Chapter 2
Stochastic Invariant Manifolds: Background and Main Contributions

The focus of this first volume is the derivation of leading-order approximations of stochastic invariant manifolds such as stochastic center manifolds by extending, to a stochastic context, the techniques described in [117, Chap. 3] and [120, Appendix A]; see Chap. 6. New properties of approximating manifolds described in terms of pullback characterization will be reported in Volume II [37, Sect. 4.1].[1] The framework set up in this way allows us, furthermore, to unify the previous approximation approaches from the literature [18, 29, 40, 103]. These features are not limited to the stochastic setting as pointed out in Volume II [37, Sect. 4.1].

In that respect and motivated by the study of stochastic bifurcations or more general phase transitions arising in nonlinear SPDEs[2] [56, 127], we first revisit in Chaps. 4 and 5 the existence and smoothness properties (Theorems 4.1, 4.2 and 5.1)—as well as the attraction properties in terms of *almost sure asymptotic completeness* (Theorem 4.3)—of families of global stochastic invariant manifolds parameterized by the noise amplitude σ, and by some control parameter λ. The latter is assumed here to vary in some interval Λ over which a uniform decomposition of the spectrum holds; see (3.11) below. The latter condition implies some uniform (partial)-dichotomy estimates that are satisfied by the linearized stochastic flow about the basic state; see (3.46).

The questions of existence and smoothness are dealt within a framework rooted in the standard Lyapunov-Perron method [19, 109, 114, 131]. The techniques follow those used for instance in [46, 92, 152, 153, 154], from which we propose a treatment adapted to the random setting inspired mainly by the works of [42, 66]. The related existence and smoothness results are essentially known, but are revisited here in order to set up the precise framework and to provide the technical tools on which we

[1] Section 4.1 in Volume II [37] concerns the approximating manifolds considered in this volume. This somewhat unconventional presentation has been adopted here in order to articulate, in a unified way, the *pullback characterization* of such manifolds as well as of the *stochastic parameterizing manifolds* considered in [37].

[2] which is the main purpose of [36].

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rely to establish the main results of this volume in Chaps. 6 and 7 as well as those presented in Volume II [37, Sect. 4.1].

Our treatment of the *asymptotic completeness problem* is inspired by the work of [46] that we adopt in the stochastic framework, and that consists of reformulating this problem as a fixed point problem under constraints; see (4.21). The latter problem is then recast as an unconstrained fixed point problem associated with a random integral operator (see 4.24) that is solved by means of the *uniform contraction mapping principle* [44, Theorems 2.1 and 2.2]. Various types of attraction properties of random invariant manifolds have been explored in the literature mostly in contexts where the associated SPDEs possess a stable self-adjoint linear part and a bounded and Lipschitz nonlinearity. For instance, in [12, Theorem 3.1], both forward and pullback exponential attractions[3] of the stochastic inertial manifold are established. Asymptotic completeness in some nth-moment has been established in [51, Theorem 2] and [57, Proposition 3.5] (with n being any integer in [51] and $n = 2$ in [57]) for stochastic inertial manifolds associated with certain types of SPDEs with respectively additive and multiplicative noise. Almost sure forward asymptotic completeness for deterministic initial data has been established in [42] for retarded SPDEs with additive noise and a stable self-adjoint linear part, adapting also the work of [46] to a stochastic context. Almost sure pullback asymptotic completeness of stochastic invariant manifolds has also been investigated in [155, Theorem 2.1] for certain type of SPDEs with nonlinearities which do not cause a loss of regularity compared to the ambient space $\mathscr{H}$.[4]

For SPDEs considered in this monograph, which in particular allow for nonlinearities causing a loss of regularity (see Chap. 3), Theorem 4.3 provides conditions under which the stochastic invariant manifolds ensured by Corollary 4.1 are almost surely forward and pullback asymptotically complete with respect to random tempered initial data; see Definition 4.3. In particular the existence of a one-parameter family of global stochastic inertial manifolds is obtained, and it is shown that the constitutive manifolds of this family attract exponentially the dynamics at a uniform rate as λ varies in Λ. The results obtained in Theorem 4.3 and Corollary 4.3 are not restricted to the case of self-adjoint linear operator and include the cases where unstable modes are present. The latter situation is particularly useful to establish in Volume II [37] that stochastic inertial manifold always constitute a stochastic PM; see [37, Theorem 4.1] whose proof relies furthermore on some elements contained in the proof of Theorem 4.3.

In Chap. 5, we present a local theory of stochastic invariant manifolds associated with the global theory described in Chap. 4. The ideas are standard but the material is detailed here again in view of the derivation of the main results regarding the approximation formulas of stochastic critical manifolds (Chap. 6) and local stochastic

[3]The exponential attraction used therein extends in a random context the classical one encountered in the theory of (deterministic) inertial manifold [79].

[4]Namely, $F\colon \mathscr{H} \to \mathscr{H}$, adopting the notations of Chap. 3. Note also that the proof of [155, Theorem 2.1] provided therein is not complete.

hyperbolic invariant manifolds (Chap. 7), as well as the related pullback characterizations discussed in Volume II [37, Sect. 4.1].

Chapters 6 and 7 are devoted to the main results concerning approximating manifolds of local stochastic critical manifolds on the one hand (Chap. 6), and local hyperbolic ones, on the other (Chap. 7). They concern the derivation of new approximation formulas of these (local) stochastic invariant manifolds. More precisely, in Chap. 6, we consider the important case for applications where some leading modes lose (once) their stability as λ varies in Λ, which is formulated as the *principle of exchange of stabilities (PES)*; see condition (6.4). It is shown in Lemma 6.1 that the latter implies the uniform spectrum decomposition assumed in previous sections. This allows us in turn to establish in Proposition 6.1, the existence of a family of *local stochastic critical manifolds* which are built—by relying on Chap. 5—as graphs over some deterministic neighborhood of the origin in the subspace spanned by the critical modes that lose their stability as λ varies.[5] By construction, these manifolds carry nonlinear dynamical information associated with the loss of the linear stability of these critical modes; see [36].

