2 Computing Exposures

An idea which can be used once is a trick. If it can be used more than once, it becomes a method.

George Pólya

What factors affect the value of a security? That is a common, and important, question asked by investors across all asset classes. Investors of fixed-income securities are no exception. Security holders take risk. They must be, at least in expectation, compensated for taking this risk. Factors that affect the value of a security, therefore, are those factors that generate risk or, in other words, return or loss. We typically call these *risk factors*. In this chapter, we will review a number of risk factors that drive fixed-income security returns. We will particularly focus on the exposure, or sensitivity, of a fixed-income security to these risk factors. These sensitivities form the backbone of portfolio analytics since they permit us to quickly understand the nature of a security's or, more generally, a portfolio's risk both on an absolute or relative basis. The base unit of examination, however, is the security. Portfolios then are merely collections of securities.

While there are many excellent sources that describe the sensitivities of fixed-income instruments, we take the time to derive a number of key measures in order to ensure consistent notation, a common understanding, and to permit a self-contained discussion. Moreover, we also hope that the reader will be exposed to a few new ideas.

2.1 A Starting Point

Terminology is useful in any profession as it permits the succinct and precise description of complicated ideas. It can, on occasion, lead to confusion when the underlying ideas are not fully understood or multiple definitions for a given term exist. The concept of duration in the field of finance appears to fall into this category.

D.J. Bolder, *Fixed-Income Portfolio Analytics*,
DOI 10.1007/978-3-319-12667-8_2

There are many different notions of duration; MacCauley, modified, effective, spread, and key-rate duration are a few commonly used examples. Moreover, sometimes duration is quoted as a sensitivity and sometimes it is described as a cash-flow weighted time to maturity of a fixed-income security. It is fair to say, therefore, that when one evokes the term *duration*, not everyone immediately shares the same understanding of what it means. Given the potential for confusion arising from a loose use of terminology, we will work correspondingly hard to thoroughly define all terms used in this text.

Our true starting point is the bond-price equation. The value, at time t of a generic fixed-income security is described as,

$$V(t, y) = \sum_{i=1}^{I} c_{t_i} \delta(t, t_i), \tag{2.1}$$

where the discount factor, $\delta(t, t_i)$, may be modelled as

$$\delta(t, t_i) = \frac{1}{(1 + y)^{t_i - t}}. \tag{2.2}$$

This is a relatively simple, but powerful, identity. It holds that the value of a generic fixed-income security is merely the sum of its discounted future cash-flows, $\{c_{t_i} : i = 1, \ldots, I\}$. In this case, we use the security's yield, y, to discount each cash-flow; as we will see in later development, this need not always be the case.[1]

2.2 Simple Yield Exposure

It is clear from Eq. (2.1) that the yield of the security plays an important role in the security's value. Should one increase y, then the present value of each cash flow becomes smaller, leading to a reduction in the current value, $V(t, y)$.[2] Conversely, decreasing the yield, y, has the effect of increasing the present value of each cash-flow and thereby increasing the security's value. In short, there is an inverse relationship between the value of a fixed-income security and its yield. With a bit

[1]In reality, market practice is almost always a bit more complicated than it appears in (2.1). There are day-count conventions that describe each of the individual $(t_i - t)$'s, compounding frequencies impacting the yield and the coupon, and settlement dates. These elements are market convention. For the purposes of this discussion, however, we will skip over many of these details unless, of course, they become important.

[2]We employ the terms *discounted cash flow* and *present value of a cash flow* interchangeably.

of calculus, we can formalize this relationship through the computation of the first derivative of the security's value with respect to a change in its yield,

$$\begin{aligned} \frac{\partial V(t,y)}{\partial y} &= \frac{\partial}{\partial y}\left(\sum_{i=1}^{I} \frac{c_{t_i}}{(1+y)^{t_i-t}}\right), \\ &= \sum_{i=1}^{I} \frac{-(t_i-t)c_{t_i}}{(1+y)^{(t_i-t)+1}}, \\ &= -\frac{1}{(1+y)}\sum_{i=1}^{I} \frac{(t_i-t)c_{t_i}}{(1+y)^{t_i-t}}. \end{aligned} \tag{2.3}$$

This expression describes the sensitivity of the value of a fixed-income security to an infinitesimal change in its yield. The form of the expression is fairly enlightening. It holds that the sensitivity depends on the sum of the time-weighted discounted cash-flows, $\{(t_i - t)c_{t_i} : i = 1, \ldots, I\}$. If one divides both sides by the security's value, one arrives at,

$$\frac{1}{V(t,y)}\frac{\partial V(t,y)}{\partial y} = -\frac{1}{V(t,y)(1+y)}\sum_{i=1}^{I} \frac{(t_i-t)c_{t_i}}{(1+y)^{(t_i-t)}}. \tag{2.4}$$

What have we done? Recall that for a small change in y (i.e., $\Delta y = y_1 - y_0$) that

$$\left.\frac{\partial V(t,y)}{\partial y}\right|_{y=y_0} \approx \frac{V(t,y_1)-V(t,y_0)}{\underbrace{y_1-y_0}_{\Delta y}}. \tag{2.5}$$

Thus, we have that

$$\underbrace{\left.\frac{1}{V(t,y)}\frac{\partial V(t,y)}{\partial y}\right|_{y=y_0}}_{\text{Eq. (2.4)}} \approx \left(\frac{1}{\Delta y}\right)\underbrace{\frac{V(t,y_1)-V(t,y_0)}{V(t,y)}}_{\text{Percentage change in } V}, \tag{2.6}$$

which implies that Eq. (2.4) is, approximately at least, a function of the percentage change in the value of our security for a small change in its yield.

