On Nonlocal Perturbations of Integral Kernels

Krzysztof Bogdan and Sebastian Sydor

Abstract We give sufficient conditions for nonlocal perturbations of integral kernels to be locally in time comparable with the original kernel.

1 Introduction

We may delete or add jumps to a Markov process by adding a nonlocal operator to its generator. We shall be concerned with estimates of the resulting, perturbed transition kernels. In fact, we consider perturbations of quite general integral kernels on space-time. We focus on perturbations by nonlocal operators, which model evolution of mass in presence of births, deaths, dislocations and delays. We are motivated by recent estimates of local, or Schrödinger, perturbations of integral kernels in [3, 6], and nonlocal perturbations of the Green functions in [11, 13].

We deal with the so-called *forward* kernels, reflecting directionality of time. The resulting perturbation and the original kernel turn out to be comparable locally in time and globally in space under an appropriate integral smallness condition on the first nontrivial term of the perturbation series. A related paper [7] studies nonlocal perturbations of the semigroup of the fractional Laplacian and related discontinuous multiplicative and additive functionals, which offer a probabilistic counterpart of our approach. We emphasize that transition and potential kernels of Markov processes are our main motivation for this work, however in what follows we do not generally impose Chapman-Kolmogorov condition on the kernels.

The paper is composed as follows. In Sect. 2 we formulate our main estimates: Theorem 1 for kernels and Theorems 2 and 3 for kernel densities. In Sect. 5 we focus

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on kernels q which are nonlocal in space but local (or instantaneous) in time. Such kernels q are not forward kernels and they require a separate treatment. We call perturbations by such q *nonlocal Schrödinger perturbations*. As usual, our approach consists in making appropriate smallness assumptions on the first nontrivial term $K_1 = KqK$ of the perturbation series. In Sect. 6 we indicate the extra work that needs to be done to verify the smallness of K_1 and apply our results in specific situations. Namely, we focus on perturbations of the transition density of the fractional Laplacian, describe the perturbations in terms of generators and fundamental solutions and illustrate the effect that the nonlocal perturbations have on jump intensity of stochastic processes.

We note that Theorems 1, 2 and 3 generalize the main estimates of [6] for Schrödinger perturbations of integral kernels. The reader may find in [6] and related paper [3] general comments on this research program, and more applications, e.g., to Weyl fractional integrals [6, Example 3] and to the potential kernel of the vector of two independent 1/2-stable subordinators [3, Example 4.1].

Considering transition probabilities, it should be noted that the perturbations considered in the present paper and [7] generally produce non-probabilistic kernels as they may increase the mass of the kernel. To preserve the mass, the generator of the perturbation should be of Lévy-type; it should involve compensation, and annihilate constant functions. There is a considerable progress in construction and estimates of transition probabilities resulting from such operators. We refer the reader to recent papers [9, 12, 14, 17], whose techniques are close to our perturbation methods, but require specific smoothness assumptions on the transition kernels and do not address the problem of growth of mass of the kernel.

2 Main Results

We first recall, after [10], some properties of kernels. Let $(E, \mathscr{E})$ be a measurable space. A kernel on E is a map K from $E \times \mathscr{E}$ to $[0, \infty]$ such that

$x \mapsto K(x, A)$ is $\mathscr{E}$-measurable for all $A \in \mathscr{E}$, and
$A \mapsto K(x, A)$ is countably additive for all $x \in E$.

Consider kernels K and J on E. The map

$$(x, A) \mapsto \int_E K(x, dy) J(y, A)$$

from $(E \times \mathscr{E})$ to $[0, \infty]$ is another kernel on E, called the *composition* of K and J, and denoted KJ. Here and below we alternatively write $\int f(x)\mu(dx) = \int \mu(dx) f(x)$. We let $K_n = K_{n-1}JK(s, x, A) = (KJ)^n K$, $n = 0, 1, \ldots$. The composition of kernels is associative, which yields the following lemma.

Lemma 1 $K_n = K_{n-1-m}JK_m$ *for all* $n \in \mathbb{N}$ *and* $m = 0, 1, \ldots, n-1$.

We define the *perturbation*, $\widetilde{K}$, of K by J, via the *perturbation series*,

$$\widetilde{K} = \sum_{n=0}^{\infty} K_n = \sum_{n=0}^{\infty} (KJ)^n K. \tag{1}$$

Of course, $K \le \widetilde{K}$, and the following *perturbation formula* holds,

$$\widetilde{K} = K + \widetilde{K}JK. \tag{2}$$

Below we prove upper bounds for $\widetilde{K}$ under additional conditions on K, J and $K_1 = KJK$.

