Chapter 2
Preliminaries

Abstract In this chapter, we introduce the functional framework and our standing hypotheses concerning the stochastic partial differential equations (SPDEs) that we will work with. We also recall some basic concepts from the random dynamical system (RDS) theory, and introduce a classical random change of variables which will be used to cast a given parameterized family of SPDEs into the RDS framework.

In this chapter, we introduce the functional framework and our standing hypotheses concerning the abstract stochastic evolution equations of type (2.1) below that we will work with. We also recall some basic concepts from the RDS theory [3, 59], and introduce a classical random change of variables [77] which will be used to cast a given parameterized family of SPDEs with abstract formulation as given by Eq. (2.1) into the RDS framework.

2.1 Stochastic Evolution Equations

We consider the following nonlinear stochastic evolution equation[1] driven by linear multiplicative white noise in the sense of Stratonovich:

$$\mathrm{d}u = \big(L_\lambda u + F(u)\big)\mathrm{d}t + \sigma u \circ \mathrm{d}W_t. \tag{2.1}$$

Here, $\{L_\lambda\}$ represents a family of linear operators parameterized by a scalar control parameter λ, $F(u)$ accounts for the nonlinear terms, W_t is a two-sided one-dimensional Wiener process, and σ is a positive constant which gives a measure of the "amplitude" of the noise. We make precise below the functional framework that we will adopt throughout this monograph. Such equations arise in various contexts such as in turbulence theory or non-equilibrium phase transitions [19, 64, 152],

[1] Throughout this monograph, we will often refer to a stochastic evolution equation of type (2.1) as an SPDE.

M.D. Chekroun et al., *Stochastic Parameterizing Manifolds and Non-Markovian Reduced Equations*, SpringerBriefs in Mathematics,
DOI 10.1007/978-3-319-12520-6_2

in the modeling of randomly fluctuating environment [14] in spatially-extended harvesting models [43,111,150,151,167,168], or simply in the modeling of parameter disturbances [20].

Assumptions about the operator L_λ. Let $(\mathscr{H}, \|\cdot\|)$ be an infinite-dimensional separable real Hilbert space. First, let us introduce a sectorial operator A on $\mathscr{H}$ [110, Definition 1.3.1] with domain

$$\mathscr{H}_1 := D(A) \subset \mathscr{H}, \tag{2.2}$$

and which has compact resolvent. In particular $\mathscr{H}_1$ is compactly and densely embedded in $\mathscr{H}$. We assume furthermore that $-A$ is stable in the sense that its spectrum satisfies $\mathrm{Re}\,\sigma(-A) < 0$.

We shall also make use of the fractional powers of A and the associated interpolated spaces between $\mathscr{H}_1$ and $\mathscr{H}$; see, e.g., [110, Sect. 1.4] and [171, Sect. 3.7]. Let $\mathscr{H}_\gamma := D(A^\gamma)$ be such an interpolated space for some $\gamma \in [0, 1]$, endowed with the norm $\|\cdot\|_\gamma$ induced by the inner product $\langle u, v\rangle_\gamma := \langle A^\gamma u, A^\gamma v\rangle_{\mathscr{H}}$; in particular $\mathscr{H}_0 = \mathscr{H}$, $\mathscr{H}_1$ corresponds to $\gamma = 1$, and $\mathscr{H}_1 \subset \mathscr{H}_\gamma \subset \mathscr{H}_0$ for $\gamma \in (0, 1)$. Note that in the sequel, $\langle\cdot,\cdot\rangle$, will be used to denote the inner-product in the ambient Hilbert space $\mathscr{H}$.

Let us introduce now

$$B_\lambda : \mathscr{H}_\gamma \to \mathscr{H} \tag{2.3}$$

a parameterized family of bounded linear operators depending continuously on λ, with here $\gamma \in [0, 1)$. In particular, $-B_\lambda A^{-\gamma}$ is bounded on $\mathscr{H}$ and according to [110, Corollary 1.4.5] the operator $-L_\lambda$ is sectorial on $\mathscr{H}$ with domain $\mathscr{H}_1$ where

$$L_\lambda := -A + B_\lambda. \tag{2.4}$$

Note that L_λ has compact resolvent by recalling that $\mathscr{H}_1$ is compactly embedded in $\mathscr{H}$ [84, Proposition II.4.25]. As a consequence, since $L_\lambda : \mathscr{H}_1 \to \mathscr{H}$ is a closed operator,[2] we have that for each λ, the spectrum of L_λ, $\sigma(L_\lambda)$, consists only of isolated eigenvalues with finite algebraic multiplicities; see [122, Theorem III-6.29] (see also [84, Corollary IV.1.19]).