The negative of this quantity has another, much more frequently used, name. It is called the *modified duration*, which we will denote as D_M. It is formerly defined as,

$$D_M = \frac{1}{V(t,y)}\frac{\partial V(t,y)}{\partial y}. \tag{2.7}$$

Equation (2.7) provides, in short, the analytic representation of a security's exposure to its yield. There is good reason that this is such a well-known and often used measure of the risk of a fixed-income security. It summarizes, in a single number, the percentage gain or loss for a fixed-income security associated with a small change in its yield. Given a 25 basis-point decrease in yields, a fixed-income security with a duration of 5 would expect to gain about 125 basis points. Conversely, a security with a duration of 0.5 would only expect to earn a profit of 12.5 basis points. This capacity to succinctly describe one's risk is extremely useful.

Another, more explicit mathematical way to understand this fact is to return to the linear approximation of the derivative in Eq. (2.6) (i.e., $\frac{\partial V}{\partial y} \approx \frac{\Delta V}{\Delta y}$) and re-arrange the terms as follows,

$$
\begin{aligned}
D_M(t,y) &= -\frac{1}{V(t,y)}\frac{\partial V(t,y)}{\partial y}, \qquad (2.8)\\
D_M(t,y) &\approx -\frac{1}{V(t,y)}\frac{\overbrace{V(t,y+\Delta y)-V(t,y)}^{\Delta V(t,y)}}{\Delta y},\\
-D_M(t,y)\Delta y &\approx \frac{\Delta V(t,y)}{V(t,y)} = r(t,\Delta y).
\end{aligned}
$$

This is another way to see the result from Eq. (2.6). In words, the product of one's modified duration and expected (or realized) yield change directly approximates the percentage change in the bond's value for a given yield movement, $r(t,\Delta y)$. This is immensely useful.

We can, of course, perform the same exercise using continuously compounded interest rates—this is merely a different model of the discount factor, $\delta(t,t_i) = e^{-y_c(t_i-t)}$. The partial derivative of the bond value with respect to the continuously compounded yield, y_c, is given as,

$$
\begin{aligned}
\frac{\partial V(t,y_c)}{\partial y_c} &= \frac{\partial}{\partial y_c}\left(\sum_{i=1}^{I} c_{t_i} e^{-y_c(t_i-t)}\right), \qquad (2.9)\\
&= -\sum_{i=1}^{I} c_{t_i}\,(t_i-t)\,e^{-y_c(t_i-t)},
\end{aligned}
$$

(continued)

whereas the modified duration has the following, relatively simple, form,

$$D_M(t, y_c) = \frac{1}{V(t, y_c)} \frac{\partial V(t, y_c)}{\partial y_c}, \tag{2.10}$$
$$= -\frac{1}{V(t, y_c)} \sum_{i=1}^{I} c_{t_i} (t_i - t) e^{-y_c(t_i - t)}.$$

One may also represent the modified duration analytically as,

$$\frac{1}{V(t, y)} \frac{\partial V(y)}{\partial y} = \lim_{\epsilon \to 0} \frac{1}{V(t, y)} \left(\frac{V(t, y + \epsilon) - V(t, y - \epsilon)}{2\epsilon} \right). \tag{2.11}$$

This comes directly from the formal definition of a partial derivative. It also suggests the following possible numerical approximation,

$$\frac{1}{V(t, y)} \frac{\partial V(y)}{\partial y} \approx \frac{1}{V(t, y)} \left(\frac{V(t, y + \epsilon) - V(t, y - \epsilon)}{2\epsilon} \right), \tag{2.12}$$

for a sufficiently small and judicious choice of ϵ.[3] Often, when the duration is numerically computed using something like Eq. (2.12), it is termed the *effective duration*.[4] This is particularly useful when the security value cannot be so easily represented as indicated in Eq. (2.1); a good example would be a security with embedded optionality such as a callable bond or a mortgage-backed security. In such a case, a numerical computation of the sensitivity may prove more convenient. We will, however, also see that such a numerical computation can be useful, even for straightforward fixed-income securities, for a complex model of the discount factor.[5]

It is always easier to understand an idea in the context of a concrete example. Consider, therefore, the US Treasury bond described in Table 2.1. Our plan is to demonstrate the application of Eq. (2.11) using this specific bond. At this point, it is useful to indicate where market conventions become important. When discounting cash-flows using Eq. (2.1), we arrive at what is called the dirty price. This is the value obtained by discounting all of one's cash-flows back to the settlement date, without accounting for accrued interest. The clean price, of course, is the dirty price

[3] This is formally termed a central finite-difference approximation. See Press et al. [1] for much more information on the numerical computation of derivatives.

[4] Caution should nevertheless be exercised as there is not, to the author's knowledge, a clear consensus in the finance universe on the definition of effective duration.

[5] In general, given that the numerical computation requires three full function valuations, it will only be employed in the absence of an analytical solution.

Table 2.1 An example bond

Characteristic	Data value
Issuer	US Treasury
ISIN	US912828NP10
Position	\$100
Coupon	1.75 %
Issue date	31 July 2010
Maturity date	31 July 2015
Settle date	10 August 2010
Next coupon date	31 January 2011
Tenor	4.980 years
Yield	1.524 %
Clean price	\$101.078
Accrued interest	\$0.048
Dirty price	*\$101.126*
Modified duration	4.747 years

This table outlines the key data values for a 5-year on-the-run US Treasury bond on 9 August 2010. This information is used to practically demonstrate the analytic and numeric computation of modified duration.