Consider a set X (the state space) with σ-algebra $\mathscr{M}$, the real line $\mathbb{R}$ (the time) equipped with the Borel sets $\mathscr{B}_{\mathbb{R}}$, and consider the space-time

$$E := \mathbb{R} \times X,$$

with the product σ-algebra $\mathscr{E} = \mathscr{B}_{\mathbb{R}} \times \mathscr{M}$. Let $\eta \in [0, \infty)$ and a function $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfy the following condition of *super-additivity*:

$$Q(u, r) + Q(r, v) \le Q(u, v) \quad \text{for all } u < r < v.$$

In particular, $Q(r, v) \le Q(u, v)$. Let J be another kernel on E. We assume that K and J are *forward* kernels, i.e. for $A \in \mathscr{E}$, $s \in \mathbb{R}$, $x \in X$,

$$K(s, x, A) = J(s, x, A) = 0 \text{ whenever } A \subseteq (-\infty, s] \times X.$$

For $r < t$ we consider the strip $S = (r, t] \times X$, and the restriction of K to S, to wit, $K(s, x, A)$, where $(s, x) \in S$ and $A \subset S$. We note that the restriction of KJ to S depends only on the restrictions of K and J. In fact we could consider $E = (r, t] \times X$ as our basic setting. This observation allows to localize our estimates in time.

In what follows we study consequences of the following assumption,

$$KJK(s, x, A) \le \int_A [\eta + Q(s, t)] K(s, x, dtdy). \tag{3}$$

Theorem 1 *Assuming (3), for all $n = 1, 2, \ldots$ and $(s, x) \in E$ we have*

$$K_n(s, x, dtdy) \le K_{n-1}(s, x, dtdy) \left[\eta + \frac{Q(s, t)}{n}\right] \tag{4}$$

$$\le K(s, x, dtdy) \prod_{l=1}^{n} \left[\eta + \frac{Q(s, t)}{l}\right]. \tag{5}$$

If $0 < \eta < 1$*, then for all* $(s, x) \in E$*,*

$$\widetilde{K}(s, x, dtdy) \leq K(s, x, dtdy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}. \tag{6}$$

If $\eta = 0$*, then for all* $(s, x) \in E$*,*

$$\widetilde{K}(s, x, dtdy) \leq K(s, x, dtdy)e^{Q(s,t)}. \tag{7}$$

Proof (3) yields (4) for $n = 1$. By induction, for $n = 1, 2, \ldots$ we have

$$\begin{aligned}
(n+1)K_{n+1}(s, x, A) &= nK_nJK(s, x, A) + K_{n-1}JK_1(s, x, A) \\
&= n\int_E K_n(s, x, dudz)(JK)(u, z, A) + \int_E (K_{n-1}J)(s, x, du_1dz_1)K_1(u_1, z_1, A) \\
&\leq n\int_E \left[\eta + \frac{Q(s, u)}{n}\right] K_{n-1}(s, x, dudz)(JK)(u, z, A) \\
&\quad + \int_E (K_{n-1}J)(s, x, du_1dz_1) \int_A [\eta + Q(u_1, t)]K(u_1, z_1, dtdy) \\
&= (n+1)\eta K_n(s, x, A) \\
&\quad + \int_E Q(s, u)K_{n-1}(s, x, dudz) \int_E J(u, z, du_1dz_1) \int_A K(u_1, z_1, dtdy) \\
&\quad + \int_E \int_{(u,\infty)\times X} K_{n-1}(s, x, dudz)J(u, z, du_1dz_1) \int_A Q(u_1, t)K(u_1, z_1, dtdy) \\
&\leq (n+1)\eta K_n(s, x, A) \\
&\quad + \int_A \int_E \int_E Q(s, u)K_{n-1}(s, x, dudz)J(u, z, du_1dz_1)K(u_1, z_1, dtdy) \\
&\quad + \int_A \int_E \int_E K_{n-1}(s, x, dudz)J(u, z, du_1dz_1)Q(u, t)K(u_1, z_1, dtdy) \\
&\leq (n+1)\eta K_n(s, x, A) \\
&\quad + \int_A Q(s, t) \int_E K_{n-1}(s, x, dudz) \int_E J(u, z, du_1dz_1)K(u_1, z_1, dtdy) \\
&= (n+1)\eta K_n(s, x, A) + \int_A Q(s, t) \int_E K_{n-1}(s, x, dudz)(JK)(u, z, dtdy) \\
&= (n+1)\int_A \left[\eta + \frac{Q(s, t)}{n+1}\right] K_n(s, x, dtdy).
\end{aligned}$$