Assumptions about the nonlinearity F. For the nonlinearity, we assume that $F : \mathscr{H}_\alpha \to \mathscr{H}$ is continuous for some $\alpha \in [0, 1)$.[3]

We assume furthermore that[4]

$$F(0) = 0, \tag{2.5}$$

[2] As a consequence of the sectorial property of $-L_\lambda$.

[3] In particular, nonlinearities including a loss of regularity compared to the ambient space $\mathscr{H}$, are allowed; see e.g. Chaps. 6 and 7 for an illustration.

[4] Near a nontrivial steady state $\overline{u}$ of some deterministic system, one can think u as some deviation from this steady state (subject to noise fluctuations), and Eq. (2.5) is then satisfied in such situations.

and in the case where F is at least C^1-smooth, the tangent map of F at 0 is assumed to be the null map, i.e.,

$$DF(0) = 0. \tag{2.6}$$

Note that in particular, according to (2.5) the noise term in (2.1) is multiplicative with respect to the basic state; see [3, p. 473] for this terminology.

Other assumptions on F will be specified when needed; see, e.g., Chaps. 3 and 5.

The spectrum of L_λ and the uniform spectrum decomposition. Recall that the spectrum $\sigma(L_\lambda)$ consists only of isolated eigenvalues with finite multiplicities. This combined with the sectorial property of $-L_\lambda$ implies that there are at most finitely many eigenvalues with a given real part. The sectorial property of $-L_\lambda$ also implies that $\operatorname{Re}\sigma(L_\lambda)$ is bounded above (see also [84, Theorem II.4.18]). These two properties of $\operatorname{Re}\sigma(L_\lambda)$ allow us in turn to label elements in $\sigma(L_\lambda)$ according to the lexicographical order:

$$\sigma(L_\lambda) = \{\beta_n(\lambda) \mid n \in \mathbb{N}^*\}, \tag{2.7}$$

such that for any $1 \leq n < n'$ we have either

$$\operatorname{Re}\beta_n(\lambda) > \operatorname{Re}\beta_{n'}(\lambda), \tag{2.8}$$

or

$$\operatorname{Re}\beta_n(\lambda) = \operatorname{Re}\beta_{n'}(\lambda), \quad \text{and} \quad \operatorname{Im}\beta_n(\lambda) \geq \operatorname{Im}\beta_{n'}(\lambda). \tag{2.9}$$

Note that we will adopt in this monograph the convention that each eigenvalue, $\beta_n(\lambda)$, is repeated according to its algebraic multiplicity.

For the material recalled in Chap. 3, we will also assume that an open interval Λ can be chosen such that the following *uniform spectrum decomposition of* $\sigma(L_\lambda)$ *holds over* Λ:

$$\sigma(L_\lambda) = \sigma_{\mathfrak{c}}(L_\lambda) \cup \sigma_{\mathfrak{s}}(L_\lambda), \; \lambda \in \Lambda, \quad \text{with} \quad \eta_{\mathfrak{c}} > \eta_{\mathfrak{s}}, \tag{2.10}$$

where

$$\begin{aligned} \eta_{\mathfrak{c}} &:= \inf_{\lambda \in \Lambda} \inf\{\operatorname{Re}\beta(\lambda) \mid \beta(\lambda) \in \sigma_{\mathfrak{c}}(L_\lambda)\}, \\ \eta_{\mathfrak{s}} &:= \sup_{\lambda \in \Lambda} \sup\{\operatorname{Re}\beta(\lambda) \mid \beta(\lambda) \in \sigma_{\mathfrak{s}}(L_\lambda)\}, \end{aligned} \tag{2.11}$$

and $\sigma_{\mathfrak{c}}(L_\lambda)$ consists of the first m eigenvalues (counting multiplicities) in $\sigma(L_\lambda)$:

$$\operatorname{card}(\sigma_{\mathfrak{c}}(L_\lambda)) = m. \tag{2.12}$$

It is interesting to note that the uniform spectrum decomposition (2.10) prevents eigenvalues in $\sigma_{\mathfrak{s}}(L_\lambda)$ from merging with eigenvalues in $\sigma_{\mathfrak{c}}(L_\lambda)$ as λ varies in Λ, while the cardinality of $\sigma_{\mathfrak{c}}(L_\lambda)$ remains fixed to be m over Λ. As a consequence, the spaces $\mathscr{H}_\alpha$ and $\mathscr{H}$ can be decomposed into L_λ-invariant subspaces in such a way that the eigensubspace associated with $\sigma_{\mathfrak{c}}(L_\lambda)$ has fixed dimension m for each $\lambda \in \Lambda$; see (2.14) below. These subspaces will be at the basis of the construction of stochastic invariant manifolds considered in later chapters.

Related L_λ-invariant subspaces. The splitting of the spectrum $\sigma(L_\lambda)$ given in (2.10) leads naturally to decompositions of the spaces $\mathscr{H}$ and $\mathscr{H}_\alpha$ into L_λ-invariant eigensubspaces

$$\mathscr{H} = \mathscr{H}^{\mathfrak{c}}(\lambda) \oplus \mathscr{H}^{\mathfrak{s}}(\lambda), \quad \mathscr{H}_\alpha = \mathscr{H}^{\mathfrak{c}}(\lambda) \oplus \mathscr{H}_\alpha^{\mathfrak{s}}(\lambda), \quad \forall\, \lambda \in \Lambda, \tag{2.13}$$

where $\mathscr{H}^{\mathfrak{c}}$ is associated with $\sigma_{\mathfrak{c}}(L_\lambda)$ while $\mathscr{H}^{\mathfrak{s}}(\lambda)$ and $\mathscr{H}_\alpha^{\mathfrak{s}}(\lambda)$ are the corresponding topological complements in $\mathscr{H}$ and $\mathscr{H}_\alpha$ respectively, the latter spaces being associated with $\sigma_{\mathfrak{s}}(L_\lambda)$; see Volume I [41, Chap. 3] for more details. These decompositions lead naturally to a partial dichotomy of the deterministic linear semigroup[5] associated with (2.1); see (2.18a)–(2.18c).

Note that since the eigenvalues are repeated according to their multiplicities, we have

$$\dim \mathscr{H}^{\mathfrak{c}}(\lambda) = m, \quad \forall\, \lambda \in \Lambda, \tag{2.14}$$

where m is the cardinality of $\sigma_c(L_\lambda)$ as given in (2.12).

Let

$$P_{\mathfrak{c}}(\lambda) : \mathscr{H} \to \mathscr{H}^{\mathfrak{c}}(\lambda), \quad P_{\mathfrak{s}}(\lambda) : \mathscr{H} \to \mathscr{H}^{\mathfrak{s}}(\lambda) \tag{2.15}$$

be the associated canonical (spectral) projectors, and we denote

$$L_\lambda^{\mathfrak{c}} := L_\lambda P_{\mathfrak{c}}(\lambda), \quad L_\lambda^{\mathfrak{s}} := L_\lambda P_{\mathfrak{s}}(\lambda). \tag{2.16}$$

Note that L_λ commutes with $P_{\mathfrak{c}}(\lambda)$ and $P_{\mathfrak{s}}(\lambda)$; see Volume I [41, Chap. 3]. As a consequence, the subspaces $\mathscr{H}^{\mathfrak{c}}(\lambda)$ and $\mathscr{H}^{\mathfrak{s}}(\lambda)$ are invariant by the semigroup e^{tL_λ}. Note also that the operator $L_\lambda^{\mathfrak{c}}$ is a bounded linear operator on $\mathscr{H}^{\mathfrak{c}}(\lambda)$, so that $e^{tL_\lambda}P_{\mathfrak{c}}$ can be extended to $t < 0$, namely $e^{tL_\lambda}P_{\mathfrak{c}}$ defines a flow on $\mathscr{H}^{\mathfrak{c}}(\lambda)$. This fact is used in the partial dichotomy estimate (2.18c) below.

