Integrating the strains with respect to the prescribed fixities yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x &= u_y = 0, \\ u_z(z) &= \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\,\rho(-g)\left(Hz - \frac{1}{2}z^2\right). \end{aligned} \tag{2.166}$$

2.4.4 Stresses Relax in a Deformed Cube of Norton Material

The domain is a single cube with edge size $L = 1$ m located in the first octant. It has three faces located on the coordinate planes and is discretized by $2 \times 2 \times 2$ cubic elements. The cube is represented by a Norton material. Poisson's ratio $\nu = 0.27$ and Young's modulus $E = 25{,}000$ MPa have been assigned, gravity is neglected via zero material density. Various additional parameters are involved in the rheological model, details are given below. Faces on the coordinate planes are sliding planes. The constant vertical displacement $w = 0.0012$ m is applied at the top face for times $t > 0$. Starting from an initial setup free of load the simulation evaluates stresses, strains, and displacements through time with output after 0.1 and 1.1 days.

Let $\boldsymbol{\sigma}$ denote the stress tensor, $\mathbf{I}$ the unit tensor,

$$\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{\mathrm{tr}\boldsymbol{\sigma}}{3}\mathbf{I} \tag{2.167}$$

the stress deviator, and

$$\sigma_{\text{eff}} = \sqrt{\frac{3}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}^{D}\sigma_{ji}^{D}} \tag{2.168}$$

the v. Mises or effective stress. The rheological model involved yields the fundamental stress/strain relationships as a system of differential equations for the creep strains

$$\frac{\partial \boldsymbol{\epsilon}^{cr}}{\partial t} = \frac{3}{2}\frac{\boldsymbol{\sigma}^{D}}{\sigma_{\text{eff}}}(N\,\sigma_{\text{eff}}^{n}) \tag{2.169}$$

and the total strains

$$\boldsymbol{\epsilon}^{tot} = \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{cr}, \tag{2.170}$$

where $\boldsymbol{\epsilon}^{el}$ denotes the elastic strains via Hooke's law. Both equations have to be solved with respect to the imposed initial and boundary conditions.

For the present example the behaviour of the Norton material is specified with the aid of the parameters

$$\begin{aligned} n &= 5, \\ N &= A\exp\left(-\frac{Q}{RT}\right), \end{aligned} \tag{2.171}$$

where $R = 8.31441\,\text{J/(mol}\cdot\text{K)}$ is the gas constant, T is the absolute temperature (we have $T = 273.15\,\text{K}$ by default), and experimental data obtained from rock salt yield

$$\begin{aligned} A &= 0.18\ \ 1/(\text{d}\cdot\text{MPa}^5), \\ Q &= 54,000\ \text{J/mol}. \end{aligned} \tag{2.172}$$

Note that day is required as unit of time and stresses have to be in MPa.

Due to the example setup the principal axes are identical to the coordinate axes and the vertical stress is the only non-zero element of the stress tensor. Therefore,

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}, \tag{2.173}$$

the trace of $\boldsymbol{\sigma}$

$$\text{tr}\boldsymbol{\sigma} = \sigma_{33}, \tag{2.174}$$

the stress deviator

$$\boldsymbol{\sigma}^{D} = \frac{\sigma_{33}}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{2.175}$$

the v. Mises or effective stress

$$\sigma_{\text{eff}} = |\sigma_{33}| \frac{\sqrt{3/2}}{3} \sqrt{1^2 + 1^2 + 2^2} = |\sigma_{33}|, \tag{2.176}$$

and the time derivative of the creep strains

$$\frac{\partial \boldsymbol{\epsilon}^{cr}}{\partial t} = \frac{N}{2} \sigma_{33}^5 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.177}$$

The entire domain is initially free of creep strains. Hence, integrating with respect to time t the creep strains become

$$\boldsymbol{\epsilon}^{cr} = \frac{N}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \int_0^t \sigma_{33}^5 \, dt. \tag{2.178}$$

The elastic strains are obtained from the stress $\boldsymbol{\sigma}$ via Hooke's law

$$\boldsymbol{\epsilon}^{el} = \frac{\sigma_{33}}{E} \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.179}$$

The total strains in terms of σ_{33} and the displacements (u_x, u_y, u_z) read

$$\begin{aligned} \boldsymbol{\epsilon}^{tot} = \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{cr} &= \begin{pmatrix} \partial u_x/\partial x & 0 & 0 \\ 0 & \partial u_y/\partial y & 0 \\ 0 & 0 & \partial u_z/\partial z \end{pmatrix} \\ &= \frac{\sigma_{33}}{E} \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{N}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \int_0^t \sigma_{33}^5 \, dt. \end{aligned} \tag{2.180}$$

Due to the simulation setup

$$\epsilon_{33}^{tot} = \frac{\partial u_z}{\partial z} = \frac{w}{L} \tag{2.181}$$

is the specified constant strain along the z-axis. Then

$$\frac{w}{L} = \frac{1}{E} \sigma_{33} + N \int_0^t \sigma_{33}^5 \, dt. \tag{2.182}$$

This integral equation is transformed into the ordinary differential equation

$$0 = \frac{1}{E} \frac{d\sigma_{33}}{dt} + N \sigma_{33}^5. \tag{2.183}$$

Separation of variables and integration yields

$$\sigma_{33}(t) = \frac{w}{L}\frac{E}{\sqrt[4]{4E^5(w/L)^4 Nt + 1}}, \tag{2.184}$$

and the strains $\partial u_x/\partial x$ and $\partial u_y/\partial y$ are obtained in terms of $\sigma_{33}(t)$

$$\begin{aligned} \epsilon_{11}^{tot}(t) &= \frac{\partial u_x}{\partial x} = -\frac{w}{2L} + \frac{1-2\nu}{2E}\sigma_{33}(t), \\ \epsilon_{22}^{tot}(t) &= \frac{\partial u_y}{\partial y} = -\frac{w}{2L} + \frac{1-2\nu}{2E}\sigma_{33}(t). \end{aligned} \tag{2.185}$$

Integrating the strains with respect to the fixities at the coordinate planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x(x,t) &= x\left(-\frac{w}{2L} + \frac{1-2\nu}{2E}\sigma_{33}(t)\right), \\ u_y(y,t) &= y\left(-\frac{w}{2L} + \frac{1-2\nu}{2E}\sigma_{33}(t)\right), \\ u_z(z,t) &= z\,\frac{w}{L} \end{aligned} \tag{2.186}$$

again in terms of $\sigma_{33}(t)$ derived above.

2.4.5 A Cube of Norton Material Creeps Under Constant Stress

The domain is a single cube with an edge size of 1 m located in the first octant. It has three faces located on the coordinate planes and is discretized by $2 \times 2 \times 2$ cubic elements. The cube is represented by a Norton material. Poisson's ratio $\nu = 0.27$ and Young's modulus $E = 25{,}000$ MPa have been assigned, gravity is neglected via zero material density. The additional parameters involved in the rheological model are identical to those of the previous example. Faces on the coordinate planes are sliding planes. The constant vertical stress $\sigma_0 = -20$ MPa is applied at the top face for times $t > 0$. Starting from an initial setup free of load the simulation evaluates stresses, strains, and displacements through time with output after 10 and 20 days.

The rheological model of the Norton material and its underlying theory have been sketched just before; we focus on the special features of the present example. Due to the setup the principal axes are identical to the coordinate axes and the specified vertical stress is the only non-zero element of the stress tensor. Therefore,

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}, \tag{2.187}$$

the trace of $\boldsymbol{\sigma}$

$$\mathrm{tr}\boldsymbol{\sigma} = \sigma_0, \tag{2.188}$$

the stress deviator

$$\boldsymbol{\sigma}^D = \frac{\sigma_0}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{2.189}$$

the v. Mises or effective stress

$$\sigma_{\mathrm{eff}} = |\sigma_0|\frac{\sqrt{3/2}}{3}\sqrt{1^2+1^2+2^2} = |\sigma_0|, \tag{2.190}$$

and the time derivative of the creep strains

$$\frac{\partial \boldsymbol{\epsilon}^{cr}}{\partial t} = \frac{N}{2}\sigma_0^5\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.191}$$

The entire domain is initially free of creep strains. Hence, integrating with respect to time t the creep strains become

$$\boldsymbol{\epsilon}^{cr} = \frac{N}{2}\sigma_0^5\begin{pmatrix} -t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 2t \end{pmatrix}. \tag{2.192}$$

The elastic strains are obtained from the stress $\boldsymbol{\sigma}$ via Hooke's law

$$\boldsymbol{\epsilon}^{el} = \frac{\sigma_0}{E}\begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.193}$$

The total strains in terms of the displacements (u_x, u_y, u_z) read

$$\boldsymbol{\epsilon}^{tot} = \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{cr} = \begin{pmatrix} \partial u_x/\partial x & 0 & 0 \\ 0 & \partial u_y/\partial y & 0 \\ 0 & 0 & \partial u_z/\partial z \end{pmatrix}. \tag{2.194}$$

The strains $\partial u_x/\partial x$, $\partial u_y/\partial y$, and $\partial u_z/\partial z$ are thus given by

$$
\begin{aligned}
\epsilon_{11}^{tot}(t) &= \frac{\partial u_x}{\partial x} = -\frac{\nu\sigma_0}{E} - \frac{N}{2}\sigma_0^5\, t,\\
\epsilon_{22}^{tot}(t) &= \frac{\partial u_y}{\partial y} = -\frac{\nu\sigma_0}{E} - \frac{N}{2}\sigma_0^5\, t,\\
\epsilon_{33}^{tot}(t) &= \frac{\partial u_z}{\partial z} = \frac{\sigma_0}{E} + N\sigma_0^5\, t.
\end{aligned}
\tag{2.195}
$$

Integrating the strains with respect to the fixities at the coordinate planes yields the displacement vector (u_x, u_y, u_z)

$$
\begin{aligned}
u_x(x,t) &= x\left(-\frac{\nu\sigma_0}{E} - \frac{N}{2}\sigma_0^5\, t\right),\\
u_y(y,t) &= y\left(-\frac{\nu\sigma_0}{E} - \frac{N}{2}\sigma_0^5\, t\right),\\
u_z(z,t) &= z\left(\frac{\sigma_0}{E} + N\sigma_0^5\, t\right).
\end{aligned}
\tag{2.196}
$$

2.4.6 A Cube of Norton Material Undergoes Tensile Strain Increasing Linearly with Time

The domain is a single cube with edge size $L = 1$ m located in the first octant. It has three faces located on the coordinate planes and is discretized by $2 \times 2 \times 2$ cubic elements. The cube is represented by a Norton material. Poisson's ratio $\nu = 0.27$ and Young's modulus $E = 2.5 \times 10^7$ MPa have been assigned, gravity is neglected via zero material density. Except from the stress exponent n, which now has $n = 2$, the values of the additional parameters involved in the rheological model are identical to those of the two previous examples. Faces on the coordinate planes are sliding planes. The vertical displacement $w_1 \cdot t$ ($w_1 = 0.0001$ m/d) increases linearly with time t, it is applied at the top face for times $t > 0$. Starting from an initial setup free of load the simulation evaluates stresses, strains, and displacements through time with output after 1.5 and 3.0 days.

The rheological model of the Norton material and its underlying theory have already been outlined before; we focus on the special features of the present example. Due to the setup the principal axes are identical to the coordinate axes and the vertical stress is the only non-zero element of the stress tensor. Therefore,

$$
\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix},
\tag{2.197}
$$

the trace of $\boldsymbol{\sigma}$

$$
\mathrm{tr}\boldsymbol{\sigma} = \sigma_{33},
\tag{2.198}
$$

the stress deviator

$$\boldsymbol{\sigma}^D = \frac{\sigma_{33}}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{2.199}$$

the v. Mises or effective stress

$$\sigma_{\text{eff}} = |\sigma_{33}| \frac{\sqrt{3/2}}{3} \sqrt{1^2 + 1^2 + 2^2} = |\sigma_{33}|, \tag{2.200}$$

and the time derivative of the (positive) creep strains

$$\frac{\partial \boldsymbol{\epsilon}^{cr}}{\partial t} = \frac{N}{2} \sigma_{33}^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.201}$$

The entire domain is initially free of creep strains. Hence, integrating with respect to time t the creep strains become

$$\boldsymbol{\epsilon}^{cr} = \frac{N}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \int_0^t \sigma_{33}^2 \, dt. \tag{2.202}$$

The elastic strains are obtained from the stress $\boldsymbol{\sigma}$ via Hooke's law

$$\boldsymbol{\epsilon}^{el} = \frac{\sigma_{33}}{E} \begin{pmatrix} -\upsilon & 0 & 0 \\ 0 & -\upsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.203}$$

The total strains in terms of σ_{33} and the displacements (u_x, u_y, u_z) read

$$\begin{aligned} \boldsymbol{\epsilon}^{tot} = \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{cr} &= \begin{pmatrix} \partial u_x/\partial x & 0 & 0 \\ 0 & \partial u_y/\partial y & 0 \\ 0 & 0 & \partial u_z/\partial z \end{pmatrix} \\ &= \frac{\sigma_{33}}{E} \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{N}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \int_0^t \sigma_{33}^2 \, dt. \end{aligned} \tag{2.204}$$

Due to the simulation setup

$$\epsilon_{33}^{tot} = \frac{\partial u_z}{\partial z} = \frac{w_1}{L} t \tag{2.205}$$

is the specified strain along the z-axis. Then

$$\frac{w_1}{L} t = \frac{1}{E} \sigma_{33} + N \int_0^t \sigma_{33}^2 \, dt. \tag{2.206}$$

This integral equation is transformed into the ordinary differential equation

$$\frac{w_1}{L} = \frac{1}{E}\frac{d\sigma_{33}}{dt} + N\sigma_{33}^2. \tag{2.207}$$

Separation of variables and integration yields

$$ENt = \sqrt{\frac{LN}{4w_1}} \ln \left| \frac{\sqrt{w_1/(LN)} + \sigma_{33}}{\sqrt{w_1/(LN)} - \sigma_{33}} \right|, \tag{2.208}$$

the vertical stress σ_{33} becomes

$$\sigma_{33}(t) = \sqrt{\frac{w_1}{LN}} \tanh(\sqrt{w_1 N/L}\; E\, t), \tag{2.209}$$

and the strains $\partial u_x/\partial x$ and $\partial u_y/\partial y$ are obtained in terms of $\sigma_{33}(t)$

$$\begin{aligned} \epsilon_{11}^{tot}(t) &= \frac{\partial u_x}{\partial x} = -\frac{w_1}{2L}t + \frac{1-2\nu}{2E}\sigma_{33}(t), \\ \epsilon_{22}^{tot}(t) &= \frac{\partial u_y}{\partial y} = -\frac{w_1}{2L}t + \frac{1-2\nu}{2E}\sigma_{33}(t). \end{aligned} \tag{2.210}$$

Integrating the strains with respect to the fixities at the coordinate planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x(x,t) &= x\left(-\frac{w_1}{2L}t + \frac{1-2\nu}{2E}\sigma_{33}(t)\right), \\ u_y(y,t) &= y\left(-\frac{w_1}{2L}t + \frac{1-2\nu}{2E}\sigma_{33}(t)\right), \\ u_z(z,t) &= z\,\frac{w_1}{L}\,t \end{aligned} \tag{2.211}$$

again in terms of $\sigma_{33}(t)$ derived above.

2.4.7 *A Cube of Norton Material Undergoes Compressive Stress Increasing Linearly with Time*

The domain is a single cube with an edge size of 1 m located in the first octant. It has three faces located on the coordinate planes and is discretized by $2 \times 2 \times 2$ cubic elements. The cube is represented by a Norton material. Poisson's ratio $\nu = 0.27$ and Young's modulus $E = 25{,}000$ MPa have been assigned, gravity is neglected via zero material density. Except from the stress exponent n, which now has $n = 5$ again, the values of the additional parameters involved in the rheological model are identical to those of the three previous examples. Faces on the coordinate planes are sliding planes. The vertical stress $\sigma_1 \cdot t$ ($\sigma_1 = -1\,\text{MPa/d}$) depends linearly on time t, it is applied at the top face for times $t > 0$. Starting from an initial setup free of load the simulation evaluates stresses, strains, and displacements through time with output after 15 and 30 days.

