Chapter 2
The Generalized Frequency Response Functions and Output Spectrum of Nonlinear Systems

2.1 Volterra Series Expansion and Frequency Response Functions

As discussed in Chap. 1, the input-output relationship for a wide class of nonlinear systems can be approximated by a Volterra series up to a sufficiently high order N as

$$y(t) = \sum_{n=1}^{N} y_n(t) \tag{2.1a}$$

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i) \mathrm{d}\tau_i \tag{2.1b}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is a real valued function of $\tau_1,\cdots,\tau_n$ known as the nth order Volterra kernel. The nth order generalized frequency response function (GFRF) is defined as

$$H_n(j\omega_1,\cdots,j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \exp\left(-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)\right) \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \tag{2.2}$$

which is the multidimensional Fourier transform of $h_n(\tau_1,\cdots,\tau_n)$. By applying the inverse Fourier transform of the nth order GFRF, (2.1b) can be written as

$$y_n(t) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j\omega_1, \cdots, j\omega_n) \prod_{i=1}^{n} U(j\omega_i) e^{j(\omega_1 + \cdots + \omega_n)t} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n$$

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which can, by denoting $\omega_n = \omega - \omega_1 - \cdots - \omega_{n-1}$, be further written as

$$y_n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{(2\pi)^{n-1}}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n-1} H_n(j\omega_1,\cdots,j\omega_{n-1},j(\omega-\omega_1-\cdots-\omega_{n-1})) \times U_n(j\omega_1,\cdots j\omega_{n-1})\mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-1}\right]e^{j\omega t}\mathrm{d}\omega$$

where $U_n(j\omega_1,\cdots,j\omega_{n-1}) = U(j\omega_1)\cdots U(j\omega_{n-1})U(j(\omega-\omega_1-\cdots-\omega_{n-1}))$. Therefore the Fourier transform of $y_n(t)$ is obtained as

$$Y_n(j\omega) = \frac{1}{(2\pi)^{n-1}}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n-1} H_n(j\omega_1,\cdots,j\omega_{n-1},j(\omega-\omega_1-\cdots-\omega_{n-1})) \times U_n(j\omega_1,\cdots,j\omega_{n-1})\mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-1} \tag{2.3}$$

which is referred to as the nth-order output spectrum. The output spectrum of the nonlinear system in (2.1a,b) can then be computed by

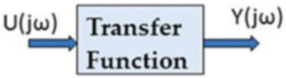
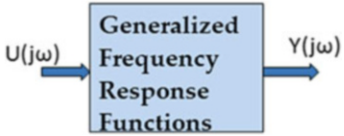
$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) \tag{2.4}$$

Note that in (2.1a,b, 2.3 and 2.4) the input signal u(t) can be any signal with a Fourier transform U(jω). The GFRFs and output spectrum of each order defined above are all referred to as frequency response functions of nonlinear systems in this book. It can be seen that the nonlinear frequency response functions defined above and associated analysis and design methods are important extension and/or natural generalization of existing theory and methods for linear systems to the nonlinear case. A simple comparison is shown in Table 2.1.

2.1.1 The Probing Method

Obviously, the output spectrum of a nonlinear system involves the computation of the GFRFs. Given the parametric model of a nonlinear system, the GFRFs can be derived by using the "harmonic probing" method (Rugh 1981), which can be traced back to Bedrosian and Rice (1971) or earlier. Examples can be seen in Peyton Jones and Billings (1989), and Billings and Peyton-Jones (1990) etc. Consider the excitation of system (2.1a,b) with an input consisting of *n* complex exponentials defined as

Table 2.1 A comparison between the frequency domain theories of linear and nonlinear systems. The nonlinear output spectrum will be discussed in more details in Chaps. 3 and 6–9

Linear systems		Nonlinear systems
		
System output: $y(t)=\int_{-\infty}^{\infty}h(\tau)u(t-\tau)d\tau$		System output: $y(t)=\int_{-\infty}^{\infty}h_1(\tau_1)u(t-\tau_1)d\tau_1+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_2(\tau_1,\tau_2)\prod_{i=1}^{2}u(t-\tau_i)d\tau_i$ $+\sum_{n=1}^{3}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i+\cdots$
Frequency response function or Transfer function $H(j\omega)=\int_{-\infty}^{\infty}h(\tau)\exp(-j(\omega\tau))d\tau$	⇨	Generalized frequency response functions (GFRFs) $H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n$
Output spectrum $Y(j\omega)=H(j\omega)U(j\omega)$		Nonlinear output spectrum $Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega+..+\omega=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$

$$u(t) = \sum_{r=1}^{n} e^{j\omega_r t} \tag{2.5}$$

The output y(t) is given as

$$y(t) = \sum_{n=1}^{N} \sum_{r_1, r_n=1}^{n} H_n(j\omega_{r_1}, \cdots, j\omega_{r_n}) e^{j\left(\omega_{r_1} + \cdots + \omega_{r_n}\right)t} \tag{2.6}$$

Then for the nonlinear system, replacing the input and output by (2.5–2.6), the nth-order GFRF can be obtained by extracting the coefficients of the term $e^{j\left(\omega_{r_1} + \cdots + \omega_{r_n}\right)t}$. For example, consider a static polynomial function,

$$y(t) = f(u(t)) = c_1 u(t) + c_2 u(t)^2 + c_3 u(t)^3 + \ldots + c_{\mathrm{n}} u(t)^{\mathrm{n}} + \ldots \tag{2.7}$$

The nth-order GFRF between the input $u(t)$ and output $y(t)$ can be derived as

$$H_n(j\omega_1, \cdots, j\omega_n) = c_n \tag{2.8}$$

To show this, the GFRFs for (2.7) can be obtained by directly applying the probing method. Note that (2.6) can be expanded as

$$\begin{aligned} y(t) = {} & n! H_n(j\omega_1, \cdots, j\omega_n) e^{j(\omega_1 + \cdots + \omega_n)t} + (n-1)! H_{n-1}(j\omega_1, \cdots, j\omega_{n-1}) e^{j(\omega_1 + \cdots + \omega_{n-1})t} \\ & + \cdots + H_1(j\omega_1) e^{j\omega_1 t} + \text{(the other terms)} \end{aligned} \tag{2.9}$$

Using (2.5) and (2.9) with n=3 in (2.7), it can be obtained that

$$\begin{aligned} y(t) &= 3! H_3(j\omega_1, \cdots, j\omega_3) e^{j(\omega_1 + \cdots + \omega_3)t} + 2H_2(j\omega_1, j\omega_2) e^{j(\omega_1 + \omega_2)t} + H_1(j\omega_1) e^{j\omega_1 t} + \text{(the other terms)} \\ &= c_1\left(e^{j\omega_1 t} + e^{j\omega_2 t} + e^{j\omega_3 t}\right) + c_2\left(e^{j\omega_1 t} + e^{j\omega_2 t} + e^{j\omega_3 t}\right)^2 + c_3\left(e^{j\omega_1 t} + e^{j\omega_2 t} + e^{j\omega_3 t}\right)^3 + \ldots \\ &\quad + c_{\mathrm{n}}\left(e^{j\omega_1 t} + e^{j\omega_2 t} + e^{j\omega_3 t}\right)^{\mathrm{n}} + \ldots \end{aligned}$$