Note also that by (2.14), the dimension of $\mathscr{H}^{\mathfrak{c}}(\lambda)$ is independent of λ as it varies in Λ, so that $\mathscr{H}^{\mathfrak{c}}(\lambda)$ is unique up to orthogonal transformations. For the sake of concision, this property has led us to suppress the λ-dependence of the subspaces given in (2.13), and of the projectors $P_{\mathfrak{c}}(\lambda)$ and $P_{\mathfrak{s}}(\lambda)$ defined in (2.15). The results are derived and presented hereafter according to this convention.

Partial-dichotomy estimates. Thanks to the uniform spectrum decomposition (2.10), for any given numbers η_1 and η_2 satisfying

$$\eta_{\mathfrak{c}} > \eta_1 > \eta_2 > \eta_{\mathfrak{s}}, \tag{2.17}$$

[5] Namely, the semigroup associated with $\mathrm{d}v = L_\lambda v \mathrm{d}t$.

there exists a constant, $K \geq 1$, such that for all $\lambda \in \Lambda$ the following *partial-dichotomy*[6] estimates hold for the semigroup generated by L_λ (see, e.g., [110, Theorems 1.5.3 and 1.5.4]):

$$\|e^{tL_\lambda} P_{\mathfrak{s}}\|_{L(\mathscr{H}_\alpha, \mathscr{H}_\alpha)} \leq K e^{\eta_2 t}, \quad t \geq 0, \tag{2.18a}$$

$$\|e^{tL_\lambda} P_{\mathfrak{s}}\|_{L(\mathscr{H}, \mathscr{H}_\alpha)} \leq K t^{-\alpha} e^{\eta_2 t}, \quad t > 0, \tag{2.18b}$$

$$\|e^{tL_\lambda} P_{\mathfrak{c}}\|_{L(\mathscr{H}, \mathscr{H}_\alpha)} \leq K e^{\eta_1 t}, \quad t \leq 0, \tag{2.18c}$$

where $L(X, Y)$ denotes the space of bounded linear operators from the Banach space X to the Banach space Y. Note that the estimate given in (2.18b) accounts for the instantaneous smoothing effects of the semigroup e^{tL_λ} for $t > 0$ from $\mathscr{H}$ to $\mathscr{H}_\alpha$ where we recall that $\mathscr{H}_\alpha$ has been imposed by the choice of the nonlinearity.

The conditions $\eta_{\mathfrak{c}} > \eta_1$ and $\eta_2 > \eta_{\mathfrak{s}}$ allow us to absorb the polynomial growth terms in the estimates (2.18a)–(2.18c) that—because of our assumptions (L_λ being not necessarily self-adjoint)—could be present in front of the exponential terms with $\eta_{\mathfrak{c}}$ (resp. $\eta_{\mathfrak{s}}$) in place of η_1 (resp. η_2). As a consequence, K in (2.18a)–(2.18c) depends on $\eta_* := \min\{\eta_{\mathfrak{c}} - \eta_1, \eta_2 - \eta_{\mathfrak{s}}\}$, and may get larger as η_* gets closer to zero in the non-self-adjoint case. Note that however, K is independent of $\lambda \in \Lambda$ in all the cases.

Remark 2.1 Throughout this monograph, the use of a random frame which moves with the cocycle, will not be required to build the stochastic PMs introduced in later sections to deal with the parameterization problem of the small spatial scales by the large ones. As a consequence, the approach presented here does not make usage of the Lyapunov spectrum and the multiplicative ergodic theory (MET) in Hilbert or Banach spaces [130, 169]. The MET is typically employed when stochastic invariant manifolds in the vicinity of a nontrivial random stationary solution are concerned; see, e.g., [33, 130, 149]. Recalling that stochastic PMs are not necessarily invariant helps understand why stochastic PMs may be "enfranchised" from the MET. Nevertheless, it remains still interesting to extend the approach of this monograph to a MET setting inclined to deal with more general noises, although such situations can be handled within the formalism introduced here as well; see [39] for more details.

