Chapter 2
Periodic Orbits of Planar Integrable Birational Maps

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Abstract A birational planar map F possessing a rational first integral preserves a foliation of the plane given by algebraic curves which, if F is not globally periodic, is given by a foliation of curves that have generically genus 0 or 1. In the genus 1 case, the group structure of the foliation characterizes the dynamics of any birational map preserving it. We will see how to take advantage of this structure to find periodic orbits of such maps.

2.1 Introduction

A planar *rational* map $F : \mathcal{U} \to \mathcal{U}$, where $\mathcal{U} \subseteq \mathbb{K}^2$ is an open set and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, is called *birational* if it has a rational inverse F^{-1}. In this chapter, we will say that a map F is *integrable* if there exists a nonconstant function $V : \mathcal{U} \to \mathbb{K}$ such that

$$V(F(x, y)) = V(x, y),$$

which is called a *first integral* or *invariant* of F. If a map F possesses a first integral V then each orbit lies in some level set of V or, in other words, the level sets of V are invariant under F.

Planar birational maps are a classical object of study in algebraic geometry and have been the focus of intense research activity in recent years (see [24] and references therein). The integrable cases appear in many contexts, from algebraic geometry and number theory to mathematical physics. This is the case of the celebrated QRT family of maps introduced in [44, 45] (see also [26]), which contains the well-known McMillan family of maps, and some of the integrable cases studied by Gumovski and Mira [30, 40]. Many maps in this family arise as special solutions, termed discrete solitons, of differential-difference equations arising in statistical mechanics. The QRT maps all have a rational first integral.

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In this chapter, we will consider only those integrable maps that have *rational first integrals*. In fact, all the examples of integrable birational maps that we know have rational first integrals, but as far as we know there is no reason for an integrable birational map to be rationally integrable. In this sense, it is interesting to recall the case given by the composition maps associated to the five-periodic Lyness recurrences. These maps are birational, and the numerical results show phase portraits compatible with the existence of first integrals, however, it has been recently proved that, generically, these maps are not rationally integrable, see [19, Theorem 19] and [14, Theorem 1 and Proposition 3].

Observe that if the first integral is a rational function,

$$V(x,y)=\frac{P(x,y)}{Q(x,y)},\tag{2.1}$$

then the map preserves the foliation[1] of $\mathcal{U}$ is given by the algebraic curves

$$\mathcal{F}=\{P(x,y)-h\,Q(x,y)=0,h\in\mathrm{Im}(V)\}.\tag{2.2}$$

We will assume that P and Q are coprime and, although it is not essential in this chapter, that V has *minimal* degree. Recall that the degree of a rational first integral is the greater of the degrees of P and Q. We say that the degree n of V is minimal if any other rational first integral of F has degree at least n. Given a rational first integral, one always can find a minimal rational first integral.

In this note, our objective is to show how to take advantage of the algebraic–geometric properties of the invariant foliation $\mathcal{F}$ to study the periodic orbits of the birational maps preserving it. Although the techniques explained in this chapter have been used to study several birational maps [4–7, 9, 26, 53, 54], to illustrate them we will refer only to a particular, but paradigmatic, example: the well-known Lyness family of maps.

Example 2.1 Lyness's maps are a 1-parametric family of birational maps given by

$$F_a(x,y)=\left(y,\frac{a+y}{x}\right),\tag{2.3}$$

These maps give the dynamical system associated to recurrence $x_{n+2}=(a+x_{n+1})/x_n$. There is a large recent literature concerning this family. In the appendix of this chapter the reader can also find a short account of references and the history of the Lyness recurrences and maps.