Extracting the coefficients of the term $e^{j\omega_1 t}$ from the equation above, it can be obtained that

$$H_1(j\omega_1) = c_1$$

Extracting the coefficients of $e^{j(\omega_1 + \omega_2)t}$, it can be obtained that

$$H_2(j\omega_1, j\omega_2) = c_2$$

Similarly, from the coefficients of $e^{j(\omega_1 + \cdots + \omega_3)t}$, it can be obtained that

$$H_3(j\omega_1, \cdots, j\omega_3) = c_3$$

Following this method, the nth-order GFRF can be obtained as

$$H_n(j\omega_1,\cdots,j\omega_n)=c_n$$

2.2 The GFRFs for NARX and NDE Models

Nonlinear systems can often be described by different parametric models. To compute the GFRFs, a parametric model of the nonlinear system under study can be given. In this book, two parametric models are focused, i.e., the Nonlinear Auto-Regressive with eXogenous input (NARX) model and Nonlinear Differential Equation (NDE) model.

The NARX model provides a unified and natural representation for a wide class of nonlinear systems, including many nonlinear models as special cases (e.g., Wiener models, Hammerstein models). For this reason, the NARX model has been extensively used in various engineering problems for system identification (Li et al. 2011), signal processing (McWhorter and Scharf 1995; Kay and Nagesha 1994), and control (Sheng and Chon 2003) etc. In practice, most systems are inherently nonlinear and can be identified to obtain a NARX model using several efficient algorithms such as the OLS method (Chen et al. 1989). The NARX model is given by

$$y(t)=\sum_{m=1}^{M}y_m(t)$$

$$y_m(t)=\sum_{p=0}^{m}\sum_{\substack{k_1,k_{p+q}=1\\ p+q=m}}^{K}c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}y(t-k_i)\prod_{i=p+1}^{p+q}u(t-k_i) \tag{2.10}$$

where $y_m(t)$ is the mth-order output of the NARX model; $\sum_{k_1,k_{p+q}=1}^{K}(\cdot)=\sum_{k_1=1}^{K}(\cdot)\cdots\sum_{k_{p+q}=1}^{K}(\cdot)$; $p+q$ is referred to as the nonlinear degree of parameter $c_{p,q}(\cdot)$, which corresponds to the $(p+q)$-degree nonlinear terms $\prod_{i=1}^{p}y(t-k_i)\prod_{i=p+1}^{p+q}u(t-k_i)$, *e.g.*, $y(t-1)^p u(t-2)^q$ (p orders in terms of the output and q orders in terms of the input), and k_i is the lag of the ith output when $i\le p$ or the $(i\text{-}p)$th input when $p<i\le m$ with the maximum lag K; $c_{0,1}(.)$ and $c_{1,0}(.)$ of nonlinear degree 1 are referred to as linear parameters, and all the other model parameters are referred to as nonlinear parameters; the model includes all the possible nonlinear combinations in terms of $y(k)$ and $u(k)$ with the maximum order M.

The NDE model can be regarded as a continuous-time version of the NARX model, which is usually obtained by physical modelling and can be given by

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}\left(k_1,\cdots,k_{p+q}\right)\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}=0 \tag{2.11}$$

where $\left.\frac{d^k x(t)}{dt^k}\right|_{k=0}=x(t)$, and all other notations take similar forms and definitions to those for the NARX model for convenience. But in the NDE model, K is the maximum order of the derivative, and $c_{p,q}(\cdot)$ for $p+q>1$ are referred to as nonlinear parameters corresponding to nonlinear terms in the model of the form $\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}$, e.g., $y(t)^p u(t)^q$.

2.2.1 Computation of the GFRFs for NARX Models

By using the probing method demonstrated above, a recursive algorithm to compute the nth-order GFRF in terms of model parameters for nonlinear systems described by the NARX model can be developed, which is given as follows (Peyton Jones and Billings 1989; Jing and Lang 2009a):

$$\begin{aligned}&\left(1-\sum_{k_1=1}^{K}c_{1,0}(k_1)\exp(-j(\omega_1+\cdots+\omega_n)k_1)\right)\cdot H_n(j\omega_1,\cdots,j\omega_n)\\&\quad=\sum_{k_1,k_n=1}^{K}c_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1k_1+\cdots+\omega_nk_n))\\&\qquad+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_n=1}^{K}c_{p,q}\left(k_1,\cdots,k_{p+q}\right)\exp\left(-j\sum_{i=1}^{q}\omega_{n-q+i}k_{p+i}\right)H_{n-q,p}\left(j\omega_1,\cdots,j\omega_{n-q}\right)\\&\qquad+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K}c_{p,0}\left(k_1,\cdots,k_p\right)H_{n,p}(j\omega_1,\cdots,j\omega_n)\end{aligned} \tag{2.12}$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1}H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\exp\left(-j(\omega_1+\cdots+\omega_i)k_p\right) \tag{2.13}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \tag{2.14}$$

Furthermore, define $H_{0,0}(\cdot)=1$, $H_{n,0}(\cdot)=0$ for n>0, $H_{n,p}(\cdot)=0$ for $n<p$, and let

$$\exp\left(\sum_{i=1}^{q}\varepsilon(p)\right)=\begin{cases}1 & q=0,p>1\\0 & q=0,p\le 1\end{cases} \tag{2.15}$$

where $\varepsilon(p)$ is a function of p, and

$$L_n(\omega_1,\cdots,\omega_n) = 1 - \sum_{k_1=1}^{K} c_{1,0}(k_1)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \tag{2.16}$$

Then (2.12) can be written more concisely as

$$H_n(j\omega_1,\cdots,j\omega_n) = \frac{1}{L_n(\omega_1\cdots\omega_n)}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\sum_{k_1,k_{p+q}=1}^{K} c_{p,q}\left(k_1,\cdots,k_{p+q}\right)e^{-j\sum_{i=1}^{q}\left(\omega_{n-q+i}k_{p+i}\right)}$$
$$\times H_{n-q,p}\left(j\omega_1,\cdots,j\omega_{n-q}\right) \tag{2.17}$$

Thus the recursive algorithm for the computation of GFRFs is (2.12 or 2.17, 2.13–2.16).

Moreover, $H_{n,p}(j\omega_1,\cdots,j\omega_\mathrm{n})$ in (2.13) can also be written as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n) = \sum_{\substack{r_1\cdots r_p = 1 \\ \sum r_i = n}}^{n-p+1} \prod_{i=1}^{p} H_{r_i}(j\omega_{X+1},\cdots,j\omega_{X+r_i})$$
$$\exp(-j(\omega_{X+1}+\cdots+\omega_{X+r_i})k_i) \tag{2.18}$$

where $X = \sum_{x=1}^{i-1} r_x$.

