

Methods of Quantitative Reconstruction of Shapes and Refractive Indices from Experimental data

Larisa Beilina, Nguyen Trung Thành, Michael V. Klibanov
and John Bondestam Malmberg

1 Introduction

We consider the problem of reconstruction of refractive indices and shapes of objects placed in the air from blind backscattered time-dependent experimental data using two-stage numerical procedure presented in [4, 6]. Our problem is a coefficient inverse problem (CIP) for the three dimensional Maxwell's equations. Experimental data were collected using a microwave scattering facility which was built at the University of North Carolina at Charlotte, USA. Our experimental data are generated using a single location of the source. The backscattered signal is measured on a part of a plane, see [5, 14] for the description of the data collection procedure.

Two-stage numerical procedure means that we combine two different methods to solve our CIP. On the first stage the approximately globally convergent method of [4] is applied in order to obtain a good first approximation for the exact solution. We have presented results of reconstruction on this stage in our recent publications [5, 14]. On the second stage the local adaptive finite element method of [1] is applied by taking the solution of the first stage as the starting point in the minimization of a Tikhonov functional in order to obtain better approximations and shapes of objects on the adaptively refined meshes. In our experiments of this chapter we image targets

L. Beilina (✉) · J. B. Malmberg

Department of Mathematical Sciences, Chalmers University of Technology
and Gothenburg University, SE-42196 Gothenburg, Sweden
e-mail: larisa@chalmers.se

J. B. Malmberg

e-mail: john.bondestam.malmberg@chalmers.se

N. T. Thành · M. V. Klibanov

Department of Mathematics and Statistics University of North Carolina at Charlotte,
Charlotte, NC 28223, USA
e-mail: tnguy152@uncc.edu

M. V. Klibanov

e-mail: mklibanv@uncc.edu

located in the air. A potential application of our work is in imaging of explosives. In this chapter we present full set of experiments which were not presented before in our previous publications [5, 6, 14] on this topic.

An outline of this chapter is as follows: In Sect. 2 we describe the first stage of the two-stage procedure. In Sect. 3 we formulate the forward and inverse problems for the second stage. Section 4 introduces the Tikhonov functional used on the second stage. Section 5 describes the adaptive algorithm on the second stage. Numerical results are presented in Sect. 6.

2 The First Stage: Approximately Globally Convergent Method

In this section we state the forward and inverse problems which we consider on the first stage. We also briefly outline the globally convergent method of [4] and present algorithm used in computations of the first stage.

Forward and Inverse Problems

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Denote the spatial coordinates by $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Let $C^{k+\alpha}$ be Hölder spaces, where $k \geq 0$ is an integer and $\alpha \in (0, 1)$. We consider the propagation of the electromagnetic wave in $\mathbb{R}^3$ generated by an incident plane wave. On the first stage we model the wave propagation by the following Cauchy problem for the scalar wave equation

$$\varepsilon_r(\mathbf{x}) \frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) + \delta(z - z_0)f(t), \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times (0, \infty), \quad (1)$$

$$u(\mathbf{x}, 0) = 0, \quad \frac{\partial u}{\partial t}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^3. \quad (2)$$

Here $f(t) \neq 0$ is the time-dependent incident plane wave at the plane $\{z = z_0\}$, u is the total wave generated by $f(t)$ and propagating along the z -axis.

Let the function $u(\mathbf{x}, t)$ represent the voltage of one component E_2 of the electric field $E(\mathbf{x}, t) = (E_1, E_2, E_3)(\mathbf{x}, t)$. In our experiments the component E_2 corresponds to the electromagnetic wave which is sent into the medium. Our mathematical model of the first stage uses only the single Eq. (1) with $u = E_2$ instead of the full Maxwell's system. We can justify such approximation since in [2] it was shown numerically that the component E_2 of the electric field E dominates two other components in the case when only E_2 is initialized at time $t = 0$. See also [4], where a similar scalar wave equation was used to work with transmitted experimental data.