Each map F_a has the first integral

$$V(x,y)=\frac{(x+1)(y+1)(x+y+a)}{xy},\tag{2.4}$$

[1] In this chapter, we say that a map F preserves a foliation of curves $\{C_h\}$ if each curve C_h is invariant under the iterates of F.

so it preserves the foliation given by

$$\mathcal{F} = \{C_h = \{(x+1)(y+1)(x+y+a) - hxy = 0\},\, h \in \mathrm{Im}(V)\}\,. \tag{2.5}$$

The chapter is structured as follows: in Sect. 2.2, we will recall the notion of genus of an algebraic curve, and we will see that if we are interested in those maps not being globally periodic, then we can consider that the curves in the foliation (2.2) have genus 0 or 1, see Corollary 2.3. In Sect. 2.3, we restrict our attention to maps having invariant curves with genus 1 (also named elliptic curves). We recall the group structure of these curves and also a result of Jogia et al. (Theorem 2.4) which relates the dynamics of a particular birational map on an invariant elliptic curve and its group operation. We will take advantage of this result to obtain a description of the periodic orbits in terms of the torsion of the curve (Eq. 2.8). In Sect. 2.4, we discuss the global dynamics of birational maps preserving a foliation given by elliptic curves C_h. First, we start by introducing and discussing the nature of the rotation number function $\theta(h)$ associated to each curve C_h. Then, we see that a typical situation occurs when there is a dense set of curves in phase space filled by p-periodic orbits of all the periods $p \geq p_0 \in \mathbb{N}$, for some integer p_0 which is sometimes computable (see Proposition 2.9 and Sect. 2.4.3).

In Sect. 2.5, as a straightforward application, we show how to address the problem of finding the curves containing periodic orbits with a prescribed period, by using the characterization of periodic orbits given by the *group law of the curve* (see Eq. (2.8)). We show the main technique by applying it to the Lyness case, as already done in [4].

In Sect. 2.6, we will see how the group structure of rational elliptic curves is strongly related to the existence of rational periodic orbits. We will recall Mazur's theorem and its dynamical implications. We also give some insight on the known results of rational periodic orbits in the Lyness case [4, 29, 54]. This section ends with a digression about why the numerical simulations of the phase portrait of birational maps preserving an elliptic foliation do not show the plethora of periodic orbits that they possess, on the contrary of what happens when general integrable diffeomorphisms are considered.

We end these notes with a comment on the genus 0 case, and with an appendix giving more information about the Lyness maps and curves.

The aim of the chapter is expository, and it is inspired in the papers of Bastien and Rogalski [4] and of Jogia et al. [33]. The reader is invited to read them, as well as their references. Another essential reference is the book of Duistermaat [26] about some algebraic–geometric aspects of QRT maps.

2.2 A First Dynamical Result: Restriction to the Genus 0 and 1 Cases

When studying the dynamics of an integrable map, a first step is to know the topology of the invariant level sets. When the level sets are algebraic curves, the natural way to study them is to consider their extension, and also the extension of the birational maps, to the complex projective space

$$\mathbb{C}\mathbf{P}^2 = \{[x : y : z] \neq [0 : 0 : 0],\ x, y, z \in \mathbb{C}\}/\sim,$$

where $[x_1 : y_1 : z_1] \sim [x_2 : y_2 : z_2]$ if and only if $[x_1 : y_1 : z_1] = \lambda[x_2 : y_2 : z_2]$ for $\lambda \neq 0$.

In this chapter, $[x : y : 1]$ denotes an affine point, corresponding with the point $(x, y) \in \mathbb{K}^2$ (where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$), and $[x : y : 0]$ denotes an infinite point. The infinite points are added to real affine algebraic curves in order to capture the asymptotic directions of possible unbounded components. See Fig. 2.2 for instance.