Table 2.2 The analytic computation

Date	Days	$t_i - t$	c_{t_i}	$\delta(t, t_i)$	$c_{t_i}\delta(t, t_i)$	$(t_i - t)c_{t_i}\delta(t, t_i)$
31 Jan 2011	173	0.473	0.875	0.9928	0.87	0.41
31 Jul 2011	355	0.973	0.875	0.9853	0.86	0.84
31 Jan 2012	539	1.473	0.875	0.9779	0.86	1.26
31 Jul 2012	721	1.973	0.875	0.9705	0.85	1.68
31 Jan 2013	905	2.473	0.875	0.9632	0.84	2.08
31 Jul 2013	1,086	2.973	0.875	0.9559	0.84	2.49
31 Jan 2014	1,270	3.473	0.875	0.9486	0.83	2.88
31 Jul 2014	1,451	3.973	0.875	0.9415	0.82	3.27
31 Jan 2015	1,635	4.473	0.875	0.9343	0.82	3.66
31 Jul 2015	1,816	4.973	100.875	0.9273	93.54	465.15
Total	n/a	n/a	n/a	n/a	*101.126*	483.720

This table outlines the computation of the bond price in Table 2.1 using its yield and then the further computation of the modified duration.

adjusted for accrued interest. One must be careful, however, when taking prices from different screens or data sources not to mix clean and dirty prices.[6]

[6]It is often the case, for example, that clean prices are returned from various software functions and, as a consequence, a bit of caution is advised.

All of the necessary computations are outlined in Table 2.2. We begin, in the first column, with the actual cash-flow dates. As this bond pays a semi-annual coupon, as of August 2010 there are ten remaining cash-flows culminating with its maturity payment on 31 July 2015. Each of these calendar dates are transformed into the number of days from the settlement date, 10 August 2010. Using these days, we proceed to divide them by 365 to generate a sequence of cash-flow times. This permits us to easily compute the sequence of discount factors,

$$\delta(t, t_i) = \frac{1}{\left(1+\frac{y}{2}\right)^{2\cdot(t_i - t)}}, \tag{2.13}$$

to account for the semi-annual compounding associated with the bond's cash-flows. Observe that each of the cash-flows is discounted using the bond's yield as the discount rate. Taking the product of the cash-flows with the discount factor, we obtain a sequence of discounted cash-flows, $\{\delta(t, t_i)c_{t_i}, i = 1, \ldots, 10\}$. The sum of this sequence is \$101.126, which coincides with the clean price outlined in Table 2.1.

The next step is the computation of the modified duration. Here we need to compute the sum of the time-weighted discounted cash-flows. This sequence, $\{(t_i - t)\delta(t, t_i)c_{t_i}, i = 1, \ldots, 10\}$, is the final column in Table 2.2. The sum of this sequence is \$483.720. Thus using the analytic expression in Eq. (2.11), we should arrive at the modified duration,

$$\begin{aligned} D_M &= -\underbrace{\frac{1}{V(t,y)}}_{\substack{\text{Dirty}\\ \text{price}}} \underbrace{\frac{1}{\left(1+\frac{y}{2}\right)}}_{\substack{\text{Adjust for}\\ \text{compounding}}} \sum_{i=1}^{I} (t_i - t)c_{t_i} \underbrace{\frac{1}{\left(1+\frac{y}{2}\right)^{2(t_i - t)}}}_{\substack{\text{Eq. (2.13):}\\ \delta(t,t_i)}}, \\ &= -\left(\frac{1}{101.126\left(1+\frac{1.524\,\%}{2}\right)}\right) 483.720, \\ &= -4.747. \end{aligned} \tag{2.14}$$

This is exactly the value of 4.747 that we expected and which is provided in Table 2.1.

We may now turn to the numerical computation of modified duration introduced in Eq. (2.11). This is a relatively straightforward computation; the tricky part, however, is the appropriate selection of the parameter, ϵ. If it is too large, then the approximation will be poor. Conversely, if it is too small, then it can lead to instability in the computation. To make this clearer, let us work through the computation with ϵ set to 0.01, which is equivalent to 100 basis points. We, therefore, start with Eq. (2.11) and follow through with the computation as,

$$D_M = \frac{1}{V(t,y)} \left(\frac{V(t, y+\epsilon) - V(t, y-\epsilon)}{2\epsilon} \right), \tag{2.15}$$

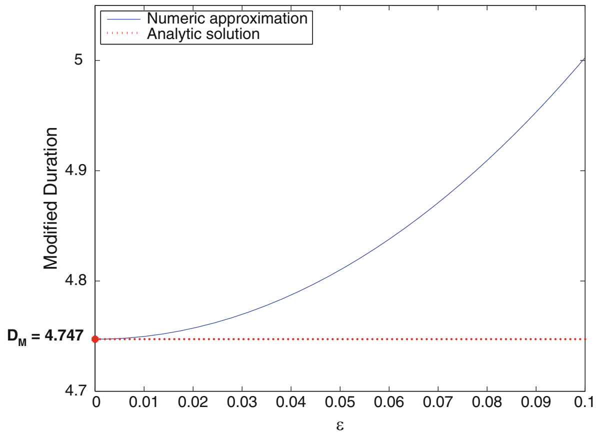


Fig. 2.1 Numerical computation of duration. This figure demonstrates the convergence of the central finite-difference approximation introduced in Eq. (2.11) to the bond described in Table 2.1. Observe that for relatively large values of ϵ, the numerical approximation is a poor estimate for the analytic value of 4.747. As ϵ tends to zero, however, it converges to the analytically computed value

$$
\begin{aligned}
&= \frac{1}{101.126}\left(\frac{V(t, y+0.01) - V(t, y-0.01)}{2 \cdot 0.01}\right), \\
&= \frac{1}{101.126}\left(\frac{96.4513 - 106.0577}{0.02}\right), \\
&= -4.750.
\end{aligned}
$$