The rheological model of the Norton material and its underlying theory have already been outlined before; we focus on the special features of the present example. Due to the setup the principal axes are identical to the coordinate axes and the specified vertical stress is the only non-zero element of the stress tensor. Therefore,

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_1 \cdot t \end{pmatrix}, \tag{2.212}$$

the trace of $\boldsymbol{\sigma}$

$$\mathrm{tr}\boldsymbol{\sigma} = \sigma_1 \cdot t, \tag{2.213}$$

the stress deviator

$$\boldsymbol{\sigma}^D = \frac{\sigma_1 \cdot t}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{2.214}$$

the v. Mises or effective stress

$$\sigma_{\text{eff}} = |\sigma_1 \cdot t| \frac{\sqrt{3/2}}{3} \sqrt{1^2 + 1^2 + 2^2} = |\sigma_1 \cdot t|, \tag{2.215}$$

and the time derivative of the creep strains

$$\frac{\partial \boldsymbol{\epsilon}^{cr}}{\partial t} = \frac{N}{2} (\sigma_1 \cdot t)^5 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.216}$$

The entire domain is initially free of creep strains. Hence, integrating with respect to time t the creep strains become

$$\boldsymbol{\epsilon}^{cr} = \frac{N}{12} \sigma_1^5 t^6 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.217}$$

The elastic strains are obtained from the stress $\boldsymbol{\sigma}$ via Hooke's law

$$\boldsymbol{\epsilon}^{el} = \frac{\sigma_1 \cdot t}{E} \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.218}$$

The total strains in terms of the displacements (u_x, u_y, u_z) read

$$\boldsymbol{\epsilon}^{tot} = \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{cr} = \begin{pmatrix} \partial u_x/\partial x & 0 & 0 \\ 0 & \partial u_y/\partial y & 0 \\ 0 & 0 & \partial u_z/\partial z \end{pmatrix}, \tag{2.219}$$

the strains $\partial u_x/\partial x$, $\partial u_y/\partial y$, and $\partial u_z/\partial z$ are thus given by

$$\begin{aligned} \epsilon_{11}^{tot}(t) &= \frac{\partial u_x}{\partial x} = -\frac{\nu\sigma_1\, t}{E} - \frac{N}{12}\sigma_1^5 t^6, \\ \epsilon_{22}^{tot}(t) &= \frac{\partial u_y}{\partial y} = -\frac{\nu\sigma_1\, t}{E} - \frac{N}{12}\sigma_1^5 t^6, \\ \epsilon_{33}^{tot}(t) &= \frac{\partial u_z}{\partial z} = \frac{\sigma_1\, t}{E} + \frac{N}{6}\sigma_1^5 t^6, \end{aligned} \tag{2.220}$$

and integration with respect to the fixities at the coordinate planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x(x,t) &= x\left(-\frac{\nu\,\sigma_1 t}{E} - \frac{N}{12}\sigma_1^5 t^6\right), \\ u_y(y,t) &= y\left(-\frac{\nu\,\sigma_1 t}{E} - \frac{N}{12}\sigma_1^5 t^6\right), \\ u_z(z,t) &= z\left(\frac{\sigma_1 t}{E} + \frac{N}{6}\sigma_1^5 t^6\right). \end{aligned} \tag{2.221}$$

2.5 Mass Transport

The Laplace transform solution method proves to be a powerful tool in solving mass transport problems. From the variety of closed form solutions available in the literature we adopted some basic examples from standard references. We made no attempt to trace back the entire material to its original sources.

2.5.1 Solute Transport Along Permeable Beams, Hydraulic and Solute Boundary Conditions of 1st and 2nd Kind

The domain comprises four parallel beams of length $L = 10\,\text{m}$ extending along the positive x-axis, each composed of $200 \times 1 \times 1$ equally sized hexahedral elements (Fig. 2.12). An isotropic permeability $k = 10^{-11}\,\text{m}^2$ holds for all beams, porosities are listed below, the liquid is incompressible with viscosity $\mu = 1\,\text{mPa}\cdot\text{s}$. The diffusion coefficient assumes the constant value $D = 10^{-4}\,\text{m}^2/\text{s}$ comprising molecular diffusion and mechanical dispersion. Gravity is neglected via zero liquid density.

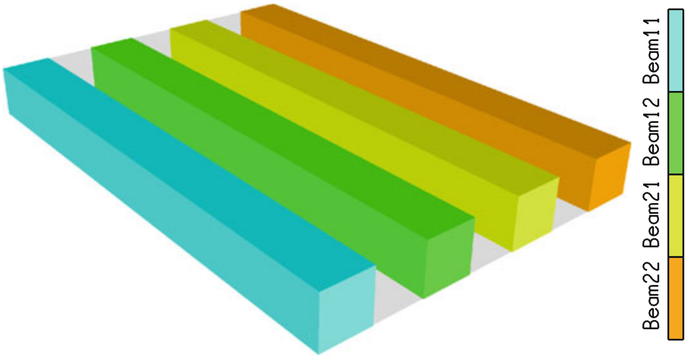

Fig. 2.12 Example setup

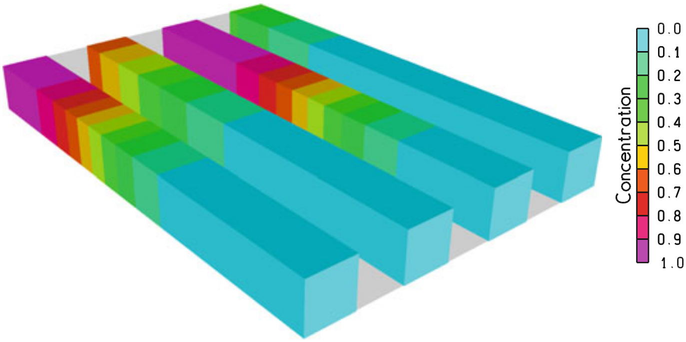

Fig. 2.13 Solute distributions after 20,000 s

Zero pressure prevails at the beams outlets ($x = L$). Hydraulic and solute boundary conditions of 1st and 2nd kind are imposed at the inlets ($x = 0$ m) and are listed below. Starting from zero initial solute concentration the simulation evaluates the transient solute distributions with output after 10,000 and 20,000 s (Fig. 2.13).

The formal solution proceeds in two steps, first to solve for the pressure and second to determine the solute distributions.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{d^2 p}{dx^2} = 0 \tag{2.222}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = ax + b. \tag{2.223}$$

Table 2.1 Example overview

	Beam11	Beam12	Beam21	Beam22
Porosity ϕ	0.6	0.4	0.4	0.6
Inlet pressure p_0	10^5 Pa	10^5 Pa		
Specific discharge q			10^{-4} m/s	10^{-4} m/s
Inlet concentration	1		1	
Solute input $j = -\phi D \nabla c$		5×10^{-6} m/s		5×10^{-6} m/s

The free constants, a and b, have to be determined from the specified boundary conditions at $x = L$ and $x = 0$ m. Hence, in case of specified inlet pressure (Beam11 and Beam12)

$$p(x) = p_0 \left(1 - \frac{x}{L}\right) \tag{2.224}$$

and by Darcy's law

$$p(x) = \frac{\mu}{k} q L \left(1 - \frac{x}{L}\right) \tag{2.225}$$

in case of specified specific discharge (Beam21 and Beam22). Employing the data given above both result in identical pressure distributions and a unique specific discharge q applied to all beams (Table 2.1).

We will next focus on the closed form solution of the transport problems, i.e. we will solve the solute transport equation subject to the imposed initial and boundary conditions. The solute distributions along Beam11 and Beam21 (c1-distributions) are based on a 1st kind solute boundary condition. Due to free outflow at $x = L$ these distributions represent those of a solute in steady linear flow downstream ($x > 0$) the source

$$c1(0, t) = 1 \quad \text{for } t > 0. \tag{2.226}$$

The formal problem is to determine the solution $c1(x, t)$ of the 1D transport equation

$$\frac{\partial c1}{\partial t} + \frac{q}{\phi} \frac{\partial c1}{\partial x} = D \frac{\partial^2 c1}{\partial x^2} \tag{2.227}$$

subject to the initial condition

$$c1(x, 0) = 0 \quad \text{for } x > 0, \tag{2.228}$$

and the boundary conditions

$$\begin{aligned} c1(0, t) &= 1 \quad \text{for } t > 0, \\ \lim_{x \to \infty} c1(x, t) &= 0 \quad \text{for } t > 0. \end{aligned} \tag{2.229}$$

Applying the Laplace transform with respect to t yields the ordinary differential equation

$$D\,\bar{c}1'' - \frac{q}{\phi}\bar{c}1' - s\,\bar{c}1 = 0, \tag{2.230}$$

where $\bar{c}1$ is the transform of $c1$, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to transformed boundary conditions. This yields

$$\bar{c}1(x,s) = \frac{1}{s}\exp\left[x\left(\frac{q}{2\phi D} - \sqrt{(\frac{q}{2\phi D})^2 + \frac{s}{D}}\right)\right]. \tag{2.231}$$

Churchill [7] outlines how to obtain the solution $c1(x,t)$ from their transform with the aid of operational calculus.

looseness-1 The solute distributions along Beam12 and Beam22 (c2-distributions) are based on a 2nd kind solute boundary condition. Due to free outflow at $x = L$ these distributions represent those of a solute in steady linear flow downstream $(x > 0)$ the source

$$\frac{\partial c2}{\partial x}(0,t) = -\frac{j}{\phi D} \quad \text{for } t > 0. \tag{2.232}$$

The formal problem is to determine the solution $c2(x,t)$ of the 1D transport equation

$$\frac{\partial c2}{\partial t} + \frac{q}{\phi}\frac{\partial c2}{\partial x} = D\frac{\partial^2 c2}{\partial x^2} \tag{2.233}$$

subject to the initial condition

$$c2(x,0) = 0 \quad \text{for } x > 0, \tag{2.234}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial c2}{\partial x}(0,t) &= -\frac{j}{\phi D} \quad \text{for } t > 0, \\ \lim_{x\to\infty} c2(x,t) &= \quad 0 \quad\;\; \text{for } t > 0. \end{aligned} \tag{2.235}$$

Applying the Laplace transform with respect to t yields the ordinary differential equation

$$D\,\bar{c}2'' - \frac{q}{\phi}\bar{c}2' - s\,\bar{c}2 = 0, \tag{2.236}$$

where $\bar{c}2$ is the transform of $c2$, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{c}2(x,s) = -\frac{j}{\phi D s}\frac{\exp\left[x\left(\frac{q}{2\phi D} - \sqrt{(\frac{q}{2\phi D})^2 + \frac{s}{D}}\right)\right]}{\frac{q}{2\phi D} - \sqrt{(\frac{q}{2\phi D})^2 + \frac{s}{D}}}. \tag{2.237}$$

The entire solution may now be obtained from the transforms of the solute distributions. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of $c1(x,t)$ and $c2(x,t)$ (Fig. 2.13).

2.5.2 *Solute Transport Along Permeable Beams with an Inert, a Decaying, and an Adsorbing Solute, Time-Dependent Boundary Conditions of 1st Kind*

The domain comprises three parallel beams of length $L = 10\,\text{m}$ extending along the positive x-axis, each composed of $200 \times 1 \times 1$ equally sized hexahedral elements (Fig. 2.14). An isotropic permeability $k = 10^{-11}\,\text{m}^2$ holds for all beams, porosities have been assigned $\phi = 0.4$. The liquid is incompressible with viscosity $\mu = 1\,\text{mPa}\cdot\text{s}$. The diffusion coefficient assumes the constant value $D = 10^{-4}\,\text{m}^2/\text{s}$ comprising molecular diffusion and mechanical dispersion. Gravity is neglected via zero liquid density. The decaying solute (half life T) undergoes first order decay with decay constant $(\ln 2)/T = 0.5 \times 10^{-4}$ 1/s. For the adsorbing solute the adsorbed mass fraction is related to the solute mass fraction by a linear equilibrium sorption model. Input data require the distribution coefficient $K_d = 6.8 \times 10^{-4}\,\text{m}^3/\text{kg}$ and the density of the solid grain ($\rho_s = 2{,}000\,\text{kg/m}^3$), for more details see below. Pressure $p_0 = 10^5$ Pa prevails at the beams inlets ($x = 0\,\text{m}$), zero pressure prevails at the beams outlets ($x = L$). Along the inlets concentration 1 is specified for times less than $t_0 = 15{,}000$ s and zero afterward. Starting from zero initial solute concentration the simulation evaluates the transient solute distributions with output after 10,000 and 20,000 s (Fig. 2.15).

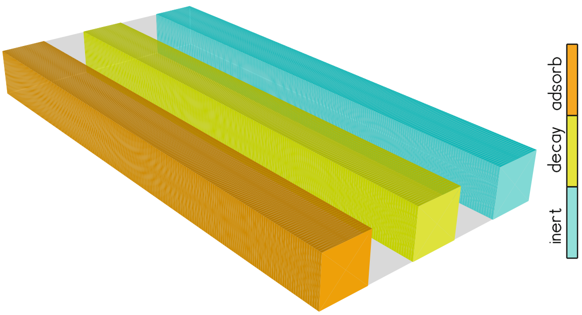

Fig. 2.14 Example setup

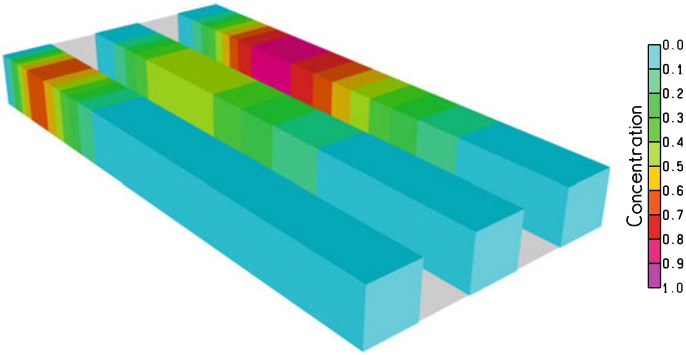

Fig. 2.15 Solute distributions after 20,000 s

The formal solution proceeds in two steps, first to solve for pressure $p(x)$ and specific discharge q and second to determine the solute distributions.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{d^2 p}{dx^2} = 0 \tag{2.238}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = p_0 \left(1 - \frac{x}{L}\right), \tag{2.239}$$

and the specific discharge q is obtained by Darcy's law

$$q = \frac{k}{\mu} \frac{p_0}{L}. \tag{2.240}$$

We will next focus on the closed form solution of the transport problems, i.e. we will solve 1D solute transport equations

$$\frac{\partial c}{\partial t} + \frac{q}{\phi} \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2} - r(x,t), \tag{2.241}$$

where $r(x,t)$ depends on the various reactions involved.

Due to free outflow at $x = L$ the distribution of the inert solute (c3-distribution) represents that of a solute in steady linear flow downstream ($x > 0$) the source

$$c3(0,t) = \begin{cases} 1 & \text{for } t_0 > t > 0, \\ 0 & \text{for } t > t_0, \end{cases} \tag{2.242}$$

and the formal problem is to determine the solution $c3(x,t)$ of the 1D transport equation

$$\frac{\partial c3}{\partial t} + \frac{q}{\phi}\frac{\partial c3}{\partial x} = D\frac{\partial^2 c3}{\partial x^2} \tag{2.243}$$

subject to the initial condition

$$c3(x,0) = 0 \quad \text{for } x > 0, \tag{2.244}$$

and the boundary conditions

$$c3(0,t) = \begin{cases} 1 & \text{for } t_0 > t > 0, \\ 0 & \text{for } t > t_0, \end{cases} \tag{2.245}$$

$$\lim_{x\to\infty} c3(x,t) = 0 \quad \text{for } t > 0. \tag{2.246}$$

Applying the Laplace transform with respect to t yields the ordinary differential equation

$$D\,\bar{c}3'' - \frac{q}{\phi}\bar{c}3' - s\,\bar{c}3 = 0, \tag{2.247}$$

where $\bar{c}3$ is the transform of $c3$, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{c}3(x,s) = \frac{1-\exp(-t_0\,s)}{s}\exp\left[x\left(\frac{q}{2\phi D} - \sqrt{(\frac{q}{2\phi D})^2 + \frac{s}{D}}\right)\right]. \tag{2.248}$$

Churchill [7] outlines how to obtain the solution $c3(x,t)$ from their transform with the aid of operational calculus.

Due to free outflow at $x = L$ the distribution of the decaying solute (c2-distribution) represents that of a solute in steady linear flow downstream ($x > 0$) the source

$$c2(0,t) = \begin{cases} 1 & \text{for } t_0 > t > 0, \\ 0 & \text{for } t > t_0, \end{cases} \tag{2.249}$$

and formal problem is to determine the solution $c2(x,t)$ of the 1D transport equation

$$\frac{\partial c2}{\partial t} + \frac{q}{\phi}\frac{\partial c2}{\partial x} = D\frac{\partial^2 c2}{\partial x^2} - \frac{\ln 2}{T}c2 \tag{2.250}$$

subject to the initial condition

$$c2(x,0) = 0 \quad \text{for } x > 0, \tag{2.251}$$

and the boundary conditions

$$c2(0,t) = \begin{cases} 1 & \text{for } t_0 > t > 0, \\ 0 & \text{for } t > t_0, \end{cases} \tag{2.252}$$

$$\lim_{x\to\infty} c2(x,t) = 0 \quad \text{for } t > 0. \tag{2.253}$$

Applying the Laplace transform with respect to t yields the ordinary differential equation

$$D\,\bar{c}2'' - \frac{q}{\phi}\bar{c}2' - (s + \frac{\ln 2}{T})\,\bar{c}2 = 0, \tag{2.254}$$

where $\bar{c}2$ is the transform of $c2$, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{c}2(x,s) = \frac{1 - \exp(-t_0\,s)}{s} \exp\left[x\left(\frac{q}{2\phi D} - \sqrt{(\frac{q}{2\phi D})^2 + \frac{\ln 2}{DT} + \frac{s}{D}}\right)\right]. \tag{2.255}$$

Following Churchill [7] again the solution $c2(x,t)$ may be obtained from their transform with the aid of operational calculus.

The transport equation associated to the distribution $c1(x,t)$ of an adsorbing solute is obtained from a mass balance of solute in the liquid and on the porous matrix. Let ρ_l denote the density of the liquid, and let ρ_s denote the density of the solid grain. Continuity of solute mass in the liquid yields

$$\frac{\partial(\phi\rho_l c1)}{\partial t} + \frac{q}{\phi}\frac{\partial(\phi\rho_l c1)}{\partial x} = D\frac{\partial^2(\phi\rho_l c1)}{\partial x^2} - a_1(x,t), \tag{2.256}$$

where $a_1(x,t)$ denotes the change of solute mass in the liquid due to interaction with the porous matrix. On the matrix the solute mass changes due to liquid/matrix-interaction again

$$\frac{\partial[(1-\phi)\rho_s s1]}{\partial t} = a_1(x,t), \tag{2.257}$$

and the adsorbed mass fraction $s1$ is related to the solute mass fraction $c1$ by the linear equilibrium sorption model

$$\rho_s s1 = (K_d\rho_s)\rho_l c1. \tag{2.258}$$

Introducing the notation

$$R = 1 + \frac{1-\phi}{\phi}K_d\rho_s, \tag{2.259}$$

yields the formal problem to determine the solution $c1(x, t)$ of the 1D transport equation

$$\frac{\partial c1}{\partial t} + \frac{q}{\phi R}\frac{\partial c1}{\partial x} = \frac{D}{R}\frac{\partial^2 c1}{\partial x^2} \tag{2.260}$$

subject to the initial condition

$$c1(x, 0) = 0 \quad \text{for } x > 0, \tag{2.261}$$

and the boundary conditions

$$c1(0, t) = \begin{cases} 1 & \text{for } t_0 > t > 0, \\ 0 & \text{for } t > t_0, \end{cases} \tag{2.262}$$

$$\lim_{x\to\infty} c1(x, t) = 0 \quad \text{for } t > 0. \tag{2.263}$$

Applying the Laplace transform with respect to t yields the ordinary differential equation

$$\frac{D}{R}\bar{c}1'' - \frac{q}{\phi R}\bar{c}1' - s\,\bar{c}1 = 0, \tag{2.264}$$

where $\bar{c}1$ is the transform of $c1$, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{c}1(x, s) = \frac{1 - \exp(-t_0\, s)}{s} \exp\left[x\left(\frac{q}{2\phi D} - \sqrt{(\frac{q}{2\phi D})^2 + \frac{sR}{D}}\right)\right]. \tag{2.265}$$

The entire solution may now be obtained from the transforms of the solute distributions. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of $c1(x, t)$, $c2(x, t)$, and $c3(x, t)$.