(5) follows from (4), (7) results from Taylor's expansion of the exponential function, and (6) follows from the Taylor series

$$(1-\eta)^{-a} = \sum_{n=0}^{\infty} \frac{\eta^n (a)_n}{n!},$$

where $0 < \eta < 1$, $a \in \mathbb{R}$, and $(a)_n = a(a+1)\cdots(a+n-1)$. □

Theorem 1 has two *finer* or *pointwise* variants, which we shall state under suitable conditions. Fix a (nonnegative) σ-finite, non-atomic measure

$$dt = \mu(dt)$$

on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and a function $k(s, x, t, A) \geq 0$ defined for $s, t \in \mathbb{R}, x \in X, A \in \mathscr{M}$, such that $k(s, x, t, dy)dt$ is a forward kernel and $(s, x) \mapsto k(s, x, t, A)$ is jointly measurable for all $t \in \mathbb{R}$ and $A \in \mathscr{M}$. Let $k_0 = k$, and for $n = 1, 2, \ldots$,

$$k_n(s, x, t, A) = \int_s^t \int_X k_{n-1}(s, x, u, dz) \int_{(u,t)\times X} J(u, z, du_1 dz_1) k(u_1, z_1, t, A) du.$$

The perturbation, $\widetilde{k}$, of k by J, is defined as

$$\widetilde{k} = \sum_{n=0}^{\infty} k_n.$$

Assume that

$$\int_s^t \int_X k(s, x, u, dz) \int_{(u,t)\times X} J(u, z, du_1 dz_1) k(u_1, z_1, t, A) du \leq [\eta + Q(s, t)] k(s, x, t, A).$$

Theorem 2 *Under the assumptions, for all $n = 1, 2, \ldots$, and $(s, x) \in E$,*

$$\begin{aligned} k_n(s, x, t, dy) &\leq k_{n-1}(s, x, t, dy) \left[\eta + \frac{Q(s,t)}{n}\right] \\ &\leq k(s, x, t, dy) \prod_{l=1}^{n} \left[\eta + \frac{Q(s,t)}{l}\right]. \end{aligned}$$

If $0 < \eta < 1$, then for all $(s, x) \in E$ and $t \in \mathbb{R}$ we have

$$\widetilde{k}(s, x, t, dy) \leq k(s, x, t, dy) \left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, *then*

$$\widetilde{k}(s, x, t, dy) \le k(s, x, t, dy) e^{Q(s,t)}.$$

We skip the proof, because it is similar to the proof of Theorem 1.
For the *finest* variant of Theorem 1, we fix a σ-finite measure

$$dz = m(dz)$$

on $(X, \mathscr{M})$. We consider function $\kappa(s, x, t, y) \ge 0$, $s, t \in \mathbb{R}$, $x, y \in X$, such that $\kappa(s, x, t, y) dt dy$ is a forward kernel and $(s, x) \mapsto k(s, x, t, y)$ is jointly measurable for all $t \in \mathbb{R}$ and $y \in X$. We call such κ a (forward) *kernel density* (see [6]). We define $\kappa_0(s, x, t, y) = \kappa(s, x, t, y)$, and

$$\kappa_n(s, x, t, y) = \int_s^t \int_X \kappa_{n-1}(s, x, u, z) \int_{(u,t)\times X} J(u, z, du_1 dz_1) \kappa(u_1, z_1, t, y)\, dz\, du\,,$$

where $n = 1, 2, \ldots$. Let $\widetilde{\kappa} = \sum_{n=0}^{\infty} \kappa_n$. For all $s < t \in \mathbb{R}$, $x, y \in X$, we assume

$$\int_s^t \int_X \kappa(s, x, u, z) \int_{(u,t)\times X} J(u, z, du_1 dz_1) \kappa(u_1, z_1, t, y) dz du \le [\eta + Q(s, t)] \kappa(s, x, t, y).$$

Theorem 3 *Under the assumptions, for* $n = 1, 2, \ldots$, $s < t$ *and* $x, y \in X$,

$$\begin{aligned}\kappa_n(s, x, t, y) &\le \kappa_{n-1}(s, x, t, y) \left[\eta + \frac{Q(s, t)}{n}\right] \\ &\le \kappa(s, x, t, y) \prod_{l=1}^{n} \left[\eta + \frac{Q(s, t)}{l}\right].\end{aligned}$$

If $0 < \eta < 1$, *then for all* $s, t \in \mathbb{R}$ *and* $x, y \in X$,

$$\widetilde{\kappa}(s, x, t, y) \le \kappa(s, x, t, y) \left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, *then*

$$\widetilde{\kappa}(s, x, t, y) \le \kappa(s, x, t, y) e^{Q(s,t)}.$$

We also skip this proof, because it is similar to that of Theorem 1.