The function $\varepsilon_r(\mathbf{x})$ in (1) represents the spatially distributed dielectric permittivity. We assume that ε_r is unknown inside a domain $\Omega \subset \mathbb{R}^3$ and is such that

$$\varepsilon_r(\mathbf{x}) \in C^\alpha(\mathbb{R}^3), \quad \varepsilon_r(\mathbf{x}) \in [1, b], \quad \varepsilon_r(\mathbf{x}) = 1 \text{ for } \mathbf{x} \in \mathbb{R}^3 \setminus \Omega, \quad (3)$$

where $b = \text{const.} > 1$. We assume that the set of admissible coefficients in (3) is known a priori. Let $\Gamma \subset \partial\Omega$ be a part of the boundary $\partial\Omega$. In our experiments the plane wave is initialized outside of the domain $\overline{\Omega}$, i.e. $\overline{\Omega} \cap \{z = z_0\} = \emptyset$.

Coefficient Inverse Problem (CIP) *Determine the function $\varepsilon_r(\mathbf{x})$ for $\mathbf{x} \in \Omega$, assuming that the following function $g(\mathbf{x}, t)$ is known for a single incident plane wave generated at the plane $\{z = z_0\}$ outside of $\overline{\Omega}$:*

$$u(\mathbf{x}, t) = g(\mathbf{x}, t), \quad \forall (\mathbf{x}, t) \in \Gamma \times (0, \infty).$$

Global uniqueness theorems for multidimensional CIPs with a single measurement are currently known only under the assumption that at least one of the initial conditions does not equal to zero in the entire domain $\overline{\Omega}$ [4, 8]. However, this is not our case and the method of Carleman estimates is inapplicable to our CIP. We simply assume that the uniqueness of our CIP holds because of numerical experiments of this chapter and presented in [5, 6, 14].

The Globally Convergent Method

Here, we briefly present the approximately globally convergent method of [4]. We perform a Laplace transform

$$\tilde{u}(\mathbf{x}, s) = \int_0^\infty u(\mathbf{x}, t) e^{-st} dt,$$

where s is a positive parameter which we call *pseudo frequency*. We assume that $s \geq \underline{s} > 0$ and denote by $\tilde{f}(s)$ the Laplace transform of $f(t)$. We assume that $\tilde{f}(s) \neq 0$ for all $s \geq \underline{s}$. Define $w(\mathbf{x}, s) := \tilde{u}(\mathbf{x}, s) / \tilde{f}(s)$. The function w satisfies the equation

$$\Delta w(\mathbf{x}, s) - s^2 \varepsilon_r(\mathbf{x}) w(\mathbf{x}, s) = -\delta(z - z_0), \quad \mathbf{x} \in \mathbb{R}^3, \quad s \geq \underline{s}. \quad (4)$$

Similarly with Theorem 2.7.2 of [4] we can prove that $w(\mathbf{x}, s) > 0$ and

$$\lim_{|\mathbf{x}| \rightarrow \infty} (w(\mathbf{x}, s) - w_0(\mathbf{x}, s)) = 0,$$

where $w_0(\mathbf{x}, s) := e^{-s|z-z_0|} / (2s)$ is a solution of Eq. (4) for the case $\varepsilon_r(\mathbf{x}) \equiv 1$, which decays to zero as $|z| \rightarrow \infty$. Next, introduce the function v by

$$v(\mathbf{x}, s) := \frac{\ln w(\mathbf{x}, s)}{s^2}$$

and substitute $w = e^{vs^2}$ into (4). Noting that $\overline{\Omega} \cap \{z = z_0\} = \emptyset$, we finally obtain the following equation for the explicit computation of the coefficient ε_r :

$$\Delta v(\mathbf{x}, s) + s^2 |\nabla v(\mathbf{x}, s)|^2 = \varepsilon_r(\mathbf{x}), \quad \mathbf{x} \in \Omega. \quad (5)$$

