ML-α-Deconvolution Model in a Bounded Domain with a Vertical Regularization

Hani Ali

Abstract In this chapter, we consider the deconvolution modified Leray alpha (ML-α-deconvolution) model with fractional filter acting only in one variable $\mathbb{A}_{3,\theta} = I + \alpha_3^{2\theta}(-\partial_3)^{2\theta}$, where $0 \le \theta \le 1$ controls the degree of smoothing in the filter. We study the global existence and uniqueness of solutions to the vertical ML-α-deconvolution model on a bounded product domain of the type $D = \Omega \times (-\pi,\pi)$, where Ω is a smooth domain with homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega \times (-\pi,\pi)$, and with periodic boundary conditions in the vertical variable. To present the model, we define the vertical Nth Van Cittert deconvolution operator by $D_{N,\theta} = \sum_{i=0}^{N}(I - \mathbb{A}_{3,\theta}^{-1})^i$. The vertical ML-$\alpha$-deconvolution model is then defined by replacing the nonlinear term in the Navier–Stokes equations $(v \cdot \nabla)v$ by $(v \cdot \nabla)D_{N,\theta}(\overline{v})$ where v is the velocity, and $\overline{v} = \mathbb{A}_{3,\theta}^{-1}(v)$ is the smoothed velocity. We adapt the ideas from (H. Ali, Approximate Deconvolution Model in a bounded domain with a vertical regularization. J Math Anal Appl **408**, 355–363 (2013)) to prove that the vertical ML-α-deconvolution model which is derived by using $\mathbb{A}_{3,\theta}$, has a unique weak solution for any $\theta > \frac{1}{2}$.

1 Introduction

In this chapter, we consider the deconvolution modified Leray alpha (ML-α-deconvolution) model with fractional filter acting only in one variable

$$\mathbb{A}_{3,\theta} := I + \alpha_3^{2\theta}(-\partial_3)^{2\theta}, \quad 0 \le \theta \le 1, \tag{1}$$

where θ controls the degree of smoothing in the filter.

This filter is less memory consuming than the classical one (see, e.g., [3, 5, 7, 8]). Moreover, there is no need to introduce artificial boundary conditions for Helmholtz operator. It was shown in [4] that the Large Eddy Simulation models which are derived by using $\mathbb{A}_{3,\theta}$ for any $\theta > \frac{1}{2}$, are well posed. Motivated by this work [4],

H. Ali (✉)
Institut Jean le Rond d'Alembert, CNRS & UPMC Université Paris 06,
UMR 7190, 75005 Paris, France
e-mail: hani.ali@etu.upmc.fr

M. G. Cojocaru et al. (eds.), *Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science,* Springer Proceedings in Mathematics & Statistics 117,
DOI 10.1007/978-3-319-12307-3_2

we study the global existence and uniqueness of solutions to the vertical ML-α-deconvolution model on a bounded product domain of the type $D = \Omega \times (-\pi, \pi)$, where Ω is a smooth domain with homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega \times (-\pi, \pi)$, and with periodic boundary conditions in the vertical variable. To present the model, we define the vertical Nth Van Cittert deconvolution operator by

$$D_{N,\theta} = \sum_{i=0}^{N} (I - \mathbb{A}_{3,\theta}^{-1})^i. \tag{2}$$

The vertical ML-α-deconvolution model is then defined, for some fixed $\theta > 0$, with a filtering radius $\alpha_3 > 0$, a kinematic viscosity $\nu > 0$, a deconvolution order $N \geq 0$, and an initial velocity v_0 as follows,

$$\partial_t v + (v \cdot \nabla) D_{N,\theta}(\overline{v}) - \nu \Delta v + \nabla p = f, \tag{3}$$

$$\nabla \cdot v = 0, \tag{4}$$

$$v(0) = v_0, \tag{5}$$

where v and p are the velocity and the pressure, $\overline{v} = \mathbb{A}_{3,\theta}^{-1}(v)$ is the smoothed velocity, and f is a forcing term.