3 Transition Kernels

Let k above (note the joint measurability) be a *transition kernel* i.e. additionally satisfy the Chapman-Kolmogorov conditions for $s < u < t, A \in \mathscr{M}$,

$$\int_X k(s, x, u, dz)k(u, z, t, A) = k(s, x, t, A).$$

We note that we do *not* assume $k(s, x, t, X) = 1$.

Following [2], we shall show that $\widetilde{k}$ is a transition kernel, too.

Lemma 2 *For all $s < u < t$, $x, y \in X$, $A \in \mathscr{M}$ and $n = 0, 1, \ldots$,*

$$\sum_{m=0}^{n} \int_X k_m(s, x, u, dz)k_{n-m}(u, z, t, A) = k_n(s, x, t, A) \tag{8}$$

Proof We note that (8) is true for $n = 0$ by fact that k is a transition kernel and satisfies the Chapman-Kolmogorov equation. Assume that $n \geq 1$ and (8) holds for $n - 1$. The sum of the first n terms on the left of (8) can be dealt with by induction:

$$\begin{aligned}
&\sum_{m=0}^{n-1} \int_X k_m(s, x, u, dz)k_{n-m}(u, z, t, A) \\
&= \sum_{m=0}^{n-1} \int_X k_m(s, x, u, dz) \int_u^t \int_X k_{n-m-1}(u, z, r, dw) \\
&\quad \int_{(r,\infty)\times X} J(r, w, dr_1 dw_1)k(r_1, w_1, t, A)dr \qquad (9) \\
&= \int_u^t \int_X \int_{(r,\infty)\times X} J(r, w, dr_1 dw_1)k(r_1, w_1, t, A) \\
&\quad \sum_{m=0}^{n-1} \int_X k_m(s, x, u, dz)k_{(n-1)-m}(u, z, r, dw)dr \\
&= \int_u^t \int_X k_{n-1}(s, x, r, dw) \int_{(r,\infty)\times X} J(r, w, dr_1 dw_1)k(r_1, w_1, t, A)dr.
\end{aligned}$$

The $(n+1)$-st term on the left of (8) is

$$\int_X k_n(s,x,u,dz)k(u,z,t,A) = \int_X \int_s^u \int_X k_{n-1}(s,x,r,dw) \int_{(r,\infty)\times X} J(r,w,dr_1dw_1)k(r_1,w_1,u,dz)k(u,z,t,A)dr \tag{10}$$

$$= \int_s^u \int_X k_{n-1}(s,x,r,dw) \int_{(r,\infty)\times X} J(r,w,dr_1dw_1)k(r_1,w_1,t,A)dr,$$

and (8) follows on adding (9) and (10). □

Lemma 3 *For all $s < u < t$, $x, y \in \mathbb{R}^d$ and $A \in \mathscr{M}$,*

$$\int_X \widetilde{k}(s,x,u,dz)\widetilde{k}(u,z,t,A) = \widetilde{k}(s,x,t,A).$$

We refer to [2, Lemma 2] for the proof, based on (8). Thus, $\widetilde{k}$ is a transition kernel.

Similarly, the function κ considered above (note the joint measurability) is called transition density if it satisfies Chapman-Kolmogorov equations pointwise. In an analogous way we then prove that $\widetilde{\kappa}$ defined above is a transition density, provided so is κ.

4 Signed Perturbation

The following discussion is modeled after [2]. We consider perturbation of K by $m(s,x,t,y)J(s,x,dtdy)$, where $m : \mathbb{R}\times X\times\mathbb{R}\times X \to [-1,1]$ is jointly measurable. If $\widetilde{K}$, our perturbation of K by J, is finite, then the perturbation series resulting from mJ is absolutely convergent, and the perturbation formula extends to this case. For instance, the perturbation of K by $-J$ is

$$\widetilde{K}^- = \sum_{n=0}^{\infty} (-1)^n (KJ)^n K,$$

and

$$\widetilde{K}^- = K - \widetilde{K}^- JK.$$

Clearly, if $\widetilde{K}^- \geq 0$, then $\widetilde{K}^- \leq K$, but the former property is delicate cf. [2, Sect. 4]. In this connection we note that if K is restricted to $S = (s,t] \times X$, then under the assumptions of Theorem 1 by (4) we have (on S)