Observe that, for a choice of $\epsilon = 0.01$, we do *not* reproduce the modified duration value of 4.747 in the analytic computation. The reason is that 0.01 represents a 100 basis-point movement in the bond yield, which is actually quite large. It should be stressed, however, that even with a 100 basis-point movement, the numerical approximation is fairly acceptable.[7]

Figure 2.1 takes the demonstration one step further and performs the numerical computation of the modified duration using a sequence of ϵ's from 0.1 (i.e., 100 basis points) to 0.0000001 (i.e., 1000th of a basis point). Observe that for relatively large values of ϵ, the numerical estimate is a poor approximation for the analytic value of 4.747. As ϵ tends to zero, however, it converges to the analytically computed value. Indeed, it appears that for values of ϵ slightly less than 0.01 (ten basis points), the numerical computation basically converges to the analytic value.

In summary, modified duration is a key fixed-income exposure to the interest-rate risk factor and may be computed analytically or numerically. Given the simplicity of

[7]Note also that an increase of 100 basis points generates a \$4.67 decrease in the price, while a 50 basis-point decrease leads to a \$4.93 basis-point rise in the price. There are two points that should be taken from this fact. First, a 100 basis-point movement in yields leads to relatively large price changes. Second, the price movement is not symmetric for an equivalent upward and downward movement in bond yields. This is due to the fact that the relationship between bond prices and yields is *not* linear. More on this point will be discussed in the next section.

the analytic approach, a numerical approximation is generally only employed when the security has complex features that preclude the use of Eq. (2.8) on page 24.

2.3 Correcting for Our Linear Approximation

Up to this point, we have only examined the concept of modified duration—one dimension of the security's exposure to its yield. As we've established, this is the percentage change in the value of a fixed-income security associated with a small change in its yield. What happens, however, when the change in yield is *not* so small? Indeed, the daily changes in market yields can occasionally be quite sizeable. In Eq. (2.1), we note that the relationship between the bond's value, $V(y)$, and its yield, y, is not linear. Indeed, the bond-price equation has a polynomial form in terms of the discount factors. Thus, we would expect some degree of non-linearity. To understand the nature of this relationship, we have selected a specific US Treasury bond and plotted its value across yields ranging from one basis point to 10 %. The results, summarized in Fig. 2.2, clearly indicate a non-linear relationship. The bond examined is an on-the-run 10-year US Treasury bond—it has a 3.5 % coupon and a 15 May 2020 maturity date giving it, as of 9 August 2010, a tenor of about 9.8 years. The straight line passing through the current clean price represents the predicted price movement stemming from the modified duration. For relatively small yield changes, this linear approximation is quite reasonable; as the yield change increases, however, the accuracy of this approximation deteriorates.

Given this non-linearity, it would seem sensible to construct a measure that attempts to capture it. Modified duration is clearly insufficient to capture the full exposure of a fixed-income security to the yield factor. Since locally the linear approximation is quite good, one reasonable approach would be to try to capture the rate of change in the linear approximation. This leads us naturally to the second

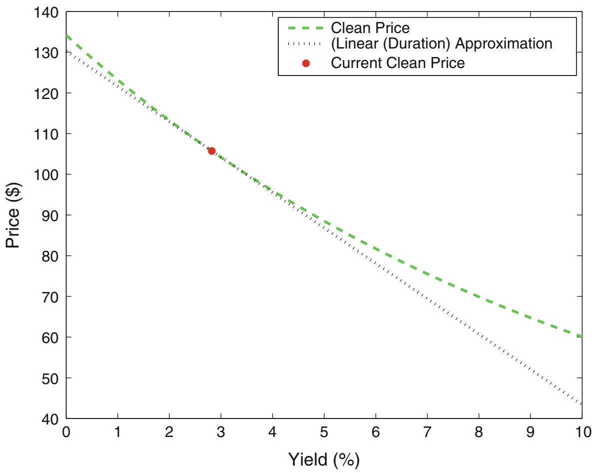


Fig. 2.2 Relationship between price and yield. This figure outlines the relationship between a bond's yield and its price for yields ranging from 0 to 10 %. The bond examined is 10-year US Treasury as of 9 August 2010—it has a 3.5 % coupon, a 15 May 2020 maturity date, and an ISIN number of US912828ND89. Observe that the relationship between these two variables is *not* linear; the modified duration is included to demonstrate the deviation from linearity

derivative of the security's value function. The second derivative of the bond-price function with respect to yield has the form,

$$
\begin{aligned}
\frac{\partial^2 V(t,y)}{\partial y^2} &= \frac{\partial^2}{\partial y^2}\left(\sum_{i=1}^{I} \frac{c_{t_i}}{(1+y)^{t_i-t}}\right), \qquad (2.16)\\
&= \frac{\partial}{\partial y}\underbrace{\left(\sum_{i=1}^{I} \frac{-(t_i-t)c_{t_i}}{(1+y)^{(t_i-t)+1}}\right)}_{\substack{\text{First derivative with}\\ \text{respect to } y}},\\
&= \sum_{i=1}^{I} \frac{(t_i-t)(t_i-t+1)c_{t_i}}{(1+y)^{(t_i-t)+2}},\\
&= \frac{1}{(1+y)^2}\sum_{i=1}^{I} \frac{(t_i-t)(t_i-t+1)c_{t_i}}{(1+y)^{t_i-t}}.
\end{aligned}
$$

Here we see the rate of change of the first derivative, which is also somewhat more difficult to interpret. If one normalizes this quantity by its current value, $V(t,y)$, as follows

$$
\frac{1}{V(t,y)}\frac{\partial^2 V(t,y)}{\partial y^2} = \frac{1}{V(t,y)(1+y)^2}\sum_{i=1}^{I} \frac{(t_i-t)(t_i-t+1)c_{t_i}}{(1+y)^{t_i-t}}, \qquad (2.17)
$$

then one arrives at a second well-known quantity: the bond convexity, which we will denote as C. The convexity measure is not as simple to apply as the modified duration and, generally speaking, acts as a correction factor for approximations made using modified duration. Interestingly, we have seen that for a simple risk factor—the market yield—we may employ multiple exposures. We will see how one applies the convexity measure in rather more detail in the next chapter.