2.2.2 *Computation of the GFRFs for NDE Models*

Similarly, the computation of the GFRFs for the NDE model can be recursively conducted in terms of model parameters as follows (Billings and Peyton-Jones1990; Jing et al. 2008e):

$$L_n(j\omega_1+\cdots+j\omega_n)\cdot H_n(j\omega_1,\cdots,j\omega_n) = \sum_{k_1,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$$
$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}\left(k_1,\cdots,k_{p+q}\right)\left(\prod_{i=1}^{q}\left(j\omega_{n-q+i}\right)^{k_{p+i}}\right)H_{n-q,p}\left(j\omega_1,\cdots,j\omega_{n-q}\right)$$
$$+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K} c_{p,0}\left(k_1,\cdots,k_p\right)H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{2.19}$$

$$H_{n,p}(\cdot) = \sum_{i=1}^{n-p+1} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p} \quad (2.20)$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1} \quad (2.21)$$

where

$$L_n(j\omega_1+\cdots+j\omega_n) = -\sum_{k_1=0}^{K} c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1} \quad (2.22)$$

Moreover, $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (2.20) can also be written as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n) = \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i = n}}^{n-p+1} \prod_{i=1}^{p} H_{r_i}(j\omega_{X+1},\cdots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{k_i} \quad (2.23)$$

where

$$X = \sum_{x=1}^{i-1} r_x \quad (2.24)$$

Similarly, for convenience in discussion, define

$$H_{0,0}(\cdot) = 1, \quad H_{n,0}(\cdot) = 0 \quad \text{for n} > 0, \quad H_{n,p}(\cdot) = 0 \quad \text{for } n < p,$$
$$\text{and} \quad \prod_{i=1}^{q}(\cdot) = \begin{cases} 1 & q = 0, p > 1 \\ 0 & q = 0, p \le 1 \end{cases} \quad (2.25)$$

Then (2.19) can be written in a more concise form as

$$H_n(j\omega_1,\cdots,j\omega_n) = \frac{1}{L_n\left(j\sum_{i=1}^{n}\omega_i\right)} \sum_{q=0}^{n}\sum_{p=0}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})$$
$$\times\left(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}}\right)H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \quad (2.26)$$

Therefore, the recursive algorithm for the computation of the GFRFs is (2.19 or 2.26, 2.25, 2.20–2.23).

Note that the GFRFs above both for the NARX and NDE models are assumed to be asymmetric. Generally, different permutations of the frequency variables

$\omega_1,\cdots,\omega_n$ may lead to different values of $H_n(j\omega_1,\cdots,j\omega_n)$. The symmetric GFRFs can be obtained as

$$H_n^{sym}(j\omega_1,\cdots,j\omega_n)=\frac{1}{n!}\sum_{\substack{\text{all the permutations}\\ \text{of }\{1,2,\ldots,n\}}}H_n(j\omega_1,\cdots,j\omega_n) \tag{2.27}$$

But for computation of nonlinear output spectrum in (2.4), asymmetric GFRFs suffice.

2.3 The GFRFs for a Single Input Double Output Nonlinear System

In many practical cases, nonlinear system models are usually described by a nonlinear state equation with a general nonlinear output function of system states. Sometimes, the output function of interest can also be a nonlinear objective function to optimize. Therefore, the computation of the GFRFs for nonlinear systems in this form would be more relevant in practice. The systems can be classified into several cases: single-input multi-output (SIMO), or multi-input and multi-output (MIMO) etc. The GFRFs for MIMO systems would be more complicated, which can be referred to Swain and Billings (2001). This section addresses a much simpler case, i.e., single-input double-output (SIDO), which is actually frequently encountered in practice. Similar results can be easily extended to the SIMO case (many multiple-degree-of-freedom mechanical systems would belong to this case).

Consider the following SIDO NARX system,

$$x(t)=\sum_{m=1}^{M_1}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K}\bar{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p}x(t-k_i)\prod_{i=p+1}^{m}u(t-k_i) \tag{2.28a}$$

$$y(t)=\sum_{m=1}^{M_2}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K}\tilde{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p}x(t-k_i)\prod_{i=p+1}^{m}u(t-k_i) \tag{2.28b}$$

where M_1, M_2 and K are all positive integers, and x(t), y(t), u(t) $\in\mathbb{R}$. Equation (2.28a) is the system state equation which is still described by a NARX model, and (2.28b) represents the system output which is a nonlinear function of state x(t) and input u(t) in a general polynomial form.

Instead of using the probing method for derivation of the GFRFs for (2.28a,b), an alternative simple method would be adopted here, since the model structure and nonlinear types are known clearly. Note that the expression of the nth-order GFRF in (2.12) for the NARX model (2.10) can be divided into three parts. That is, those arising from pure input nonlinear terms $H_{n_u}(\cdot)$ corresponding to the first part in the right side of (2.12), those from cross product nonlinear terms $H_{n_{uy}}(\cdot)$ corresponding to the second part in the right side of (2.12), and those from pure output nonlinear terms $H_{n_y}(\cdot)$ corresponding to the last part of (2.12). For clarity, (2.12) can also be written as

$$H_n(j\omega_1,\cdots,j\omega_n) = \left(H_{n_u}(\cdot) + H_{n_{uy}}(\cdot) + H_{n_y}(\cdot)\right)/L_n(j(\omega_1 + \cdots + \omega_n)) \quad (2.29)$$

Equation (2.29) shows clearly that different categories of nonlinearities produce different contribution to the system GFRFs. Hence, when deriving the GFRFs of a nonlinear system, what can be done is to combine the different contributions from different nonlinearities without directly using the probing method. This property can be used for the derivation of the GFRFs for (2.28a,b).

To this aim, (2.28a,b) can be regarded as a system of one input u(t) and two outputs x(t) and y(t). Therefore, there are two sets of GFRFs for (2.28a,b) corresponding to the two input-output relationships between the input u(t) and two outputs x(t) and y(t) respectively. Considering the GFRFs from input u(t) to output x(t), there are three categories of nonlinearities as mentioned before. Therefore, the *n*th-order GFRF from input u(t) to output x(t) denoted by $H_n^x(j\omega_1,\cdots,j\omega_n)$ can be directly determined which is the same as (2.12–2.17), i.e.,

$$H_n^x(j\omega_1,\cdots,j\omega_n) = \frac{H_{n_u}^x(j\omega_1,\cdots,j\omega_n) + H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n) + H_{n_x}^x(j\omega_1,\cdots,j\omega_n)}{L_n(j(\omega_1+\cdots+\omega_n))} \quad (2.30)$$

where, $L_n(j(\omega_1+\cdots+\omega_n)) = 1 - \sum_{k_1=1}^{K} \bar{c}_{1,0}(k_1)\exp(-j(\omega_1+\cdots+\omega_n)k_1)$

$$H_{n_u}^x(j\omega_1,\cdots,j\omega_n) = \sum_{k_1,k_n=0}^{K} \bar{c}_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1 k_1 + \cdots + \omega_n k_n)) \quad (2.31a)$$

$$H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n) = \sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} \bar{c}_{p,q}\left(k_1,\cdots,k_{p+q}\right)$$
$$\times\exp\left(-j\left(\omega_{n-q+1}k_{p+1}+\cdots+\omega_n k_{p+q}\right)\right)H_{n-q,p}\left(j\omega_1,\cdots,j\omega_{n-q}\right) \quad (2.31b)$$

$$H_{n_x}^x(j\omega_1,\cdots,j\omega_n) = \sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K} \bar{c}_{p,0}\left(k_1,\cdots,k_p\right)H_{n,p}(j\omega_1,\cdots,j\omega_n) \quad (2.31c)$$

$$H_{n,p}(j\omega_1,\cdots,j\omega_n) = \sum_{i=1}^{n-p+1} H_i^x(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)$$
$$\times\exp\left(-j(\omega_1+\cdots+\omega_i)k_p\right) \quad (2.31d)$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = H_n^x(j\omega_1,\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \tag{2.31e}$$

