Chapter 2
Priority vector and consistency

> The average man's judgment is so poor, he runs a risk every time he uses it.
>
> Ed Howe

It is important to reflect on the fact that in the previous chapter, almost unconsciously, a number of very restrictive assumptions were imposed. Let us summarize them within one sentence, where the assumptions are highlighted in italic.

A *single* decision maker is *perfectly rational* and can *precisely* express his preferences on *all pairs* of *independent* alternatives and criteria using *positive real numbers*.

Some of these assumptions had already been relaxed in Saaty's original works, and some others were relaxed later. In this and in the next chapter we shall present the ways in which these assumptions have been relaxed in the literature to provide the users of the AHP with a more flexible method. Everytime one assumption is relaxed, the previous box will be presented again and the assumption at stake emphasized in boldface. We are now ready to depart from a normative view on the AHP (how decisions should be made in a perfect world) to adopt a more descriptive view (how decisions are actually made).

2.1 Priority vector

We have seen that one pivotal step in the AHP is the derivation of a *priority vector* for each pairwise comparison matrix. Note that if each entry a_{ij} of the matrix is exactly the *ratio* between two weights w_i and w_j, then all the columns of $\mathbf{A}$ are proportional one another and consequently the weight vector is equal to any normalized

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column of **A** (see the matrices in Chapter 1). In this case the information contained in the matrix **A** can be perfectly synthesized in **w** and there is no loss of information. However, we do not even bother dwelling on this case and technique to derive the weights, since it is hardly ever the case that a decision maker is so accurate and rational to give exactly the entries as ratios between weights. In this, and in the next section on consistency, we shall investigate how the AHP can cope with irrational pairwise comparisons. Let us then represent again the box with the relaxed assumption now in boldface.

A *single* decision maker is ***perfectly rational*** and can *precisely* express his preferences on *all pairs* of *independent* alternatives and criteria using *positive real numbers*.

When the entries of the matrix **A** are not obtained exactly as ratios between weights, there does *not* exist a weight vector which perfectly synthesize the information in **A**. Nonetheless, since the AHP cannot make it without the weight vectors, it is necessary to devise some smart ways of estimating a 'good' priority vector. Several methods for eliciting the priority vector $\mathbf{w} = (w_1, \ldots, w_n)^T$ have been proposed in the literature. Each method is just a rule for synthesizing pairwise comparisons into a rating, and mathematically is a function $\tau : \mathbb{R}^{n\times n}_{>} \to \mathbb{R}^{n}_{>}$. Clearly, different methods might lead to different priority vectors, except when the entries of the matrix are representable as ratios between weights, in which case all methods shall lead to the same vector **w**. Needless to say, in the case of perfect rationality, the same vector **w** obtained with any method must be such that $(w_i/w_j)_{n\times n} = \mathbf{A}$.

2.1.1 Eigenvector method

The most popular method to estimate a priority vector is that proposed by Saaty himself, according to which the priority vector should be the principal eigenvector of **A**. In linear algebra it is often called the Perron-Frobenius eigenvector, from the homonymic theorem [70]. The method stems from the following observation. Taking a matrix **A** whose entries are exactly obtained as ratios between weights and multiplying it by **w** one obtains

$$\mathbf{A}\mathbf{w} = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \ldots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \ldots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \ldots & w_n/w_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ \vdots \\ nw_n \end{pmatrix} = n\mathbf{w}.$$

From linear algebra, we know that a formulation of the kind $\mathbf{A}\mathbf{w} = n\mathbf{w}$ implies that n and **w** are an *eigenvalue* and an *eigenvector* of **A**, respectively [1]. Moreover, by

[1] A short overview of eigenvector theory in the AHP can be found in the Appendix.

knowing that the other eigenvalue of $\mathbf{A}$ is 0, and has multiplicity $(n-1)$, then we know that n is the largest eigenvalue of $\mathbf{A}$. Hence, if the entries of $\mathbf{A}$ are ratios between weights, then the weight vector is the eigenvector of $\mathbf{A}$ associated with the eigenvalue n. Saaty proposed to extend this result to all pairwise comparison matrices by replacing n with the more generic maximum eigenvalue of $\mathbf{A}$. That is, vector $\mathbf{w}$ can be obtained from any pairwise comparison matrix $\mathbf{A}$ as the solution of the following equation system,

$$\begin{cases} \mathbf{A}\mathbf{w} = \lambda_{\max}\mathbf{w} \\ \mathbf{w}^T\mathbf{1} = 1 \end{cases}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{A}$, and $\mathbf{1} = (1,\ldots,1)^T$. Although this problem can easily be solved by mathematical software and also spreadsheets, its interpretation remains cumbersome for practitioners.