2.2 Random Dynamical Systems

In this section, we recall the definitions of metric dynamical systems (MDSs) and random dynamical systems (RDSs), and specify—in a measure-theoretic sense—the canonical MDS associated with the Wiener process in Eq. (2.1) which will be used throughout this monograph. The interested readers are referred to [3, 51, 59] for more

[6] The partial aspect of the dichotomy is explained when η_1 and η_2 share the same sign which is allowed by (2.10). In that case, the distinction is made on the magnitude of the rate of contraction (or expansion) associated with $dv = L_\lambda v dt$; otherwise the concept matches the classical one of exponential dichotomy found in the literature; see, e.g., [110, 171].

details, and to [44] for an intuitive and "physically-oriented" presentation of these concepts.

Metric dynamical system. A family of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *metric dynamical system* if the following conditions are satisfied:

(i) $(t, \omega) \mapsto \theta_t\omega$ is $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}; \mathscr{F})$-measurable, where $\mathscr{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$, and $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}$ denotes the σ-algebra generated by the direct product of elements of $\mathscr{B}(\mathbb{R})$ and $\mathscr{F}$;
(ii) $\{\theta_t\}$ satisfies the one-parameter group property, i.e., $\theta_0 = \mathrm{Id}_\Omega$, and $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$;
(iii) $\mathbb{P}$ is invariant with respect to θ_t for all $t \in \mathbb{R}$, i.e., $(\theta_t)_*\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$, where $(\theta_t)_*\mathbb{P}$ is the *push-forward measure* of $\mathbb{P}$ by θ_t, defined by $(\theta_t)_*(F) := \mathbb{P}(\theta_{-t}(F))$, for all $F \in \mathscr{F}$.

Continuous random dynamical system. Given a separable Hilbert space $(H, |\cdot|_H)$ with the associated Borel σ-algebra denoted by $\mathscr{B}(H)$, a continuous *random dynamical system* acting on H over an MDS, $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, is a $(\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F} \otimes \mathscr{B}(H); \mathscr{B}(H))$-measurable mapping

$$S\colon \mathbb{R}^+ \times \Omega \times H \to H, \quad (t, \omega, \xi) \mapsto S(t, \omega)\xi,$$

which satisfies the following properties:

(i)′ $S(0, \omega) = \mathrm{Id}_H$, for all $\omega \in \Omega$,
(ii)′ S satisfies the *perfect cocycle property*, i.e.,

$$S(t + s, \omega) = S(t, \theta_s\omega) \circ S(s, \omega), \quad \forall\, t, s \in \mathbb{R}^+, \text{ and } \omega \in \Omega,$$

(iii)′ $S(t, \omega)\colon H \to H$ is continuous for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

We are now in position to introduce the aforementioned MDS associated with the Wiener process. Let us first recall the canonical MDS $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, associated with the Wiener process; see, e.g., [3, Appendices A.2 and A.3] and [51, Chap. 1]. Here the sample space Ω consists of the sample paths of a two-sided one-dimensional Wiener process W_t taking zero value at $t = 0$, that is,

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) \mid \omega(0) = 0\};$$

$\mathscr{F}$ is the Borel σ-algebra associated with the Wiener process; $\mathbb{P}$ is the classical Wiener measure on Ω; and for each $t \in \mathbb{R}$, the map $\theta_t\colon (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ is the measure preserving transformation defined by:

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t). \tag{2.19}$$

In order the solution operator associated with Eq. (2.1) to satisfy the perfect cocycle property given in (ii)′ above, we will restrict our consideration to some subset in Ω of full measure which is also θ_t-invariant for all $t \in \mathbb{R}$. In the following, we will identify such a subset, and introduce the restriction of the canonical MDS to this subset.

In that respect, let us consider the following scalar Langevin equation:

$$dz + z\,dt = \sigma\,dW_t. \tag{2.20}$$

It is known that this equation possesses a unique stationary solution $z_\sigma(\theta_t\omega)$—the stationary Ornstein-Uhlenbeck (OU) process—whose main properties are in particular recalled in the following lemma.