For simplicity, we consider the domain $D = \{x \in \mathbb{R}^3, x_1^2 + x_2^2 < d, \ -\pi < x_3 < \pi\}$ with 2π periodicity with respect to x_3. Therefore, the deconvolution model in this chapter is chosen to model the flow through a cylinder or a pipe with periodic boundary conditions with respect to x_3. We note that the filter is acting only in the vertical variable, that is why it is possible to require the periodicity only in x_3. Moreover, we consider the unfiltered function with homogeneous Dirichlet boundary conditions on the boundary $\partial D = \partial\Omega \times (-\pi, \pi)$. These boundary conditions of the unfiltered function are supposed to be the same as the filtered ones, in order to prevent from introducing artificial boundary conditions. In order to state our main result, let us define the following spaces:

$$L^2(D) := \left\{v \in L^2(D)^3, 2\pi\text{-periodic in } x_3\right\}, \tag{6}$$

$$H := \left\{v \in L^2(D), \text{ such that } \nabla \cdot v = 0 \text{ and } v \cdot n = 0 \text{ on } \partial\Omega \times (-\pi, \pi)\right\}, \tag{7}$$

$$V := \left\{v \in H, \text{ such that } \nabla v \in L^2(D) \text{ and } v = 0 \text{ on } \partial\Omega \times (-\pi, \pi)\right\}. \tag{8}$$

Next, we give a definition of what is called a weak solution to the vertical ML-α-deconvolution model.

Definition 1 Let $f \in L^2(0, T; H)$ and $v_0 \in H$. For any $0 \leq \theta \leq 1$ and $0 \leq N < \infty$, the couple (v, p) is called a weak solution to (3)–(5) if

$$v \in \mathcal{C}_w(0, T; H) \cap L^2(0, T; V), \tag{9}$$

and the couple (v, p) fulfills

$$\begin{aligned}\int_0^T \langle \partial_t v, \varphi\rangle - \langle D_{N,\theta}(\overline{v}) \otimes v, \nabla\varphi\rangle + \nu\langle \nabla v, \nabla\varphi\rangle + \langle \nabla p, \varphi\rangle \, dt \\ = \int_0^T \langle f, \varphi\rangle \, dt \qquad \text{for all } \varphi \in \mathcal{C}_c^\infty([0, T] \times D).\end{aligned} \tag{10}$$

Moreover,

$$v(0) = v_0. \tag{11}$$

Our main result is the following.

Theorem 1 *Assume $f \in L^2(0, T; H)$ and $v_0 \in H$. let $0 \leq N < \infty$ be a given and fixed number and let $\theta > \frac{1}{2}$. Then problem (3)–(5) has a unique weak solution.*

This result holds also true on the whole space $\mathbb{R}^3$ and on the torus $\mathbb{T}_3$. The vertical ML-α-deconvolution with $N = 0$ becomes the modified Leray alpha (ML-α) model of turbulence [2, 6] with filter acting only in one variable. Consequently, Theorem 1 gives us also existence and uniqueness of solutions to the vertical ML-α model of turbulence on the bounded domain D. Other α models, with partial filter, will be reported in a forthcoming paper.

2 Notation and Auxiliary Result

In this section, we introduce relevant function spaces and we recall an auxiliary result used in the proof of the main result.

Let $1 < p \leq +\infty$ and $1 < q \leq +\infty$. We denote by $L_v^q L_h^p(D) = L^q((-\pi, +\pi); L^p(\Omega))$ the space of functions g such that $(\int_{-\pi}^{+\pi} (\int_\Omega |g(x_1, x_2, x_3)|^p dx_1 dx_2)^{q/p} dx_3)^{1/q} < +\infty$.

We denote by $\|v\|_2 := \int_D v \cdot v \, dx$ the usual norm in $L^2(D)^3$.

The following lemma will play an important role [4].