2.1.2 Geometric mean method

Another widely used method to estimate the priority vector is the *geometric mean method*, proposed by Crawford and Williams [43]. According to this method each component of $\mathbf{w}$ is obtained as the geometric mean of the elements on the respective row divided by a normalization term so that the components of $\mathbf{w}$ eventually add up to 1,

$$w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}} \Bigg/ \underbrace{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}}_{\text{normalization term}}. \tag{2.1}$$

Example 2.1. Let us take into account the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1/2 & 1/4 & 3 \\ 2 & 1 & 1/2 & 2 \\ 4 & 2 & 1 & 2 \\ 1/3 & 1/2 & 1/2 & 1 \end{pmatrix} \tag{2.2}$$

for which, by using (2.1), one obtains

$$\mathbf{w} \approx (0.119, 0.208, 0.454, 0.219)^T$$

2.1. Prove that, if $a_{ij} = w_i/w_j \ \forall i,j$, then the geometric mean method (2.1) returns the vector $\mathbf{w}$ whose ratios between components are the elements of $\mathbf{A}$.

By looking at (2.1) it is apparent that the geometric mean method is very appealing for practical applications since, in contrast to the eigenvector method, the weights can be expressed as analytic functions of the entries of the matrix. Furthermore, even the final weights of the whole hierarchy can be expressed as analytic expressions of the entries of all the matrices in the hierarchy. This is particularly important since it opens avenues to perform efficiently some sensitivity analysis. Moreover, on a more mathematical note, it is interesting to note that the vector **w** obtained with this method, can equivalently be obtained as the argument minimizing the following optimization problem

$$\begin{aligned} \underset{(w_1,\dots,w_n)}{\text{minimize}} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} (\ln a_{ij} + \ln w_j - \ln w_i)^2 \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i = 1, \quad w_i > 0 \, \forall i \end{aligned} \tag{2.3}$$

2.2. Prove that the argument optimizing (2.3) is the same vector (up to multiplication by a suitable scalar) which could be obtained with the geometric mean method.

This optimization problem has some interpretations, the following being quite straightforward. We know that, in the case of perfect rationality, $a_{ij} = w_i/w_j \, \forall i,j$. Indeed, it is fair to consider $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - w_i/w_j)^2$ as a distance between **A** and the matrix $(w_i/w_j)_{n\times n}$ associated with the weight vector **w**. Another metric can be found by using the natural logarithm ln, which is a monotone increasing function, thus obtaining $\sum_{i=1}^{n}\sum_{j=1}^{n}(\ln a_{ij} - \ln(w_i/w_j))^2$. The rest is done by observing that the logarithm of a quotient is the difference of the logarithms. Then the minimization problem (2.3) is introduced to find a suitable priority vector associated to a 'close' consistent approximation $(w_i/w_j)_{n\times n}$ of the matrix **A**.

2.1.3 Other methods and discussion ⋆

A large number of alternative methods to compute the priority vector have been proposed in the literature. Choo and Wedley [40] listed 18 different methods and proposed a numerical and comparative study. Lin [82] reconsidered and simplified their framework. Another comparative study was offered by Ishizaka and Lusti [73]. Instead, Cook and Kress [41] presented a more axiomatic analysis where some desirable properties were stated. From all these studies it appears that, besides the eigenvector and the geometric mean method, other two methods have gained some popularity.

- The so-called *least squares method* where the priority vector is the argument solving the following optimization problem

$$\begin{aligned} &\underset{(w_1,\dots,w_n)}{\text{minimize}} && \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}-\frac{w_i}{w_j}\right)^2 \\ &\text{subject to} && \sum_{i=1}^{n} w_i = 1, \quad w_i > 0\,\forall i\,. \end{aligned} \tag{2.4}$$

  In spite of its elegance, this optimization problem can have local minimizers where the optimization algorithms get trapped. For a discussion on this method and its solutions the reader can refer to Bozóki [23].
- The other one is the *normalized columns method* which requires the normalization of all the columns of $\mathbf{A}$ so that the elements add up to 1 before the arithmetic means of the rows are taken and normalized to add up to 1 to yield the weights $w_1,\dots,w_n$. This is the simplest method but lacks solid theoretical foundation.