Again, we can perform the same exercise using continuously compounded interest rates. The second partial derivative of the bond price with respect to y is given as,

$$
\begin{aligned}
\frac{\partial V^2(t,y)}{\partial y^2} &= \frac{\partial^2}{\partial y^2}\left(\sum_{i=1}^{I} c_{t_i} e^{-y(t_i-t)}\right), \qquad (2.18)\\
&= \sum_{i=1}^{I} c_{t_i}\,(t_i-t)^2\, e^{-y(t_i-t)}.
\end{aligned}
$$

(continued)

Convexity, in a continuously compounded setting, is thus defined as,

$$\frac{1}{V(t,y)}\frac{\partial^2 V(t,y)}{\partial y^2} = \frac{1}{V(t,y)}\sum_{i=1}^{I} c_{t_i}\,(t_i - t)^2\, e^{-y(t_i-t)}. \tag{2.19}$$

Once again, we observe that continuous compounding gives rise to more convenient mathematics.

2.4 Time Exposure

We have been careful to indicate that the value of a bond is a function of two arguments: time and yield. Modified duration and convexity provide a basis for understanding the exposure of our bond to changes in the yield, but these measures are silent on the implications associated with changes in the first argument: time. This is easily corrected. To better understand the sensitivity, or exposure, of a bond to changes in time, we need only compute the partial derivative of our bond-value function with respect to time. In principle, this is a straightforward exercise, although we need to recall a simple property of logarithms to isolate the t term. In particular, we need to remember that a function of the form $(1+y)^x$ can be alternatively written as $e^{x\ln(1+y)}$. With that in mind, the partial derivative of our bond-value function with respect to time is,

$$\begin{aligned}
\frac{\partial V(t,y)}{\partial t} &= \frac{\partial}{\partial t}\left(\sum_{i=1}^{I}\frac{c_{t_i}}{(1+y)^{t_i-t}}\right), \qquad (2.20)\\
&= \frac{\partial}{\partial t}\left(\sum_{i=1}^{I}\frac{c_{t_i}}{e^{(t_i-t)\ln(1+y)}}\right),\\
&= \frac{\partial}{\partial t}\left(\sum_{i=1}^{I} c_{t_i} e^{-(t_i-t)\ln(1+y)}\right),\\
&= \sum_{i=1}^{I} c_{t_i}\ln(1+y)e^{-(t_i-t)\ln(1+y)},\\
&= \ln(1+y)\underbrace{\left(\sum_{i=1}^{I}\frac{c_{t_i}}{(1+y)^{t_i-t}}\right)}_{V(t,y)},\\
&= \ln(1+y)V(t,y).
\end{aligned}$$

If we recall that $\ln(1+y) \approx y$ for small values of y, then

$$\frac{\partial V(t,y)}{\partial t} \approx yV(t,y). \tag{2.21}$$

To compute, therefore, a kind of time duration for our bond, we need only to divide both sides by $V(t,y)$ arriving at,

$$\begin{aligned} D_t &= \frac{1}{V(t,y)}\frac{\partial V(t,y)}{\partial t}, \\ &= \frac{1}{\cancel{V(t,y)}}\underbrace{\ln(1+y)\cancel{V(t,y)}}_{\text{Eq. (2.20)}}, \\ &\approx y. \end{aligned} \tag{2.22}$$

In simple words, therefore, the exposure of a fixed-income security to the passage of time—also a risk factor—is well approximated by its yield. This is related to the notion of carry.

Using the same heuristic notion of a derivative as in the previous discussion (i.e., $\frac{\partial V}{\partial t} \approx \frac{\Delta V}{\Delta t}$) and re-arrange the terms as follows,[8]

$$\begin{aligned} D_t &= \frac{1}{V(t,y)}\frac{\partial V(t,y)}{\partial t}, \\ \underbrace{D_t}_{\approx y} &\approx \frac{1}{V(t,y)}\frac{\overbrace{V(t+\Delta t,y)-V(t,y)}^{\Delta V(t,y)}}{\Delta t}, \\ y\Delta t &\approx \frac{\Delta V(t,y)}{V(t,y)}. \end{aligned} \tag{2.23}$$

This is not a terribly surprising result, but it is nonetheless encouraging that this basic framework provides a consistent and sensible answer. Moreover, the notion of time sensitivity is an important aspect of performance analysis and, as such, we will making extensive use of Eq. (2.23) in the coming chapters.

[8]The product of the security's yield and the time interval approximates the return associated with the movement of time.