Lemma 1 *There exists a positive constant $C > 0$ such that, for any $s > \frac{1}{2}$ and for any smooth enough divergence-free vector fields u, v, and w, the following estimate holds,*

$$|((u \cdot \nabla)v, w)| \leq C \|u\|_2^{\frac{1}{2}} \|\nabla u\|_2^{\frac{1}{2}} \left(\|\nabla v\|_2^{1-\frac{1}{2s}} \|\partial_3^s \nabla v\|_2^{\frac{1}{2s}} + \|\nabla v\|_2 \right) \|w\|_2^{\frac{1}{2}} \|\nabla w\|_2^{\frac{1}{2}}.$$

3 The Vertical Filter and the Vertical Deconvolution Operator

In this section, we record some properties of the vertical filter and of the vertical deconvolution operator. Let v be a smooth function of the form $v = \sum_{k_3 \in \mathbb{Z} \setminus \{0\}} c_{k_3}(x_1, x_2) e^{\mathrm{i} k_3 x_3}$. The action of the vertical filter on $v(x) =$

$\sum_{k_3\in\mathbb{Z}\setminus\{0\}} c_{k_3}(x_1,x_2)e^{\mathrm{i}\,k_3x_3}$ can be written as $\mathbb{A}_{3,\theta}(v) = \sum_{k_3\in\mathbb{Z}\setminus\{0\}} \mathcal{A}_\theta(k_3)c_{k_3}(x_1,x_2)$ $e^{\mathrm{i}\,k_3x_3}$, where the symbol with respect to x_3 of the vertical filter is given by

$$\mathcal{A}_\theta(k_3) = \left(1+\alpha^{2\theta}|k_3|^{2\theta}\right). \tag{12}$$

Therefore, by using the Parseval's identity with respect to x_3 we get,

$$\|\mathbb{A}_{3,\theta}^{\frac{1}{2}}v\|_2^2 = \|v\|_2^2 + \alpha^{2\theta}\|\partial_3^\theta v\|_2^2 = \left(\mathbb{A}_{3,\theta}v, v\right). \tag{13}$$

The deconvolution operator $D_{N,\theta} = \sum_{i=0}^{N}(I-\mathbb{A}_{3,\theta}^{-1})^i$ is constructed by using the vertical filter with fractional regularization (1). For a fixed $N>0$ and for $\theta=1$, we recover a vertical operator form from the Van Cittert deconvolution operator.

A straightforward calculation yields

$$D_{N,\theta}\left(\sum_{k_3\in\mathbb{Z}\setminus\{0\}} c_{k_3}(x_1,x_2)e^{\mathrm{i}\,k_3x_3}\right) = \sum_{k_3\in\mathbb{Z}\setminus\{0\}} \mathcal{D}_{N,\theta}(k_3)c_{k_3}(x_1,x_2)e^{\mathrm{i}\,k_3x_3}, \tag{14}$$

where for $k_3\in\mathbb{Z}\setminus\{0\}$ and $\theta\geq 0$, $\mathcal{D}_{N,\theta}(k_3)$ verifies:

$$\mathcal{D}_{0,\theta}(k_3) = 1, \tag{15}$$

$$1 \leq \mathcal{D}_{N,\theta}(k_3) \leq N+1 \quad \text{for each } N>0, \tag{16}$$

$$\text{and } \mathcal{D}_{N,\theta}(k_3) \leq \mathcal{A}_{3,\theta} \quad \text{for a fixed } \alpha>0. \tag{17}$$

From the previous hypothesis, one can prove the following Lemma by adapting the results summarized in the isotropic case in [1]:

Lemma 2 *For all $s\geq -1$, $\theta\geq 0$, $k_3\in\mathbb{Z}\setminus\{0\}$ and for each $N>0$, there exists a constant $C>0$ such that for all v sufficiently smooth we have*

$$\|v\|_{s,2} \leq \|D_{N,\theta}(v)\|_{s,2} \leq (N+1)\|v\|_{s,2}, \tag{18}$$

$$\|\mathbb{A}_{3,\theta}{}^{-\frac{1}{2}}D_{N,\theta}^{\frac{1}{2}}(v)\|_{s,2} \leq \|v\|_{s,2}, \tag{19}$$

$$\|\mathbb{A}_{3,\theta}^{-\frac{1}{2}}(v)\|_{s,2}^2 = \|\overline{v}\|_{s,2}^2 + \alpha_3^{2\theta}\|\partial_3^\theta\overline{v}\|_{s,2}^2. \tag{20}$$

4 Sketch of the Proof of the Main Result

We briefly present the main ideas of the proof of Theorem 1. The proof follows from the following a priori estimates with a Galerkin method.

For further information, we refer the reader to [1, 4] and the references therein.