Example 2.2. Consider the pairwise comparison matrix (2.2) already used to illustrate the geometric mean method. Then, the matrix with normalized columns and the priority vector are the following, respectively,

$$\begin{pmatrix} 3/22 & 1/8 & 1/9 & 3/8 \\ 6/22 & 2/8 & 2/9 & 2/8 \\ 12/22 & 4/8 & 4/9 & 2/8 \\ 1/22 & 1/8 & 2/9 & 1/8 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 21/163 \\ 42/163 \\ 84/163 \\ 16/163 \end{pmatrix}.$$

Nevertheless, in spite of the great variety of methods, it is safe to say that the eigenvector and the geometric mean method have been the most used and therefore it is convenient to confine further discussions to these two. Saaty and Vargas [115] claimed the superiority of the eigenvector method and concluded that:

> In fact it is the only method that should be used when the data are not entirely consistent in order to make the best choice of alternative.

Saaty and Hu [109] proposed a theorem claiming the necessity of the eigenvector method, and Saaty [105] also proposed ten reasons for not using other methods. Fichtner [53] proposed some axioms and showed that the eigenvector method is the only one satisfying them. Curiously, supporters of the geometric mean method have used similar arguments. For instance, Barzilai at al. [11] proposed another axiomatic framework and proved that the geometric mean method is the only one which satisfies his axioms. Seemingly, the existence of two axiomatic frameworks leading to different conclusions suggest that the choice of the method depends on what set of properties we want the method to satisfy. Supporters of the geometric mean method also gave precise statements on the use of this method and, to summarize one of his papers, Barzilai [9] wrote:

> We establish that *the geometric mean is the only method* for deriving weights from multiplicative pairwise comparisons which satisfies fundamental consistency requirements.

Bana e Costa and Vansnick [7] also moved a criticism against the eigenvector method based on what they called the *condition of order preservation* (COP). The COP states that, if x_i more strongly dominates x_j than x_k does with x_l, it means that $a_{ij} > a_{kl}$, and then it is natural to expect that the priority vector be such that $w_i/w_j > w_k/w_l$. Formally,

$$a_{ij} > a_{kl} \Rightarrow \frac{w_i}{w_j} > \frac{w_k}{w_l} \quad \forall i,j,k,l.$$

Bana e Costa and Vansnick showed some examples of cases where, given a pairwise comparison matrix **A**, the eigenvector method does not return a priority vector satisfying the COP, although there exists a set of other vectors satisfying it.

On a similar note, a recent discovery related to what economists call Pareto efficiency. The reasonable idea behind this is suggested also by (2.3) and (2.4) and is that, having estimated the priority vector **w**, the matrix $(w_i/w_j)_{n\times n}$ should be as near as possible to the original preferences in **A**. Blanquero *et al.* [19] showed that, if **w** is estimated by the eigenvector method, in some cases there exists a vector $\mathbf{v} = (v_1,\dots,v_n)^T \neq \mathbf{w}$ such that

$$\left|\frac{v_i}{v_j} - a_{ij}\right| \leq \left|\frac{w_i}{w_j} - a_{ij}\right| \quad \forall i,j.$$

The fact that $\mathbf{w} \neq \mathbf{v}$ implies that the inequality is strict for some i,j. To summarize, this means that there can be vectors which are closer than the eigenvector to the preferences expressed in **A**. At the time of writing this manuscript, it seems that in some cases the differences between **v** and **w** can be relevant [24].

2.2 Consistency

A perfectly rational decision maker should be able to state his pairwise preferences exactly, i.e. $a_{ij} = w_i/w_j \ \forall i,j$. So, let us consider the ramifications of this condition on the entries of the pairwise comparison matrix **A**. If we write $a_{ij}a_{jk}$ and apply the condition $a_{ij} = w_i/w_j \ \forall i,j$, then we can derive the following

$$a_{ij}a_{jk} = \frac{w_i}{w_j}\frac{w_j}{w_k} = \frac{w_i}{w_k} = a_{ik}.$$

Hence, we discovered that, if all the entries of the pairwise comparison matrix **A** satisfy the condition $a_{ij} = w_i/w_j \ \forall i,j$, then the following condition holds [2],

$$a_{ik} = a_{ij}a_{jk} \quad \forall i,j,k, \tag{2.5}$$

[2] As we will see, the 'if' condition is in fact an 'if and only if'.

which means that each direct comparison a_{ik} is exactly confirmed by all indirect comparisons $a_{ij}a_{jk}\ \forall j$. Formally, a decision maker able to give perfectly consistent pairwise comparisons does not contradict himself. A matrix for which this transitivity condition holds is called *consistent*.