2.5 Key-Rate Exposures

The notion of modified duration is quite useful, but it has some limitations. The principal limitation is that, in a typical fixed-income portfolio, one generally holds a collection of bonds with varying tenors. If one holds n bonds in one's portfolio with durations $\{D_{M_i}, i = 1, \ldots, n\}$ and market-value weights $\{\omega_i, i = 1, \ldots, n\}$, we straightforwardly define the duration of the portfolio as,

$$D_{M_p} = \sum_{i=1}^{n} \omega_i D_{M_i}. \tag{2.24}$$

Quite simply, the modified duration of the portfolio is the weighted-average modified duration of the instruments in the portfolio.[9] The duration of the portfolio gives one an insight into the sensitivity of that portfolio to a constant change across all yields in the portfolio. It is, however, not always the case that all yields move in an identical manner across the entire yield curve. Quite often, 2-year yields move in a different way compared to, say, 5-year yields or 10-year yields. Figure 2.3 provides a graphical example of the change in the UST yield curve from 31 July 2010 to 31 August 2010. It shows an example of relatively modest yield movement in the short end of the curve and coincident reduction and flattening of the curve beyond about 4 years.

Briefly put, yields at different tenors move in different ways. To understand the sensitivity of our portfolio to changes in yields at different parts of the yield curve, therefore, we require a more *local* measure of yield changes. Local in this context

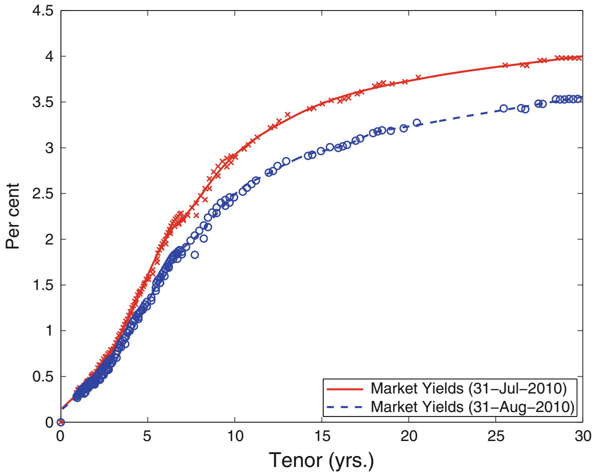


Fig. 2.3 Typical yield-curve movements. The underlying graph describes the change in the UST yield curve from 31 July 2010 to 31 August 2010. This is a clear example of situation where, over a given period of time, not all yields change in a parallel fashion

[9]This computation is based on the fact that modified duration—essentially a derivative—is a linear quantity and, as such, can be averaged.

means restricted to a smaller sector of the yield curve. The obvious solution to determining the exposure of one's portfolio to changes in 2-year yields is to compute something like the 2-year duration of the portfolio. In other words, to compute the portfolio's mathematical exposure to movements in the 2-year yield. Determination of such a sensitivity should, in principle, be possible.

Exactly for which yields one should compute such an exposure is a very natural question. Should we stop at the 2-year sector? Clearly not. There are a range of possible points along the sovereign yield curve that may be of interest. One's exact choice of *key* yield points will depend on the needs of the user, although a reasonable, and defensible, approach is to select key rates to coincide with areas of market liquidity (i.e., on-the-run sovereign bonds) to permit hedging of one's exposure to these sectors. Very specific tenors—such as the 1-year and 9-month yield or the 2-year and 1-month yield—are probably to be avoided.

For a more concrete perspective on the computation of bond (and portfolio) exposures to specific tenors along the yield curve, we consider an example. Figure 2.4 outlines the US Treasury curve as of 11 October 2010 and highlights eight different yield points across the curve: the 6-month, 2-year, 5-year, 7-year, 10-year, 15-year, 20-year, and 30-year yields. It is important to repeat that there is nothing magic about this specific choice of eight yields and that each analyst—hopefully after consultation with his or her portfolio-manager colleagues—must decide on the set of *key* yields across the yield curve for which exposures are required.

Having defined a set of key yield tenors, the next step is the determination of the sensitivity of one's fixed-income instrument to a change in this key rate. To understand how such a sensitivity might be computed, imagine that only *one* of these key yields, or rates, moves by, say, 50 basis points? Figure 2.5 highlights just such a 50 basis-point movement in the 10-year UST rate. Observe that all of the other key rates remain unchanged, but that the intermediate yield points are

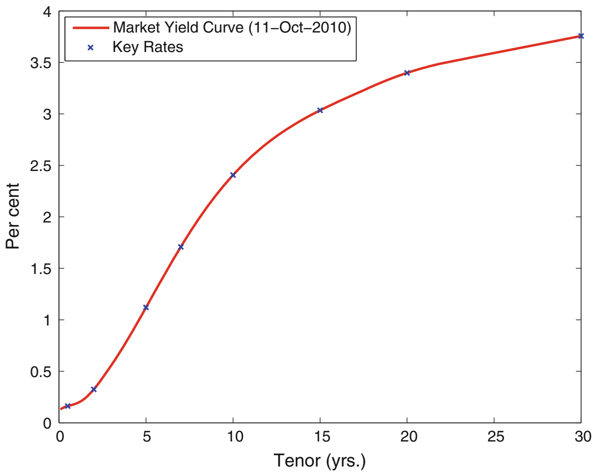


Fig. 2.4 Selection of key rates. This figure highlights eight different yield points across the US Treasury curve on 11 October 2010