Next, we eliminate the unknown coefficient ε_r from (5) by differentiating with respect to s on both sides of (5). Denote by $q(\mathbf{x}, s) := \frac{\partial v}{\partial s}(\mathbf{x}, s)$. Then

$$v(\mathbf{x}, s) = - \int_s^\infty q(\mathbf{x}, \tau) d\tau = - \int_s^{\bar{s}} q(\mathbf{x}, \tau) d\tau + V(\mathbf{x}),$$

where $\bar{s} > \underline{s}$. We call the function $V(\mathbf{x}) = v(\mathbf{x}, \bar{s})$ the “tail function” and define it by

$$V(\mathbf{x}) = \frac{\ln w(\mathbf{x}, \bar{s})}{\bar{s}^2}. \quad (6)$$

From (5) we obtain the following equation for two unknown functions q and V

$$\begin{aligned} \Delta q(\mathbf{x}, s) - 2s^2 \nabla q(\mathbf{x}, s) \cdot \int_s^{\bar{s}} \nabla q(\mathbf{x}, \tau) d\tau + 2s^2 \nabla V(\mathbf{x}) \cdot \nabla q(\mathbf{x}, s) \\ + 2s \left| \int_s^{\bar{s}} \nabla q(\mathbf{x}, \tau) d\tau \right|^2 - 4s \nabla V(\mathbf{x}) \cdot \int_s^{\bar{s}} \nabla q(\mathbf{x}, \tau) d\tau + 2s |\nabla V(\mathbf{x})|^2 = 0, \end{aligned} \quad (7)$$

for $\mathbf{x} \in \Omega$, $s \in (\underline{s}, \bar{s})$.

To find the tail function V we use an iterative procedure presented in the next section, see [5, 14] for details of this procedure. The function q satisfies the following boundary condition

$$q(\mathbf{x}, s) = \psi(\mathbf{x}, s), \quad \mathbf{x} \in \partial\Omega, \quad (8)$$

where $\psi(\mathbf{x}, s) = \frac{\partial}{\partial s} \left[\frac{\ln \varphi(\mathbf{x}, s)}{s^2} \right]$ with $\varphi(\mathbf{x}, s) = \int_0^\infty g(\mathbf{x}, t) e^{-st} dt / \tilde{f}(s)$.

Iterative Procedure and Description of the Approximate Globally Convergent Algorithm

In our iterative procedure we divide the pseudo frequency interval $(\underline{s}, \bar{s})$ into N sub-intervals $\bar{s} = s_0 > s_1 > \dots > s_N = \underline{s}$ of the step size h such that $s_n - s_{n+1} = h$. We approximate the function q by a piecewise constant function with respect s , $q(\mathbf{x}, s) \approx q_n(\mathbf{x})$, $s \in (s_n, s_{n+1}]$, $n = 1, \dots, N$ and set $q_0 \equiv 0$. Next, we multiply the equation for q_n , obtained by the restriction of (7) to $s \in (s_n, s_{n+1}]$, by the Carleman Weight Function $\exp[\Lambda(s - s_{n+1})]$, where $\Lambda \gg 1$ is a large parameter chosen in the computations, and integrate with respect to s over every pseudo frequency interval $[s_n, s_{n+1}]$. Finally, we get a system of elliptic equations for functions q_n ,

$$\begin{aligned} \Delta q_n(\mathbf{x}) + A_{1,n} \nabla q_n(\mathbf{x}) \cdot (\nabla V_n(\mathbf{x}) - \nabla \overline{q_{n-1}}(\mathbf{x})) \\ = A_{2,n} |\nabla q_n(\mathbf{x})|^2 + A_{3,n} (|\nabla \overline{q_{n-1}}(\mathbf{x})|^2 + |\nabla V_n(\mathbf{x})|^2 - 2 \nabla V_n(\mathbf{x}) \cdot \nabla \overline{q_{n-1}}(\mathbf{x})), \end{aligned} \quad (9)$$

where $A_{i,n}, i = 1, 2, 3$, are some coefficients defined in [4] which can be computed analytically, and $\nabla \overline{q_{n-1}} = h \sum_{j=0}^{n-1} \nabla q_j$. The tail function $V = V_n$ is approximated iteratively, see algorithm below. The discretized version of the boundary condition (8) is given by

$$q_n(\mathbf{x}) = \psi_n(\mathbf{x}) := \frac{1}{h} \int_{s_n}^{s_{n-1}} \psi(\mathbf{x}, s) ds \approx \frac{1}{2} [\psi(\mathbf{x}, s_n) + \psi(\mathbf{x}, s_{n-1})], \mathbf{x} \in \partial \Omega. \quad (10)$$