$$\begin{aligned}\widetilde{K}^- &= [K - KJK] + [(KJ)^2K - (KJ)^3K] - \cdots \\ &\geq \sum_{n=0,\,2,\ldots} \left(1 - \eta - \frac{Q(s,t)}{n+1}\right)(KJ)^nK \geq \frac{1-\eta}{2}K,\end{aligned}$$

provided $Q(s,t) \leq (1-\eta)/2$ and we also have (on S)

$$\begin{aligned}\widetilde{K}^- &= K - [KJK - (KJ)^2K] - [(KJ)^3K - (KJ)^4K] - \cdots \\ &\leq K - \sum_{n=1,\,3,\ldots} \left(1 - \eta - \frac{Q(s,t)}{n+1}\right)(KJ)^nK \leq K,\end{aligned} \tag{11}$$

provided $Q(s,t) \leq 2(1-\eta)$. Chapman-Kolmogorov equations allow to propagate this for *transition* kernels k as follows. If $s = u_0 < u_1 < \cdots < u_{n-1} < u_n = t$ and $Q(u_{l-1}, u_l) \leq (1-\eta)/2$ for $l = 1, 2, \ldots, n$, then

$$\begin{aligned}\widetilde{k}(s,x,t,A) &= \int_X \ldots \int_X \widetilde{k}(s,x,u_1,dz_1)\widetilde{k}(u_1,z_1,u_2,dz_2)\ldots\widetilde{k}(u_{n-1},z_{n-1},t,A) \\ &\geq \left(\frac{1-\eta}{2}\right)^n \int_X \ldots \int_X k(s,x,u_1,dz_1)k(u_1,z_1,u_2,dz_2)\ldots \\ &\qquad\qquad k(u_{n-1},z_{n-1},t,A) \\ &= \left(\frac{1-\eta}{2}\right)^n k(s,x,t,A).\end{aligned} \tag{12}$$

If $Q(s,t) \leq h(t-s)$ for a function h, and $h(0^+) = 0$, then global nonnegativity and lower bounds for $\widetilde{k}^-$ easily follow, and so

$$0 \leq \widetilde{k}^- \leq k.$$

Analogous results hold pointwise for *transition densities* κ (we skip details).

We remark that estimates of transition kernels give bounds for the corresponding resolvent and potential operators provided we also have bounds for large times (see [4, Lemma 7] and (25) in this connection).

5 Nonlocal Schrödinger Perturbations

The results of the preceding sections do not allow for $q(s,x,dtdy)$ concentrated on $\{s\} \times X \subset E$. In fact there is some evidence that kernels concentrated on $[t,\infty) \times X$ rather than on $(t,\infty) \times X$ require special attention, see [3, Examples 4.4and 4.5]. In this section we give results for special, instantaneous perturbations q nonlocal in space.

Let $\delta_s(B) = \mathbb{1}_B(s)$ denote the Dirac measure at $s \in \mathbb{R}$. Assume that kernel q on $(E, \mathscr{E})$ is instantaneous in time, i.e. $q(s,x,dtdy) = q(s,x,dtdy)\mathbb{1}_{t=s}$ or $q(s,x,dtdy) = j(s,x,dy)\delta_s(dt)$, where $j(s,x,dy) = q(s,x,\mathbb{R}\times dy)$.

Theorem 4 *If* $KqK(s,x,A) \le \int\limits_A [\eta + Q(s,t)]K(s,x,dtdy)$*, then*

$$K_n(s,x,dtdy) \le K_{n-1}(s,x,dtdy)\left[\eta + \frac{Q(s,t)}{n}\right], \tag{13}$$

$$\le K(s,x,dtdy)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right], \tag{14}$$

for all $n = 1, 2, \ldots$*, and* $(s,x) \in E$*. If* $0 < \eta < 1$*, then for all* $(s,x) \in E$*,*

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}. \tag{15}$$

If $\eta = 0$*, then for all* $(s,x) \in E$*,*

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)e^{Q(s,t)}. \tag{16}$$

We skip the proof, because it is similar to those given in previous sections. We shall also give, without proofs, two pointwise variants of Theorem 4.