Any real affine algebraic curve can be extended to $\mathbb{C}\mathbf{P}^2$ by the formal process of *homogenization*. For instance, any Lyness curve

$$C_h = \{(x + 1)(y + 1)(x + y + a) - hxy = 0\} \subset \mathbb{R}^2$$

where $x, y \in \mathbb{R}$ extends to $\mathbb{C}\mathbf{P}^2$ as

$$\widetilde{C_h} := \{(x + z)(y + z)(x + y + az) - hxyz = 0,\ x, y, z \in \mathbb{C}\}.$$

Notice also that any birational map in $\mathbb{R}^2$ extends formally to a polynomial map in $\mathbb{C}\mathbf{P}^2$. For instance, the Lyness map $F_a(x, y) = (y, (a + x)/y)$ extends formally to

$$\widetilde{F_a}([x : y : z]) = [xy : az^2 + yz : xz],$$

except for the points $[x : 0 : 0]$, $[0 : y : 0]$, and $[0 : -a : 1]$ (see also the alternative description given by Eq. (2.7) in Sect. 2.3), where $x, y, z \in \mathbb{C}$.

Any algebraic curve $\widetilde{C}$ in $\mathbb{C}\mathbf{P}^2$ is a Riemann surface characterized by its *genus*, [34]. On any irreducible component of a curve in $\mathbb{C}\mathbf{P}^2$, the genus g is related to the *degree* d by the *degree-genus formula*:

$$g = \frac{(d-1)(d-2)}{2} - \sum_{p \in \mathrm{Sing}(C)} \frac{m_p(m_p - 1)}{2},$$

where m_p stands for the multiplicity of any possible singular ordinary point. Recall that a singular point is called ordinary when all the tangents at the point are distinct and that, given an irreducible curve, it is always possible to find a birationaly equivalent curve with only ordinary multiple points, so that the above formula gives the genus.

In this chapter, we will say that an invariant foliation has *generic genus* g if the genus has constant value g on the irreducible components of $\{P - hQ\}$, except maybe for a finite set of values of $h \in \mathrm{Im}(V)$ for which the genus is lower. This is a common situation. The reader is addressed, for instance, to Pettigrew and Roberts [43] for a characterization of the singular curves corresponding to a biquadratic foliation that generalizes the classical elliptic QRT foliations. We will assume that in our foliations (2.2) the genus is generic.

Next, we will see that if one expects to obtain a rich dynamics of a birational map preserving a foliation $\{C_h\}$, where C_h are irreducible curves, then one has to restrict attention to those maps that preserve foliations of generic genus 0 or 1, because any birational map F preserving a foliation of generic genus greater or equal than 2 is a globally periodic map, that is, there exists $p \in \mathbb{N}$ such that $F^p(x, y) = (x, y)$ for all (x, y) where F is defined. This fact is a consequence of the following two classical results.

Theorem 2.1 (Montgomery, [42]) *Any pointwise periodic homeomorphism in a connected metric space, locally homeomorphic to $\mathbb{R}^n$, is globally periodic.*

The next one is an adaptation to our context of the Hurwitz automorphisms theorem which states that any compact Riemann surface with genus $g > 1$ admits at most $84(g-1)$ conformal automorphisms, that is, homeomorphisms of the surface onto itself which preserve the local structure; see [21, 22]. In our context, Hurwitz's theorem can be stated as follows, [33]:

Theorem 2.2 (Hurwitz, 1893) *The group of birational maps on a nonsingular algebraic curve of genus $g > 1$ is finite, and of order less or equal than $84(g-1)$.*

The above result states that any birational map preserving *a particular* nonsingular curve of genus $g \geq 2$ must be periodic (on the curve) with a period bounded by $84(g-1)$.

Corollary 2.3 ([20]) *A birational map in $\mathcal{U} \subseteq \mathbb{K}^2$ (where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$) preserving a foliation of nonsingular curves $\{C_h\} \subseteq \mathcal{U}$ that have generic genus $g > 1$, must be globally periodic.*

Proof If the foliation $\{C_h\}$ has generic genus $g > 1$, then there exists an open set $\mathcal{V} \subseteq \mathcal{U}$ foliated by curves of genus g. By Hurwitz's theorem on each of these curves the map must be periodic, so F is pointwise periodic on $\mathcal{V}$. Therefore, by Montgomery's theorem, F must be globally periodic on the whole $\mathcal{V}$. Since F is rational, and so the global periodicity is characterized by some formal polynomial identities, then it must be periodic on the whole $\mathbb{K}^2$ except at the points where its iterates are not well-defined. □

In summary, from a dynamic viewpoint it makes sense to restrict our attention to birational maps preserving foliations of algebraic curves with genus 0 or 1.