Similarly, consider the GFRFs from input u(t) to output y(t). There are also three categories of nonlinearities in terms of input u(t) and output x(t) (similar to those from input u(t) to output x(t)), and there is one linear output y(t). Note that there are no nonlinearities in terms of y(t), and all the nonlinearities come from input u(t) and output x(t). For this reason, the GFRFs from u(t) to y(t) are dependent on the GFRFs from u(t) to x(t). Therefore, in this case the *n*th-order GFRF from input u(t) to output y(t) denoted by $H_n^y(j\omega_1,\cdots,j\omega_n)$ is,

$$\begin{aligned} H_n^y(j\omega_1,\cdots,j\omega_n) &= H_{n_u}^y(j\omega_1,\cdots,j\omega_n) + H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n) \\ &\quad + H_{n_x}^y(j\omega_1,\cdots,j\omega_n) \end{aligned} \tag{2.32}$$

where the corresponding terms in (2.32) are

$$H_{n_u}^y(j\omega_1,\cdots,j\omega_n) = \sum_{k_1,k_n=0}^{K} \tilde{c}_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1 k_1+\cdots+\omega_n k_n)) \tag{2.33a}$$

$$\begin{aligned} H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n) = &\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} \tilde{c}_{p,q}\left(k_1,\cdots,k_{p+q}\right) \\ &\times\exp\left(-j\left(\omega_{n-q+1}k_{p+1}+\cdots+\omega_n k_{p+q}\right)\right)H_{n-q,p}\left(j\omega_1,\cdots,j\omega_{n-q}\right) \end{aligned} \tag{2.33b}$$

$$H_{n_x}^y(j\omega_1,\cdots,j\omega_n) = \sum_{p=1}^{n}\sum_{k_1,k_p=0}^{K} \tilde{c}_{p,0}\left(k_1,\cdots,k_p\right)H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{2.33c}$$

Note that p is counted from 1 in (2.33c), different from (2.31c) where p is counted from 2, and $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (2.33b, c) is the same as that in (2.31b–d) because the nonlinearities in (2.28b) have no relationship with y(t) but x(t). The results here are developed in a very straightforward manner and provide a concise analytical expression for the GFRFs of the system in (2.28a,b).

Similar results can be obtained for the following SIDO NDE system

$$\sum_{m=1}^{M_1}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K} \bar{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p}\frac{d^{k_i}x(t)}{dt^{k_i}}\prod_{i=p+1}^{m}\frac{d^{k_i}u(t)}{dt^{k_i}} = 0 \tag{2.34a}$$

$$\sum_{m=1}^{M_2}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K} \tilde{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p}\frac{d^{k_i}x(t)}{dt^{k_i}}\prod_{i=p+1}^{m}\frac{d^{k_i}u(t)}{dt^{k_i}} = y(t) \tag{2.34b}$$

where $x(\mathrm{t})$, $y(\mathrm{t})$, $u(\mathrm{t}) \in \mathbb{R}$. System (2.34a,b) has similar notations and structure as system (2.28a,b). It can be regarded as an NDE model with two outputs x(t) and y(t), and one input u(t). Hence, following the same idea, the GFRFs for the relationship from u(t) to y(t) are given as

$$H_n^y(j\omega_1,\cdots,j\omega_n) = H_{n_u}^y(j\omega_1,\cdots,j\omega_n) + H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n) + H_{n_x}^y(j\omega_1,\cdots,j\omega_n) \tag{2.35}$$

where

$$H_{n_u}^y(j\omega_1,\cdots,j\omega_n) = \sum_{k_1,k_n=0}^{K} \widetilde{c}_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n} \tag{2.36a}$$

$$H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n) = \sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} \widetilde{c}_{p,q}\left(k_1,\cdots,k_{p+q}\right) \times \left(j\omega_{n-q+1}\right)^{k_{p+1}}\cdots(j\omega_n)^{k_{p+q}} H_{n-q,p}\left(j\omega_1,\cdots,j\omega_{n-q}\right) \tag{2.36b}$$

$$H_{n_x}^y(j\omega_1,\cdots,j\omega_n) = \sum_{p=1}^{n}\sum_{k_1,k_p=0}^{K} \widetilde{c}_{p,0}\left(k_1,\cdots,k_p\right) H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{2.36c}$$

$$H_{n,p}(\cdot) = \sum_{i=1}^{n-p+1} H_i^x(j\omega_1,\cdots,j\omega_i) H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p} \tag{2.36d}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = H_n^x(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1} \tag{2.36e}$$

where $H_n^x(j\omega_1,\cdots,j\omega_n)$ is the nth-order GFRF from u(t) to x(t), which is the same as that given in (2.19 or 2.26, 2.25, 2.20–2.23).

Example 2.1 Consider the following nonlinear system,

$$\begin{aligned} &mx(t-2) + a_1x(t-1) + a_2x^2(t-1) + a_3x^3(t-1) + kx(t) = u(t) \\ &y(t) = a_1x(t-1) + a_2x^2(t-1) + a_3x^3(t-1) + kx(t) \end{aligned} \tag{2.37}$$

which can be written into the form of model (2.28a,b) with parameters K=2, $\overline{c}_{1,0}(2) = -m/k, \overline{c}_{1,0}(1) = -a_1/k,\ \overline{c}_{2,0}(11) = -a_2/k, \overline{c}_{3,0}(111) = -a_3/k, \overline{c}_{0,1}(0) = 1/k$ $\widetilde{c}_{1,0}(1) = a_1, \widetilde{c}_{2,0}(11) = a_2, \widetilde{c}_{3,0}(111) = a_3, \widetilde{c}_{1,0}(0) = k$, and all the other parameters are zero. The GFRFs can be computed according to (2.12–2.16). For example,

$$H_{1_u}^x(j\omega_1) = \sum_{k_1=0}^{2} \overline{c}_{0,1}(k_1)\exp(-j\omega_1k_1) = \overline{c}_{0,1}(0) = 1/k, \quad H_{1_u}^y(j\omega_1) = 0,$$

Because there are no input nonlinearities and cross nonlinearities, thus
$H_{n_u}^x(j\omega_1,\cdots,j\omega_n) = 0$ and $H_{n_u}^y(j\omega_1,\cdots,j\omega_n) = 0$ for n>1
$H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n) = 0$ and $H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n) = 0$ for all n