We also note that the first term on the right hand side of (9) is negligible compared to the other terms since $|A_{2,n}| \leq C/\Lambda$ for sufficiently large Λ , where $C > 0$ is a certain constant. Thus, we set $A_{2,n} |\nabla q_n(\mathbf{x})|^2 \equiv 0$. The system of elliptic Eq. (9) with boundary conditions (10) is solved sequentially starting from $n = 1$. To solve it we use following algorithm:

Globally Convergent Algorithm

- Compute the first tail function V_0 using the asymptotic behavior of v and q as $s \rightarrow \infty$ (see [5] for details). Set $q_0 \equiv 0$.
- For $n = 1, 2, \dots, N$
 1. Set $q_{n,0} = q_{n-1}, V_{n,1} = V_{n-1}$
 2. For $i = 1, 2, \dots, m_n$
 - Find $q_{n,i}$ by solving (9)–(10) with $V_n := V_{n,i}$.
 - Compute $v_{n,i} = -h q_{n,i} - \overline{q_{n-1}} + V_{n,i}$.
 - Compute $\varepsilon_{r,n,i}$ via (5). Then solve the forward problem (1–2) with the new computed coefficient $\varepsilon_r := \varepsilon_{r,n,i}$, compute $w := w_{n,i}$ and update the tail $V_{n,i+1}$ by (6).
 3. Set $q_n = q_{n,m_n}, \varepsilon_{r,n} = \varepsilon_{r,n,m_n}, V_n = V_{n,m_n+1}$ and go to the next frequency interval $[s_{n+1}, s_n]$ if $n < N$. If $n = N$, then stop.

Stopping criteria of this algorithm with respect to i and n are derived computationally and is presented in [5, 14]. We denote the solution obtained at the first stage by $\varepsilon_{r, \text{glob}}$.

3 Statement of Forward and Inverse Problems on the Second Stage

On the second stage we model the electromagnetic wave propagation in an isotropic and nonmagnetic space with permeability $\mu = 1$ in $\mathbb{R}^3$, and with the dimensionless coefficient $\varepsilon_r(\mathbf{x})$, which describes the spatially distributed dielectric permittivity of the medium. We consider the following Cauchy problem in the model problem for the electric field $E(\mathbf{x}, t) = (E_1, E_2, E_3)(\mathbf{x}, t)$

$$\varepsilon_r(\mathbf{x}) \frac{\partial^2 E}{\partial t^2}(\mathbf{x}, t) + \nabla \times (\nabla \times E(\mathbf{x}, t)) = j(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times (0, T),$$

Table 1 Object names and description

Object number	Name of the object
1	A piece of oak
2	A piece of pine
3	The same as #2, but at a different distance to the measurement plane
4	A metallic sphere
5	A metallic cylinder
6	The same as #5, but at a different distance to the measurement plane
7	Blind target (oak)
8	Blind target (metallic block)
9	Blind targets, consists of two targets #10 and #11
10	Blind target, upper target of #9
11	Blind target, lower target of #9
12	Blind target
13	Blind targets, consists of two targets #14 and #15
14	Blind target, upper target of #13
15	Blind target, lower target of #13
16	Blind target, doll, air inside
17	Blind target, doll, metal inside
18	Blind target, doll, sand inside

$$\nabla \cdot (\varepsilon_r(\mathbf{x})E(\mathbf{x}, t)) = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times (0, T), \quad (11)$$

$$E(\mathbf{x}, 0) = 0, \quad \frac{\partial E}{\partial t}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^3.$$

In the above equation $j(\mathbf{x}, t) = (0, \delta(z - z_0)f(t), 0)$, where $f(t) \neq 0$ is the time-dependent waveform of the incident plane wave. This wave propagates along the z -axis and is incident at the plane $\{z = z_0\}$.

We assume that the coefficient ε_r of Eq. (11) is the same as in (3). Let again $\Gamma \subset \partial\Omega$ be a part of the boundary $\partial\Omega$.

