

Preface

With his famous painting “The Treachery of Images” as duplicated in the previous page, René Magritte coined in an essential way the fact that as realistic as possible a representation of a given reality may be, fundamental differences are irreducible: “...And yet, could you stuff my pipe?...” It is actually the Magritte’s treachery of images that any mathematical model has to deal with. The same is true when it comes to small-scale parameterization and model reduction, or more generally to meta-modeling: as much as the challenge may seem unreachable, it remains desirable to reduce the “treachery of images” as far as possible for the purpose of deriving reduced systems able to faithfully mimic the main features of the original model’s dynamics. This second monograph of a two-volume series is aimed to propose one such reduction procedure for nonlinear stochastic partial differential equations (SPDEs) driven by white noise. It can be read independently of Volume I [41], the main material in the first volume being recalled in Chaps. 2 and 3 below.

In comparison to Volume I, a point of view still *pathwise* from the theory of random dynamical systems (RDS), but more global in the phase space, is adopted here. In particular, we have in mind solutions that evolve not necessarily close to the criticality, such as in Volume I. The purpose of this second volume is to introduce a novel approach to deal with the parameterization problem of the small spatial scales by the large ones for stochastic PDEs, along with the effective derivation of the related reduced systems.

This approach relies on *stochastic parameterizing manifolds* (PMs) which are random and non-necessarily invariant manifolds aiming to provide—in a mean square sense—approximate slaving relationships between the small and large spatial scales. More precisely, given a realization ω of the noise and a low-mode truncation of the SPDE solution driven by ω , a stochastic PM provides an approximate parameterization of the high modes by the low ones so that the unexplained high-mode energy is reduced—in an L^2 -in-time sense—when this parameterization is applied. Stochastic PMs can be viewed as a substitute to stochastic inertial manifolds that complement prior notions of (stochastic) approximate inertial manifolds (AIMs). As a byproduct for instance, the distinction between two

types of PM-based parameterizations is simply monitored by an easily computable scalar quantity that takes its values between zero and unity; the so-called *parameterization defect* naturally related to the energy contained in the high modes.

The central theme of Chap. 4 is then the introduction of backward–forward systems that give potentially access to such stochastic PMs as *pullback limits* depending on the time history of approximations of the dynamics of the low modes. These approximations are either simply built from the stochastic linear part of the low-mode dynamics or from more elaborated functions of the latter. As shown in the subsequent chapters for the broad class of stochastic PDEs described in Chap. 2, the resulting pullback limits can be efficiently determined in practice under the form of analytic formulas or pseudocodes. These practical features lead in turn to an operational procedure for the derivation of stochastic reduced equations that convey *noise-induced* memory effects which are shown to play a central role in our approach to reach good statistical modeling skills. The role of these memory effects become particularly prominent when the separation of timescales between the resolved and unresolved variables is not as sharp as required by other parameterization methods; see Chaps. 6 and 7.

The formalism adopted in Chap. 4 allows us furthermore to build bridges with the rigorous approximation formulas of stochastic center manifolds or other stochastic invariant manifolds considered in Volume I, and recalled in Chap. 3 of this monograph. The resulting pullback characterization via backward–forward systems provides a novel interpretation of such objects in terms of flows which allows us, furthermore, to unify the previous approximation approaches to stochastic center manifolds from the literature.

The stochastic PMs obtained by the procedure described in Chap. 4 are not subject to a spectral gap condition such as encountered in the classical theory of stochastic invariant manifolds as reviewed in Volume I; see [41, Theorems 4.1 and 4.3]. Instead, stochastic PMs can be determined under weaker *non-resonance conditions*: for any given set of resolved modes for which their self-interactions through the nonlinear terms do not vanish when projected against a given unresolved mode e_n , it is required that some specific linear combinations of the corresponding eigenvalues dominate the eigenvalue associated with e_n .

Chapter 5 presents a systematic procedure for the derivation of stochastic reduced systems, given a realization ω of the noise that drives the original SPDE and the corresponding small-scale parameterization provided by a stochastic PM. These reduced systems take the form of non-Markovian stochastic differential equations (SDEs) involving random coefficients that convey noise-induced memory effects via the history of the Wiener path $W_t(\omega)$. These random coefficients come from the nonlinear interactions between the low modes, embedded in the “noise bath,” and follow typically non-Gaussian statistics while exhibiting an exponential decay of correlations whose rate depends explicitly on gaps arising in the aforementioned non-resonance conditions.

In Chaps. 6 and 7, it is finally shown on a stochastic Burgers-type equation, that such PM-based reduced systems can achieve very good performance in reproducing the main statistical features of the dynamics on the low modes, such as the

autocorrelation and the probability functions of the corresponding amplitudes. In particular, it is illustrated that the modeling of the large excursions exhibited by the latter can be reproduced with high-accuracy, even when the amount of noise is significant and the separation of time scales is weak. Such a success is attributed to the ability of the underlying stochastic PM to capture, for a given realization and as time flows, the noise-driven transfer of energy to the small spatial scales through the nonlinear term.

Finally, we mention that the proposed framework to address (for SPDEs) the problem of approximate parameterizations of the small spatial scales by the large ones, has been intentionally articulated for the case of linear multiplicative noise (also known as parameter noise), in order to present the main ideas in a simple stochastic context. We emphasize that this framework is not limited to that case and actually extends to SPDEs driven by multidimensional noise, either multiplicative or additive and possibly degenerate; we refer to [39] for extensions to SPDEs driven by additive noise forcing finitely many modes. Similarly, PMs can be defined and efficiently computed in the deterministic setting as discussed in Sect. 4.5 below and further illustrated in [38] for the design of low-dimensional suboptimal controllers of nonlinear parabolic PDEs.

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