2.3 The Elliptic Case: Dynamics on Invariant Curves Through Its Group Structure

In this chapter, we will concentrate our attention on those birational maps that preserve a foliation of algebraic curves $\{C_h\}$ of generic genus 1. Recall that a projective algebraic curve of genus 1 is called an *elliptic curve*. Any elliptic curve has an associated group structure [34, 50, 51]. In this section, we will see that in the case that $\{C_h\}$ is generically given by elliptic curves, then the group structure of the elliptic foliation characterizes the dynamics of any birational map preserving it.

First, we recall the group structure associated with an elliptic curve $C \in \mathbb{C}\mathbf{P}^2$, the so called *chord-tangent group* law. Given two points P and Q in C, we define the addition $P + Q$ in the following way:

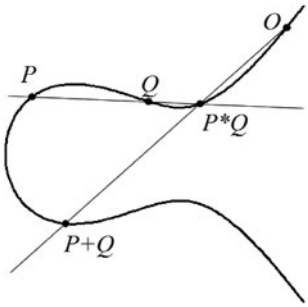

Fig. 2.1 Group law with an affine neutral element $\mathcal{O}$

1. Select a point $\mathcal{O} \in C$ to be the neutral element of the inner addition.
2. Take the chord passing through P and Q (the tangent line if $P = Q$). It will always intersect C at a unique third point denoted by $P * Q$. This is because the curves of genus 1 are birationally homeomorphic to *smooth cubic* curves, [50, Proposition 3.1].
3. The point $P + Q$ is then defined as $\mathcal{O} * (P * Q)$, see Fig. 2.1.

The curve endowed with this inner addition $(C, +, \mathcal{O})$ is an *abelian group* [51].

A brief comment on notation: typically algebraic curves are defined on $\mathbb{K}^2$, or on $\mathbb{K}P^2$, where $\mathbb{K}$ is the field of coefficients. In this chapter, this field will be mainly $\mathbb{R}$ or $\mathbb{C}$ (or $\mathbb{Q}$ in Sect. 2.6). The notation $C(\mathbb{K})$ or $C/\mathbb{K}$ denotes an elliptic curve C which has at least one point $\mathcal{O}$ with coordinates in $\mathbb{K}$. In this chapter, unless we explicitly state the contrary, we will assume that C stands for a *real* curve.

The relationship between the dynamics of a birational map preserving an elliptic curve and its group structure is given by the following adaptation of a result of Jogia et al. [33, Theorem 3], that will be referred as the *JRV theorem* from now on. In [33], the result is stated for birational maps leaving invariant an elliptic curve expressed in a certain Weierstrass normal form (see [34, 50] but especially [51, Sect. I.3]). This adaptation is immediately obtained by using the isomorphism with this normal form.

Theorem 2.4 (Jogia et al. [33]) *Let F be a birational map over a field $\mathbb{K}$, not of characteristic 2 or 3, that preserves an elliptic curve $C(\mathbb{K})$. Then, there exists a point $Q \in C(\mathbb{K})$ such that the map can be expressed in terms of the group law $+$ on $C(\mathbb{K})$ as either*

(i) $F_{|C(\mathbb{K})} : P \mapsto P + Q$, *or*
(ii) $F_{|C(\mathbb{K})} : P \mapsto i(P) + Q$, *where i is an automorphism of possible order (period) 2, 4, 3 or 6, and the map F has the same order (period) as i.*

We will give an easier dynamical interpretation of the above result, but first we will illustrate it.