Proof Multiplying (3) with $D_{N,\theta}(\overline{v})$ integrating over time from 0 to t, for all $t\in[0,T]$, and using standard manipulations lead to the a priori estimate

$$\begin{aligned}\frac{1}{2}\|\mathbb{A}_\theta^{-\frac{1}{2}}D_{N,\theta}^{\frac{1}{2}}(v)\|_2^2 + \nu\int_0^t \|\nabla\mathbb{A}_\theta^{-\frac{1}{2}}D_{N,\theta}^{\frac{1}{2}}(v)\|_2^2\,ds \\ = \int_0^t \langle\mathbb{A}_\theta^{-\frac{1}{2}}D_{N,\theta}^{\frac{1}{2}}(f), \mathbb{A}_\theta^{-\frac{1}{2}}D_{N,\theta}^{\frac{1}{2}}(v)\rangle\,ds + \frac{1}{2}\|\mathbb{A}_\theta^{-\frac{1}{2}}D_{N,\theta}^{\frac{1}{2}}(v_0)\|_2^2.\end{aligned} \tag{21}$$

By using the duality norm combined with Young inequality and inequality (19), we conclude from (21) that

$$\begin{aligned}\sup_{t\in[0,T]} \|\mathbb{A}_\theta^{-\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}}(v)\|_2^2 + \nu \int_0^T \|\nabla \mathbb{A}_\theta^{-\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}}(v)\|_2^2\, dt \\ \leq \|v_0\|_2^2 + \frac{C}{\nu}\int_0^T \|f\|_2^2\, dt.\end{aligned} \tag{22}$$

We deduce from (22) and (20) that

$$\overline{v} \text{ and } \partial_3^\theta \overline{v} \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{23}$$

Thus, it follows from (18) that

$$D_{N,\theta}(\overline{v}^n) \text{ and } \partial_3^\theta D_{N,\theta}(\overline{v}^n) \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{24}$$

Multiplying (3) with v we conclude that

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \nu\|\nabla v\|_2^2 \leq \left|\left((v\cdot\nabla)D_{N,\theta}(\overline{v}), v\right)\right| + |\langle f, v\rangle|\,. \tag{25}$$

For $\theta > \frac{1}{2}$ we have

$$\begin{aligned}\left|\left((v\cdot\nabla)D_{N,\theta}(\overline{v}), v\right)\right| &\leq C\|v\|_2\|\nabla v\|_2 \\ &\quad\times \left(\|\nabla D_{N,\theta}(\overline{v})\|_2^{1-\frac{1}{2\theta}} \|\partial_3^\theta \nabla D_{N,\theta}(\overline{v})\|_2^{\frac{1}{2\theta}} + \|\nabla D_{N,\theta}(\overline{v})\|_2\right) \\ &\leq C\left(\|\nabla D_{N,\theta}(\overline{v})\|_2^{2-\frac{1}{\theta}} \|\partial_3^\theta \nabla D_{N,\theta}(\overline{v})\|_2^{\frac{1}{\theta}} + \|\nabla D_{N,\theta}(\overline{v})\|_2^2\right) \\ &\quad\times \|v\|_2^2 + \frac{\nu}{4}\|\nabla v\|_2^2,\end{aligned} \tag{26}$$

where we have used Lemma 1 and the Young inequality.

The second term in right hand side of (25) is estimated by

$$|\langle f, v\rangle| \leq C\|f\|_2\|\nabla v\|_2 \leq C\|f\|_2^2 + \frac{\nu}{4}\|\nabla v\|_2^2. \tag{27}$$

Thus, (26) and (27) lead to the conclusion that

$$\begin{aligned}&\frac{d}{dt}\|v\|_2^2 + \nu\|\nabla v\|_2^2 \leq \\ &\quad C\left(\|\nabla D_{N,\theta}(\overline{v})\|_2^{2-\frac{1}{\theta}} \|\partial_3^\theta \nabla D_{N,\theta}(\overline{v})\|_2^{\frac{1}{\theta}} + \|\nabla D_{N,\theta}(\overline{v})\|_2^2\right)\|v\|_2^2 + C\|f\|_2^2.\end{aligned} \tag{28}$$