2.5.3 A Transient 2D Solute Distribution

Given length $L = 1$ m the domain represents the rectangle $[0, 2L] \times [-0.75L, 0.75L]$ located in the x-y-plane and subdivided into $80 \times 60 \times 1$ cubic elements. A permeable material represents the porous medium, with isotropic permeability $k = 10^{-11}\,\text{m}^2$ and porosity $\phi = 0.5$. The liquid is incompressible with viscosity $\mu = 1\,\text{mPa}\cdot\text{s}$, the diffusion coefficient assumes the constant value $D = 3 \times 10^{-6}\,\text{m}^2/\text{s}$ comprising molecular diffusion and mechanical dispersion. Gravity is neglected via zero liquid density. Pressure $p_0 = 2 \times 10^4$ Pa at the liquid inlet ($x = 0$ m) and zero pressure the

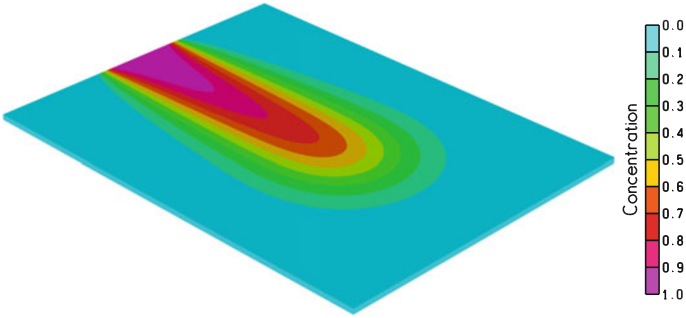

Fig. 2.16 Solute distribution after 7,000 s

liquid outlet ($x = 2L$) generate steady-state 1D flow along the x-axis. At the liquid inlet a non-zero solute concentration is specified along a line segment of the y-axis. Given $a = 0.15$ m and $b = 0.25$ m the specified inlet concentration reads

$$g(y) = \begin{cases} 0 & \text{for } y \leq -b, \\ \dfrac{b+y}{b-a} & \text{for } -b \leq y \leq -a, \\ 1 & \text{for } -a \leq y \leq a, \\ \dfrac{b-y}{b-a} & \text{for } a \leq y \leq b, \\ 0 & \text{for } b \leq y. \end{cases} \tag{2.266}$$

Starting from zero initial solute concentration the simulation evaluates the transient solute distribution $c(x, y, t)$ with output after 3,500 and 7,000 s (Fig. 2.16).

The formal solution proceeds in two steps, first to solve for pressure $p(x)$ and specific discharge q and second to determine the solute distributions.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{d^2 p}{dx^2} = 0 \tag{2.267}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = p_0 \left(1 - \frac{x}{2L}\right), \tag{2.268}$$

and the specific discharge q is obtained by Darcy's law

$$q = \frac{k}{\mu} \frac{p_0}{2L}. \tag{2.269}$$

We will next focus on the closed form solution of the transport problem and solve the 2D solute transport equation with the aid of successive integral transforms as described by Leij and Dane [8]. The formal problem is to determine the solution $c(x, y, t)$ of the 2D transport equation

$$\frac{\partial c}{\partial t} + \frac{q}{\phi}\frac{\partial c}{\partial x} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) \tag{2.270}$$

subject to the initial condition

$$c(x, y, 0) = 0 \quad \text{for } x, y > 0, \tag{2.271}$$

and the boundary conditions

$$\begin{aligned} c(0, y, t) &= g(y) && \text{for } t > 0,\\ \lim_{x\to\infty} c(x, y, t) &= 0 && \text{for } t > 0,\\ \lim_{y\to\infty} c(x, y, t) &= 0 && \text{for } t > 0,\\ \lim_{y\to-\infty} c(x, y, t) &= 0 && \text{for } t > 0. \end{aligned} \tag{2.272}$$

Applying the Laplace transform with respect to t yields the differential equation

$$s\,\bar{c} + \frac{q}{\phi}\frac{\partial \bar{c}}{\partial x} = D\left(\frac{\partial^2 \bar{c}}{\partial x^2} + \frac{\partial^2 \bar{c}}{\partial y^2}\right), \tag{2.273}$$

where $\bar{c}$ is the Laplace transform of c, and s is the transformation parameter. The boundary conditions become

$$\begin{aligned} \bar{c}(0, y, s) &= \frac{g(y)}{s},\\ \lim_{x\to\infty} \bar{c}(x, y, s) &= 0,\\ \lim_{y\to\infty} \bar{c}(x, y, s) &= 0,\\ \lim_{y\to-\infty} \bar{c}(x, y, s) &= 0. \end{aligned} \tag{2.274}$$

Applying next the Fourier transform with respect to y yields the ordinary differential equation

$$D\bar{C}'' - \frac{q}{\phi}\bar{C}' - (s + Dr^2)\bar{C} = 0, \tag{2.275}$$

where $\bar{C}$ is the Fourier transform of $\bar{c}$, r is the Fourier transformation parameter, and the prime denotes the derivative with respect to x. The boundary conditions read

$$\begin{aligned} \bar{C}(0, r, s) &= \frac{G(r)}{s},\\ \lim_{x\to\infty} \bar{C}(x, r, s) &= 0, \end{aligned} \tag{2.276}$$

where $G(r)$ is the Fourier transform of $g(y)$. The ordinary differential equation above has to be solved with respect to the twofold transformed boundary conditions.

This yields

$$\bar{C}(x,r,s) = \frac{G(r)}{s}\exp\left[x\left(\frac{q}{2\phi D} - \sqrt{(\frac{q}{2\phi D})^2 + \frac{s}{D} + r^2}\right)\right]. \tag{2.277}$$

The solution in the x, y, t domain will be obtained from their transforms, the invers Laplace transformation is carried out first. Knowing (e.g. Abramowitz and Stegun [9]) the inverse Laplace transform

$$L^{-1}\{\exp(-x\sqrt{s/D})\} = \frac{x\ \exp(-x^2/(4Dt))}{2(\pi D t^3)^{1/2}}, \tag{2.278}$$

it follows with the aid of the property on substitution

$$\begin{aligned} L^{-1}&\left\{\exp\left(-x\sqrt{\frac{1}{D}[(\frac{1}{4D}(\frac{q}{\phi})^2 + Dr^2) + s]}\right)\right\} \\ &= \frac{x\exp(-x^2/(4Dt))}{2(\pi D t^3)^{1/2}}\exp\left(-[\frac{1}{4D}(\frac{q}{\phi})^2 + Dr^2]\,t\right). \end{aligned} \tag{2.279}$$

The Fourier transform $C(x,r,t)$ of the solute concentration is thus obtained by the convolution theorem

$$C(x,r,t) = \frac{x}{(4\pi D)^{1/2}}\int_0^t \frac{G(r)\exp(-Dr^2t')}{(t')^{3/2}}\exp\left(-\frac{(x-t'q/\phi)^2}{4Dt'}\right)dt'. \tag{2.280}$$

The last step of the solution procedure is the application of the invers Fourier transform. $G(r)$ has the invers $g(y)$, and knowing the invers Fourier transform

$$F^{-1}\{\exp(-Dr^2t')\} = \frac{\exp(-y^2/(4Dt'))}{(2Dt')^{1/2}} \tag{2.281}$$

the convolution theorem of the Fourier transformation yields

$$\begin{aligned} F^{-1}\{G(r)\exp(-Dr^2t')\} &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{g(v)}{(2Dt')^{1/2}}\exp(-\frac{(y-v)^2}{4Dt'})\,dv \\ &= \int_{-b}^{-a}\frac{(2\pi)^{-1/2}}{(2Dt')^{1/2}}\frac{b+v}{b-a}\exp(-\frac{(y-v)^2}{4Dt'})\,dv \\ &+ \int_{-a}^{a}\frac{(2\pi)^{-1/2}}{(2Dt')^{1/2}}\exp(-\frac{(y-v)^2}{4Dt'})\,dv \\ &+ \int_{a}^{b}\frac{(2\pi)^{-1/2}}{(2Dt')^{1/2}}\frac{b-v}{b-a}\exp(-\frac{(y-v)^2}{4Dt'})\,dv. \end{aligned} \tag{2.282}$$

The integrals involved may be evaluated by elementary analytical methods. The solution $c(x, y, t)$ of the 2D transport problem takes the form

$$
\begin{aligned}
c(x,y,t) = {} & \frac{x}{4(\pi D)^{1/2}} \cdot \int_0^t \exp\Big(-\frac{(x - t'q/\phi)^2}{4Dt'}\Big) \\
& \times \Big\{ \Big[\mathrm{erf}\Big(\frac{b+y}{(4Dt')^{1/2}}\Big) - \mathrm{erf}\Big(\frac{a+y}{(4Dt')^{1/2}}\Big)\Big]\frac{b+y}{b-a} \\
& + \Big[\exp\Big(\frac{-(b+y)^2}{4Dt'}\Big) - \exp\Big(\frac{-(a+y)^2}{4Dt'}\Big)\Big]\frac{(4Dt')^{1/2}\pi^{-1/2}}{b-a} \\
& + \Big[\mathrm{erf}\Big(\frac{a+y}{(4Dt')^{1/2}}\Big) + \mathrm{erf}\Big(\frac{a-y}{(4Dt')^{1/2}}\Big)\Big] \\
& + \Big[\mathrm{erf}\Big(\frac{b-y}{(4Dt')^{1/2}}\Big) - \mathrm{erf}\Big(\frac{a-y}{(4Dt')^{1/2}}\Big)\Big]\frac{b-y}{b-a} \\
& + \Big[\exp\Big(\frac{-(b-y)^2}{4Dt'}\Big) - \exp\Big(\frac{-(a-y)^2}{4Dt'}\Big)\Big]\frac{(4Dt')^{1/2}\pi^{-1/2}}{b-a} \Big\} t'^{-3/2} dt'.
\end{aligned}
\tag{2.283}
$$

The remaining integral was evaluated numerically, the Romberg integration scheme may conveniently be employed. For the numerical evaluation of the error function see [4].

2.6 Hydrothermal Processes

Heat transport in a moving liquid is the subject of this section. Closed form solutions may be obtained from corresponding mass transport problems. We present two examples, for the underlying theory see Bear [10].

2.6.1 A Transient 1D Temperature Distribution in a Moving Liquid

The domain is a rectanglular beam of length $L = 10\,\mathrm{m}$ extending along the positive x-axis. It is discretized by $100 \times 1 \times 1$ equally sized hexahedral elements. A permeable material represents the porous medium, with isotropic permeability $k = 10^{-11}\,\mathrm{m}^2$ and porosity $\phi = 0.1$. The liquid is incompressible and has viscosity $\mu = 1\,\mathrm{mPa \cdot s}$. Densities, heat capacities, and thermal conductivities of liquid and solid grain are given below, gravity has explicitly been neglected (Table 2.2).

Pressure $p_0 = 10^5\,\mathrm{Pa}$ at the liquid inlet ($x = 0\,\mathrm{m}$) and zero pressure the liquid outlet ($x = L$) generate steady-state 1D flow along the x-axis. At the liquid inlet a constant temperature $T_0 = 10\,°\mathrm{C}$ is specified for times $t > 0$. Starting from zero initial temperature the simulation evaluates the transient temperature distribution $T(x, t)$ with output after 10,000 and 20,000 s (Fig. 2.17).

Table 2.2 Example overview

	Liquid	Solid grain
Density	$\rho_l = 1000\ \text{kg/m}^3$	$\rho_s = 2000\ \text{kg/m}^3$
Specific heat capacity	$c_l = 1100\ \text{J/(kg}\cdot\text{K)}$	$c_s = 250\ \text{J/(kg}\cdot\text{K)}$
Thermal conductivity	$\lambda_l = 10\ \text{W/(m}\cdot\text{K)}$	$\lambda_s = 50\ \text{W/(m}\cdot\text{K)}$

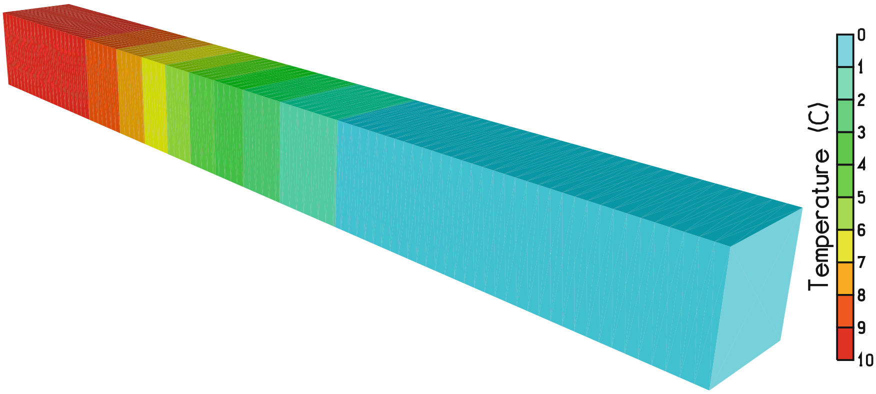

Fig. 2.17 Temperature distribution after 20,000 s

The formal solution proceeds in two steps, first to solve for pressure $p(x)$ and specific discharge q and second to determine the temperature distribution.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{\partial^2 p}{\partial x^2} = 0 \tag{2.284}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = p_0 \left(1 - \frac{x}{L}\right), \tag{2.285}$$

and the specific discharge q is obtained by Darcy's law

$$q = \frac{k}{\mu}\frac{p_0}{L}. \tag{2.286}$$

We will next focus on the closed form solution of the heat transport problem. Based on the setup of the present example the heat transport equation reads

$$(\phi\rho_l c_l + (1-\phi)\rho_s c_s)\frac{\partial T}{\partial t} + (\phi\rho_l c_l)\frac{q}{\phi}\frac{\partial T}{\partial x} = (\phi\lambda_l + (1-\phi)\lambda_s)\frac{\partial^2 T}{\partial x^2}. \tag{2.287}$$

Introducing the notation

$$w = \frac{\phi\rho_l c_l}{\phi\rho_l c_l + (1-\phi)\rho_s c_s}\frac{q}{\phi}, \qquad (2.288)$$
$$\chi = \frac{\phi\lambda_l + (1-\phi)\lambda_s}{\phi\rho_l c_l + (1-\phi)\rho_s c_s},$$

the heat transport equation becomes

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial x} = \chi\frac{\partial^2 T}{\partial x^2}. \qquad (2.289)$$

Due to free outflow at $x = L$ the formal problem is to determine the solution $T(x,t)$ of the above heat transport equation subject to the initial condition

$$T(x,0) = 0 \quad \text{for } x > 0, \qquad (2.290)$$

and the boundary conditions

$$\begin{aligned} T(0,t) &= T_0 \quad \text{for } t > 0, \\ \lim_{x\to\infty} T(x,t) &= 0 \quad \text{for } t > 0. \end{aligned} \qquad (2.291)$$

Applying the Laplace transform with respect to t yields the ordinary differential equation

$$\chi\bar{T}'' - w\bar{T}' - s\bar{T} = 0, \qquad (2.292)$$

where $\bar{T}$ is the transform of T, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to transformed boundary conditions. This yields

$$\bar{T}(x,s) = \frac{T_0}{s}\exp\left[x\left(\frac{w}{2\chi} - \sqrt{(\frac{w}{2\chi})^2 + \frac{s}{\chi}}\right)\right]. \qquad (2.293)$$

The solution may now be obtained from the transform of the temperature distribution, Churchill [7] outlines how to proceed with the aid of operational calculus. We note, that the present example is well suited for numerical inversion. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of the temperature distribution $T(x,t)$ (Fig. 2.17).

Table 2.3 Example overview

	Liquid	Solid grain
Density	$\rho_l = 1000\ \mathrm{kg/m^3}$	$\rho_s = 2000\ \mathrm{kg/m^3}$
Specific heat capacity	$c_l = 1100\ \mathrm{J/(kg \cdot K)}$	$c_s = 250\ \mathrm{J/(kg \cdot K)}$
Thermal conductivity	$\lambda_l = 0.5\ \mathrm{W/(m \cdot K)}$	$\lambda_s = 2.0\ \mathrm{W/(m \cdot K)}$

2.6.2 A Transient 2D Temperature Distribution in a Moving Liquid

Given length $L = 1\,\mathrm{m}$ the domain represents the rectangle $[0, 2L] \times [-0.75L, 0.75L]$ located in the x-y-plane and subdivided into $80 \times 60 \times 1$ cubic elements. A permeable material represents the porous medium, with isotropic permeability $k = 10^{-11}\,\mathrm{m}^2$ and porosity $\phi = 0.1$. The liquid is incompressible and has viscosity $\mu = 1\,\mathrm{mPa \cdot s}$. Densities, heat capacities, and thermal conductivities of liquid and solid grain are given below, gravity has explicitly been neglected (Table 2.3).