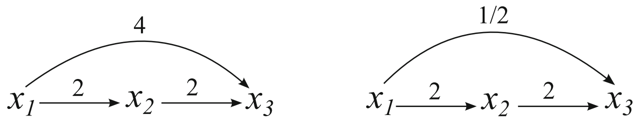
Example 2.3. Consider the characteristic ‘weight’ of three stones x_1, x_2, x_3. If the decision maker says that x_1 is three times heavier than x_3 ($a_{13} = 3$), and then also says that x_1 is two times heavier than x_2 ($a_{12} = 2$), and x_2 is also two times heavier than x_3 ($a_{23} = 2$), then he contradicts himself, because he directly states that $a_{13} = 3$, but indirectly states that the value of a_{13} should be $a_{12}a_{23} = 2 \cdot 2 = 4$ and not 3.

Evidently the whole reasoning can be translated into the language of pairwise comparison matrices.

Example 2.4. Consider this other example with the two pairwise comparison matrices

$$\begin{pmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1/2 \\ 1/2 & 1 & 2 \\ 2 & 1/2 & 1 \end{pmatrix}$$

for which we have the two diagrams in Figure 2.1 respectively.



(a) The matrix is consistent (b) The matrix is inconsistent

Fig. 2.1: Examples of consistent and inconsistent transitivities.

Being consistent is seldom possible because many factors can determine the emergency of inconsistencies. For instance, the decision maker might be asked to use integer numbers and their reciprocals; in this case if $a_{ij} = 3$ and $a_{jk} = 1/2$ it is impossible to find a consistent value for a_{ik}. Moreover, the number of independent transitivities (i, j, k) in a matrix of order n is equal to $\binom{n}{3}$, thus evidencing the difficulty of being fully consistent.

Example 2.5. In a matrix of order 6, there are $\binom{6}{3} = 20$ independent transitivities; that is the number of possible assignments of values to i, j, k such that $1 \leq i < j < k \leq 6$. In a matrix of order 4, there are $\binom{4}{3} = 4$ transitivities. They are $(1,2,3)$, $(1,2,4)$, $(1,3,4)$ and $(2,3,4)$.

In spite of the difficulty in being fully transitive, it is undeniable that consistency is a desirable property. In fact, an inconsistent matrix could be a symptom of the decision maker’s incapacity or inexperience in the field. Additionally, it is

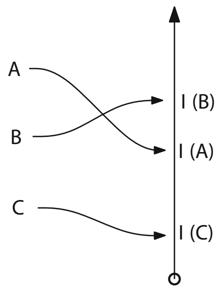


Fig. 2.2: An inconsistency index can be seen as a 'thermometer', which takes pairwise comparison matrices as inputs and evaluates how inconsistent the judgments are.

possible to envision that violations of the condition of consistency (2.5) can be of different extent and gravity and imagine inconsistency as a gradual notion. Hence, on the ground that a matrix should deviate as less as possible from the condition of transitivity, a number of inconsistency indices have been proposed in the literature to quantify this deviation. Formally, an *inconsistency index* is a function mapping pairwise comparison matrices into the real line (see Figure 2.2 for an oversimplification).

There exist various inconsistency indices in the literature and this variety is in part justified by the fact that the condition of consistency can be formulated in many *equivalent* ways. Among them, it is the case to reckon the following four:

i) $a_{ik} = a_{ij}a_{jk} \; \forall i,j,k$,
ii) There exists a vector $(w_1,\ldots,w_n)^T$ such that $a_{ij} = w_i/w_j \; \forall i,j$,
iii) The columns of $\mathbf{A}$ are proportional, i.e. $\mathbf{A}$ has rank one,
iv) The pairwise comparison matrix $\mathbf{A}$ has its maximum eigenvalue, $\lambda_{\max}$, equal to n.

In this section we explore some inconsistency indices, each inspired by one of these equivalent consistency conditions.

2.2.1 Consistency index and consistency ratio

According to the result that given a pairwise comparison matrix $\mathbf{A}$, its maximum eigenvalue, $\lambda_{\max}$, is equal to n if and only if the matrix is consistent (and greater than n otherwise), Saaty [99] proposed the *Consistency Index*

$$CI(\mathbf{A}) = \frac{\lambda_{\max} - n}{n-1}. \tag{2.6}$$

However, numerical studies showed that the expected value of CI of a random matrix of size $n+1$ is, on average, greater than the expected value of CI of a random matrix of order n. Consequently, CI is not fair in comparing matrices of different orders and needs to be rescaled.