Integrating (28) over time from 0 to T and using Gronwall's Lemma and (24) lead to the following estimate

$$\sup_{t\in[0,T]} \|v\|_2^2 + \nu\int_0^T \|\nabla v\|_2^2\, dt \leq C. \tag{29}$$

We deduce from (29) that

$$v \in L^\infty(0, T; H) \cap L^2(0, T; V). \tag{30}$$

Finally, we check the question of the uniqueness of the solution. Let $\theta > \frac{1}{2}$ and let (v_1, p_1) and (v_2, p_2) be any weak solutions of (3)–(5) on the interval $[0, T]$, with initial values $v_1(0)$ and $v_2(0)$. Let us denote by $\delta v = v_2 - v_1$ and $\delta D_{N,\theta}(\overline{v}) = D_{N,\theta}(\overline{v}_2) - D_{N,\theta}(\overline{v}_1)$. We subtract the equation for v_1 from the equation for v_2 and test it with δv, we formally get:

$$\begin{aligned} &\frac{d}{dt}\|\delta v\|_2^2 + \nu\|\nabla \delta v\|_2^2 \\ &\quad \le C\|\nabla D_{N,\theta}(\overline{v}_1)\|_2^{2-\frac{1}{\theta}}\|\partial_3^\theta \nabla D_{N,\theta}(\overline{v}_1)\|_2^{\frac{1}{\theta}}\|\delta v\|_2^2 + C\|v_2\|_2^2\|\nabla v_2\|_2^2\|\delta v\|_2^2 \end{aligned} \tag{31}$$

where we have used Lemma 1, the Young inequality and the fact that $\|\nabla \delta D_{N,\theta}(\overline{v})\|_2 \le C\|\nabla \delta v\|_2$ and $\|\nabla \partial_3^\theta \delta D_{N,\theta}(\overline{v})\|_2 \le C\|\nabla \delta v\|_2$.

Since $\|\nabla D_{N,\theta}(\overline{v}_1)\|_2^{2-\frac{1}{\theta}}\|\partial_3^\theta \nabla D_{N,\theta}(\overline{v}_1)\|_2^{\frac{1}{\theta}} + \|v_2\|_2^2\|\nabla v_2\|_2^2 \in L^1([0, T])$, we conclude by using Gronwall's inequality the continuous dependence of the solutions on the initial data in the $L^\infty(0, T; H)$ norm. In particular, if $\delta v_0 = 0$ then $\delta v = 0$ and the solutions are unique for all $t \in [0, T]$. □

References

1. Ali, H.: On the High Accuracy ML-α-Deconvolution Turbulence Model. Under revision in Analysis and Applications
2. Ali, H.: Ladder theorem and length scale estimates for the modified Leray-alpha model of turbulence. Commun. Math. Sci. **2,** 477–491 (2012)
3. Ali, H.: On a critical Leray-α model of turbulence. Nonlinear Anal.: Real World Appl. **14,** 1536–1584 (2013)
4. Ali, H.: Approximate deconvolution model in a bounded domain with vertical regularization. J. Math. Anal. Appl. **408,** 355–363 (2013)
5. Ali, H.: Theory for the rotational deconvolution model of turbulence with fractional regularization. Appl. Anal. **93,** 339–355 (2014)
6. Ilyin, A.A., Lunasin, E.M., Titi, E.S.: A modified Leray-alpha subgrid-scale model of turbulence. Nonlinearity **19,** 879–897 (2006)
7. Layton, W., Lewandowski, R.: A high accuracy Leray-deconvolution model of turbulence and its limiting behavior. Anal. Appl. **6,** 1–27 (2008)
8. Neda, M., Rebholz, L.G., Layton, W., Manica, C.C.: Numerical analysis and computational testing of a high-accuracy Leray-deconvolution model of turbulence. Numer. Methods Part. Differ. Equ. **24,** 555–582 (2008)

http://www.springer.com/978-3-319-12306-6

Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science

Cojocaru, M.; Kotsireas, I.S.; Makarov, R.; Melnik, R.V.N.; Shodiev, H. (Eds.)

2015, XXIII, 555 p. 145 illus., 102 illus. in color., Hardcover
ISBN: 978-3-319-12306-6