Pressure $p_0 = 2 \times 10^4\,\mathrm{Pa}$ at the liquid inlet ($x = 0\,\mathrm{m}$) and zero pressure the liquid outlet ($x = 2L$) generate steady-state 1D flow along the x-axis. At the liquid inlet a non-zero temperature is specified along a line segment of the y-axis. Given temperature $T_0 = 10\,°\mathrm{C}$ as well as $a = 0.15\,\mathrm{m}$ and $b = 0.25\,\mathrm{m}$ the specified inlet temperature reads

$$g(y) = \begin{cases} 0 & \text{for } y \le -b, \\ T_0 \dfrac{b+y}{b-a} & \text{for } -b \le y \le -a, \\ T_0 & \text{for } -a \le y \le a, \\ T_0 \dfrac{b-y}{b-a} & \text{for } a \le y \le b, \\ 0 & \text{for } b \le y. \end{cases} \tag{2.294}$$

Starting from zero initial temperature the simulation evaluates the transient temperature distribution $T(x, y, t)$ with output after 3,500 and 7,000 s (Fig. 2.18).

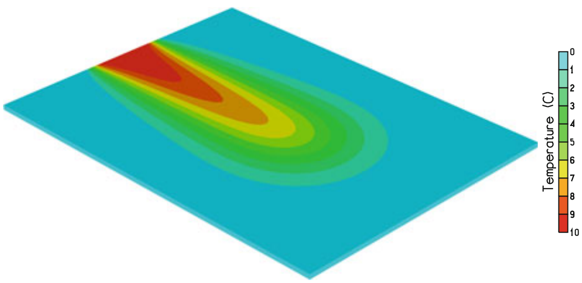

Fig. 2.18 Temperature distribution after 7,000 s

The formal solution proceeds in two steps, first to solve for pressure $p(x)$ and specific discharge q and second to determine the solute distributions.

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{d^2 p}{dx^2} = 0 \tag{2.295}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = p_0 \left(1 - \frac{x}{2L}\right), \tag{2.296}$$

and the specific discharge q is obtained by Darcy's law

$$q = \frac{k}{\mu} \frac{p_0}{2L}. \tag{2.297}$$

We will next focus on the closed form solution of the heat transport problem. Based on the setup of the present example the heat transport equation reads

$$\begin{aligned} (\phi\rho_l c_l + (1-\phi)\rho_s c_s)\frac{\partial T}{\partial t} + (\phi\rho_l c_l)\frac{q}{\phi}\frac{\partial T}{\partial x} \\ = (\phi\lambda_l + (1-\phi)\lambda_s)\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \end{aligned} \tag{2.298}$$

Introducing the notation

$$\begin{aligned} w &= \frac{\phi\rho_l c_l}{\phi\rho_l c_l + (1-\phi)\rho_s c_s}\frac{q}{\phi}, \\ \chi &= \frac{\phi\lambda_l + (1-\phi)\lambda_s}{\phi\rho_l c_l + (1-\phi)\rho_s c_s}, \end{aligned} \tag{2.299}$$

the heat transport equation becomes

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial x} = \chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{2.300}$$

The formal problem is to determine the solution $T(x, y, t)$ of the above heat transport equation subject to the initial condition

$$T(x, y, 0) = 0 \quad \text{for } x, y > 0, \tag{2.301}$$

and the boundary conditions

$$
\begin{aligned}
T(0, y, t) &= g(y) \quad \text{for } t > 0, \\
\lim_{x \to \infty} T(x, y, t) &= 0 \quad \text{for } t > 0, \\
\lim_{y \to \infty} T(x, y, t) &= 0 \quad \text{for } t > 0, \\
\lim_{y \to -\infty} T(x, y, t) &= 0 \quad \text{for } t > 0.
\end{aligned}
\tag{2.302}
$$

The closed form solution of this problem will be obtained with the aid of successive Laplace and Fourier transforms as described by Leij and Dane [8]. Applying the Laplace transform with respect to t yields the differential equation

$$
s\bar{T} + w\frac{\partial \bar{T}}{\partial x} = \chi\left(\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2}\right),
\tag{2.303}
$$

where $\bar{T}$ is the Laplace transform of T, and s is the transformation parameter. The boundary conditions become

$$
\begin{aligned}
\bar{T}(0, y, s) &= \frac{g(y)}{s}, \\
\lim_{x \to \infty} \bar{T}(x, y, s) &= 0, \\
\lim_{y \to \infty} \bar{T}(x, y, s) &= 0, \\
\lim_{y \to -\infty} \bar{T}(x, y, s) &= 0.
\end{aligned}
\tag{2.304}
$$

Applying next the Fourier transform with respect to y yields the ordinary differential equation

$$
\chi\bar{U}'' - w\bar{U}' - (s + \chi r^2)\bar{U} = 0,
\tag{2.305}
$$

where $\bar{U}$ is the Fourier transform of $\bar{T}$, r is the Fourier transformation parameter, and the prime denotes the derivative with respect to x. The boundary conditions read

$$
\begin{aligned}
\bar{U}(0, r, s) &= \frac{G(r)}{s}, \\
\lim_{x \to \infty} \bar{U}(x, r, s) &= 0,
\end{aligned}
\tag{2.306}
$$

where $G(r)$ is the Fourier transform of $g(y)$. The ordinary differential equation above has to be solved with respect to the twofold transformed boundary conditions. This yields

$$
\bar{U}(x, r, s) = \frac{G(r)}{s}\exp\left[x\left(\frac{w}{2\chi} - \sqrt{(\frac{w}{2\chi})^2 + \frac{s}{\chi} + r^2}\right)\right].
\tag{2.307}
$$

The solution in the x, y, t domain will be obtained from their transforms, the invers Laplace transformation is carried out first. Knowing (e.g. Abramowitz and Stegun [9]) the inverse Laplace transform

$$L^{-1}\{\exp(-x\sqrt{s/\chi})\} = \frac{x\ \exp(-x^2/(4\chi t))}{2(\pi\chi t^3)^{1/2}}, \tag{2.308}$$

it follows with the aid of the property on substitution

$$\begin{aligned} &L^{-1}\left\{\exp\left(-x\sqrt{\frac{1}{\chi}[(\frac{1}{4\chi}(w)^2+\chi r^2)+s]}\right)\right\} \\ &= \frac{x\exp(-x^2/(4\chi t))}{2(\pi\chi t^3)^{1/2}}\exp\left(-[\frac{1}{4\chi}(w)^2+\chi r^2]t\right). \end{aligned} \tag{2.309}$$

The Fourier transform $U(x, r, t)$ of the temperature is thus obtained by the convolution theorem

$$U(x, r, t) = \frac{x}{(4\pi\chi)^{1/2}}\int_0^t \frac{G(r)\exp(-\chi r^2 t')}{(t')^{3/2}}\exp\left(-\frac{(x-t'w)^2}{4\chi t'}\right)dt'. \tag{2.310}$$

The last step of the solution procedure is the application of the invers Fourier transform. $G(r)$ has the invers $g(y)$, and knowing the invers Fourier transform

$$F^{-1}\{\exp(-\chi r^2 t')\} = \frac{\exp(-y^2/(4\chi t'))}{(2\chi t')^{1/2}} \tag{2.311}$$

the convolution theorem of the Fourier transformation yields

$$\begin{aligned} F^{-1}\{G(r)\exp(-\chi r^2 t')\} &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{g(v)}{(2\chi t')^{1/2}}\exp(-\frac{(y-v)^2}{4\chi t'})\,dv \\ &= T_0\int_{-b}^{-a}\frac{(2\pi)^{-1/2}}{(2\chi t')^{1/2}}\frac{b+v}{b-a}\exp(-\frac{(y-v)^2}{4\chi t'})\,dv \\ &\quad + T_0\int_{-a}^{a}\frac{(2\pi)^{-1/2}}{(2\chi t')^{1/2}}\exp(-\frac{(y-v)^2}{4\chi t'})\,dv \\ &\quad + T_0\int_{a}^{b}\frac{(2\pi)^{-1/2}}{(2\chi t')^{1/2}}\frac{b-v}{b-a}\exp(-\frac{(y-v)^2}{4\chi t'})\,dv. \end{aligned} \tag{2.312}$$

The integrals involved may be evaluated by elementary analytical methods. Employing w and χ as defined above the solution $T(x, y, t)$ of the 2D heat transport problem takes the form

$$
\begin{aligned}
T(x,y,t) = \quad & \frac{T_0\,x}{4(\pi\chi)^{1/2}} \quad \cdot \quad \int_0^t \exp(-\frac{(x-t'w)^2}{4\chi t'}) \\
& \times \Big\{ [\mathrm{erf}(\frac{b+y}{(4\chi t')^{1/2}}) \;-\; \mathrm{erf}(\frac{a+y}{(4\chi t')^{1/2}})]\frac{b+y}{b-a} \\
& + [\exp(\frac{-(b+y)^2}{4\chi t'}) - \;\exp(\frac{-(a+y)^2}{4\chi t'})]\frac{(4\chi t')^{1/2}\pi^{-1/2}}{b-a} \\
& + [\mathrm{erf}(\frac{a+y}{(4\chi t')^{1/2}}) \;+\; \mathrm{erf}(\frac{a-y}{(4\chi t')^{1/2}})] \\
& + [\mathrm{erf}(\frac{b-y}{(4\chi t')^{1/2}}) \;-\; \mathrm{erf}(\frac{a-y}{(4\chi t')^{1/2}})]\frac{b-y}{b-a} \\
& + [\exp(\frac{-(b-y)^2}{4\chi t'}) - \;\exp(\frac{-(a-y)^2}{4\chi t'})]\frac{(4\chi t')^{1/2}\pi^{-1/2}}{b-a} \Big\} t'^{-3/2} dt'.
\end{aligned}
\tag{2.313}
$$

The remaining integral was evaluated numerically, the Romberg integration scheme may conveniently be employed. For the numerical evaluation of the error function see [4].

2.7 Hydromechanical Coupling

The presence of a fluid pressure affects the mechanical load on the porous matrix. This interaction constitutes the subject of Biot's theory, see Biot [11] or see Jaeger and Cook [6]. We note, that the setup of our time-dependent problems has been adopted from Kolditz et al. [12].

2.7.1 A Permeable Elastic Beam Deforms Under Steady-State Internal Liquid Pressure

The domain is a rectangular beam of length $L = 1\,\mathrm{m}$ extending along the positive x-axis. It has three faces located on the coordinate planes and is discretized by $20 \times 2 \times 2$ cubic elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 25{,}000\,\mathrm{MPa}$, and Biot number equal one. An isotropic permeability of $10^{-12}\,\mathrm{m}^2$ and zero porosity is assumed for the material, liquid viscosity is $1\,\mathrm{mPa \cdot s}$ and gravity is neglected via zero material and liquid densities. The face $x = 0$ is free, all other faces of the beam are sliding planes. The simulation comprises one time step with pressure $p_1 = 1\,\mathrm{MPa}$ at $x = L$ and zero pressure applied at $x = 0$.

The formal solution proceeds in two steps, first to solve for the pressure distribution $p(x)$ and second to determine stresses, strains and displacements (Fig. 2.19).

The Laplace equation is the governing equation describing the steady-state pressure distribution. It reads

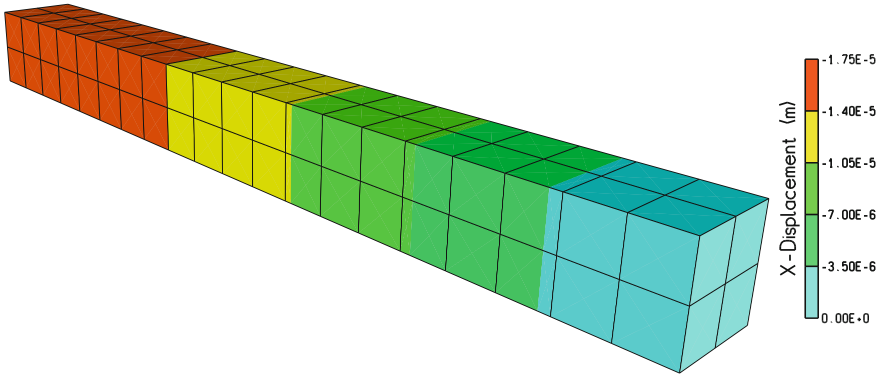

Fig. 2.19 X-Displacements

$$\frac{d^2p}{dx^2} = 0 \tag{2.314}$$

for 1D flow along the x-axis. The pressure distribution $p(x)$ is given by

$$p(x) = p_1 \cdot \frac{x}{L}, \tag{2.315}$$

which satisfies the specified pressure boundary conditions at $x = 0$ and $x = L$.

Let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}). \tag{2.316}$$

It is satisfied by zero shear, if the stresses σ_{22} and σ_{33} are functions of x only and the horizontal stress σ_{11} satisfies

$$\frac{d}{dx}(\sigma_{11} - p) = 0. \tag{2.317}$$

The face $x = 0$ is free of load, hence, integration gives

$$\sigma_{11} = p = p_1 \frac{x}{L}. \tag{2.318}$$

Due to the y- and z-fixities along the front, rear, top, and bottom, and with principal axes equal to coordinate axes, Hooke's law gives for the strains

$$\begin{aligned} E \cdot \epsilon_{11} &= \sigma_{11} - \nu\,(\sigma_{22} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{22} &= \sigma_{22} - \nu\,(\sigma_{11} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{33} &= \sigma_{33} - \nu\,(\sigma_{11} + \sigma_{22}), \end{aligned} \tag{2.319}$$

and therefore

$$\begin{aligned}\sigma_{22} = \sigma_{33} &= \frac{\nu}{1-\nu}\,\sigma_{11} = \frac{\nu}{1-\nu}\,p = \frac{p_1}{L}\frac{\nu}{1-\nu}x,\\ \epsilon_{11} &= \frac{1}{E}\left(1-\frac{2\nu^2}{1-\nu}\right)\sigma_{11} = \frac{p_1}{E\,L}\left(1-\frac{2\nu^2}{1-\nu}\right)x.\end{aligned} \tag{2.320}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned}u_x(x) &= \frac{p_1}{2\,E\,L}\left(1-\frac{2\nu^2}{1-\nu}\right)(x^2-L^2),\\ u_y &= 0,\\ u_z &= 0.\end{aligned} \tag{2.321}$$

2.7.2 A Permeable Elastic Square Deforms Under Constant Internal Liquid Pressure

The domain represents the unit square $[0, 1] \times [0, 1]$ in the x-y-plane. It has three faces located on the coordinate planes and is discretized by $5 \times 5 \times 2$ equally sized hexahedral elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 10{,}000\,\text{MPa}$, and Biot number equal one. An isotropic permeability of $10^{-11}\,\text{m}^2$ and zero porosity is assumed for the material, liquid viscosity is $1\,\text{mPa}\cdot\text{s}$ and gravity is neglected via zero material and liquid densities. Top and bottom as well as the lateral faces on the coordinate planes are sliding planes. The simulation comprises one time step applying a constant liquid pressure $p_0 = 0.8\,\text{MPa}$ at the bottom of the domain.

The formal solution proceeds in two steps, first to solve for the pressure distribution and then to evaluate stresses, strains and displacements. However, due to the setup the pressure p has constant value p_0 throughout the entire domain, and we will focus on the mechanical aspects of the problem.

Let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}). \tag{2.322}$$

It is satisfied by zero shear and constant stresses. Due to the setup

$$\sigma_{11} = \sigma_{22} = p_0 \tag{2.323}$$

and with principal axes equal to coordinate axes, Hooke's law gives for the strains

$$
\begin{aligned}
0 = \epsilon_{33} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] = \frac{1}{E}(\sigma_{33} - 2\nu \cdot p_0),\\
\epsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] = (1 - \nu - 2\nu^2)\frac{p_0}{E},\\
\epsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] = (1 - \nu - 2\nu^2)\frac{p_0}{E}.
\end{aligned}
\tag{2.324}
$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$
\begin{aligned}
u_x(x) &= (1 - \nu - 2\nu^2)\frac{p_0}{E}x,\\
u_y(y) &= (1 - \nu - 2\nu^2)\frac{p_0}{E}y,\\
u_z &= 0.
\end{aligned}
\tag{2.325}
$$

2.7.3 A Permeable Elastic Cube Deforms Under Constant Internal Liquid Pressure

The domain is a cube with an edge size of 1 m. It has three faces located on the coordinate planes and is discretized by $4 \times 4 \times 4$ cubic elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 10{,}000$ MPa, and Biot number equal one. An isotropic permeability of $10^{-10}\,\mathrm{m}^2$ and zero porosity is assumed for the material, liquid viscosity is $1\,\mathrm{mPa \cdot s}$ and gravity is neglected via zero material and liquid densities. The faces on the coordinate planes are sliding planes. The simulation comprises one time step applying a constant liquid pressure $p_0 = 20$ MPa at the top of the domain.

The formal solution proceeds in two steps, first to solve for the pressure distribution and then to evaluate stresses, strains and displacements. However, due to the setup the pressure p has constant value p_0 throughout the entire domain, and we will focus on the mechanical aspects of the problem.