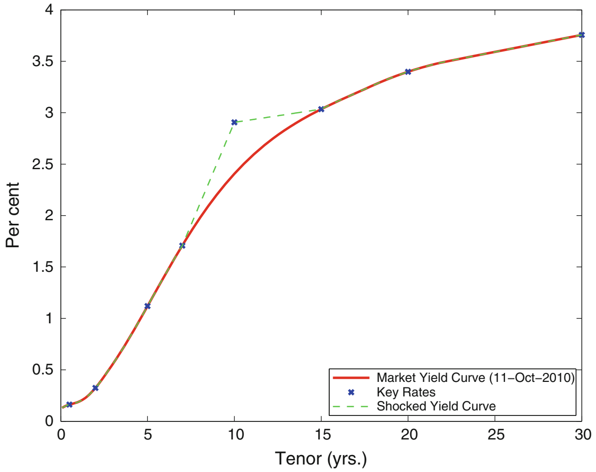


Fig. 2.5 Perturbing a key-rate. This figure demonstrates the impact of a 50 basis-point movement in the 10-year UST rate. All of the other rates remain unchanged, except for the intermediate yield points between the nearest adjacent non-perturbed key rates and the new perturbed value. These values are linearly interpolated to create a tent-like shape around the perturbed rate

linearly interpreted between the nearest adjacent non-perturbed key rates and the new perturbed value for the 10-year rate. The end result looks something like a tent.

The next task is to transform the image in Fig. 2.5 into a concrete mathematical expression. Naturally, this requires us to return to the bond-price equation. Once again, we perform a slight modification of the original expression in Eq. (2.1). In this case, instead of discounting all cash-flows at a single yield, we discount each cash-flow with its distinct yield. This leads us to the following expression:

$$V(t, y_{t_1}, \ldots, y_{t_n}) = \sum_{i=1}^{n} \frac{c_{t_i}}{(1 + y_{t_i})^{t_i - t}}. \tag{2.25}$$

Formally, therefore, $\{y_{t_i}, i = 1, .., n\}$ denotes the set of individual yields associated with each individual cash-flow. To compute the sensitivity of our bond price to a change in the kth yield, we start, as usual, with the partial derivative with respect to y_{t_k} for $k \in \{1, \ldots, n\}$:

$$\begin{aligned} \frac{\partial V(t, y_{t_1}, \ldots, y_{t_n})}{\partial y_k} &= \frac{\partial}{\partial y_{t_k}} \left(\sum_{i=1}^{n} \frac{c_{t_i}}{(1 + y_{t_i})^{t_i - t}} \right), \\ &= \frac{-(t_k - t)\, c_{t_k}}{(1 + y_{t_k})^{(t_k - t) + 1}} \end{aligned} \tag{2.26}$$

In this case, the resulting derivative is *not* a sum, but rather a single term.[10] This is the equivalent of modified duration, but only for a given area of the curve.

[10]This is because all terms where $i \neq k$ are, by definition, zero. That is, they do not contribute to the derivative.

Dividing both sides of Eq. (2.26) by $V(t, y_{t_1}, \ldots, y_{t_n})$ transforms this derivative into something resembling the duration concepts seen earlier in this chapter. Indeed, the resulting expression is generally termed the *key-rate* duration,

$$
\begin{aligned}
D_{t_k} &= \frac{1}{V(t, y_{t_1}, \ldots, y_{t_n})} \frac{\partial V(t, y_{t_1}, \ldots, y_{t_n})}{\partial y_{t_k}}, \\
&= \frac{-(t_k - t)\, c_{t_k}}{V(t, y_{t_1}, \ldots, y_{t_n})(1 + y_{t_k})^{(t_k - t)+1}}.
\end{aligned} \tag{2.27}
$$

We denote the key-rate duration, therefore, as D_{t_k} representing the sensitivity of one's fixed-income instrument, or portfolio, to a change in y_{t_k}.

The use of the key-rate duration is conceptually identical to the use of the modified duration.[11] The following approximation, similar in spirit to Eq. (2.8), demonstrates this point in mathematical terms,

$$
\begin{aligned}
D_{t_k} &\approx -\frac{1}{V(t, y_{t_1}, \ldots, y_{t_n})} \\
&\quad \times \frac{\overbrace{V(t, y_{t_1}, \cdots, y_{t_k} + \Delta y_{t_k}, \cdots, y_{t_n}) - V(t, y_{t_1}, \ldots, y_{t_n})}^{\Delta V(t, y_{t_1}, \ldots, y_{t_n})}}{\Delta y_{t_k}}, \\
-D_{t_k} \Delta y_{t_k} &\approx \frac{\Delta V(t, y_{t_1}, \ldots, y_{t_n})}{V(t, y_{t_1}, \ldots, y_{t_n})}.
\end{aligned} \tag{2.28}
$$

The key-rate duration is basically the *exposure* of the bond to a small change in the kth yield. The local nature of this measure of portfolio sensitivity makes it a very useful supplement to the modified duration, which provides a more global view of yield-curve sensitivity.

The careful reader has probably noticed that these analytic computations only make sense when the security's cash flows coincide precisely with the desired key-rate tenors. This is quite unlikely and probably impossible for a large portfolio of fixed-income securities. In reality, key-rate durations are determined numerically—a description of a sensible algorithm for their computation is found in the underlying shaded box. We will nonetheless continue to use the analytic development, because it provides useful insight into the notion of a key-rate duration and permits easy comparison with the other exposures developed in this chapter.

[11] In words, the product of the key-rate duration and an expected or realized change in the kth yield approximates the approximate percentage change in the value of one's fixed-income instrument.