Lemma 2.1 *There exists a subset Ω^* of Ω which is of full measure and is θ_t-invariant for all $t \in \mathbb{R}$, i.e.,*

$$\mathbb{P}(\Omega^*) = 1, \quad \textit{and} \quad \theta_t(\Omega^*) = \Omega^* \quad \forall\, t \in \mathbb{R}; \tag{2.21}$$

and the following properties hold on Ω^*:

(i) *For each $\omega \in \Omega^*$, $t \mapsto W_t(\omega)$ is γ-Hölder for any $\gamma \in (0, 1/2)$.*
(ii) *$t \mapsto W_t(\omega)$ has sublinear growth:*

$$\lim_{t\to\pm\infty} \frac{W_t(\omega)}{t} = 0, \quad \forall\, \omega \in \Omega^*. \tag{2.22}$$

(iii) *For each $\omega \in \Omega^*$, $t \mapsto z_\sigma(\theta_t\omega)$ is γ-Hölder for any $\gamma \in (0, 1/2)$, and can be written as:*

$$\begin{aligned} z_\sigma(\theta_t\omega) &= -\sigma \int_{-\infty}^{0} e^\tau W_\tau(\theta_t\omega)\,d\tau \\ &= -\sigma \int_{-\infty}^{0} e^\tau W_{\tau+t}(\omega)\,d\tau + \sigma W_t(\omega), \quad t \in \mathbb{R}, \quad \omega \in \Omega^*. \end{aligned} \tag{2.23}$$

(iv) *The following growth control relations are satisfied:*

$$\lim_{t\to\pm\infty} \frac{z_\sigma(\theta_t\omega)}{t} = 0, \textit{ and } \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z_\sigma(\theta_\tau\omega)\,d\tau = 0, \quad \forall\, \omega \in \Omega^*. \tag{2.24}$$

Proof See [41, Sect. 3.2]. □

Now, let Ω^* be the θ_t-invariant subset of Ω as given in Lemma 2.1, and $\mathscr{F}_{\Omega^*}$ be the trace σ-algebra of $\mathscr{F}$ with respect to Ω^*, i.e.,

$$\mathscr{F}_{\Omega^*} := \{F \cap \Omega^* \mid F \in \mathscr{F}\}.$$

It can be checked that $(t, \omega) \to \theta_t \omega$ is $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}_{\Omega^*}; \mathscr{F}_{\Omega^*})$-measurable; see for instance [33, Lemma 3.2]. It follows that $(\Omega^*, \mathscr{F}_{\Omega^*}, \mathbb{P}_{\Omega^*}, \{\theta_t\}_{t\in\mathbb{R}})$ forms an MDS, where $\mathbb{P}_{\Omega^*}$ denotes the restriction of $\mathbb{P}$ on Ω^*.

To simplify the notations and the presentation, we will denote hereafter the new sample space $(\Omega^*, \mathscr{F}_{\Omega^*}, \mathbb{P}_{\Omega^*})$ as $(\Omega, \mathscr{F}, \mathbb{P})$; and we will work with this restricted MDS, $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, without confusion with the original MDS. For the sake of concision, we will often omit mentioning such an underlying MDS, thus identifying an RDS with its cocycle part.

2.3 Cohomologous Cocycles and Random Evolution Equations

In order to set up the original SPDE (2.1) within the RDS framework, we will make use of a smooth cohomology, which consists of a stationary coordinate change on the phase space $\mathscr{H}_\alpha$. For the treatment, we follow here more specifically the approach of [77], and we refer to Volume I [41, Sect. 3.3] for more details. In that respect, we transform Eq. (2.1) into an evolution equation with random coefficients which helps simplify the analysis of the dynamics associated with the SPDE.

Let us first introduce the following standard change of variables:

$$v(t) = e^{-z_\sigma(\theta_t\omega)} u(t), \tag{2.25}$$

where z_σ is the OU process defined in (2.23).

Note that by the Itô formula (cf. [156, Theorem 4.1.2]), the stochastic process $e^{-z_\sigma(\theta_t\omega)}$ satisfies

$$\begin{aligned} \mathrm{d}e^{-z_\sigma(\theta_t\omega)} &= \Big(z_\sigma(\theta_t\omega)e^{-z_\sigma(\theta_t\omega)} + \frac{\sigma^2}{2}e^{-z_\sigma(\theta_t\omega)}\Big)\mathrm{d}t - \sigma e^{-z_\sigma(\theta_t\omega)}\mathrm{d}W_t \\ &= z_\sigma(\theta_t\omega)e^{-z_\sigma(\theta_t\omega)}\mathrm{d}t - \sigma e^{-z_\sigma(\theta_t\omega)} \circ \mathrm{d}W_t, \end{aligned} \tag{2.26}$$

where the second equality above follows from the conversion between the Itô and Stratonovich integrals; cf. [128, Theorem 2.3.11].