Let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$
0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}). \tag{2.326}
$$

It is satisfied by zero shear and constant stresses. Due to the setup

$$
\sigma_{11} = \sigma_{22} = \sigma_{33} = p_0 \tag{2.327}
$$

and with principal axes equal to coordinate axes, Hooke's law gives for the strains

$$
\begin{aligned}
\epsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] = (1 - 2\nu)\frac{p_0}{E}, \\
\epsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] = (1 - 2\nu)\frac{p_0}{E}, \\
\epsilon_{33} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] = (1 - 2\nu)\frac{p_0}{E}.
\end{aligned}
\tag{2.328}
$$

Integrating the strains with respect to the fixities at the coordinate planes yields the displacement vector (u_x, u_y, u_z)

$$
\begin{aligned}
u_x(x) &= (1 - 2\nu)\frac{p_0}{E}x, \\
u_y(y) &= (1 - 2\nu)\frac{p_0}{E}y, \\
u_z(z) &= (1 - 2\nu)\frac{p_0}{E}z.
\end{aligned}
\tag{2.329}
$$

2.7.4 A Permeable Elastic Cuboid Undergoes Static Load Due to Gravity and Hydrostatic Liquid Pressure

The domain is a cuboid of height $H = 30\,\mathrm{m}$ and edges parallel to the x-y-z coordinate axes. It is discretized by an irregular mesh of hexahedral elements. The domain is composed of four groups of isotropic permeable materials with zero porosity. Liquid viscosity is $1\,\mathrm{mPa \cdot s}$ and $\rho_l = 1019.368\,\mathrm{kg/m^3}$ is the liquid density. Each of the material groups has been assigned solid density $\rho_s = 3058.104\,\mathrm{kg/m^3}$, Poisson's ratio $\nu = 0.25$, Young's modulus $E = 10{,}000\,\mathrm{MPa}$, and Biot number equal one. Zero pressure is applied at the top face $z = H$. This face is free, all other faces are sliding planes. The simulation comprises one time step to establish the hydrostatic pressure distribution as well as the mechanical load.

The formal solution proceeds in two steps, first to solve for the pressure distribution and second to determine stresses, strains, and displacements.

The simulation setup employs a prescribed zero pressure at the top ($z = H$), therefore the pressure distribution is hydrostatic, does not depend on the coordinates x and y and is given by

$$
p(z) = \rho_l\, g\,(H - z), \tag{2.330}
$$

where $g = 9.81\,\mathrm{m/s^2}$ is the magnitude of gravity, and z is the vertical coordinate extending from 0 to H.

Let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$
0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}) - (0, 0, \rho_s\, g). \tag{2.331}
$$

It is satisfied by zero shear, if pressure p and the horizontal stresses σ_{11} and σ_{22} are functions of the vertical coordinate z only and the vertical stress σ_{33} satisfies

$$\frac{d\sigma_{33}}{dz} = \rho_s\, g + \frac{dp}{dz} = (\rho_s - \rho_l)\, g. \tag{2.332}$$

The face $z = H$ is free, hence, integration gives

$$\sigma_{33} = (\rho_s - \rho_l)\,(-g)\,(H - z). \tag{2.333}$$

Assuming that there is no horizontal displacement anywhere we have for the horizontal strains

$$\epsilon_{11} = \epsilon_{22} = 0. \tag{2.334}$$

Then, with principal axes equal to coordinate axes, Hooke's law gives

$$\begin{aligned} 0 &= \sigma_{11} - \nu(\sigma_{22} + \sigma_{33}), \\ 0 &= \sigma_{22} - \nu(\sigma_{11} + \sigma_{33}), \\ E\,\epsilon_{33} &= \sigma_{33} - \nu(\sigma_{11} + \sigma_{22}). \end{aligned} \tag{2.335}$$

Solving for σ_{11}, σ_{22}, and the vertical strain ϵ_{33} yields

$$\begin{aligned} \sigma_{11} = \sigma_{22} &= \frac{\nu}{1-\nu}\sigma_{33} = \frac{\nu}{1-\nu}(\rho_s - \rho_l)(-g)(H-z), \\ \epsilon_{33} &= \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)(\rho_s - \rho_l)(-g)(H-z) \end{aligned} \tag{2.336}$$

in terms of the vertical coordinate. Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x &= u_y = 0, \\ u_z(z) &= \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)(\rho_s - \rho_l)(-g)\left(Hz - \frac{z^2}{2}\right). \end{aligned} \tag{2.337}$$

2.7.5 A Permeable Elastic Beam Deforms Under Transient Internal Liquid Pressure. Specified Boundary Conditions are Time-Dependent and of 1st Kind

The domain is a rectangular beam of length $L = 1\,\text{m}$ extending along the positive x-axis. It has three faces located on the coordinate planes and is discretized by hexahedral elements with section $0 <= x <= 0.6\,L$ composed of $10 \times 1 \times 1$ and section $0.6\,L <= x <= L$ composed of $60 \times 1 \times 1$ elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.2$, Young's modulus $E = 27{,}000\,\text{Pa}$, and Biot number equal one. An isotropic permeability $k = 10^{-10}\,\text{m}^2$ and zero porosity is assumed for the material, liquid viscosity is $\mu = 1\,\text{mPa}\cdot\text{s}$, and gravity is neglected via zero material and liquid densities. The face $x = 0$ is free, all other faces of the beam are sliding planes. Zero pressure has been specified at the face $x = 0$, the pressure $p_1 \cdot t$ ($p_1 = 100\,\text{Pa/s}$) increases linearly with time t, it is applied at the face $x = L$ for times $t > 0$. Starting from zero initial pressure the simulation evaluates the transient pressure distribution $p(x, t)$ as well as stresses, strains, and displacements with output after 5 and 10 s (Fig. 2.20).

Let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}). \tag{2.338}$$

It is satisfied by zero shear, if the stresses σ_{22} and σ_{33} are functions of x only and the horizontal stress σ_{11} satisfies

$$\frac{\partial}{\partial x}(\sigma_{11} - p) = 0. \tag{2.339}$$

With respect to prescribed boundary conditions at $x = 0\,\text{m}$ the last equation yields

$$\sigma_{11} = p. \tag{2.340}$$

Due to the y- and z-fixities along the front, rear, top, and bottom and with principal axes equal to coordinate axes, Hooke's law gives for the strains

$$\begin{aligned} E \cdot \epsilon_{11} &= \sigma_{11} - \nu\,(\sigma_{22} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{22} &= \sigma_{22} - \nu\,(\sigma_{11} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{33} &= \sigma_{33} - \nu\,(\sigma_{11} + \sigma_{22}). \end{aligned} \tag{2.341}$$

In terms of pressure $p(x, t)$ the non-zero stresses and strains take the form

$$\begin{aligned} \sigma_{22} = \sigma_{33} &= \frac{\nu}{1-\nu}\,\sigma_{11} = \frac{\nu}{1-\nu}\,p, \\ \epsilon_{11} &= \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\sigma_{11} = \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)p. \end{aligned} \tag{2.342}$$

Let (u_x, u_y, u_z) denote the displacement vector and q the specific discharge via Darcy's law

$$q = -\frac{k}{\mu}\nabla p. \tag{2.343}$$

Conservation of momentum yields

$$\nabla \cdot \left(q + \frac{\partial}{\partial t}(u_x, u_y, u_z)\right) = 0. \tag{2.344}$$

For the present 1D example this reduces to

$$\begin{aligned} 0 &= -\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial t}\frac{\partial u_x}{\partial x} \\ &= -\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2} + \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{\partial p}{\partial t} \end{aligned} \tag{2.345}$$

the 1D pressure conduction equation, which has to be solved subject to the initial condition

$$p(x,0) = 0 \qquad \text{for } 0 \le x \le L, \tag{2.346}$$

and the boundary conditions

$$\begin{aligned} p(0,t) &= 0 \qquad \text{for } t > 0, \\ p(L,t) &= p_1\, t \quad \text{for } t > 0, \end{aligned} \tag{2.347}$$

arising from the problem setup. Once that the pressure distribution $p(x,t)$ has been found, the remaining non-zero stresses and strains may be obtained.

It is therefore sufficient to solve the pressure conduction equation with respect to the imposed initial and boundary conditions. Introducing the notation

$$\chi = \frac{kE}{\mu} \bigg/ \left(1 - \frac{2\nu^2}{1-\nu}\right) \tag{2.348}$$

the formal problem is to determine the solution $p(x,t)$ of the parabolic equation

$$\frac{\partial p}{\partial t} = \chi \frac{\partial^2 p}{\partial x^2} \tag{2.349}$$

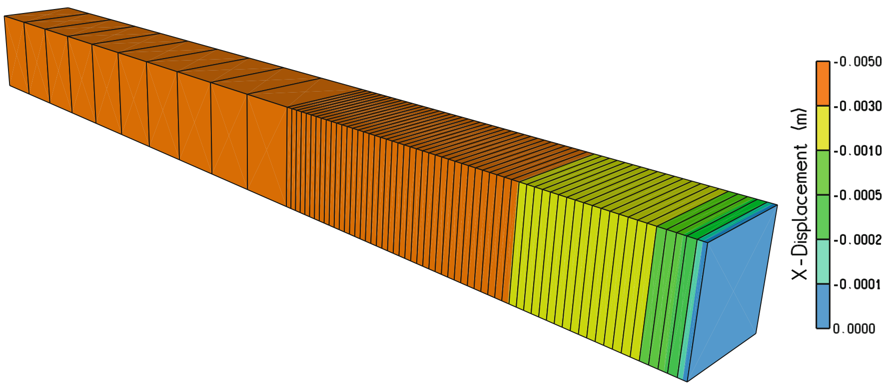

Fig. 2.20 X-Displacements after 10 s

subject to the initial and boundary conditions cited above. Applying the Laplace transform with respect to t yields the ordinary differential equation

$$\chi\, \bar{p}'' - s\, \bar{p} = 0, \tag{2.350}$$

where $\bar{p}$ is the transform of p, the prime denotes the derivative with respect to x, and s is the transformation parameter. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{p}(x, s) = p_1 \frac{\sinh(\sqrt{s/\chi}\, x)}{s^2 \sinh(\sqrt{s/\chi}\, L)}. \tag{2.351}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z), and this also holds for the Laplace transforms. Because $\sigma_{11} = p$ the transform of the only non-zero displacement $u_x(x, t)$ becomes

$$\bar{u}_x(x, s) = \frac{p_1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right) \frac{\cosh(\sqrt{s/\chi}\, x) - \cosh(\sqrt{s/\chi}\, L)}{s^2 \sqrt{s/\chi}\, \sinh(\sqrt{s/\chi}\, L)}. \tag{2.352}$$

The entire solution may now be obtained from the transforms of pressure and x-displacement. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of $p(x, t)$ and $u_x(x, t)$ (Fig. 2.20).

2.7.6 A Permeable Elastic Beam Deforms Under Transient Internal Liquid Pressure. Specified Boundary Conditions are Time-Dependent and of 1st and 2nd Kind

The domain is a rectangular beam of length $L = 1\,\text{m}$ extending along the positive x-axis. It has three faces located on the coordinate planes and is discretized by hexahedral elements with section $0 <= x <= 0.6\,L$ composed of $10 \times 1 \times 1$ and section $0.6\,L <= x <= L$ composed of $60 \times 1 \times 1$ elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.2$, Young's modulus $E = 27{,}000\,\text{Pa}$, and Biot number equal one. An isotropic permeability $k = 10^{-10}\,\text{m}^2$ and zero porosity is assumed for the material, liquid viscosity is $\mu = 1\,\text{mPa}\cdot\text{s}$, and gravity is neglected via zero material and liquid densities. The face $x = 0$ is free, all other faces of the beam are sliding planes. Zero pressure has been specified at the face $x = 0$, the specific discharge $q_1 \cdot t$ ($q_1 = 7.6 \times 10^{-5}\,\text{m/s}^2$) increases linearly with time t, it is applied at the face $x = L$ for times $t > 0$ and acts as a source to the domain. Starting from zero initial pressure the simulation evaluates the transient pressure distribution $p(x, t)$ as well as stresses, strains, and displacements with output after 5 and 10 s (Fig. 2.21).

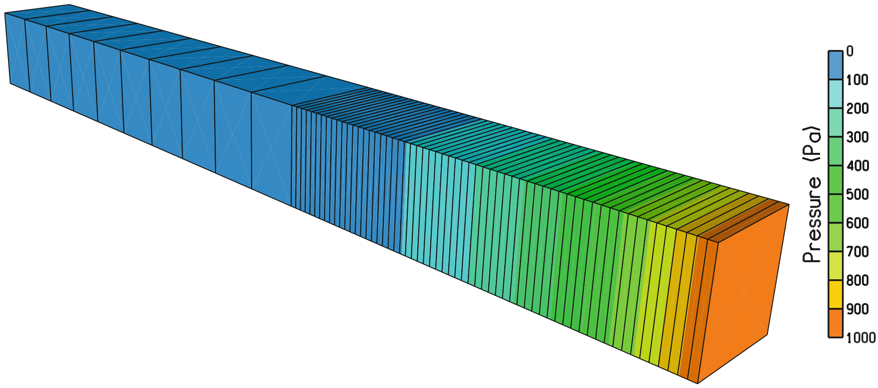

Fig. 2.21 Pressure distribution after 10 s

Let $\boldsymbol{\sigma}$ denote the stress tensor and **I** the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}). \tag{2.353}$$

It is satisfied by zero shear, if the stresses σ_{22} and σ_{33} are functions of x only and the horizontal stress σ_{11} satisfies

$$\frac{\partial}{\partial x}(\sigma_{11} - p) = 0. \tag{2.354}$$

With respect to prescribed boundary conditions at $x = 0\,\mathrm{m}$ the last equation yields

$$\sigma_{11} = p. \tag{2.355}$$

Due to the y- and z-fixities along the front, rear, top, and bottom and with principal axes equal to coordinate axes, Hooke's law gives for the strains

$$\begin{aligned} E \cdot \epsilon_{11} &= \sigma_{11} - \nu\,(\sigma_{22} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{22} &= \sigma_{22} - \nu\,(\sigma_{11} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{33} &= \sigma_{33} - \nu\,(\sigma_{11} + \sigma_{22}). \end{aligned} \tag{2.356}$$

In terms of pressure $p(x, t)$ the non-zero stresses and strains take the form

$$\begin{aligned} \sigma_{22} = \sigma_{33} &= \frac{\nu}{1-\nu}\,\sigma_{11} = \frac{\nu}{1-\nu}\,p, \\ \epsilon_{11} &= \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\sigma_{11} = \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)p. \end{aligned} \tag{2.357}$$

Let (u_x, u_y, u_z) denote the displacement vector and q the specific discharge via Darcy's law

$$q = -\frac{k}{\mu}\nabla p. \tag{2.358}$$

Conservation of momentum yields

$$\nabla \cdot (q + \frac{\partial}{\partial t}(u_x, u_y, u_z)) = 0. \tag{2.359}$$

For the present 1D example this reduces to

$$\begin{aligned} 0 &= -\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial t}\frac{\partial u_x}{\partial x} \\ &= -\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2} + \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{\partial p}{\partial t} \end{aligned} \tag{2.360}$$

the 1D pressure conduction equation, which has to be solved subject to the initial condition

$$p(x, 0) = 0 \qquad \text{for } 0 \le x \le L, \tag{2.361}$$

and the boundary conditions

$$\begin{aligned} p(0, t) &= 0 && \text{for } t > 0, \\ \frac{\partial p}{\partial x}(L, t) &= q_1 \frac{\mu}{k} t && \text{for } t > 0, \end{aligned} \tag{2.362}$$

arising from the problem setup. Once that the pressure distribution $p(x, t)$ has been found, the remaining non-zero stresses and strains may be obtained.

It is therefore sufficient to solve the pressure conduction equation with respect to the imposed initial and boundary conditions. Introducing the notation

$$\chi = \frac{kE}{\mu} \Big/ \left(1 - \frac{2\nu^2}{1-\nu}\right) \tag{2.363}$$

the formal problem is to determine the solution $p(x, t)$ of the parabolic equation

$$\frac{\partial p}{\partial t} = \chi \frac{\partial^2 p}{\partial x^2} \tag{2.364}$$

subject to the initial and boundary conditions cited above. Applying the Laplace transform with respect to t yields the ordinary differential equation

$$\chi\, \bar{p}'' - s\, \bar{p} = 0, \tag{2.365}$$

where $\bar{p}$ is the transform of p, the prime denotes the derivative with respect to x, and s is the transformation parameter. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{p}(x, s) = q_1 \frac{\mu}{k} \frac{\sinh(\sqrt{s/\chi}\; x)}{s^2 \sqrt{s/\chi}\; \cosh(\sqrt{s/\chi}\; L)}. \tag{2.366}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z), and this also holds for the Laplace transforms. Because $\sigma_{11} = p$ the transform of the only non-zero displacement $u_x(x, t)$ becomes

$$\bar{u}_x(x, s) = q_1 \frac{\cosh(\sqrt{s/\chi}\, x) - \cosh(\sqrt{s/\chi}\, L)}{s^3 \cosh(\sqrt{s/\chi}\, L)}. \tag{2.367}$$

The entire solution may now be obtained from the transforms of pressure and x-displacement. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of $p(x, t)$ and $u_x(x, t)$.

2.7.7 Biot's 1D Consolidation Problem: Squeezing of a Pressurized Column Causes the Liquid to Discharge from the Domain

The domain is a rectangular beam of length $L = 1\,\text{m}$ extending along the positive x-axis. It has three faces located on the coordinate planes and is discretized by hexahedral elements with section $0 <= x <= 0.6\,L$ composed of $10 \times 1 \times 1$ and section $0.6\,L <= x <= L$ composed of $60 \times 1 \times 1$ elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.2$, Young's modulus $E = 30{,}000\,\text{Pa}$, and Biot number equal one. An isotropic permeability $k = 10^{-10}\,\text{m}^2$ and zero porosity is assumed for the material, liquid viscosity is $\mu = 1\,\text{mPa} \cdot \text{s}$ and gravity is neglected via zero material and liquid densities. Except from the face $x = L$ all faces of the beam are sliding planes. At the face $x = L$ pressure and mechanical boundary conditions have explicitly been assigned: a compressive stress of 1,000 Pa acts in negative x-direction and pressure is assigned zero for times $t > 0$. Starting from initial equilibrium, i.e. pressure $p_i = 1{,}000\,\text{Pa}$ and zero mechanical stress, the simulation evaluates the transient pressure distribution $p(x, t)$ as well as stresses, strains, and displacements with output after 5 and 10 s (Fig. 2.22).