Regarding the output nonlinear terms,

$$H_{1_x}^{x}(j\omega_1) = 0,$$

$$\begin{aligned}
H_{2_x}^{x}(j\omega_1, j\omega_2) &= \sum_{p=2}^{2} \sum_{k_1, k_p=1}^{2} \overline{c}_{p,0}\big(k_1, \cdots, k_p\big) H_{2,p}(j\omega_1, j\omega_2) \\
&= \sum_{k_1, k_p=1}^{2} \overline{c}_{2,0}(k_1, k_2) H_{2,2}(j\omega_1, j\omega_2) \\
&= \sum_{k_1, k_p=1}^{2} \overline{c}_{2,0}(k_1, k_2) H_1^x(j\omega_1) H_{1,1}(j\omega_2) \exp(-j\omega_1 k_2) \\
&= \sum_{k_1, k_p=1}^{2} \overline{c}_{2,0}(k_1, k_2) H_1^x(j\omega_1) H_1^x(j\omega_2) \exp(-j\omega_2 k_1) \exp(-j\omega_1 k_2) \\
&= -\frac{a_2}{k} H_1^x(j\omega_1) H_1^x(j\omega_2) \exp(-j\omega_2) \exp(-j\omega_1)
\end{aligned}$$

$$\begin{aligned}
H_{1_x}^{y}(j\omega_1) &= \sum_{k_1}^{2} \widetilde{c}_{1,0}(k_1) H_{1,1}(j\omega_1) = \sum_{k_1}^{2} \widetilde{c}_{1,0}(k_1) H_1^x(j\omega_1) \exp(-j\omega_1 k_1) \\
&= a_1 H_1^x(j\omega_1) \exp(-j\omega_1) + k H_1^x(j\omega_1)
\end{aligned}$$

$$\begin{aligned}
H_{2_x}^{y}(j\omega_1, j\omega_2) &= \sum_{p=1}^{2} \sum_{k_1, k_p=0}^{2} \widetilde{c}_{p,0}\big(k_1, \cdots, k_p\big) H_{2,p}(j\omega_1, j\omega_2) \\
&= \sum_{k_1=0}^{2} \widetilde{c}_{1,0}(k_1) H_{2,1}(j\omega_1, j\omega_2) + \sum_{k_1, k_2=0}^{2} \widetilde{c}_{2,0}(k_1, k_2) H_{2,2}(j\omega_1, j\omega_2) \\
&= \sum_{k_1=0}^{2} \widetilde{c}_{1,0}(k_1) H_2^x(j\omega_1, j\omega_2) \exp(-j(\omega_1 + \omega_2) k_1) \\
&\quad + \sum_{k_1, k_2=0}^{2} \widetilde{c}_{2,0}(k_1, k_2) H_1^x(j\omega_1) H_1^x(j\omega_2) \exp(-j\omega_2 k_1) \exp(-j\omega_1 k_2) \\
&= k H_2^x(j\omega_1, j\omega_2) + a_1 H_2^x(j\omega_1, j\omega_2) \exp(-j(\omega_1 + \omega_2) k_1) \\
&\quad + a_2 H_1^x(j\omega_1) H_1^x(j\omega_2) \exp(-j\omega_2) \exp(-j\omega_1)
\end{aligned}$$

Note that

$$\begin{aligned}
L_n(j(\omega_1 + \cdots + \omega_n)) &= 1 - \sum_{k_1=1}^{2} \overline{c}_{1,0}(k_1) \exp(-j(\omega_1 + \cdots + \omega_n) k_1) \\
&= 1 + \frac{a_1}{k} \exp(-j(\omega_1 + \cdots + \omega_n)) + \frac{m}{k} \exp(-j2(\omega_1 + \cdots + \omega_n))
\end{aligned}$$

Hence, by following similar process as above, the GFRFs for x(t) and y(t) can all be computed recursively up to any high orders. For example,

$$H_1^x(j\omega_1) = \frac{H_{1_u}^x(j\omega_1) + H_{1_{ux}}^x(j\omega_1) + H_{1_x}^x(j\omega_1)}{L_1(j\omega_1)} = \frac{1/k}{1 + \frac{a_1}{k}\exp(-j\omega_1) + \frac{m}{k}\exp(-j2\omega_1)}$$

$$H_2^x(j\omega_1, j\omega_2) = \frac{H_{2_u}^x(j\omega_1, j\omega_2) + H_{2_{ux}}^x(j\omega_1, j\omega_2) + H_{2_x}^x(j\omega_1, j\omega_2)}{L_2(j(\omega_1 + \omega_2))}$$

$$= \frac{-\frac{a_2}{k}H_1^x(j\omega_1)H_1^x(j\omega_2)\exp(-j\omega_2)\exp(-j\omega_1)}{1 + \frac{a_1}{k}\exp(-j(\omega_1 + \omega_2)) + \frac{m}{k}\exp(-j2(\omega_1 + \omega_2))}$$

$$H_1^y(j\omega_1) = H_{1_u}^y(j\omega_1) + H_{1_{ux}}^y(j\omega_1) + H_{1_x}^y(j\omega_1) = k + a_1H_1^x(j\omega_1)\exp(-j\omega_1)$$

$$H_2^y(j\omega_1, j\omega_2) = H_{2_u}^y(j\omega_1, j\omega_2) + H_{2_{ux}}^y(j\omega_1, j\omega_2) + H_{2_x}^y(j\omega_1, j\omega_2)$$

$$= a_1H_2^x(j\omega_1, j\omega_2)\exp(-j(\omega_1 + \omega_2)) + a_2H_1^x(j\omega_1)$$

$$\times H_1^x(j\omega_2)\exp(-j\omega_2)\exp(-j\omega_1)$$

It can be verified that the first order GFRFs are the frequency response functions in z-space of the linear parts of model (2.37).

Example 2.2 Consider a nonlinear mechanical system shown in Fig. 2.1.

The output property of the spring satisfies $A = kx$, the damper $F = a_1\dot{x} + a_3\dot{x}^3$, and the active unit is described by $F = a_2\dot{x}^2$. u(t) is the external input force. Therefore, the system dynamics is

$$m\ddot{x} = -kx - a_1\dot{x} - a_2\dot{x}^2 - a_3\dot{x}^3 + u(t) \tag{2.38a}$$

with the transmitted force measured on the base as the output

$$y(t) = a_1\dot{x} + a_2\dot{x}^2 + a_3\dot{x}^3 + kx(t) \tag{2.38b}$$

It can be seen that the continuous time model (2.38a,b) is similar in structure to the discrete time model (2.37) in Example 2.1. Therefore, similar results regarding the frequency response functions as demonstrated in Examples 2.1 for the discrete time model (2.37) can be obtained readily for system (2.38a,b).

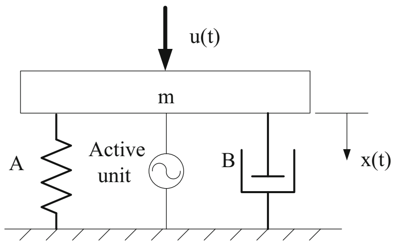


Fig. 2.1 A mechanical system

2.4 The Frequency Response Functions for Block-Oriented Nonlinear Systems

Block-oriented nonlinear systems such as Hammerstein and Wiener models are composed by a cascade combination of a linear dynamic model and a static (memoryless) nonlinear function. Theoretically, any nonlinear systems which have a Volterra expansion can be represented by a finite sum of Wiener models with sufficient accuracy (Korenberg 1982; Boyd and Chua 1985). The Wiener model is shown to be a reasonable model for many chemical and biological processes (Zhu 1999; Kalafatis et al. 1995; Hunter and Korenberg 1986). The magneto-rheological (MR) damping systems can also be well approximated by a Hammerstein model (Huang et al. 1998). Applications of these block-oriented models can be found in many areas such as mechanical systems (Huang et al. 1998), control systems (Bloemen et al. 2001), communication systems (Wang et al. 2010), chemical processes (Kalafatis et al. 1995), and biological systems (Hunter and Korenberg 1986).