Formally, we also have that

$$\mathrm{d}v = \mathrm{d}\big(e^{-z_\sigma(\theta_t\omega)}u\big) = u \circ \mathrm{d}e^{-z_\sigma(\theta_t\omega)} + e^{-z_\sigma(\theta_t\omega)} \circ \mathrm{d}u. \tag{2.27}$$

Then, by using Eqs. (2.1) and (2.26) into the above equation, we find after simplification that v satisfies formally the following random evolution equation (hereafter referred to as an RPDE):

$$\frac{\mathrm{d}v}{\mathrm{d}t} = L_\lambda v + z_\sigma(\theta_t\omega)v + G(\theta_t\omega, v), \tag{2.28}$$

where $G(\omega, v) := e^{-z_\sigma(\omega)}F(e^{z_\sigma(\omega)}v)$.

Assume that for any λ and any $(\mathscr{F}; \mathscr{B}(\mathscr{H}_\alpha))$-measurable random initial datum $v_0(\omega)$, there exists a unique classical solution $v_{\lambda,v_0(\omega)}(t,\omega) := v_\lambda(t,\omega; v_0(\omega))$ of Eq. (2.28) which is $\big(\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}; \mathscr{B}(\mathscr{H}_\alpha)\big)$-measurable; see, e.g., Volume I [41, Proposition 3.1] for conditions. We can then define for each λ, an RDS generated by Eq. (2.28), S_λ, as follows:

$$S_\lambda \colon \mathbb{R}^+ \times \Omega \times \mathscr{H}_\alpha \to \mathscr{H}_\alpha, \quad (t,\omega,v_0) \mapsto S_\lambda(t,\ \omega)v_0 := v_\lambda(t,\omega; v_0). \tag{2.29}$$

Let us now define the mapping $\widehat{S}_\lambda \colon \mathbb{R}^+ \times \Omega \times \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ via

$$\widehat{S}_\lambda(t,\ \omega)u_0 := \mathfrak{D}^{-1}(\theta_t\omega) \circ S_\lambda(t,\omega) \circ \mathfrak{D}(\omega)u_0, \tag{2.30}$$

where $\mathfrak{D}$ is the random smooth transformation acting on the space $\mathscr{H}_\alpha$ defined by $\mathfrak{D}(\omega)\xi := \xi e^{-z_\sigma(\omega)}$ with its inverse given by $\mathfrak{D}^{-1}(\omega)\xi := \xi e^{z_\sigma(\omega)}$, and the symbol, $\circ$, denotes the basic composition operation between self-mappings on $\mathscr{H}_\alpha$. The mapping $\widehat{S}_\lambda$ thus defined is clearly measurable, and defines an RDS acting on $\mathscr{H}_\alpha$.

By a solution to the SPDE (2.1) with initial datum $u_0 \in \mathscr{H}_\alpha$, we always mean a process $u_\lambda(t,\omega; u_0) := e^{z_\sigma(\theta_t\omega)} v_\lambda(t,\omega; u_0 e^{-z_\sigma(\omega)})$, where v_λ is a classical solution of the RPDE (2.28) with initial datum $u_0 e^{-z_\sigma(\omega)}$. In that sense, the RDS, $\widehat{S}_\lambda$, provides solutions to Eq. (2.1) since $u_\lambda(t,\omega; u_0) = \widehat{S}_\lambda(t,\ \omega)u_0$.

Springer

http://www.springer.com/978-3-319-12519-0

Stochastic Parameterizing Manifolds and Non-Markovian Reduced Equations
Stochastic Manifolds for Nonlinear SPDEs II
Chekroun, M.D.; Liu, H.; Wang, S.
2015, XVII, 129 p. 12 illus., 11 illus. in color., Softcover
ISBN: 978-3-319-12519-0