Coefficient Inverse Problem (CIP) Suppose that the coefficient ε_r satisfies 11. Determine the function $\varepsilon_r(\mathbf{x})$, $\mathbf{x} \in \Omega$, assuming that the following function $g(\mathbf{x}, t)$ is known for a single incident plane wave:

$$E(\mathbf{x}, t) = g(\mathbf{x}, t), \quad \forall (\mathbf{x}, t) \in \Gamma \times (0, T). \quad (12)$$

In (12) the function g models time-dependent measurements of the electromagnetic field at the part Γ of the boundary $\partial\Omega$ of the domain Ω where coefficient ε_r is unknown. The uniqueness of the above CIP in the multidimensional case is currently known only if we will consider in (11) Gaussian function $\delta_\theta(z - z_0)$ centered around

Table 2 Stage 1: Computed n^{comp} and directly measured refractive indices of dielectric targets of Table 1 together with both measurement and computational errors as well as the average error

Target number	1	2	3	7	13	14	15	16	18	Average error
Blind	No	No	No	Yes	Yes	Yes	Yes	Yes	Yes	
Measured/	2.11	1.84	1.84	2.14	See 14, 15	1.84	2.14	1.89	2.10	
Error	19 %	18 %	18 %	28 %		18 %	28 %	30 %	26 %	23 %
Rec. Test 1	1.92	1.8	1.81	1.83	1.98/1.96	1.98	1.96	1.86	1.92	
Error	9.0 %	2.2 %	1.6 %	14.5 %		7.6 %	8.4 %	1.6 %	8.6 %	6.7 %
Rec. Test 2	2.08	2.01	2.07	2.22	2.21/2.03	2.21	2.03	1.83	2.20	
Error	1.4 %	9.2 %	12.5 %	3.7 %		20.1 %	5.1 %	3.2 %	4.8 %	7.5 %
Rec. Test 3	2.03	1.96	1.65	2.10	2.20/2.13	2.20	2.13	1.85	2.05	
Error	3.8 %	6.5 %	10.3 %	1.9 %		19.6 %	0.5 %	2.1 %	2.4 %	6.7 %
Rec. Test 4	2.02	2.01	2.02	2.03	2.08/2.06	2.08	2.06	1.97	2.02	
Error	4.3 %	9.2 %	9.8 %	5.1 %		13 %	3.7 %	4.2 %	3.8 %	6.6 %

Table 3 Stage 1: Computed appearing dielectric constant for metallic targets of Table 1

Object	4	5	6	8	9	10	11	12	17
Blind	No	No	No	Yes	Yes	Yes	Yes	Yes	Yes
Rec. Test 1	14.37	16.93	16.45	25.0	12.66/13.1	12.66	13.1	13.61	13.56
Rec. Test 2	15.18	23.33	21.99	25.0	40.53/41.78	40.53	41.78	14.13	14.05
Rec. Test 3	7.59	10.76	10.86	19.55	11.07/12.10	11.07	12.10	8.12	7.89
Rec. Test 4	15.0	15.0	15.0	15.0	13.53/14.06	13.53	14.06	15.0	14.33

z_0 , which approximates the function $\delta(z - z_0)$, or if at least one of initial conditions in (11) is not zero. See e.g., [10, 12]. We again assume that uniqueness holds for our CIP.

In our computer simulations of Sect. 6 the incident field has only one nonzero component E_2 . This component propagates along the z -axis until it reaches the target, where it is scattered. In our experiment this component E_2 corresponds to the electromagnetic pulse which is sent into the medium. The other two components of electric field, E_1 and E_3 , are generated by the computation of the problem (11) with the known value of $\varepsilon_{\text{r, glob}}$ which we have obtained as the solution of the approximate globally convergent algorithm of Sect. 2 on the first stage of our two-stage numerical procedure.