Example 2.2 The Lyness curves $\widetilde{C_h} := \{(x+z)(y+z)(x+y+az) - hxyz = 0\} \subset \mathbb{C}\mathbf{P}^2$, are elliptic except for $h \in \{0, a-1, h_c^{\pm}\}$, where

$$h_c^{\pm} := \frac{2\,a^2 + 10\,a - 1 \pm (4\,a+1)\sqrt{4\,a+1}}{2\,a}. \tag{2.6}$$

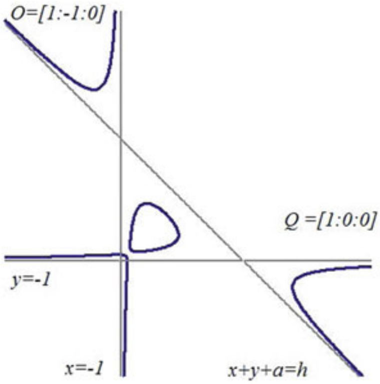

Fig. 2.2 A typical real Lyness curve $C_h = \{(x+1)(y+1)(x+y+a) - hxy = 0\}$, for $h > h_c^+$. Adding the infinite points $[1:0:0]$, $[0:1:0]$, $[1:-1:0]$, the displayed curve is isomorphic to $\mathbb{S}^1 \times \mathbb{Z}/(2)$

An interesting fact is that for all values of h the curves $\widetilde{C_h}$ contain the infinity points $[1:0:0]$, $[0:1:0]$, $[1:-1:0]$, see Fig. 2.2. An straightforward computation (or a geometrical interpretation) shows that, setting $\mathcal{O} := [1:-1:0]$, for any elliptic level h the map $\widetilde{F}_a([x:y:z]) = [xy : az^2 + yz : xz]$ can be written as:

$$\widetilde{F}_{a|\widetilde{C_h}}([x:y:z]) = [x:y:z] + [1:0:0]. \tag{2.7}$$

The nonelliptic levels correspond to curves of genus 0, and on those levels the map $\widetilde{F}_{a|\widetilde{C_h}}$ is conjugate to a Möbius transformation, see [29, Sect. 3.1].

The JRV theorem (Theorem 2.4) has the following dynamical interpretations given in Corollaries 2.5 and 2.7 below.

Corollary 2.5 *Let F be a birational map preserving a real foliation of algebraic curves $\{C_h\} \subset \mathcal{U} \in \mathbb{R}^2$ of generic genus 1. Then, on each invariant elliptic curve C_h, either F or F^2 are conjugate to a rotation.*

The above corollary is a direct consequence of Theorem 2.4 and the following result (a direct consequence of Corollary 2.3.1 in Chapter V.2 of [49]), based on the fact that every real elliptic curve (adding, if necessary, some infinite points in the real projective space) can be seen as either one or two closed simple curves, and that the inner sum can be easily represented as the usual Lie group operation of $\mathbb{S}^1$ or $\mathbb{S}^1 \times \mathbb{Z}/(2)$.

Proposition 2.6 *There is a continuous isomorphism between any nonsingular elliptic curve $(C(\mathbb{R}), +, \mathcal{O})$ and either the Lie group $\mathbb{S}^1 \times \mathbb{Z}/(2) = \{e^{it} : t \in [0, 2\pi)\} \times \{1, -1\}$ if $\Delta(C) > 0$, or $\mathbb{S}^1 = \{e^{it} : t \in [0, 2\pi)\}$ if $\Delta(C) < 0$, with the operation in $\mathbb{S}^1$ being given by $u \cdot z = uz$, where $\Delta(C)$ is the discriminant of the Weierstrass equation associated to $C(\mathbb{R})$.*

Observe that if F is a birational map preserving an elliptic curve $(C, +, \mathcal{O})$ whose dynamics corresponds to case (ii) of Theorem 2.4, then all the points in C give rise to periodic orbits. If the dynamics corresponds to case (i), then $F^n_{|C}(P) = P + nQ$, and we observe that P gives rise to a p-periodic orbit if and only if

$$pQ = \mathcal{O}. \tag{2.8}$$

In other words, in case (i) of Theorem 2.4 the curve is filled by periodic orbits of F if and only if Q is a *finite order* point of the group $(C, +, \mathcal{O})$, also called a *torsion* point (the torsion of a group G, denoted by $\mathrm{Tor}(G)$ is the set of its finite order elements). The following result characterizes the dynamics of birational maps on particular elliptic curves.