Frequently-used block-oriented nonlinear models include Wiener model, Hammerstein model and Wiener-Hammerstein model etc. This section establishes frequency response functions for these nonlinear models under assumption that the nonlinear part allows a polynomial approximation as given in (2.7), which is then extended to a more general case.

2.4.1 *Frequency Response Functions of Wiener Systems*

The GFRFs and nonlinear output spectrum are developed for Wiener systems firstly, and then extended to other models. Consider the Wiener model given by

$$u(t) = g^{\circ} r(t) \text{ and } y(t) = f(u(t)) \tag{2.39a, b}$$

where "$\circ$" represents the convolution operator, $g(t)$ is the impulse response of the linear part, and $f(u(t))$ is the static nonlinear part of the system. The linear part is defined as a stable SISO system, which can be described by parametric FIR/IIR models or nonparametric models (See Fig. 2.2).

Note that the GFRFs for (2.39b) are given in (2.8). Equation (2.39a) can be written as

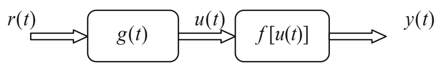


Fig. 2.2 The Wiener model, where $g(t)$ denotes the linear part and $f[\bullet]$ represents the static nonlinear function, both of which could be parametric or nonparametric (Jing 2011)

$$U(\mathrm{j}\omega) = G(\mathrm{j}\omega)R(\mathrm{j}\omega)$$

where $U(\mathrm{j}\omega)$, $G(\mathrm{j}\omega)$ and $R(\mathrm{j}\omega)$ are the corresponding Fourier transforms of $u(t)$, $g(t)$ and $r(t)$ respectively. Using (2.3–2.4), the nth-order output spectrum of (2.39a,b) can be obtained as

$$\begin{aligned}
Y_n(j\omega) &= \frac{1}{(2\pi)^{n-1}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} c_n U_n(j\omega_1, \cdots, j\omega_{n-1}) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} \\
&= \frac{1}{(2\pi)^{n-1}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} c_n \prod_{i=1}^{n} (G(j\omega_i) R(j\omega_i)) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} \\
&= \frac{1}{(2\pi)^{n-1}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} \left(c_n \prod_{i=1}^{n} G(j\omega_i) \right) \prod_{i=1}^{n} R(j\omega_i) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} \\
&= \frac{c_n}{(2\pi)^{n-1}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} \left(\prod_{i=1}^{n} G(j\omega_i) \right) \prod_{i=1}^{n} R(j\omega_i) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1}
\end{aligned} \tag{2.40}$$

where $\omega_n = \omega - \omega_1 - \cdots - \omega_{n-1}$. Comparing the structure of (2.40) with (2.3) gives

$$H_n(j\omega_1, \cdots, j\omega_n) = c_n \prod_{i=1}^{n} G(j\omega_i) \tag{2.41}$$

With the GFRFs given by (2.41), the output spectrum of Wiener system (2.39a,b) can therefore be computed under any input signal with spectrum $R(\mathrm{j}\omega)$ based on (2.3) and (2.41) as,

$$\begin{aligned}
Y_n(j\omega) = \frac{c_n}{(2\pi)^{n-1}} &\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} \\
&\left(\prod_{i=1}^{n-1} G(j\omega_i) R(j\omega_i) \right) G(j(\omega - \omega_1 - \cdots - \omega_{n-1})) \\
&\times R(j(\omega - \omega_1 - \cdots - \omega_{n-1})) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1}
\end{aligned}$$

Let $\Pi(j\omega_i) = G(j\omega_i)R(j\omega_i)$, then $Y_1(j\omega_\mathrm{i}) = c_1 \Pi(j\omega_i)$, which represents the linear dynamics of the Wiener system. The equation above can be further written as

$$Y_n(j\omega) = \frac{c_n}{(2\pi)^{n-1}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} \left(\prod_{i=1}^{n-1} \Pi(j\omega_i) \right)$$
$$\Pi(j(\omega - \omega_1 - \cdots - \omega_{n-1}))\mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} \tag{2.42}$$

It can be seen that the nth-order nonlinear output spectrum of Wiener model (2.39a, b) is completely dependent on the frequency response of system linear part. Given the frequency response of the linear part (which can be nonparametric), higher order output spectra can be computed immediately. On the other hand, given higher order output spectra, the linear part of the system could also be estimated consequently. These will be discussed later. The overall output spectrum is a combination of all different order output spectra. Clearly, the nonlinear frequency response functions obtained above can provide an effective insight into the analytical analysis and design of Wiener systems in the frequency domain. Note that the magnitude bound of system output spectrum often provides a useful insight into system dynamics at different frequencies and also into the relationship between frequency response functions and model parameters (Jing et al. 2007a, 2008b, d). With the GFRFs developed above, the bound characteristics of the output spectrum of Wiener system (2.39a,b) can be investigated readily. It is known that output frequencies of nonlinear systems are always more complicated than linear systems including sub- or super-harmonics and inter-modulations (Jing et al. 2010). The GFRFs and output spectrum above could also shed light on the analysis of output frequency characteristics of Wiener-type nonlinear systems.

2.4.2 *The GFRFs of Wiener-Hammerstein or Hammerstein Systems*

The Wiener-Hammerstein model can be described by

$$u(t) = g^{\circ}r(t);\ x(t) = f(u(t)) \text{ and } \ y(t) = p^{\circ}x(t) \tag{2.43a, b, c}$$

where p and g denote the linear parts following and preceding nonlinear function $f(\cdot)$ (See Fig. 2.3).

Consider the subsystem from $r(t)$ to $x(t)$, which is the Wiener model in (2.39a,b). According to (2.40–2.41), the nth-order output spectrum of this subsystem is

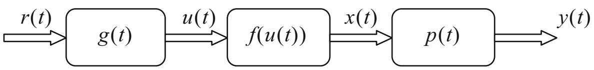


Fig. 2.3 The Wiener-Hammerstein model, where $g(t)$ and $p(t)$ denote the linear parts and $f(\bullet)$ represents the static nonlinear function, g-f is actually a Wiener sub-system and f-p is a Hammerstein sub-system (Jing 2011)

$$X_n(j\omega) = \frac{1}{(2\pi)^{n-1}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} \left(c_n \prod_{i=1}^{n} G(j\omega_i) \right) \prod_{i=1}^{n} R(j\omega_i) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1}$$

where $\omega_n = \omega - \omega_1 - \cdots - \omega_{n-1}$. Then the nth-order output spectrum of (2.43a–c) is

$$\begin{aligned} Y_n(j\omega) &= X_n(j\omega)P(j\omega) \\ &= \frac{c_n P(j\omega)}{(2\pi)^{n-1}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} \left(\prod_{i=1}^{n} G(j\omega_i) \right) \prod_{i=1}^{n} R(j\omega_i) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} \end{aligned} \tag{2.44}$$

where $P(\mathrm{j}\omega)$ is the Fourier transform of $p(\mathrm{t})$. Therefore, the nth-order GFRF for (2.43a–c) is

$$H_n(j\omega_1, \cdots, j\omega_n) = c_n P(j\omega) \prod_{i=1}^{n} G(j\omega_i) \tag{2.45a}$$