The *Consistency Ratio*, CR, is the rescaled version of CI. Given a matrix of order n, CR can be obtained dividing CI by a real number RI_n (*Random Index*) which is nothing else but an estimation of the average CI obtained from a large enough set of randomly generated matrices of size n. Hence,

$$CR(\mathbf{A}) = \frac{CI(\mathbf{A})}{RI_n} \tag{2.7}$$

Estimated values for RI_n are reported in Table 2.1. Note that the generation of random matrices requires the definition of a bounded scale where the entries take values, for instance the interval $[1/9, 9]$. According to Saaty [100], in practice one should accept matrices with values $CR \leq 0.1$ and reject values greater than 0.1. A value of $CR = 0.1$ means that the judgments are 10% as inconsistent as if they had been given randomly.

n	3	4	5	6	7	8	9	10
RI_n	0.5247	0.8816	1.1086	1.2479	1.3417	1.4057	1.4499	1.4854

Table 2.1: Values of RI_n [3].

Example 2.6. Consider the pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 9 & 1 \\ 1/2 & 1 & 1/3 & 1/6 \\ 1/9 & 3 & 1 & 2 \\ 1 & 6 & 1/2 & 1 \end{pmatrix}. \tag{2.8}$$

It can be calculated that its maximum eigenvalue is $\lambda_{\max} = 5.28$. Using the formula for CI, we obtain $CI(\mathbf{A}) = 0.42667$. Dividing it by RI_4 one obtains $CR(\mathbf{A}) \approx 0.48$ which is significantly greater than the threshold 0.1. In a decision problem it is common practice to ask the decision maker to revise his judgments until a value of CR smaller than 0.1 is reached.

2.2.2 *Index of determinants*

The index of determinants was proposed by Peláez and Lamata [93] and comes from the following property of a matrix of order three. Expanding the determinant of a

3×3 real matrix one obtains

$$\det(\mathbf{A}) = \frac{a_{13}}{a_{12}a_{23}} + \frac{a_{12}a_{23}}{a_{13}} - 2.$$

If $\mathbf{A}$ is not consistent, then $a_{13} \neq a_{12}a_{23}$ and $\det(\mathbf{A}) > 0$, because, in general, $\frac{a}{b} + \frac{b}{a} - 2 > 0 \ \forall a \neq b,\ a,b > 0$.

It is possible to generalize this result to matrices of order greater than three and define this inconsistency index as the average of the determinants of all the possible submatrices $\mathbf{T}_{ijk}$ of a given pairwise comparison matrix, constructed as follow,

$$\mathbf{T}_{ijk} = \begin{pmatrix} 1 & a_{ij} & a_{ik} \\ a_{ji} & 1 & a_{jk} \\ a_{ki} & a_{kj} & 1 \end{pmatrix}, \quad \forall i < j < k.$$

The number of so constructed submatrices is $\binom{n}{3} = \frac{n!}{3!(n-3)!}$. The result is an index whose value is the average inconsistency computed for all the submatrices $\mathbf{T}_{ijk}$ $(i < j < k)$

$$CI^*(\mathbf{A}) = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \underbrace{\left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 \right)}_{\det(\mathbf{T}_{ijk})} \Bigg/ \binom{n}{3}. \tag{2.9}$$

Example 2.7. Consider the matrix $\mathbf{A}$ in (2.8). It is then possible to calculate the average of the determinants of all the submatrices $\mathbf{T}_{ijk}$ with $i < j < k$.

$$\begin{aligned} CI^*(\mathbf{A}) =& \frac{\det \overbrace{\begin{pmatrix} 1 & 2 & 9 \\ 1/2 & 1 & 1/3 \\ 1/9 & 3 & 1 \end{pmatrix}}^{\mathbf{T}_{123}} + \cdots + \det \overbrace{\begin{pmatrix} 1 & 1/3 & 1/6 \\ 3 & 1 & 2 \\ 6 & 1/2 & 1 \end{pmatrix}}^{\mathbf{T}_{234}}}{4} \\ =& (11.5741 + 1.3333 + 16.0556 + 34.0278)/4 = 15.7477. \end{aligned}$$

Interestingly, CI^* is proportional to another inconsistency index called c_3 [28]. The coefficient c_3 of the characteristic polynomial of a pairwise comparison matrix was proposed to act as an inconsistency index by Shiraishi and Obata [122] and Shiraishi *et al.* [123, 124]. By definition, the characteristic polynomial[3] of a matrix $\mathbf{A}$ has the following form

$$P_{\mathbf{A}}(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n,$$

with $c_1, \ldots, c_n$ that are real numbers and λ the unknown. Shiraishi *et al.* [123] proved that, if $c_3 < 0$, then the matrix cannot be fully consistent. In fact, this is evident if one reckons that—in light of the Perron-Frobenius theorem—the only possible

[3] See appendix on eigenvalues and eigenvectors

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