Fix a (nonnegative) σ-finite, non-atomic measure

$$dt = \mu(dt)$$

on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and a function $k(s,x,t,A)$ defined for $s < t$, $x \in X$, $A \in \mathscr{M}$, such that $k(s,x,t,dy)dt$ is a forward kernel and $(s,x) \mapsto k(s,x,t,A)$ is jointly measurable for all $t \in \mathbb{R}$ and $A \in \mathscr{M}$. Let $k_0 = k$, and for $n = 1, 2, \ldots$,

$$k_n(s,x,t,A) = \int_s^t \int_X k_{n-1}(s,x,u,dz)\int_X j(u,z,dw)k(u,w,t,A)du.$$

The perturbation, $\widetilde{k}$, of k by q, is defined as

$$\widetilde{k} = \sum_{n=0}^{\infty} k_n.$$

Assume that

$$\int\limits_s^t \int\limits_X k(s,x,u,dz)\int\limits_X j(u,z,dw)k(u,w,t,A)du \le [\eta + Q(s,t)]k(s,x,t,A).$$

Theorem 5 *Under the assumptions, for all $n = 1, 2, \ldots$, and $(s, x) \in E$,*

$$k_n(s, x, t, dy) \le k_{n-1}(s, x, t, dy)\left[\eta + \frac{Q(s,t)}{n}\right],$$
$$\le k(s, x, t, dy)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right].$$

If $0 < \eta < 1$, then for all $(s, x) \in E$,

$$\widetilde{k}(s, x, t, dy) \le k(s, x, t, dy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, then for all $(s, x) \in E$,

$$\widetilde{k}(s, x, t, dy) \le k(s, x, t, dy)e^{Q(s,t)}.$$

For the *finest* variant of Theorem 4, we fix a σ-finite measure

$$dz = m(dz)$$

on $(X, \mathscr{M})$. We consider function $\kappa(s, x, t, y) \ge 0$, $s, t \in \mathbb{R}$, $x, y \in X$, such that $\kappa(s, x, t, y)dtdy$ is a forward kernel and $(s, x) \mapsto k(s, x, t, y)$ is jointly measurable for all $t \in \mathbb{R}$ and $y \in X$. Let $\kappa_0(s, x, t, y) = \kappa(s, x, t, y)$, and

$$\kappa_n(s, x, t, y) = \int_s^t \int_X \kappa_{n-1}(s, x, u, z) \int_X j(u, z, dw)\kappa(u, w, t, y)\, dz\, du\,,$$

where $n = 1, 2, \ldots$. We assume that for all $s < t \in \mathbb{R}$ and $x, y \in X$,

$$\int_s^t \int_X \kappa(s, x, u, z) \int_X j(u, z, dw)\kappa(u, w, t, y)dzdu \le [\eta + Q(s, t)]\kappa(s, x, t, y).$$

Theorem 6 *Under the assumptions, for $n = 1, 2, \ldots$, $s < t$, $x, y \in X$,*

$$\kappa_n(s, x, t, y) \le \kappa_{n-1}(s, x, t, y)\left[\eta + \frac{Q(s,t)}{n}\right],$$
$$\le \kappa(s, x, t, y)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right].$$

If $0 < \eta < 1$*, then for all* $s < t$ *and* $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \le \kappa(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$*, then for all* $s < t$ *and* $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \le \kappa(s,x,t,y)e^{Q(s,t)}.$$

If k (κ) above is a transition kernel (transition density), then $\widetilde{k}$ is so, too. The proof is the same as in Sect. 3, and shall be skipped. We can also study perturbations by signed $q(s,x,dtdy) = j(s,x,dy)\delta_s(dt)$ with analogous conclusions as in Sect. 4.

6 Application

Verification of our assumptions on KqK requires work. Here is a case study. Let $\alpha \in (0,2)$. Consider the convolution semigroup of functions defined as

$$p_t(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ixu}e^{-t|u|^\alpha}du \quad \text{for} \quad t > 0,\ x \in \mathbb{R}^d. \tag{17}$$

The semigroup is generated by the fractional Laplacian $\Delta^{\alpha/2}$ [1]. By (17),

$$p_t(x) = t^{-\frac{d}{\alpha}}p_1(t^{-\frac{1}{\alpha}}x).$$

By subordination [1] we see that $p_t(x)$ is decreasing in $|x|$:

$$p_t(x) \ge p_t(y) \quad \text{if} \quad |x| \le |y|. \tag{18}$$

We write $f(a,\ldots,z) \approx g(a,\ldots,z)$ if there is a number $0 < C < \infty$ independent of $a,\ldots,z$, i.e. a *constant*, such that $C^{-1}f(a,\ldots,z) \le g(a,\ldots,z) \le Cf(a,\ldots,z)$ for all $a,\ldots,z$. We have (see, e.g., [5]),

$$p_t(x) \approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}}. \tag{19}$$

Noteworthy, $t^{-\frac{d}{\alpha}} \le t/|x|^{d+\alpha}$ iff $t \le |x|^\alpha$. We observe the following property:

$$\text{If} \quad |x| \approx |y|, \quad \text{then} \quad p_t(x) \approx p_t(y).$$

We denote

$$p(s,x,t,y) = p_{t-s}(y-x), \quad x, y \in \mathbb{R}^d, s < t.$$

This p is the transition density of the standard isotropic α-stable Lévy process (Y_t, P^x) in $\mathbb{R}^d$ with the Lévy measure $\nu(dz) = c|z|^{-d-\alpha}dz$, and generator $\Delta^{\alpha/2}$.