Let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}). \tag{2.368}$$

It is satisfied by zero shear, if the stresses σ_{22} and σ_{33} are functions of x only and the horizontal stress σ_{11} satisfies

$$\frac{\partial}{\partial x}(\sigma_{11} - p) = 0. \tag{2.369}$$

With respect to the initial and prescribed boundary conditions at $x = L$ the last equation yields

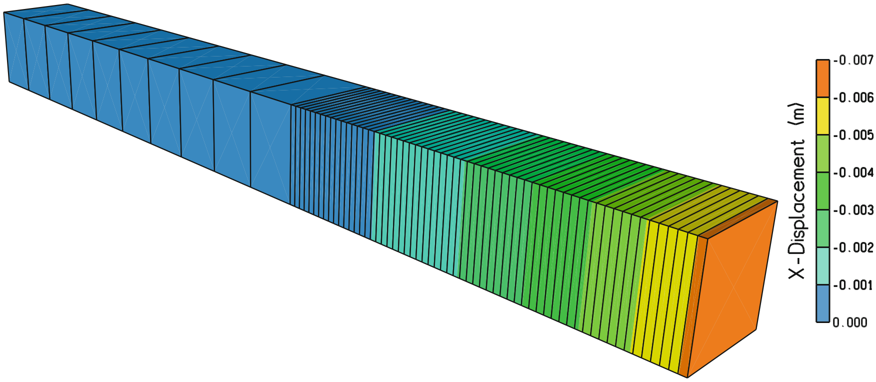

Fig. 2.22 X-Displacements after 10 s

$$p = \sigma_{11} + p_i. \tag{2.370}$$

Due to the y- and z-fixities along the front, rear, top, and bottom and with principal axes equal to coordinate axes, Hooke's law gives for the strains

$$\begin{aligned} E \cdot \epsilon_{11} &= \sigma_{11} - \nu\,(\sigma_{22} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{22} &= \sigma_{22} - \nu\,(\sigma_{11} + \sigma_{33}), \\ 0 = E \cdot \epsilon_{33} &= \sigma_{33} - \nu\,(\sigma_{11} + \sigma_{22}). \end{aligned} \tag{2.371}$$

In terms of the stress $\sigma_{11}(x, t)$ the remaining non-zero stresses and strains take the form

$$\begin{aligned} \sigma_{22} = \sigma_{33} &= \frac{\nu}{1-\nu}\sigma_{11}, \\ \epsilon_{11} &= \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\sigma_{11}. \end{aligned} \tag{2.372}$$

Let (u_x, u_y, u_z) denote the displacement vector and q the specific discharge via Darcy's law

$$q = -\frac{k}{\mu}\nabla p. \tag{2.373}$$

Conservation of momentum yields

$$\nabla \cdot (q + \frac{\partial}{\partial t}(u_x, u_y, u_z)) = 0. \tag{2.374}$$

For the present 1D example this reduces to

$$\begin{aligned} 0 &= -\frac{k}{\mu}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial t}\frac{\partial u_x}{\partial x} \\ &= -\frac{k}{\mu}\frac{\partial^2 \sigma_{11}}{\partial x^2} + \frac{1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{\partial \sigma_{11}}{\partial t} \end{aligned} \tag{2.375}$$

the 1D pressure conduction equation, which has to be solved subject to the initial condition

$$\sigma_{11}(x, 0) = 0 \quad \text{for } \; 0 \le x \le L, \tag{2.376}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x}(0, t) &= \;\;\, 0 \qquad \text{for } t > 0, \\ \sigma_{11}(L, t) \;\; &= -p_i \qquad \text{for } t > 0, \end{aligned} \tag{2.377}$$

arising from the problem setup. Once that the stress $\sigma_{11}(x, t)$ has been found, the pressure and the remaining non-zero stresses and strains may be obtained.

It is therefore sufficient to solve the pressure conduction equation with respect to the imposed initial and boundary conditions. Introducing the notation

$$\chi = \frac{kE}{\mu} \Big/ \left(1 - \frac{2\nu^2}{1-\nu}\right) \tag{2.378}$$

the formal problem is to determine the solution $\sigma_{11}(x, t)$ of the parabolic equation

$$\frac{\partial \sigma_{11}}{\partial t} = \chi \frac{\partial^2 \sigma_{11}}{\partial x^2} \tag{2.379}$$

subject to the initial and boundary conditions cited above. Applying the Laplace transform with respect to t yields the ordinary differential equation

$$\chi\, \bar{\sigma}_{11}'' - s\, \bar{\sigma}_{11} = 0, \tag{2.380}$$

where $\bar{\sigma}_{11}$ is the transform of σ_{11}, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{\sigma}_{11}(x, s) = -p_i \frac{\cosh(\sqrt{s/\chi}\, x)}{s\, \cosh(\sqrt{s/\chi}\, L)}. \tag{2.381}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z), and this also holds for the Laplace transforms. The transform of the only non-zero displacement $u_x(x, t)$ becomes

$$\bar{u}_x(x,s) = -\frac{p_i}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{\sinh(\sqrt{s/\chi}\,x)}{s\,\sqrt{s/\chi}\,\cosh(\sqrt{s/\chi}\,L)}. \tag{2.382}$$

The entire solution may now be obtained from the transforms of stress σ_{11} and x-displacement. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of $\sigma_{11}(x,t)$ and $u_x(x,t)$.

2.8 Thermomechanics

Temperature changes cause thermal strains affecting the mechanical load. The formal solution of the subsequent examples always proceeds in two steps: first to solve for the temperature, and then to evaluate stresses, strains, and displacements. Various ideas already outlined in previous sections will appear again.

2.8.1 An Elastic Beam Deforms Due to an Instant Temperature Change

The domain is a rectangular beam of length $L = 1$ m extending along the positive x-axis. It has three faces located on the coordinate planes and is discretized by $20 \times 2 \times 2$ cubic elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 25{,}000$ MPa, zero heat capacity, and thermal expansion $\alpha = 3 \times 10^{-5}$ 1/K. Gravity is neglected via zero material density. The face $x = 0$ is free, all other faces of the beam are sliding planes. The simulation starts from the initial temperature $T_0 = 0\,°$C and comprises one time step applying an instant temperature change with temperature $T_1 = 1\,°$C at $x = L$ and zero temperature T_0 at $x = 0$.

The formal solution proceeds in two steps, first to solve for the temperature distribution $T(x)$ and then to evaluate stresses, strains, and displacements (Fig. 2.23).

The Laplace equation is the governing equation describing the steady-state temperature distribution. It reads

$$\frac{d^2T}{dx^2} = 0 \tag{2.383}$$

for 1D heat flow along the x-axis, hence, the temperature distribution is given by

$$T(x) = (T_1 - T_0)\frac{x}{L} + T_0. \tag{2.384}$$

For the closed form solution of the mechanical problem note, that due to the simulation setup, the entire system is free of shear and the principal axes coincide with the coordinate axes. The constitutive equations relate the strains ϵ_{11}, ϵ_{22}, ϵ_{33}, (in x-, y-,

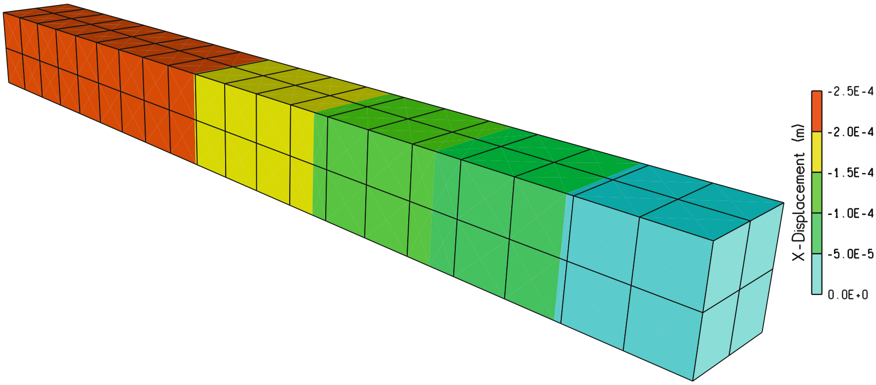

Fig. 2.23 X-Displacements

and z-direction, respectively) and the associated stresses σ_{11}, σ_{22}, and σ_{33} via

$$\begin{aligned}
\epsilon_{11} - \alpha(T(x) - T_0) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})],\\
\epsilon_{22} - \alpha(T(x) - T_0) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})], \qquad (2.385)\\
\epsilon_{33} - \alpha(T(x) - T_0) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})].
\end{aligned}$$

Due to the setup the x-direction is free of stress, y- and z-direction are free of strain, therefore,

$$\begin{aligned}
\sigma_{11} &= 0,\\
\epsilon_{22} = \epsilon_{33} &= 0. \qquad (2.386)
\end{aligned}$$

Hence, due to the change in temperature, the remaining non-zero stresses and strains become

$$\begin{aligned}
\sigma_{22} &= -\alpha\,\frac{E}{1-\nu}\,(T(x) - T_0),\\
\sigma_{33} &= -\alpha\,\frac{E}{1-\nu}\,(T(x) - T_0), \qquad (2.387)\\
\epsilon_{11} &= \alpha\,\frac{1+\nu}{1-\nu}\,(T(x) - T_0).
\end{aligned}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$
\begin{aligned}
u_x(x) &= \alpha \frac{1+\nu}{1-\nu} \frac{T_1 - T_0}{2L} (x^2 - L^2), \\
u_y &= u_z = 0.
\end{aligned}
\tag{2.388}
$$

2.8.2 An Elastic Square Deforms Due to an Instant Temperature Change

The domain represents the unit square $[0, 1] \times [0, 1]$ in the x-y-plane. It has three faces located on the coordinate planes and is discretized by $5 \times 5 \times 2$ equally sized hexahedral elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 25{,}000\,\mathrm{MPa}$, zero heat capacity, and thermal expansion $\alpha = 4 \times 10^{-5}$ 1/K. Gravity is neglected via zero material density. Top and bottom as well as the lateral faces on the coordinate planes are sliding planes. The simulation starts from the initial temperature $T_0 = 0\,°\mathrm{C}$ and comprises one time step applying an instant temperature change to $T_1 = 1\,°\mathrm{C}$ at the bottom of the domain.

The formal solution proceeds in two steps, first to solve for the temperature distribution and then to evaluate stresses, strains, and displacements. However, due to the setup the temperature change has constant value $T_1 - T_0$ throughout the entire domain, and we will focus on the mechanical aspects of the problem.

Due to the simulation setup the entire system is free of shear and the principal axes coincide with the coordinate axes. The constitutive equations relate the strains $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}$, (in x-, y-, and z-direction, respectively) and the associated stresses σ_{11}, σ_{22}, σ_{33} via

$$
\begin{aligned}
\epsilon_{11} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] = 0, \\
\epsilon_{22} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] = 0, \\
\epsilon_{33} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] = 0.
\end{aligned}
\tag{2.389}
$$

Due to the setup x- and y-direction are free of stress, and the z-direction is free of strain, therefore,

$$
\begin{aligned}
\sigma_{11} &= \sigma_{22} = 0, \\
\epsilon_{33} &= 0.
\end{aligned}
\tag{2.390}
$$

Hence, due to the change in temperature from T_0 to T_1 the remaining non-zero strains and stresses become

$$
\begin{aligned}
\sigma_{33} &= -\alpha\,(T_1 - T_0)\,E, \\
\epsilon_{11} &= \epsilon_{22} = (1+\nu)\,\alpha\,(T_1 - T_0).
\end{aligned}
\tag{2.391}
$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x(x) &= (1+\nu)\,\alpha\,(T_1 - T_0)\,x, \\ u_y(y) &= (1+\nu)\,\alpha\,(T_1 - T_0)\,y, \\ u_z &= 0. \end{aligned} \tag{2.392}$$

2.8.3 An Elastic Cube Deforms Due to an Instant Temperature Change

The domain is a cube with an edge size of 1 m. It has three faces located on the coordinate planes and is discretized by $4 \times 4 \times 4$ cubic elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E =$ 25,000 MPa, zero heat capacity, and thermal expansion $\alpha = 5 \times 10^{-5}$ 1/K. Gravity is neglected via zero material density. The faces on the coordinate planes are sliding planes. The simulation starts from the initial temperature $T_0 = 0\,°\mathrm{C}$ and comprises one time step applying an instant temperature change to $T_1 = -40\,°\mathrm{C}$ at the top of the domain.

The formal solution proceeds in two steps, first to solve for the temperature distribution and then to evaluate stresses, strains, and displacements. However, due to the setup the temperature change has constant value $T_1 - T_0$ throughout the entire domain, and we will focus on the mechanical aspects of the problem.

Due to the simulation setup the entire system is free of shear, and the principal axes coincide with the coordinate axes.

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0 \tag{2.393}$$

are the principal stresses in x-, y-, and z-direction, respectively. The constitutive equations yield for the associated strains ϵ_{11}, ϵ_{22}, and ϵ_{33}

$$\begin{aligned} \epsilon_{11} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] = 0, \\ \epsilon_{22} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] = 0, \\ \epsilon_{33} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] = 0. \end{aligned} \tag{2.394}$$

Therefore, due to the change in temperature from T_0 to T_1 the strains become

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \alpha(T_1 - T_0). \tag{2.395}$$

Integrating the strains with respect to the fixities at the coordinate planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned} u_x(x) &= \alpha\,(T_1 - T_0)\,x, \\ u_y(y) &= \alpha\,(T_1 - T_0)\,y, \\ u_z(z) &= \alpha\,(T_1 - T_0)\,z. \end{aligned} \tag{2.396}$$

2.8.4 An Elastic Cuboid Undergoes Load Due to Gravity and Instant Temperature Change

The domain is a cuboid of height $H = 30\,\mathrm{m}$ and edges parallel to the x-y-z coordinate axes. It is discretized by an irregular mesh of hexahedral elements. The cuboid is represented by four groups of elastic materials, where each has been assigned density $\rho = 2038.736\,\mathrm{kg/m^3}$, Poisson's ratio $\nu = 0.25$, Young's modulus $E = 5{,}000\,\mathrm{MPa}$, zero heat capacity, and thermal expansion $\alpha = 5 \times 10^{-6}\,1/\mathrm{K}$. Gravity is applied in negative z-direction, $g = 9.81\,\mathrm{m/s^2}$ is the magnitude of gravity. The bottom and the lateral faces are sliding planes, the top face is free. The simulation starts from the initial temperature $T_0 = 10\,°\mathrm{C}$ and comprises one time step applying an instant temperature change with temperature $T_1 = 4\,°\mathrm{C}$ at the top ($z = H$) and temperature T_0 at $z = 0$.

The formal solution proceeds in two steps, first to solve for the temperature distribution $T(x)$ and then to evaluate stresses, strains, and displacements.

The Laplace equation is the governing equation describing the steady-state temperature distribution. It reads

$$\frac{d^2T}{dz^2} = 0 \tag{2.397}$$

for 1D heat flow along the z-axis, hence the temperature distribution is given by

$$T(z) = (T_1 - T_0)\frac{z}{H} + T_0. \tag{2.398}$$

Next we focus on the closed form solution of the mechanical problem. Let $\boldsymbol{\sigma}$ denote the stress tensor. The equation of mechanical equilibrium

$$0 = \nabla \cdot \boldsymbol{\sigma} - (0, 0, \rho\, g) \tag{2.399}$$

is satisfied by zero shear, if the horizontal stresses σ_{11} and σ_{22} are functions of z only and the vertical stress σ_{33} satisfies

$$\frac{d\sigma_{33}}{dz} = \rho\, g. \tag{2.400}$$

The face $z = H$ is free, hence, integration gives

$$\sigma_{33} = \rho(-g)(H - z). \tag{2.401}$$

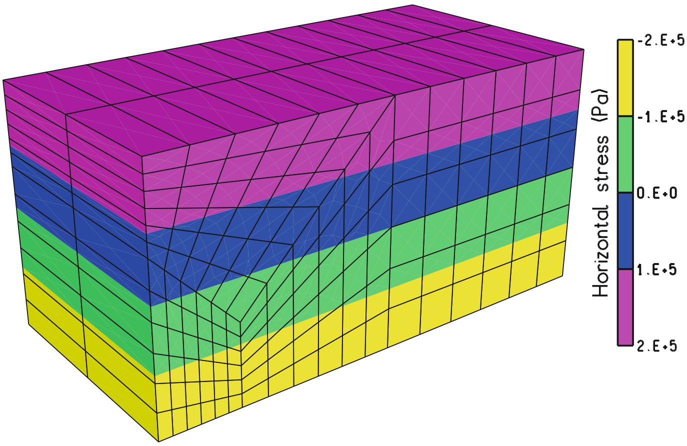

Fig. 2.24 Horizontal stress

Due to the simulation setup there is no horizontal displacement anywhere, hence, for the horizontal strains

$$\epsilon_{11} = \epsilon_{22} = 0. \tag{2.402}$$

Then, with principal axes equal to coordinate axes, the constitutive equations give

$$\begin{aligned}
\epsilon_{11} - \alpha(T(z) - T_0) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})],\\
\epsilon_{22} - \alpha(T(z) - T_0) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})],\\
\epsilon_{33} - \alpha(T(z) - T_0) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})].
\end{aligned} \tag{2.403}$$

Solving for σ_{11}, σ_{22}, (Fig. 2.24) and the vertical strain ϵ_{33} yields

$$\begin{aligned}
\sigma_{11} = \sigma_{22} &= -\alpha \frac{E}{1-\nu}(T_1 - T_0)\frac{z}{H} + \frac{\nu}{1-\nu}\rho\,(-g)(H-z),\\
\epsilon_{33} &= \frac{1+\nu}{1-\nu}\alpha\,(T_1 - T_0)\frac{z}{H} + \left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{1}{E}\rho\,(-g)(H-z)
\end{aligned} \tag{2.404}$$

in terms of the vertical coordinate. Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned}
u_x &= u_y = 0,\\
u_z(z) &= \frac{1+\nu}{1-\nu}\alpha(T_1 - T_0)\frac{z^2}{2H} + \left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{1}{E}\rho(-g)\left(zH - \frac{1}{2}z^2\right).
\end{aligned} \tag{2.405}$$

2.8.5 An Elastic Beam Deforms Due to a Transient Temperature Change. Temperature Boundary Conditions are Time-Dependent and of 1st Kind

The domain is a rectangular beam of length $L = 1\,\mathrm{m}$ extending along the positive x-axis. It has three faces located on the coordinate planes and is discretized by hexahedral elements with section $0 \le x \le 0.6\,L$ composed of $10 \times 1 \times 1$ and section $0.6\,L \le x \le L$ composed of $60 \times 1 \times 1$ elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 25{,}000\,\mathrm{MPa}$, and density $\rho = 2{,}000\,\mathrm{kg/m^3}$. Thermal conductivity $\lambda = 2.7\,\mathrm{W/(m \cdot K)}$, heat capacity $c = 0.45\,\mathrm{J/(kg \cdot K)}$, and thermal expansion $\alpha = 3 \times 10^{-4}\,\mathrm{1/K}$ have been assigned. Gravity is neglected via explicit assignment. The face $x = 0$ is free, all other faces of the beam are sliding planes. Zero temperature has been specified at the face $x = 0$, the temperature $T_1 \cdot t$ ($T_1 = 1\,°\mathrm{C/s}$) increases linearly with time t, it is applied at the face $x = L$ for times $t > 0$. Starting from zero initial temperature the simulation evaluates the transient temperature distribution $T(x, t)$ as well as stresses, strains, and displacements with output after 5 and 10 s.