Noting that $\omega = \omega_1 + \cdots + \omega_n$ in (2.45a), the equation above can be written as

$$H_n(j\omega_1, \cdots, j\omega_n) = c_n P(j\omega_1 + \cdots + j\omega_n) \prod_{i=1}^{n} G(j\omega_i) \tag{2.45b}$$

Using (2.45b) and noting the Hammerstein model is only a special case ($g(t) = 1$) of the Wiener-Hammerstein model, the GFRFs of Hammerstein systems can be obtained immediately as

$$H_n(j\omega_1, \cdots, j\omega_n) = c_n P(j\omega_1 + \cdots + j\omega_n) \tag{2.46}$$

Note that, the GFRFs and output spectrum of block-oriented nonlinear systems are derived by employing the structure property of the nonlinear output spectrum defined in (2.3) and the structure information of block-oriented models. The resulting frequency response functions are expressed into analytical functions of model parameters, which are not restricted to a specific input but allow any form of input signals. However, many existing frequency-domain results for nonlinear analysis require a specific sinusoidal input signal (Alleyne and Hedrick 1995; Gelb and Velde 1968; Nuij et al. 2006; Schmidt and Tondl 1986; Huang et al. 1998; Baumgartner and Rugh 1975; Krzyzak 1996; Crama and Schoukens 2001; Bai 2003). In the GFRFs, the relationships among the output spectrum, the GFRFs, the system nonlinear parameters, and also the linear dynamics of the system are demonstrated clearly. With these frequency response functions, bound characteristics of the output spectrum and output frequency characteristics etc can all be studied by following the methods in Jing et al. (2006, 2007a, 2008b, d, 2010).

2.4.3 Extension to a More General Polynomial Case

The static nonlinear function of block-oriented models discussed above is only a univariate polynomial function. A more general case is studied in this section. Since both the Wiener and Hammerstein models are special cases, consider the following Wiener-Hammerstein model,

$$u(t) = g^{\circ}r(t);\;\; x(t) = f(u(t), r(t)) \text{ and } y(t) = p^{\circ}x(t) \tag{2.47a, b, c}$$

where the nonlinear function is defined as a more general multivariate polynomial function as

$$f(u(t), r(t)) = \sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1, k_m=0}^{K} c_{p,m-p}(k_1, \cdots, k_m)\prod_{i=1}^{p}\frac{\mathrm{d}^{k_i}u(t)}{\mathrm{d}t^{k_i}}\prod_{i=p+1}^{m}\frac{\mathrm{d}^{k_i}r(t)}{\mathrm{d}t^{k_i}} \tag{2.47d}$$

where M is the maximum nonlinear degree of the polynomial, K is the maximum order of the derivative and $c_{p,m-p}(k_1, \cdots, k_m)$ is the coefficient of a term $\prod_{i=1}^{p}\frac{\mathrm{d}^{k_i}u(t)}{\mathrm{d}t^{k_i}}\prod_{i=p+1}^{m}\frac{\mathrm{d}^{k_i}r(t)}{\mathrm{d}t^{k_i}}$ in the polynomial.

Obviously, the univariate polynomial function in (2.7) is only a special case of the general form (2.47d). That is, if letting $c_{p,0}(\underbrace{0, \cdots, 0}_{p}) = c_p$ for p=1,2,... and the other coefficients in (2.47d) are zero, then (2.47d) will become (2.7). Obviously, (2.47a–d) can represent a wider class of nonlinear systems. For example, if (2.47d) is of sufficiently high degree and includes all possible linear and nonlinear combinations of input $u(t)$ and its derivatives of sufficiently high orders, it will be an equivalent nonlinear IIR model of the Volterra-type nonlinear systems (Kotsios 1997). The following results can be obtained.

Proposition 2.1 The nth-order GFRF of Wiener-Hammerstein system (2.47a–d) is given by

$$\begin{aligned}H_n(j\omega_1, \cdots, j\omega_n) = P(j\omega_1 + \cdots + j\omega_n)(H_{n_r}(j\omega_1, \cdots, j\omega_n)\\ + H_{n_{ru}}(j\omega_1, \cdots, j\omega_n) + H_{n_u}(j\omega_1, \cdots, j\omega_n))\end{aligned} \tag{2.48}$$

Similarly, for Wiener systems with a general polynomial function (2.47d) it is given by

$$H_n(j\omega_1, \cdots, j\omega_n) = H_{n_r}(j\omega_1, \cdots, j\omega_n) + H_{n_{ru}}(j\omega_1, \cdots, j\omega_n) + H_{n_u}(j\omega_1, \cdots, j\omega_n) \tag{2.49}$$

and for Hammerstein systems with a general polynomial function (2.47d) it is

$$H_n(j\omega_1,\cdots,j\omega_n) = P(j\omega_1 + \cdots + j\omega_n)H_{n_r}(j\omega_1,\cdots,j\omega_n) \tag{2.50}$$

where $H_{n_r}(j\omega_1,\cdots,j\omega_n)$, $H_{n_{ru}}(j\omega_1,\cdots,j\omega_n)$ and $H_{n_u}(j\omega_1,\cdots,j\omega_n)$ are given by

$$H_{n_r}(j\omega_1,\cdots,j\omega_n) = \sum_{k_1,k_n=0}^{K} c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n} \tag{2.51a}$$

$$H_{n_{ru}}(j\omega_1,\cdots,j\omega_n) = \sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}\left(k_1,\cdots,k_{p+q}\right)\left(j\omega_{n-q+1}\right)^{k_{p+1}}\cdots(j\omega_n)^{k_{p+q}}$$

$$\times H_{n-q,p}\left(j\omega_1,\cdots,j\omega_{n-q}\right) \tag{2.51b}$$

$$H_{n_u}(j\omega_1,\cdots,j\omega_n) = \sum_{p=1}^{n}\sum_{k_1,k_p=0}^{K} c_{p,0}\left(k_1,\cdots,k_p\right)H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{2.51c}$$

$$H_{n,p}(\cdot) = G(j\omega_1)H_{n-1,p-1}(j\omega_2,\cdots,j\omega_n)(j\omega_1)^{k_p} \tag{2.51d}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = \begin{cases} G(j\omega_1)(j\omega_1)^{k_1} & \text{n} = 1 \\ 0 & \text{else} \end{cases} \tag{2.51e}$$

Proof See Sect. 2.6.