Table 4 Stage 1: Best estimates of size of the targets of Table 1 out of the four tests. Lengths are given in meters

Object	Type of shape	x-length		y-length		z-length	
		Estimated	True	Estimated	True	Estimated	True
1	Prism	0.064	0.041	0.092	0.081	0.044	0.041
2	Prism	0.072	0.057	0.076	0.097	0.044	0.057
3	Prism	0.072	0.057	0.076	0.087	0.044	0.057
4	Sphere	0.056	0.103	0.056	0.103	0.032	0.103
5	Cylinder	0.048	0.056	0.092	0.110	0.048	0.056
6	Cylinder	0.048	0.056	0.092	0.110	0.044	0.056
7	Prism	0.068	0.060	0.096	0.113	0.044	0.033
8	Prism	0.056	0.055	0.100	0.101	0.056	0.025
10	Prism	0.068	0.055	0.080	0.101	0.072	0.025
11	Prism	0.076	0.058	0.084	0.041	0.064	0.041
12	Prism	0.056	0.058	0.080	0.041	0.036	0.041
14	Prism	0.072	0.096	0.108	0.058	0.048	0.053
15	Prism	0.080	0.113	0.052	0.060	0.056	0.033
16	Doll	0.056	0.080	0.060	0.110	0.048	0.080
17	Doll, Metal	0.064	0.080 0.043	0.088	0.110 0.039	0.040	0.080
18	Doll, Sand-filled	0.060	0.080	0.060	0.110	0.052	0.080

Domain Decomposition Finite Element/finite Difference Method

To solve the problem (11) numerically we choose a bounded domain G . This domain is chosen such that $\Omega \subset G$, where Ω is the domain where we reconstruct the unknown function ε_r .

In our computations of the second stage we use the domain decomposition finite element/finite difference method of [2]. To do that we decompose G as $G = \Omega_{\text{FEM}} \cup \Omega_{\text{FDM}}$ with $\Omega_{\text{FEM}} = \Omega$. In the computations, a finite element method is used in Ω_{FEM} and a finite difference method is used in Ω_{FDM} . See details in [2].

Using (3) we have that

$$\begin{aligned}\varepsilon_r(\mathbf{x}) &\geq 1, \text{ for } \mathbf{x} \in \Omega_{\text{FEM}}, \\ \varepsilon_r(\mathbf{x}) &= 1, \text{ for } \mathbf{x} \in \Omega_{\text{FDM}}.\end{aligned}$$

As in [2] in our computations we used the following stabilized model problem with parameter $\xi \geq 1$:

$$\varepsilon_r(\mathbf{x}) \frac{\partial^2 E}{\partial t^2}(\mathbf{x}, t) + \nabla \times (\nabla \times E(\mathbf{x}, t)) - \xi \nabla (\nabla \cdot (\varepsilon_r(\mathbf{x}) E(\mathbf{x}, t))) = 0, \quad (\mathbf{x}, t) \in G \times (0, T), \quad (13)$$

$$E(\mathbf{x}, 0) = 0, \quad \frac{\partial E}{\partial t}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in G. \quad (14)$$

The boundary conditions for it are defined in (15–18). We choose the domains Ω and G such that

$$\begin{aligned} \Omega &= \Omega_{\text{FEM}} = \{\mathbf{x} = (x, y, z) : -a < x < a, -b < y < b, -c < z < c'\}, \\ G &= \{\mathbf{x} = (x, y, z) : -A < x < A, -B < y < B, -C < z < z_0\}, \end{aligned}$$

with numbers $0 < a < A$, $0 < b < B$, and $-C < -c < c' < z_0$. We set $\Omega_{\text{FDM}} = G \setminus \Omega_{\text{FEM}}$. Denote by

$$\partial_1 G := \overline{G} \cap \{z = z_0\}, \quad \partial_2 G := \overline{G} \cap \{z = -C\}, \quad \partial_3 G := \partial G \setminus (\partial_1 G \cup \partial_2 G).$$

The backscattering side of Ω is $\Gamma = \partial\Omega \cap \{z = c'\}$. Next, define $\partial_i G_T := \partial_i G \times (0, T)$, $i = 1, 2, 3$. Let $t_1 \in (0, T)$ be a number, and we assume that the function $f(t) \in C[0, t_1]$ and $f(t) = 0$, for $t > t_1$.