The heat conduction equation is the governing equation describing the transient temperature distribution. It reads

$$\rho c \frac{\partial T}{\partial t} = \lambda \nabla \cdot \nabla T. \tag{2.406}$$

Introducing the notation

$$\chi = \frac{\lambda}{\rho c} \tag{2.407}$$

the present 1D problem is governed by the parabolic equation

$$\frac{1}{\chi} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \tag{2.408}$$

the initial condition

$$T(x, 0) = 0 \quad \text{for } 0 \le x \le L, \tag{2.409}$$

and the boundary conditions imposed at the beam ends

$$\begin{aligned} T(0, t) &= \quad 0 && \text{for } t > 0, \\ T(L, t) &= T_1 \cdot t && \text{for } t > 0. \end{aligned} \tag{2.410}$$

We will shown next that the solution of the mechanical problem may be obtained in terms of the temperature distribution.

For the closed form solution of the mechanical problem note, that due to the simulation setup, the entire system is free of shear and the principal axes coincide with the coordinate axes. The constitutive equations relate the strains ϵ_{11}, ϵ_{22}, ϵ_{33}, (in x-, y-, and z-direction, respectively) and the associated stresses σ_{11}, σ_{22}, and σ_{33} via

$$\begin{aligned}\epsilon_{11} - \alpha T(x,t) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})],\\ \epsilon_{22} - \alpha T(x,t) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})],\\ \epsilon_{33} - \alpha T(x,t) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})].\end{aligned} \tag{2.411}$$

Due to the setup the x-direction is free of stress, y- and z-direction are free of strain, therefore,

$$\begin{aligned}\sigma_{11} &= 0,\\ \epsilon_{22} &= \epsilon_{33} = 0.\end{aligned} \tag{2.412}$$

Hence, due to the change in temperature, the remaining non-zero stresses and strains become

$$\begin{aligned}\sigma_{22} = \sigma_{33} &= -\alpha \frac{E}{1-\nu} T(x,t),\\ \epsilon_{11} &= \alpha \frac{1+\nu}{1-\nu} T(x,t).\end{aligned} \tag{2.413}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$\begin{aligned}u_x(x) &= \alpha \frac{1+\nu}{1-\nu} \int_L^x T(x', t) dx',\\ u_y &= u_z = 0.\end{aligned} \tag{2.414}$$

Once that the temperature distribution $T(x,t)$ has been found, the entire solution of the thermomechanical problem may thus be obtained. It is therefore sufficient to solve the 1D heat conduction equation with respect to the imposed initial and boundary conditions cited above.

Applying the Laplace transform with respect to time t yields the ordinary differential equation

$$\chi \bar{T}'' - s\bar{T} = 0, \tag{2.415}$$

where $\bar{T}$ is the transform of $T(x,t)$, the prime denotes the derivative with respect to x, and s is the transformation parameter. This equation has to be solved with respect to the transformed boundary conditions. This yields the transform of the temperature

$$\bar{T}(x,s) = T_1 \frac{\sinh(\sqrt{s/\chi}\, x)}{s^2 \sinh(\sqrt{s/\chi}\, L)}, \tag{2.416}$$

and the transform of the only non-zero displacement $u_x(x,t)$ becomes

$$\bar{u}_x(x,s) = \alpha \frac{1+\nu}{1-\nu} T_1 \frac{\cosh(\sqrt{s/\chi}\, x) - \cosh(\sqrt{s/\chi}\, L)}{s^2 \sqrt{s/\chi}\, \sinh(\sqrt{s/\chi}\, L)}. \tag{2.417}$$

The entire solution may now be obtained from the transforms of temperature and x-displacement. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of temperature $T(x,t)$ and the entire mechanical load.

2.8.6 Elastic Beams Deform Due to a Transient Temperature Change. Temperature Boundary Conditions are Time-Dependent and of 2nd Kind

The domain is composed of two beams in parallel (Beam1 and Beam2) extending along the positive x-axis, each $L = 25\,\text{m}$ long and subdivided into $25 \times 1 \times 1$ cubic elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 25{,}000\,\text{MPa}$, and density $\rho = 2{,}000\,\text{kg/m}^3$. Thermal conductivity $\lambda = 1.1574074\,\text{W/(m}\cdot\text{K)}$, thermal expansion $\alpha = 3 \times 10^{-4}$ 1/K, and heat capacities $c_1 = 0.01\,\text{J/(kg}\cdot\text{K)}$ and $c_2 = 0.02\,\text{J/(kg}\cdot\text{K)}$ have been assigned to Beam1 and Beam2, respectively. Gravity is neglected via explicit assignment. The faces $x = L$ are free, all other faces of the beams are sliding planes. No-flow boundary conditions prevail at the $x = 0\,\text{m}$ faces. A specific heat flow is prescribed at $x = L$ for times $t > 0$. It acts as heat source to the domain and increases linearly with time via $q_{th1} \cdot t$, where $q_{th1} = 0.385802\,\text{W/(d}\cdot\text{m}^2)$ has been assumed. Starting from zero initial temperature the simulation evaluates the transient temperature distributions as well as stresses, strains, and displacements with output after 0.045 and 0.09 days (Figs. 2.25 and 2.26).

The formal solution proceeds in two steps, first to solve for the temperature distributions and then to evaluate stresses, strains, and displacements.

Let c denote any of c_1 or c_2. The heat conduction equation is the governing equation describing the transient temperature distribution. It reads

$$\rho c \frac{\partial T}{\partial t} = \lambda \nabla \cdot \nabla T. \tag{2.418}$$

Introducing the notation

$$\chi = \frac{\lambda}{\rho c} \tag{2.419}$$

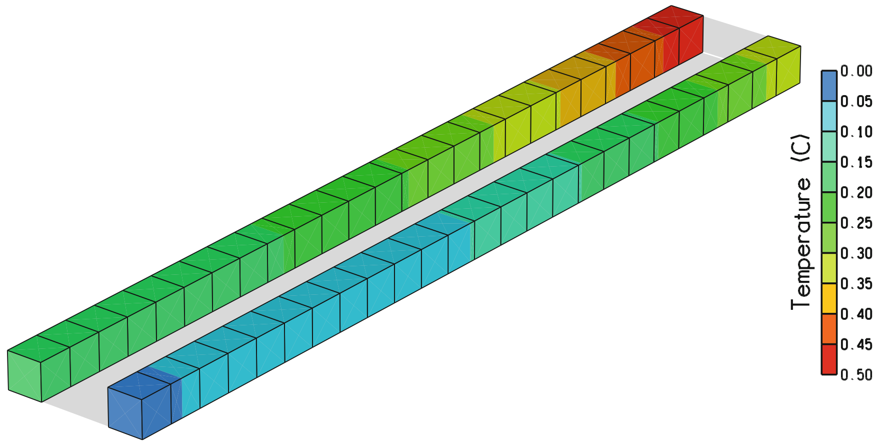

Fig. 2.25 Temperature distributions after 0.09 days

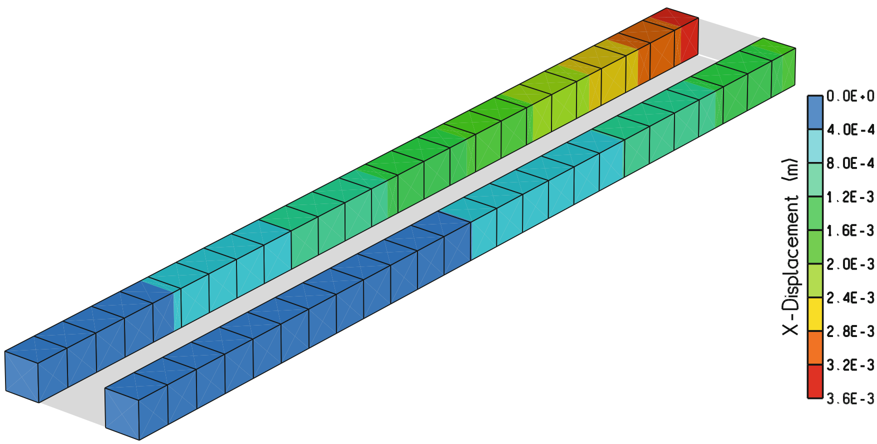

Fig. 2.26 X-Displacements after 0.09 days

the present 1D problems are governed by the parabolic equation

$$\frac{1}{\chi}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \tag{2.420}$$

the initial condition

$$T(x, 0) = 0 \quad \text{for } 0 \leq x \leq L, \tag{2.421}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial T}{\partial x}(0,t) &= 0 \qquad\qquad \text{for } t > 0, \\ \lambda\frac{\partial T}{\partial x}(L,t) &= q_{th1}\cdot t \quad \text{for } t > 0. \end{aligned} \tag{2.422}$$

The closed form solution of the above problem is given by Carslaw and Jaeger [3], who arrive at the series representation

$$T(x,t) = \frac{8q_{th1}\sqrt{\chi t^3}}{\lambda}\sum_{n=0}^{\infty}\left[i^3\text{erfc}\frac{(2n+1)L-x}{2\sqrt{\chi t}} + i^3\text{erfc}\frac{(2n+1)L+x}{2\sqrt{\chi t}}\right] \tag{2.423}$$

where i^3erfc denotes the third repeated integral of the complementary error function. See [4] for its numerical evaluation.

For the closed form solution of the mechanical problem note, that due to simulation setup, the entire system is free of shear and the principal axes coincide with the coordinate axes. The constitutive equations relate the strains ϵ_{11}, ϵ_{22}, ϵ_{33}, (in x-, y-, and z-direction, respectively) and the associated stresses σ_{11}, σ_{22}, and σ_{33} via

$$\begin{aligned} \epsilon_{11} - \alpha T(x,t) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22}+\sigma_{33})], \\ \epsilon_{22} - \alpha T(x,t) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11}+\sigma_{33})], \\ \epsilon_{33} - \alpha T(x,t) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11}+\sigma_{22})]. \end{aligned} \tag{2.424}$$

Due to the setup the x-direction is free of stress, y- and z-direction are free of strain, therefore,

$$\begin{aligned} \sigma_{11} &= 0, \\ \epsilon_{22} &= \epsilon_{33} = 0. \end{aligned} \tag{2.425}$$

Hence, due to the temperature change $T(x,t)$, the remaining non-zero stresses and strains become

$$\begin{aligned} \sigma_{22} &= -\alpha\frac{E}{1-\nu}T(x,t), \\ \sigma_{33} &= -\alpha\frac{E}{1-\nu}T(x,t), \\ \epsilon_{11} &= \alpha\frac{1+\nu}{1-\nu}T(x,t). \end{aligned} \tag{2.426}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z)

$$u_z = u_y = 0, \tag{2.427}$$

$$u_x(x) = \alpha \frac{1+\nu}{1-\nu} \frac{16 q_{th1} \chi t^2}{\lambda} \cdot \sum_{n=0}^{\infty} \left[i^4 \mathrm{erfc} \frac{(2n+1)L - x}{2\sqrt{\chi t}} - i^4 \mathrm{erfc} \frac{(2n+1)L + x}{2\sqrt{\chi t}} \right],$$

where i^4erfc denotes the 4th repeated integral of the complementary error function. See [4] for its numerical evaluation.

2.8.7 Stresses Relax in a Cube of Norton Material Undergoing an Instant Temperature Change

The domain is a single cube with edge size $L = 1$ m located in the first octant. It has three faces located on the coordinate planes and is discretized by $2 \times 2 \times 2$ cubic elements. The cube is represented by a Norton material. Poisson's ratio $\nu = 0.27$, Young's modulus $E = 25{,}000$ MPa, zero heat capacity, and thermal expansion $\alpha = 4 \times 10^{-5}$ 1/K have been assigned, gravity is neglected via zero material density. Various additional parameters are involved in the rheological behaviour, details are given below. Faces on the coordinate planes and the top face are sliding planes. The simulation starts from an initial setup free of load and an initial temperature $T_0 = 27\,°$C. It applies an instant temperature change to $T_1 = 47\,°$C at the top of the domain and evaluates stresses, strains, and displacements through time with output after 0.5 and 2 days.

Let $\boldsymbol{\sigma}$ denote the stress tensor, $\mathbf{I}$ the unit tensor,

$$\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{\mathrm{tr}\boldsymbol{\sigma}}{3}\mathbf{I} \tag{2.428}$$

the stress deviator, and

$$\sigma_{\mathrm{eff}} = \sqrt{\frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij}^D \sigma_{ji}^D} \tag{2.429}$$

the v. Mises or effective stress. The rheological model involved yields the fundamental stress/strain relationships as a system of differential equations for the creep strains

$$\frac{\partial \boldsymbol{\epsilon}^{cr}}{\partial t} = \frac{3}{2} \frac{\boldsymbol{\sigma}^D}{\sigma_{\mathrm{eff}}} (N\, \sigma_{\mathrm{eff}}^n) \tag{2.430}$$

and the total strains

$$\boldsymbol{\epsilon}^{tot} = \boldsymbol{\epsilon}^{th} + \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{cr}, \tag{2.431}$$

where $\boldsymbol{\epsilon}^{th}$ denotes the thermal strains and $\boldsymbol{\epsilon}^{el}$ the elastic strains via Hooke's law. Both equations have to be solved with respect to the imposed initial and boundary conditions.

For the present example the behaviour of the Norton material is specified with the aid of the parameters

$$\begin{aligned} n \quad &= 5, \\ N(T) &= A \exp\left(-\frac{Q}{RT}\right), \end{aligned} \tag{2.432}$$

where $R = 8.31441\,\text{J/(mol} \cdot \text{K)}$ is the gas constant, T is the absolute temperature, and experimental data obtained from rock salt yield

$$\begin{aligned} A &= 0.18\ 1/(\text{d} \cdot \text{MPa}^5), \\ Q &= 54,000\ \text{J/mol}. \end{aligned} \tag{2.433}$$

Note that day is required as unit of time and stresses have to be in MPa.