We consider nonnegative jointly Borelian $j(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$, and we define the norm

$$\|j\| := \left(\sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d} |j(z, w)|dw\right) \vee \left(\sup_{w\in\mathbb{R}^d} \int_{\mathbb{R}^d} |j(z, w)|dz\right).$$

Lemma 4 *There are* $\eta \in [0, 1)$ *and* $c < \infty$ *such that*

$$\int_s^t du \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw\, p(s, x, u, z)j(z, w)p(u, w, t, y) \le [\eta + c(t-s)]p(s, x, t, y), \quad (20)$$

if $\|j\| < \infty$, $|j(z, w)| \le \varepsilon|w - z|^{-d-\alpha}$ *and* $\varepsilon > 0$ *is sufficiently small.*

Proof Denote $I = p(s, x, u, z)j(z, w)p(u, w, t, y)$. Consider three sets $A_1 = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |z - y| \le 4\}$, $A_2 = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |w - x| \le 4|z - x|\}$ and $B = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |z - x| \le \frac{1}{3}|y - x|,\ |w - y| \le \frac{1}{3}|y - x|\}$. The union of A_1, A_2 and B gives the whole of $\mathbb{R}^d$.

If $|z - y| \le 4|w - y|$, then $p(u, w, t, y) \le c_1 p(u, z, t, y)$, and by (18),

$$\begin{aligned}
\int_s^t du \iint_{A_1} dzdw\, I &\le c_1 \int_s^t du \iint_{A_1} dzdw\, p(s, x, u, z)j(z, w)p(u, z, t, y) \\
&\le c_1\|j\| \int_s^t du \int_{\mathbb{R}^d} dz\, p(s, x, u, z)p(u, z, t, y) \\
&= c_1\|j\|(t-s)p(s, x, t, y),
\end{aligned}$$

which is satisfactory, see (4). The case of A_2 is similar. For B we first consider the case $t - s \le 2|y - x|^\alpha$, and we obtain

$$\begin{aligned}
\int_s^t du \iint_B dzdw\, I &\le \int_s^t du \iint_B dzdw\, p(s, x, u, z)\varepsilon|w - z|^{-d-\alpha}p(u, w, t, y) \\
&\le 3^{d+\alpha}\varepsilon \int_s^t du \iint_B dzdw\, p(s, x, u, z)|y - x|^{-d-\alpha}p(u, w, t, y)
\end{aligned}$$

$$\leq 3^{d+\alpha}\varepsilon \int_s^t du \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dzdw\, p(s,x,u,z)p(u,w,t,y)$$
$$= 3^{d+\alpha}\varepsilon |y-x|^{-d-\alpha}(t-s) \approx 3^{d+\alpha}\varepsilon p(s,x,t,y).$$

In the case $t-s > 2|y-x|^\alpha$ we obtain

$$\int_s^t du \iint_B dzdw\, I = \int_s^{\frac{s+t}{2}} du \iint_B dzdw\, p(s,x,u,z)j(z,w)p(u,w,t,y)$$
$$+ \int_{\frac{s+t}{2}}^t du \iint_B dzdw\, p(s,x,u,z)j(z,w)p(u,w,t,y)$$
$$\leq \int_s^{\frac{s+t}{2}} du \iint_B dzdw\, p(s,x,u,z)j(z,w)(t-u)^{-\frac{d}{\alpha}}$$
$$+ \int_{\frac{s+t}{2}}^t du \iint_B dzdw\, (u-s)^{-\frac{d}{\alpha}} j(z,w)p(u,w,t,y)$$
$$\leq \int_s^{\frac{s+t}{2}} du \iint_B dzdw\, p(s,x,u,z)j(z,w)\left(\frac{t-s}{2}\right)^{-\frac{d}{\alpha}}$$
$$+ \int_{\frac{s+t}{2}}^t du \iint_B dzdw\, \left(\frac{t-s}{2}\right)^{-\frac{d}{\alpha}} j(z,w)p(u,w,t,y)$$
$$\leq 2^{\frac{d}{\alpha}}\|j\|(t-s)^{-\frac{d}{\alpha}}(t-s) \approx 2^{\frac{d}{\alpha}}\|j\|(t-s)p(s,x,t,y).$$

We can take $\eta = 3^{d+\alpha}\varepsilon$ and $c = c_1\|j\| + 2^{d/\alpha}\|j\|$ in (20). □

In what follows, $\widetilde{p}$ denotes the perturbation of p by $q(s,x,dtdy) = j(x,y)\delta_s(dt)dy$, and $\widetilde{p}^-$ is the perturbation of p by $-q$. In view of Theorem 6 and (12) we obtain the following result.