Since (2.39b) is a special case of (2.47d) ($c_{p,0}(0,\cdots,0)=c_p$ and the others zero in (2.47d)), the nth-order GFRF in (2.41) for Wiener model (2.39a,b) can be shown to be a special case of (2.51a–e). That is, only the parameter $c_{n,0}(\underbrace{0,\cdots,0}_{n}) = c_n$ is not zero and the others are zero in (2.51a–e). Therefore,

$$H_n(j\omega_1,\cdots,j\omega_n)=H_{n_r}(j\omega_1,\cdots,j\omega_n)+H_{n_{ru}}(j\omega_1,\cdots,j\omega_n)+H_{n_u}(j\omega_1,\cdots,j\omega_n) \tag{2.52a}$$

$$H_{n_r}(j\omega_1,\cdots,j\omega_n) = 0, \quad H_{n_{ru}}(j\omega_1,\cdots,j\omega_n) = 0 \tag{2.52b, c}$$

$$H_{n_u}(j\omega_1,\cdots,j\omega_n) = c_{n,0}(0,\cdots,0)H_{n,n}(j\omega_1,\cdots,j\omega_n) \tag{2.52d}$$

$$\begin{aligned} H_{n,n}(\cdot) &= G(j\omega_1)H_{n-1,n-1}(j\omega_2,\cdots,j\omega_n)(j\omega_1)^0 \\ &= G(j\omega_1)G(j\omega_2)\cdots G(j\omega_{n-1})H_{1,1}(j\omega_n) \end{aligned} \tag{2.52e}$$

$$H_{1,1}(j\omega_n) = G(j\omega_n) \tag{2.52f}$$

The nth-order GFRF for (2.39a,b) can now be obtained from (2.52a–f) which is exactly (2.41).

Although the nonlinear frequency response functions above are all developed for continuous time system models, it would be easy to extend them to discrete time systems. In this section, analytical frequency response functions including generalized frequency response functions (GFRFs) and nonlinear output spectrum of block-oriented nonlinear systems are developed, which can demonstrate clearly the relationship between frequency response functions and model parameters, and also the dependence of frequency response functions on the linear part of the model. The nonlinear part of these models can be a more general multivariate polynomial function. These fundamental results provide a significant insight into the analysis and design of block-oriented nonlinear systems. Effective algorithms can also be developed for the estimation of nonlinear output spectrum and for parametric or nonparametric identification of nonlinear systems, which can be referred to Jing (2011).

2.5 Conclusions

The computation of the GFRFs and/or output spectrum for a given nonlinear system described by NARX, NDE or Block-oriented models is a fundamental task for nonlinear analysis in the frequency domain. This chapter summarizes the results for the computation of the GFRFs and output spectrum for several frequently-used parametric models, and shows that the GFRFs are the explicit functions of model parameters (of different nonlinear degrees), which can be regarded as an important extension of the transfer function concept of linear systems to the nonlinear case.

2.6 Proof of Proposition 2.1

As the Wiener system is a sub-system of the Wiener-Hammerstein system in (2.47a–d), the GFRFs for the sub-Wiener system can be derived firstly and then it will be easily extended to the other block-oriented models as demonstrated in Sect. 2.41–2.42. To derive the GFRFs for Wiener systems with the general polynomial function (2.47d), i.e.,

$$u(t) = g^{\circ} r(t) \quad \text{and} \quad y(t) = f(u(t), r(t)) \tag{A1, A2}$$

$$f(u(t), r(t)) = \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{k_1, k_m=0}^{K} c_{p,m-p}(k_1, \cdots, k_m) \prod_{i=1}^{p} \frac{\mathrm{d}^{k_i} u(t)}{\mathrm{d}t^{k_i}} \prod_{i=p+1}^{m} \frac{\mathrm{d}^{k_i} r(t)}{\mathrm{d}t^{k_i}} \tag{A3}$$

the model can be regarded as a nonlinear differential equation model with two outputs $u(t)$ and $y(t)$, and one input $r(t)$. Note that the frequency response function from the input $r(t)$ to the intermediate output $u(t)$ is the Fourier transform of the impulse response function g(t), i.e., $G(\mathrm{j}\omega)$, which is a linear dynamics; while the frequency responses from the input $r(t)$ to the output $y(t)$ involve nonlinear dynamics. The latter are the GFRFs to be derived. The terms in the polynomial function $f(u(t),r(t))$ can be categorized into three groups, i.e., pure input terms $c_{0,m}(k_1, \cdots, k_m) \prod_{i=1}^{m} \frac{\mathrm{d}^{k_i} r(t)}{\mathrm{d}t^{k_i}}$, pure output terms $c_{p,0}(k_1, \cdots, k_p) \prod_{i=1}^{p} \frac{\mathrm{d}^{k_i} u(t)}{\mathrm{d}t^{k_i}}$, and input-output cross terms $c_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p}\frac{\mathrm{d}^{k_i}u(t)}{\mathrm{d}t^{k_i}}\prod_{i=p+1}^{m}\frac{\mathrm{d}^{k_i}r(t)}{\mathrm{d}t^{k_i}}$ $(0<p<m)$. Therefore, by applying the probing method, each group of terms corresponds to a specific form of contributions to the *n*th-order GFRF of Wiener system (A1–A3), which can be written as

$$H_n(j\omega_1,\cdots,j\omega_n) = H_{n_r}(j\omega_1,\cdots,j\omega_n) + H_{n_{ru}}(j\omega_1,\cdots,j\omega_n) + H_{n_u}(j\omega_1,\cdots,j\omega_n) \tag{A4}$$

where $H_{n_r}(j\omega_1,\cdots,j\omega_n)$ represents the contribution from the pure input terms and similar notations are used for the other terms. This is a special case of the system studied in Sect. 2.3 or Jing et al. (2008c). Therefore following the method there, the equations in (2.51a–e) can be obtained.

Similarly, the corresponding GFRFs for the Hammerstein model and Wiener-Hammerstein model with the general polynomial function defined in (2.47d) can be derived respectively. Note that only input nonlinearity is involved in the Hammerstein model. The extended polynomial function for the Hammerstein model can be written as

$$x(t) = f(r(t)) = \sum_{m=1}^{M}\sum_{k_1,k_m=0}^{K} c_{0,m}(k_1,\cdots,k_m)\prod_{i=1}^{m}\frac{d^{k_i}r(t)}{dt^{k_i}} \tag{A5}$$

Following the same line, the nth-order GFRF for the extended Hammerstein model is given by

$$H_n(j\omega_1,\cdots,j\omega_n) = P(j\omega_1+\cdots+j\omega_n)\sum_{k_1,k_n=0}^{K} c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n} \tag{A6}$$

The nth-order GFRF for the Wiener-Hammerstein model (2.47a–d) can be obtained as

$$H_n(j\omega_1,\cdots,j\omega_n) = P(j\omega_1+\cdots+j\omega_n)(H_{n_r}(j\omega_1,\cdots,j\omega_n) + H_{n_{ru}}(j\omega_1,\cdots,j\omega_n) + H_{n_u}(j\omega_1,\cdots,j\omega_n)) \tag{A7}$$

where $H_{n_r}(j\omega_1,\cdots,j\omega_n)$, $H_{n_{ru}}(j\omega_1,\cdots,j\omega_n)$ and $H_{n_u}(j\omega_1,\cdots,j\omega_n)$ are given by (2.51a–e). This completes the proof. ∎

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