Due to the example setup the principal axes are identical to the coordinate axes and the vertical stress is the only non-zero element of the stress tensor. Therefore,

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}. \tag{2.434}$$

the trace of $\boldsymbol{\sigma}$

$$tr\boldsymbol{\sigma} = \sigma_{33}, \tag{2.435}$$

the stress deviator

$$\boldsymbol{\sigma}^D = \frac{\sigma_{33}}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{2.436}$$

the v. Mises or effective stress

$$\sigma_{\text{eff}} = |\sigma_{33}| \frac{1}{3} \sqrt{3/2} \sqrt{1^2 + 1^2 + 2^2} = |\sigma_{33}|, \tag{2.437}$$

and the time derivative of the creep strains

$$\frac{\partial \boldsymbol{\epsilon}^{cr}}{\partial t} = \frac{N(T_1)}{2} \sigma_{33}^5 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.438}$$

The entire domain is initially free of creep strains. Hence, integrating with respect to time t the creep strains become

$$\boldsymbol{\epsilon}^{cr} = \frac{N(T_1)}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \int_0^t \sigma_{33}^5 \, dt. \tag{2.439}$$

Due to the simulation setup the thermal strains read

$$\boldsymbol{\epsilon}^{th} = \alpha(T_1 - T_0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.440}$$

and the elastic strains are obtained from the stress $\boldsymbol{\sigma}$ via Hooke's law.

$$\boldsymbol{\epsilon}^{el} = \frac{\sigma_{33}}{E} \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.441}$$

The total strains in terms of σ_{33} and the displacements (u_x, u_y, u_z) read

$$\begin{aligned} \boldsymbol{\epsilon}^{tot} = \boldsymbol{\epsilon}^{th} + \boldsymbol{\epsilon}^{el} + \boldsymbol{\epsilon}^{cr} &= \begin{pmatrix} \partial u_x/\partial x & 0 & 0 \\ 0 & \partial u_y/\partial y & 0 \\ 0 & 0 & \partial u_z/\partial z \end{pmatrix} \\ &= \alpha(T_1 - T_0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad + \frac{\sigma_{33}}{E} \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad + \frac{N(T_1)}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \int_0^t \sigma_{33}^5 \, dt. \end{aligned} \tag{2.442}$$

Due to the simulation setup

$$\epsilon_{33}^{tot} = \frac{\partial u_z}{\partial z} = 0 \tag{2.443}$$

is the specified zero strain along the z-axis. Then

$$-\alpha(T_1 - T_0) = \frac{1}{E}\sigma_{33} + N(T_1) \int_0^t \sigma_{33}^5 \, dt. \tag{2.444}$$

This integral equation is transformed into the ordinary differential equation

$$0 = \frac{1}{E} \frac{d\sigma_{33}}{dt} + N(T_1)\sigma_{33}^5. \tag{2.445}$$

Separation of variables and integration yields

$$\sigma_{33}(t) = \frac{-E\alpha(T_1 - T_0)}{\sqrt[4]{4E^5[\alpha(T_1 - T_0)]^4 N(T_1)\, t + 1}}, \tag{2.446}$$

and the strains $\partial u_x/\partial x$ and $\partial u_y/\partial y$ are obtained in terms of $\sigma_{33}(t)$

$$\begin{aligned} \epsilon_{11}^{tot}(t) &= \frac{\partial u_x}{\partial x} = \frac{3}{2}\alpha(T_1 - T_0) + \frac{1-2v}{2E}\sigma_{33}(t), \\ \epsilon_{22}^{tot}(t) &= \frac{\partial u_y}{\partial y} = \frac{3}{2}\alpha(T_1 - T_0) + \frac{1-2v}{2E}\sigma_{33}(t). \end{aligned} \tag{2.447}$$

Integrating the strains with respect to the specified fixities yields the displacement vector (u_x, u_y, u_z) in terms of $\sigma_{33}(t)$ derived above

$$\begin{aligned} u_x(x,t) &= x\left[\frac{3}{2}\alpha(T_1 - T_0) + \frac{1-2v}{2E}\sigma_{33}(t)\right], \\ u_y(y,t) &= y\left[\frac{3}{2}\alpha(T_1 - T_0) + \frac{1-2v}{2E}\sigma_{33}(t)\right], \\ u_z(z,t) &= 0. \end{aligned} \tag{2.448}$$

2.9 Thermo-Hydro-Mechanical Coupling

Both, the presence of a liquid pressure as well as temperature changes affect the mechanical behaviour of the porous matrix; we present a steady-state and a transient problem. The underlying theory may be found in the references cited above.

2.9.1 A Permeable Elastic Cuboid Deforms Due to Gravity, Internal Liquid Pressure, and Instant Temperature Change

The domain is a cuboid of height $H = 30\,\text{m}$ and edges parallel to the x-y-z coordinate axes. It is discretized by an irregular mesh of hexahedral elements. The cuboid is represented by four groups of elastic materials, where each has been assigned density $\rho_s = 2038.736\,\text{kg/m}^3$, Poisson's ratio $\nu = 0.25$, Young's modulus $E = 10{,}000\,\text{MPa}$, thermal expansion $\alpha = 3 \times 10^{-6}\,1/\text{K}$, zero heat capacity, zero porosity, and Biot number equal one. Liquid density is $\rho_l = 1019.368\,\text{kg/m}^3$, gravity is applied in negative z-direction, $g = 9.81\,\text{m/s}^2$ is the magnitude of gravity. Zero pressure is applied at the top face $z = H$. This face is free, all other faces are sliding planes. The simulation starts from the initial temperature $T_0 = 0\,°\text{C}$ and comprises one time step applying an instant temperature increase to $T_1 = 2.5\,°\text{C}$ throughout the entire domain. The simulation evaluates the pressure distribution, the temperature, and the mechanical load.

The formal solution proceeds in three steps, first to solve for the temperature, next to evaluate the pressure distribution $p(z)$, and finally to determine stresses, strains and displacements. However, due to the setup the temperature change has constant value $T_1 - T_0$ throughout the entire domain, and we will focus on the hydromechanical aspects of the problem.

The simulation setup employs a prescribed zero pressure at the top ($z = H$), therefore the pressure distribution is hydrostatic, does not depend on the coordinates x and y and is given by

$$p(z) = \rho_l\, g\, (H - z). \tag{2.449}$$

For the closed form solution of the mechanical problem let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$0 = \nabla \cdot (\boldsymbol{\sigma} - p\, \mathbf{I}) - (0, 0, \rho_s\, g). \tag{2.450}$$

It is satisfied by zero shear, if pressure p and the horizontal stresses σ_{11} and σ_{22} are functions of the vertical coordinate z only and the vertical stress σ_{33} satisfies

$$\frac{d\sigma_{33}}{dz} = \rho_s\, g + \frac{dp}{dz} = (\rho_s - \rho_l)\, g. \tag{2.451}$$

The face $z = H$ is free, hence, integration gives

$$\sigma_{33} = (\rho_s - \rho_l)\, (-g)\, (H - z). \tag{2.452}$$

Due to the simulation setup there is no horizontal displacement anywhere, hence, for the horizontal strains

$$\epsilon_{11} = \epsilon_{22} = 0. \tag{2.453}$$

Then, with principal axes equal to coordinate axes, the constitutive equations give

$$\begin{aligned}
\epsilon_{11} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})],\\
\epsilon_{22} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})],\\
\epsilon_{33} - \alpha(T_1 - T_0) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})].
\end{aligned} \tag{2.454}$$

Solving for σ_{11}, σ_{22}, and the vertical strain ϵ_{33} yields

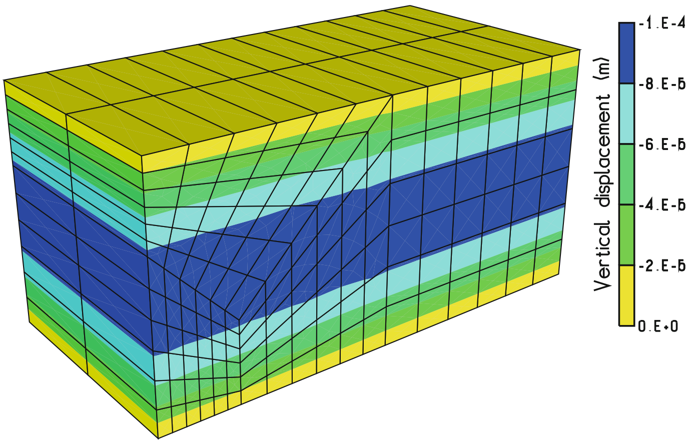

Fig. 2.27 Vertical displacements

$$\sigma_{11} = \sigma_{22} = -\alpha \frac{E}{1-\nu}(T_1 - T_0) \ + \ \frac{\nu}{1-\nu}(\rho_s - \rho_l)(-g)(H-z), \qquad (2.455)$$

$$\epsilon_{33} = \frac{1+\nu}{1-\nu}\alpha(T_1 - T_0) \ + \ \left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{1}{E}(\rho_s - \rho_l)(-g)(H-z)$$

in terms of the vertical coordinate. Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z) (Fig. 2.27).

$$\begin{aligned} u_x &= u_y = 0, \\ u_z(z) &= \frac{1+\nu}{1-\nu}\alpha(T_1 - T_0)z \\ &\quad + \left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{1}{E}(\rho_s - \rho_l)(-g)\left(zH - \frac{1}{2}z^2\right). \end{aligned} \qquad (2.456)$$

2.9.2 A Permeable Elastic Beam Deforms Due to Cooling Liquid Injection

The domain is a rectangular beam of length $L = 10\,\mathrm{m}$ extending along the positive x-axis. It is discretized by $100 \times 1 \times 1$ equally sized hexahedral elements. The solid material has been selected elastic with Poisson's ratio $\nu = 0.25$, Young's modulus $E = 5 \times 10^9\,\mathrm{Pa}$, thermal expansion $\alpha = 1 \times 10^{-6}\,1/\mathrm{K}$, and Biot number equal one. An isotropic permeability $k = 10^{-11}\,\mathrm{m}^2$ and porosity $\phi = 0.1$ is assumed for the material. The liquid is incompressible and has viscosity $\mu = 1\,\mathrm{mPa \cdot s}$. Densities, heat capacities, and thermal conductivities of liquid and solid grain are given below (Table 2.4), gravity has explicitly been neglected.

Table 2.4 Example overview

	Liquid	Solid
Density	$\rho_l = 1000\ \text{kg/m}^3$	$\rho_s = 2000\ \text{kg/m}^3$
Specific heat capacity	$c_l = 1100\ \text{J/(kg}\cdot\text{K)}$	$c_s = 250\ \text{J/(kg}\cdot\text{K)}$
Thermal conductivity	$\lambda_l = 10\ \text{W/(m}\cdot\text{K)}$	$\lambda_s = 50\ \text{W/(m}\cdot\text{K)}$

The face $x = L$ is free, all other faces of the beam are sliding planes. Pressure $p_0 = 10^5$ Pa at the liquid inlet ($x = 0$ m) and zero pressure the liquid outlet ($x = L$) generate steady-state 1D flow along the x-axis. At the liquid inlet a constant temperature $T_0 = -10\,°\text{C}$ is specified for times $t > 0$. Starting from zero initial temperature the simulation evaluates the transient temperature distribution $T(x, t)$ as well as stresses, strains, and displacements with output after 10,000 and 20,000 s.

The formal solution proceeds in three steps, first to solve for pressure $p(x)$ and specific discharge q, next to evaluate the temperature distribution $T(x, t)$, and finally to determine stresses, strains, and displacements (Figs. 2.28 and 2.29).

For incompressible liquids Darcy's law and continuity equation yield the Laplace equation as the governing equation describing the steady-state pressure distribution. It reads

$$\frac{d^2 p}{dx^2} = 0 \tag{2.457}$$

for 1D flow along the x-axis, hence, the pressure is given by

$$p(x) = p_0(1 - \frac{x}{L}), \tag{2.458}$$

and the specific discharge q is obtained by Darcy's law

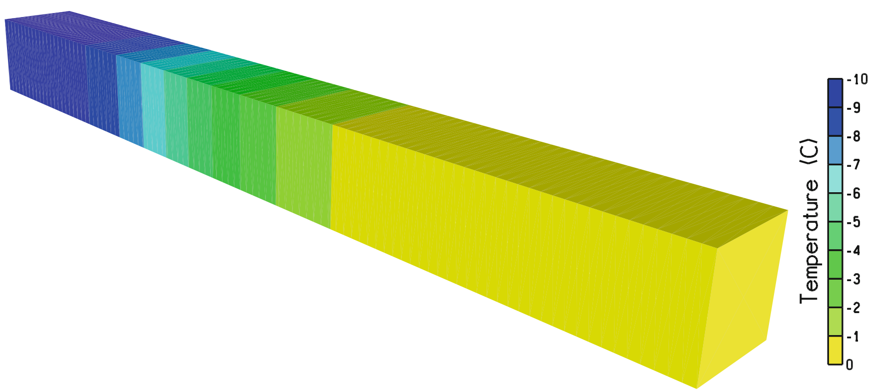

Fig. 2.28 Temperature distribution after 20,000 s

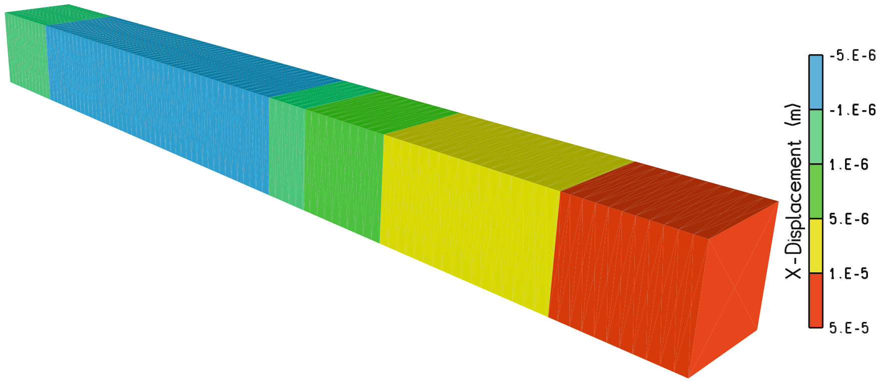

Fig. 2.29 X-Displacements after 20,000 s

$$q = \frac{k}{\mu}\frac{p_0}{L}. \tag{2.459}$$

We will next focus on the closed form solution of the heat transport problem. Based on the setup of the present example the heat transport equation reads

$$(\phi\rho_l c_l + (1-\phi)\rho_s c_s)\frac{\partial T}{\partial t} + (\phi\rho_l c_l)\frac{q}{\phi}\frac{\partial T}{\partial x} = (\phi\lambda_l + (1-\phi)\lambda_s)\frac{\partial^2 T}{\partial x^2}. \tag{2.460}$$

Introducing the notation

$$\begin{aligned} w &= \frac{\phi\rho_l c_l}{\phi\rho_l c_l + (1-\phi)\rho_s c_s}\frac{q}{\phi}, \\ \chi &= \frac{\phi\lambda_l + (1-\phi)\lambda_s}{\phi\rho_l c_l + (1-\phi)\rho_s c_s}, \end{aligned} \tag{2.461}$$

the heat transport equation becomes

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial x} = \chi\frac{\partial^2 T}{\partial x^2}. \tag{2.462}$$

Due to free outflow at $x = L$ the formal problem is to determine the solution $T(x, t)$ of the above heat transport equation subject to the initial condition

$$T(x, 0) = 0 \quad \text{for} \quad x > 0, \tag{2.463}$$

and the boundary conditions

$$\begin{aligned} T(0, t) &= T_0 \quad \text{for} \quad t > 0, \\ \lim_{x\to\infty} T(x, t) &= 0 \quad \text{for} \quad t > 0. \end{aligned} \tag{2.464}$$

Applying the Laplace transform with respect to t yields the ordinary differential equation

$$\chi \bar{T}'' - w\bar{T}' - s\bar{T} = 0, \tag{2.465}$$

where $\bar{T}$ is the transform of T, s is the transformation parameter, and the prime denotes the derivative with respect to x. This equation has to be solved with respect to the transformed boundary conditions. This yields

$$\bar{T}(x, s) = \frac{T_0}{s} \exp\left[x\left(\frac{w}{2\chi} - \sqrt{(\frac{w}{2\chi})^2 + \frac{s}{\chi}}\right)\right]. \tag{2.466}$$

The temperature distribution $T(x, t)$ may now be obtained from their transform, Churchill [7] outlines how to proceed with the aid of operational calculus.

For the mechanical aspects of the problem we assume that the temperature distribution $T(x, t)$ is already known from the above. Let $\boldsymbol{\sigma}$ denote the stress tensor and $\mathbf{I}$ the unit tensor. Employing Biot's simplified theory (i.e. Biot number equal one) the equation of mechanical equilibrium reads

$$0 = \nabla \cdot (\boldsymbol{\sigma} - p\,\mathbf{I}). \tag{2.467}$$

It is satisfied by zero shear, if the stresses σ_{22} and σ_{33} are functions of x only and the stress σ_{11} satisfies

$$\frac{\partial}{\partial x}(\sigma_{11} - p) = 0. \tag{2.468}$$

The face $x = L$ is free of load, hence, integration gives

$$\sigma_{11} = p(x) = p_0(1 - \frac{x}{L}). \tag{2.469}$$

With principal axes equal to coordinate axes, the constitutive equations give for the strains

$$\begin{aligned} \epsilon_{11} - \alpha T(x, t) &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})], \\ \epsilon_{22} - \alpha T(x, t) &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})], \\ \epsilon_{33} - \alpha T(x, t) &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})]. \end{aligned} \tag{2.470}$$

By the problem setup there is

$$\epsilon_{22} = \epsilon_{33} = 0 \tag{2.471}$$

due to the y- and z-fixities along the front, rear, top, and bottom of the beam. In terms of pressure $p(x)$ and temperature $T(x, t)$ the remaining non-zero stresses and strains become

$$\sigma_{22} = \sigma_{33} = \frac{1}{1-\nu}\left[\nu p(x) - \alpha E T(x,t)\right],$$
$$\epsilon_{11} = \left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{p(x)}{E} + \frac{1+\nu}{1-\nu}\alpha T(x,t). \tag{2.472}$$

Integrating the strains with respect to the fixities at the sliding planes yields the displacement vector (u_x, u_y, u_z). The only non-zero displacement $u_x(x,t)$ becomes

$$u_x(x,t) = \left(1 - \frac{2\nu^2}{1-\nu}\right)\frac{p_0}{E}\left(x - \frac{x^2}{2L}\right) + \frac{1+\nu}{1-\nu}\alpha\int_0^x T(x',t)dx'. \tag{2.473}$$

The already known Laplace transform $\bar{T}(x,s)$ of the temperature serves to evaluate the last integral over the temperature distribution. We have for the transformed integral

$$\begin{aligned} L\left\{\int_0^x T(x',t)dx'\right\} &= \int_0^x \bar{T}(x',s)dx' \\ &= \frac{T_0}{s}\frac{\exp\left\{x\left(\frac{w}{2\chi} - \sqrt{(\frac{w}{2\chi})^2 + \frac{s}{\chi}}\right)\right\} - 1}{\frac{w}{2\chi} - \sqrt{(\frac{w}{2\chi})^2 + \frac{s}{\chi}}}, \end{aligned} \tag{2.474}$$

and the last expression, as well as the Laplace transform of the temperature itself, are well suited for numerical inversion. The numerical inversion scheme outlined in the introductory section may easily be applied to give the required values of the temperature $T(x,t)$ and the entire mechanical load (Fig. 2.29).

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http://www.springer.com/978-3-319-11893-2

Thermo-Hydro-Mechanical-Chemical Processes in
Fractured Porous Media: Modelling and Benchmarking
Closed-Form Solutions
Kolditz, O.; Shao, H.; Wang, W.; Bauer, S. (Eds.)
2015, XIV, 313 p. 156 illus., 142 illus. in color.,
Hardcover
ISBN: 978-3-319-11893-2