Corollary 1 *If* (20) *holds with* $0 \leq \eta < 1$*, then for* $s,t \in \mathbb{R}$*,* $x,y \in \mathbb{R}^d$*,*

$$\widetilde{p}(s,x,t,y) \leq p(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+c(t-s)/\eta}, \tag{21}$$

and

$$p(s,x,t,y)\left(\frac{1-\eta}{2}\right)^{1+2c(t-s)/(1-\eta)} \le \widetilde{p}^{\,-}(s,x,t,y) \le p(s,x,t,y).$$

If $j(z,w)=j(w,z)$, then the estimates agree with those obtained in [8].

We shall verify that $\widetilde{p}$ is the fundamental solution of $\Delta^{\alpha/2}+q$, i.e.

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \widetilde{p}(s,x,t,y)[\partial_t+\Delta_y^{\alpha/2}+j(x,y)]\phi(t,y)dydt = -\phi(s,x), \tag{22}$$

provided (20) holds with $0 \le \eta < 1$. Here and below $s\in\mathbb{R}$, $x\in\mathbb{R}^d$, and ϕ is a smooth compactly supported function on $\mathbb{R}\times\mathbb{R}^d$. By (17) (see also [5]),

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} p(s,x,t,y)[\partial_t+\Delta_y^{\alpha/2}]\phi(t,y)dydt = -\phi(s,x). \tag{23}$$

We denote $P(s,x,dt,dy) = p(s,s,t,y)dtdy$, $(L\phi)(s,x) = \partial_t\phi(s,x)+\Delta_y^{\alpha/2}\phi(s,x)$ and $\widetilde{P}(s,x,dt,dy)=\widetilde{p}(s,x,t,y)dtdy$. By (23), $PL\phi=-\phi$. By (1) and (21),

$$\widetilde{P}(L+q)\phi = PL\phi + \sum_{n=1}^{\infty}(Pq)^n PL\phi + \sum_{n=0}^{\infty}(Pq)^{n+1}\phi = -\phi, \tag{24}$$

where the series converge absolutely. This proves (22). We see that the argument is quite general, and hinges only on the convergence of the series.

We now return to the setting of Theorem 5 to illustrate the influence of the perturbation on jump intensity of Markov processes. We consider k being the transition probability of a Lévy process $(X_t)_{t\ge 0}$ on $\mathbb{R}^d$ [15]. Let $\nu(dy)$ be the Lévy measure, i.e. the jump intensity of (X_t). We have $k(s,x,t,A)=\rho_{t-s}(A-x)$, where $t>s$ and ρ_t is the distribution of X_t. Let μ be a finite measure on $\mathbb{R}^d$ and $q(s,x,dtdy)=\mu(dy-x)\delta_s(dt)$ for $s<t$. By induction we verify that

$$k_n(s,x,t,dy) = \frac{(t-s)^n}{n!}\rho_{t-s}*\mu^{*n}(dy-x).$$

Therefore,

$$\widetilde{k}(s,x,t,dy) = \rho_{t-s}*\sum_{n=0}^{\infty}\frac{(t-s)^n}{n!}\mu^{*n}(dy-x)$$

cf. [7], and so

$$e^{-(t-s)|\mu|}\widetilde{k}(s,x,t,dy) \tag{25}$$

is the transition probability of a Lévy process with the Lévy measure $\nu+\mu$. Thus, perturbing k by q adds jumps and some mass to (X_t), and perturbing by $-q$ reduces jumps and mass of (X_t), as long as $\nu - \mu$ is nonnegative. This is sometimes called P. Meyer's procedure of adding/removing jumps in probability literature.

We like to note that *subtracting* jumps may destroy our (local in time, global in space) comparability of k and $\widetilde{k}^{-}$. Indeed, we can make $\nu(dz) - \mu(dz)$ a compactly supported Lévy measure, whose transition probability has a different, superexponential decay in space (compare [16, Lemma 2] and (19)). This sheds some light on the smallness assumption on ε in Lemma 4 and Corollary 1.

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