Corollary 2.7 *Let F be a birational map preserving a real elliptic curve $(C(\mathbb{R}), +, \mathcal{O})$, named C from now on, such that its dynamics is given by $F_{|C}(P) = P + Q$, where $Q \in C$. Then*

(i) If $Q \in \mathrm{Tor}(C)$, then all the orbits in C are periodic.
(ii) If $Q \notin \mathrm{Tor}(C)$, then the orbits of F fill densely the connected components of C.

2.4 Global Dynamics on Elliptic Foliations.

The JRV theorem ensures that the action of a birational map on a particular elliptic curve is linear. However, the behavior in the whole phase plane is a little bit more complex. The typical situation occurs when there is a dense set of curves filled by p-periodic orbits of all the periods $p \geq p_0 \in \mathbb{N}$, for some integer p_0. This integer p_0 is sometimes computable if the rotation set $\{\theta(h),\, h \in \mathrm{Im}(V)\}$ is known. In this section, we describe the reason for this behavior and we give an example of how to compute the set of periods of a particular map. We also will see that if a particular subinterval I in the rotation set is known, then it is possible to construct a number P such that the map F contains at least all the periods $p > P$.

2.4.1 The Rotation Number Function and Its Nature

2.4.1.1 Piecewise Continuity

From this point, we will assume that the invariant foliation of irreducible curves $\{C_h\}$ obtained from (2.2) is given by *real* curves which are generically elliptic. Also we will assume that F preserves each connected component of the invariant real elliptic curves (on the contrary, we can study F^2). Finally, we will also assume that the action of our birational maps F falls within case (i) of Theorem 2.4. Under these assumptions, Corollary 2.5 ensures that on each curve C_h the map F is conjugate to a rotation. So, we can consider a *rotation number* $\theta(h)$ associated to each level set h, or equivalently to each curve C_h.

Of course this rotation number function $\theta(h)$ can be constant. In this case, we say that the map F is *rigid*. For instance, if $a = 1$ then the Lyness map F_1 is globally five-periodic, thus $\theta(h) = 1/5$ for all $h \in \mathrm{Im}(V)$, where V is given in (2.4).

If $\theta(h)$ is not constant then it is possible to prove that this rotation number function is piecewise continuous. This is because when the irreducible components of (2.2) are generically elliptic, any birational map F (or F^2) can be thought as a family of homeomorphisms in the circle, which is piecewise continuous in the parameter h. By using the fact that the rotation number function of a continuous family of maps of $\mathbb{S}^1$ (in the $\mathcal{C}^0$ topology) is continuous, and taking into account that, in principle, there could be levels $h \in \text{Im}(V)$ corresponding to curves in the *forbidden set* of F, the piecewise continuity of $\theta(h)$ is achieved.

2.4.1.2 Piecewise Analyticity and Existence of Lie Symmetries

In fact, $\theta(h)$ is a piecewise analytic function in the domain $h \in \text{Im}(V)$. This is because if the irreducible curves C_h in (2.2) are generically elliptic, then it is possible to construct an isomorphism (piecewise analytic in h) between them and a new foliation of Weierstrass curves, so that the corresponding associated map F_W defined on this foliation has the same rotation number function as F. These Weierstrass curves can be parameterized using the Weierstrass $\wp$ function (see [34, 50], and also [26, 53]). Using this parametrization, and the fact that $\wp$ satisfies a certain differential equation, it is always possible to give an integral expression of the rotation number function, from which the piecewise analyticity of it can be deduced. This approach has been introduced in [28] (and later developed in [4]) to study the rotation number function associated to a Lyness map, and has been successfully applied to study the periods of other birational maps in the successive papers of Bastien and Rogalski, and others [5–9].