We then derive in Theorem 6.1 and Corollary 6.1, *explicit random approximation formulas* to the leading order of these local stochastic critical manifolds about the origin. These stochastic critical manifolds are built naturally as graphs over a fixed number of critical modes, which lose their stability as λ varies. More precisely, the corresponding approximating manifolds are obtained as graphs—over some λ-independent neighborhood $\mathscr{N}$ of zero in the subspace $\mathscr{H}^{\mathfrak{c}}$ spanned by the critical modes—of the following one-parameter family of random functions:

$$\mathfrak{I}_\lambda(\xi, \omega) = \int_{-\infty}^{0} e^{\sigma(k-1)W_t(\omega)\mathrm{Id}} e^{-tL_\lambda} P_{\mathfrak{s}} F_k(e^{tL_\lambda}\xi)\,\mathrm{d}t, \quad \xi \in \mathscr{N}, \tag{AF}$$

where F_k denotes the leading-order nonlinear terms of order k, L_λ the corresponding parameterized linear part, $P_{\mathfrak{s}}$ the projector upon the non-critical modes, and $W_t(\omega)$ the Wiener path associated with the realization ω of the noise with amplitude σ.

It is worth mentioning at this stage that the random approximation formulas such as (AF), contrast with the deterministic ones proposed in [18] and [40] for certain types of SPDEs. In particular, the nonlinearity considered in [18] consists of a bilinear term, $B(u, u)$, while it consists of power nonlinearity, u^k, with $k \geq 2$ in [40]. The error bounds for the approximation of the local random invariant manifold function $h(u, \omega)$ provided in both [18] and [40] are of the same order as $\|u\|$ and are valid with large probability, and for sufficiently small u; see [40, Lemma 4.10] and footnote 3 in [37, Chap. 4] for [18, Theorem 7].

The class of nonlinear SPDEs of type (3.1) considered below contains the SPDEs dealt with in [18, 40] as special cases. In contrast with the deterministic approximation formulas obtained in [18, 40], the approximations derived hereafter are genuine random polynomial functions, which approximate almost surely the local random

[5] See Definition 6.1 for more details.

critical manifolds and provide (random) Taylor approximations of these manifolds to the leading order; see Corollary 6.1. More precisely, a priori error estimates are derived in a general setting which are of order $o(\|u\|_\alpha^k)$ if the nonlinear term, $F(u)$, is such that $\|F(u)\| = O(\|u\|_\alpha^k)$ for some integer $k \geq 2$;[6] see again Theorem 6.1 and Corollary 6.1 for precise statements of these results.

Approximation formulas such as given by (AF) are then extended to the case of stochastic hyperbolic manifolds in Chap. 7, which allows for the low-dimensional subspace $\mathscr{H}^{\mathfrak{c}}$ to contain a combination of critical modes, and modes that remain stable as λ varies in some interval Λ. In that respect, relaxation of the conditions on the spectrum under which the Lyapunov-Perron integral $\mathfrak{I}_\lambda$ exists, are identified. In particular, when L_λ is self-adjoint, it is shown that $\mathfrak{I}_\lambda$ exists if a non-resonance condition (NR) is satisfied: For any given set of resolved modes for which their self-interactions (through the leading-order nonlinear term F_k) do not vanish when projected against an unresolved mode e_n, it is required that some specific linear combinations of the corresponding eigenvalues dominate the eigenvalue associated with e_n.

We turn now to the organization of this first volume. In Chap. 3, we introduce the class of SPDEs considered throughout this monograph and describe the main assumptions among which a uniform decomposition of the spectrum of the linear part constitutes a key ingredient in most of the proofs presented hereafter. We also recall some basic concepts from RDS theory [1], and cast such SPDEs into the RDS framework by a classical random change of variables leading to random partial differential equations (RPDEs). The existence, uniqueness, and measurability properties of classical solutions to such RPDEs are recalled in Proposition 3.1. To make the expository as much self-contained as possible, the proof and some related results concerning the mild solutions to these RPDEs are presented in Appendix A.

In Chap. 4, we revisit the existence and attraction properties of global random/stochastic invariant manifolds within a framework that is suitable for the derivation of certain results regarding the stochastic parameterizing manifolds introduced in Volume II [37]; see, e.g., [37, Theorem 4.1]. We first derive the existence and smoothness of such manifolds for the transformed RPDEs in Theorems 4.1 and 4.2. The corresponding results for the original SPDEs are presented in Corollaries 4.1 and 4.2. Finally, the almost sure forward-and-pullback asymptotic completeness of these manifolds is examined in Theorem 4.3 and Corollary 4.3.

In Chap. 5, we relax the global Lipschitz condition on the nonlinear term, and derive accordingly the existence of local stochastic invariant manifolds for SPDEs; see Theorem 5.1 and Corollary 5.1. Chapter 6 is devoted to the main results of this first volume regarding the approximation formulas of local stochastic critical manifolds for SPDEs, as summarized in Theorem 6.1 and Corollary 6.1. Rigorous error estimates to the leading order are in particular derived. These results are then extended to the case of (local) stochastic hyperbolic manifolds in Chap. 7.

[6] Here $\|\cdot\|_\alpha$ denotes a norm on a space of functions more regular than those of the ambient space $\mathscr{H}$; see Chap. 3.

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