Then, the boundary conditions for (13–14) are:

$$E(\mathbf{x}, t) = (0, f(t), 0), \quad (\mathbf{x}, t) \in \partial_1 G \times (0, t_1], \quad (15)$$

$$\frac{\partial E}{\partial n}(\mathbf{x}, t) = -\frac{\partial E}{\partial t}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial_1 G \times (t_1, T), \quad (16)$$

$$\frac{\partial E}{\partial n}(\mathbf{x}, t) = -\frac{\partial E}{\partial t}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial_2 G_T, \quad (17)$$

$$\frac{\partial E}{\partial n}(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial_3 G_T, \quad (18)$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative. Conditions (16), (17) are first order absorbing boundary conditions [9]. In [2] it was shown that the numerical solution of the problem (13–18) approximates well the solution of the original Maxwell's equations for $\xi = 1$ when discontinuities in the function ε_r are not large.

The model problem (13), (14), (15–18) can be also rewritten as

$$\begin{aligned} \varepsilon_r(\mathbf{x}) \frac{\partial^2 E}{\partial t^2}(\mathbf{x}, t) + \nabla (\nabla \cdot E(\mathbf{x}, t)) - \nabla \cdot (\nabla E(\mathbf{x}, t)) - \xi \nabla (\nabla \cdot (\varepsilon_r(\mathbf{x}) E(\mathbf{x}, t))) &= 0, \quad (\mathbf{x}, t) \in G \times (0, T), \quad (19) \end{aligned}$$

$$E(\mathbf{x}, 0) = 0, \quad \frac{\partial E}{\partial t}(\mathbf{x}, 0) = 0, \quad (\mathbf{x}, t) \in G, \quad (20)$$

$$E(\mathbf{x}, t) = (0, f(t), 0), \quad (\mathbf{x}, t) \in \partial_1 G \times (0, t_1], \quad (21)$$

$$\frac{\partial E}{\partial n}(\mathbf{x}, t) = -\frac{\partial E}{\partial t}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial_1 G \times (t_1, T), \quad (22)$$

$$\frac{\partial E}{\partial n}(\mathbf{x}, t) = -\frac{\partial E}{\partial t}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial_2 G_T, \quad (23)$$

$$\frac{\partial E}{\partial n}(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial_3 G_T. \quad (24)$$

Here, we used the well-known transformation $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla \cdot (\nabla E)$. We refer to [2] for details of the numerical solution of the forward problem (19–24).

4 Tikhonov Functional

We define now Γ_1 as the extension of the backscattering side Γ up to the boundary $\partial_3 G$ of the domain G i.e.,

$$\Gamma_1 = \{\mathbf{x} = (x, y, z) : -A < x < A, -B < y < B, z = c'\}.$$

We define also G' as the part of the rectangular prism G which lies between the two planes Γ_1 and $\{z = -C\}$:

$$G' = \{\mathbf{x} = (x, y, z) : -A < x < A, -B < y < B, -Z < z < c'\}.$$

Denote by $Q_T = G' \times (0, T)$, and $S_T = \partial G' \times (0, T)$.

In our CIP we have the data g in (12) only on Γ . These data are complemented on the rest of the boundary $\partial G'$ of the domain G' by simulated data using the immersing procedure of [6]. Thus, we can approximately get the function $\tilde{g}$:

$$\tilde{g}(\mathbf{x}, t) = E(\mathbf{x}, t), \quad (\mathbf{x}, t) \in S_T. \quad (25)$$

We solve our inverse problem as an optimization problem. To do so we minimize the Tikhonov functional:

$$F(E, \varepsilon_r) := \frac{1}{2} \int_{S_T} (E(\mathbf{x}, t) - \tilde{g}(\mathbf{x}, t))^2 z_\delta(t) d\sigma_{\mathbf{x}} dt + \frac{1}{2} \gamma \int_G (\varepsilon_r(\mathbf{x}) - \varepsilon_{r, \text{glob}}(\mathbf{x}))^2 d\mathbf{x}, \quad (26)$$

where $\gamma > 0$ is the regularization parameter, and $\varepsilon_{r, \text{glob}}$ is the computed coefficient which we have obtained on the first stage via the globally convergent method.

Here, z_δ is used to ensure the compatibility conditions at $\overline{Q}_T \cap \{t = T\}$ for the adjoint problem. We define the function z_δ such that

Inverse Problems and Applications

Beilina, L. (Ed.)

2015, X, 164 p. 44 illus., 41 illus. in color., Hardcover

ISBN: 978-3-319-12498-8