An alternative proof of the piecewise analyticity of $\theta(h)$ comes from the fact that if the invariant curves C_h in (2.2) are generically elliptic, then it is possible to construct a vector field X such that the map F can be seen as the flow of this vector field at certain time $\tau(h)$. Such a vector field is called a *Lie Symmetry* of F. A Lie Symmetry of a map F in $\mathbb{R}^n$ is a *vector field* X such that F maps any orbit of the differential system

$$\dot{\boldsymbol{x}} = X(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n \tag{2.9}$$

into another orbit of the system.

The existence of Lie Symmetries is an important issue in the theory of discrete integrability, see for instance [31]. From a dynamical viewpoint, this importance is clear in the case of integrable diffeomorphisms. In this case, the dynamics of the maps are in practice one-dimensional, and the existence of a Lie Symmetry whose orbits are preserved by F implies that this one-dimensional dynamics is linear on each orbit. The next results illustrate this fact.

Theorem 2.8 ([16]) *Let $F : \mathcal{U} \subseteq \mathbb{R}^n \to \mathcal{U}$ be a diffeomorphism having a Lie symmetry X, and let γ be an orbit of X, preserved by F (i.e., $F : \gamma \to \gamma$). Then the dynamics of F restricted to γ is either: (1) conjugate to a rotation with rotation number given by τ/T, where T is the period of γ and τ is defined by the equation $F(p) = \varphi(\tau, p)$, where φ denotes the flow of X; or (2) conjugate to a translation of the line; or (3) it is constant; according to whether γ is homeomorphic to $\mathbb{S}^1$, $\mathbb{R}$, or a point, respectively.*

If F is an integrable map and it possesses a Lie Symmetry X, this vector field is also integrable and shares the same first integral with F. However, in our case the curves in (2.2) would also be integral curves of any possible symmetry X. Since each connected component C_h of any curve in (2.2) is diffeomorphic to $\mathbb{S}^1$ (adding, possibly some infinite points), it is a periodic orbit of X (or a compactification of X) with period $T(h)$, see also [13]. Hence, Theorem 2.8 guarantees that

$$\theta(h) = \frac{\tau(h)}{T(h)}. \tag{2.10}$$

The regularity of the rotation number function is, then, a consequence of the regularity of the flow of X.

Again, the existence of the Lie Symmetry of a birational map preserving an elliptic foliation can be proved using the associated Weierstrass foliation associated to curve C_h, see [26, Sect. 2.6.3]. The Lie Symmetry approach was used to study the rotation number function of the Lyness map and to prove a conjecture about its monotonicity established by Zeeman [54]. This was done by Beukers and Cushman in the relevant paper [10]. This approach has been also applied to study the rotation number function and the set of periods of the extension of the Lyness map in $\mathbb{R}^3$ [15], and also to study a birational integrable map arising in the study of 2-periodic Gumovski-Mira type maps [18].

2.4.2 An Infinite Number of Periods

Taking into account the above considerations, we can prove the following result:

Proposition 2.9 *A birational map F preserving a generically real elliptic foliation $\{C_h\}$ is either rigid, or there are an infinite number of possible periods and a dense set of curves in the phase space filled with periodic orbits.*

Proof Let $\mathcal{E} = \{h$ such that the curves C_h are elliptic$\}$. From the JRV theorem on each the curve C_h with $h \in \mathcal{E}$ the map F (or F^2) is conjugate to a rotation. From the considerations in Sect. 2.4.1 (see also [26, Lemma 8.1.5]) the rotation number function $\theta(h)$ is piecewise continuous for $h \in \mathcal{E}$.

Since $\{C_h\}$ is generically elliptic, if F is not rigid, then there exists a nonempty open interval I such that $I \subseteq \{\text{Image}(\theta(h)), h \in \mathcal{E}\}$. Then, for any irreducible fraction $q/p \in I$, there exists a value of $h \in \mathcal{E}$ such that $\theta(h) = q/p$, hence an invariant real elliptic curve C_h which is full of periodic orbits of of minimal period p. □

2.4.3 Toward a Constructive Characterization of the Set of Periods

It is interesting to notice that if a rotation interval I containing some of the values of $\theta(h)$ is known, then it is always possible to compute a value P such that $q/p \in I$ for all $p > P$, hence characterizing at least an infinite number of periods in the set of periods of F. One tool to construct a (nonoptimal) number P is the following result.

Lemma 2.1 ([15]) *Consider an open interval* (c, d) *with* $0 \le c < d$*; denote by* $p_1 = 2, p_2 = 3, p_3, \ldots, p_n, \ldots$ *the set of all the prime numbers, ordered following the usual order. Also consider the following natural numbers:*

- *Let* p_{m+1} *be the smallest prime number satisfying that* $p_{m+1} > \max(3/(d-c), 2)$
- *Given any prime number* p_n, $1 \le n \le m$, *let* s_n *be the smallest natural number such that* $p_n^{s_n} > 4/(d-c)$
- *Set* $P := p_1^{s_1-1} p_2^{s_2-1} \cdots p_m^{s_m-1}$.

Then, for any $r > P$ *there exists an irreducible fraction* q/r *such that* $q/r \in (c, d)$.

In the Example 2.3, we will illustrate how to apply the above result and the known facts on the rotation number function to compute effectively some set of minimal periods appearing in a particular Lyness map F_a. Prior to stating this example we recall some basic facts. When $a > 0$ the first integral of F_a, given in (2.4), has a global minimum in $Q^+ := \{(x, y), x, y > 0\}$, located at the fixed point of F_a, given by (x_c, x_c) where $x_c = (1 + \sqrt{1+4a})/2$. This minimum corresponds to the nonelliptic level $h_c = (x_c + 1)^3/x_c$. With respect to the rotation number function, it is known that for $a > 0$

$$\theta_c = \lim_{h \to h_c^+} \theta(h) = \frac{1}{2\pi} \arccos\left(\frac{1}{1+\sqrt{1+4a}}\right),$$

see [4, 54]. Also it was conjectured in [54], and proved in [10], that when $0 < a < 1$, then $\theta(h)$ is a strictly increasing continuous function in (h_c, ∞) and strictly decreasing when $1 < a < \infty$. Moreover, in [4] it was proved (strongly using the elliptic nature of the Lyness curves) that

$$\lim_{h_c \to \infty} \theta(h) = \frac{1}{5}.$$

The case $a = 1$ corresponds to the globally five-periodic case with $\theta(h) \equiv 1/5$.

In summary, the interval

$$I^+ := (\min(\theta_c, 1/5), \max(\theta_c, 1/5))$$

gives the optimal rotation interval for the orbits in Q^+ when $a > 0$ and $a \ne 1$.

Example 2.3 When $a = 10$, the optimal rotation interval for the orbits in Q^+ is

$$I^+ := \left(\frac{1}{5}, \frac{1}{2\pi} \arccos\left(\frac{1}{1+\sqrt{41}}\right)\right).$$

Using the notation introduced in Lemma 2.1 we have $m = 27$, and $p_1 = 2, s_1 = 7$; $p_2 = 3, s_2 = 4$; $p_3 = 5, s_3 = 2$; $p_4 = 7, s_4 = 2$; $p_5 = 11, s_5 = 2$; $p_6 = 13, s_6 = 1$; $p_7 = 17, s_7 = 1$; ... ; $p_{27} = 103, s_{27} = 1$. Hence

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