In practice, it is generally quite difficult and inconvenient to analytically compute key-rate durations. Instead a numerical approximation is typically employed. This essentially involves a base sovereign yield curve—computed using your favourite method or a technique borrowed from Chap. 5—and a central finite-difference approximation. This is the key input, but the algorithm nonetheless requires a number of distinct steps. For a given key rate tenor, k, and choice of security it involves

1. determining the cash-flows of your security from its coupon and maturity date;
2. transforming your sovereign yield curve into a zero-coupon curve using a simple bootstrapping technique;[12]
3. computing the central value of the fixed-income security from the zero-coupon curve and the previously determined cash-flows—call this V;
4. shocking upwards the desired key-rate, at the desired key tenor k, on your base sovereign yield curve by a small amount, ϵ;
5. transforming—again using a bootstrap—the shocked yield curve into a correspondingly shocked zero coupon curve;
6. repricing your security with the shocked zero-coupon curve–call this V^+; and
7. perturbing the original sovereign curve downwards, again at the desired key tenor, by ϵ, determining the associated zero-coupon curve, and recomputing the new security value, V^-.

This provides all of the required ingredients for the final computation. The key-rate duration, for the selected key tenor, is thus approximated by the central finite-difference technique as,

$$D_{t_k} \approx \frac{1}{V}\left(\frac{V^+ - V^-}{2\epsilon}\right). \tag{2.29}$$

This is a tedious exercise and must be repeated for each security and each key-rate tenor (i.e., $k = 1, \ldots, \kappa$). Fortunately, such tedious tasks are easily organized into a computer program and happily delegated to your computer's CPU.

Simultaneous use of both modified and key-rate durations would be much easier if we could establish a link between these two measures. It turns out, in fact, that

[12] For more information on this technique, see Chap. 5.

such a link does exist. If we take the sum of all n key-rate durations, we arrive at the following expression

$$\sum_{k=1}^{n} \frac{\partial V(t, y_{t_1}, \ldots, y_{t_n})}{\partial y_{t_k}} = \sum_{k=1}^{n} \frac{-(t_k - t)\, c_{t_k}}{(1 + y_{t_k})^{(t_k - t)+1}}. \tag{2.30}$$

This looks slightly familiar. If we proceed to divide both sides by $V(t, y_{t_{1,\ldots,y_{t_k}}})$, then

$$\begin{aligned}
\frac{1}{V(t, y_{t_{1,\ldots,y_{t_k}}})} \sum_{k=1}^{n} \frac{\partial P(t, y_{t_1}, \ldots, y_{t_n})}{\partial y_k} &= \underbrace{\frac{1}{V(t, y_{t_{1,\ldots,y_{t_k}}})} \sum_{k=1}^{n} \frac{-(t_k - t)\, c_{t_k}}{\left(1 + \boxed{y_{t_k}}\right)^{(t_k - t)+1}}}_{\substack{\text{Sum of the} \\ \text{key-rate durations}}}, \\
&\approx \underbrace{\frac{1}{V(t, y_{t_{1,\ldots,y_{t_k}}})} \sum_{k=1}^{n} \frac{-(t_k - t)\, c_{t_k}}{\left(1 + \boxed{y}\right)^{(t_k - t)+1}}}_{\substack{\text{Modified} \\ \text{duration}}}.
\end{aligned} \tag{2.31}$$

The consequence is that the sum of the key rate durations is virtually identical to the expression for modified duration derived in Eq. (2.3). The only difference is that the individual discount factors in the sum are a sequence of values $\{y_{t_k}, k = 1, \ldots, n\}$ for the key-rate duration, but only a single value, y, for the modified duration. These need not be equal, of course, but generally they will be quite close. Consequently, the sum of the key-rate durations is a close approximation to the modified duration.

This relationship essentially permits us to sum the exposures from the key-rate durations and equate it to the exposure arising from the modified duration. More specifically, we can try to equate Eqs. (2.4) and (2.28) as,

$$-\sum_{k=1}^{n} D_{t_k} \Delta y_{t_k} \approx -D_M \Delta y, \tag{2.32}$$

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\overbrace{V(t, y_{t_1}, \cdots, y_{t_k} + \Delta y_{t_k}, \cdots, y_{t_n}) - V(t, y_{t_1}, \ldots, y_{t_n})}^{\Delta V(t, y_{t_1}, \ldots, y_{t_n})}}{V(t, y_{t_1}, \ldots, y_{t_n})} \\
\approx \frac{\overbrace{V(t, y + \Delta y) - V(t, y)}^{\Delta V(t,y)}}{V(t, y)}.
\end{aligned}$$

The power of the key-rate duration is that it generalizes the idea of the yield risk factor. Using modified duration, each individual security yield is a risk factor. When we employ key-rate durations, however, we use a generic set of key-rate risk factors for all securities in a given portfolio and strategic benchmark. This is very powerful.

2.5.1 A Word of Caution

Some caution is nevertheless required in the use of Eq. (2.32) as there are basically *two* sources of approximation. The sum of the key-durations may indeed be quite close to the modified duration, but the individual yield changes at the key-rate sectors need not be consistent.

To see how this might work, let's look at another example. Imagine that we have a UST bond with a modified duration of 5.07 years and the curve moves as described by Fig. 2.6. Over this period, the yield of this bond happily increases by 42 basis points. This would lead us to approximate the percentage change in the bond's value due to this yield movement as $-5.07 \cdot 42 = -212.9$ basis points.

Figure 2.6 and Table 2.3 also handily provide us with all of the key-rate durations and yield changes over this period. Performing the computation exactly as it appears in Eq. (2.32) with these inputs, however, we arrive at a rather different answer. Specifically, we predict a loss of -182.0 basis points amounting to a difference of approximately 30 basis points with the value stemming from the modified duration computation.

What is going on? Essentially, the changes in the yield curve were relatively modest at the short end of the curve with an 8 and 18 basis-point widening at the 6-month and 2-year sectors, respectively. Yields widened by 33 basis points at the 5-year point and 56 basis points at the 7-year sector. If we compute a key-rate weighted average of the total yield movement, therefore, it amounts to only about

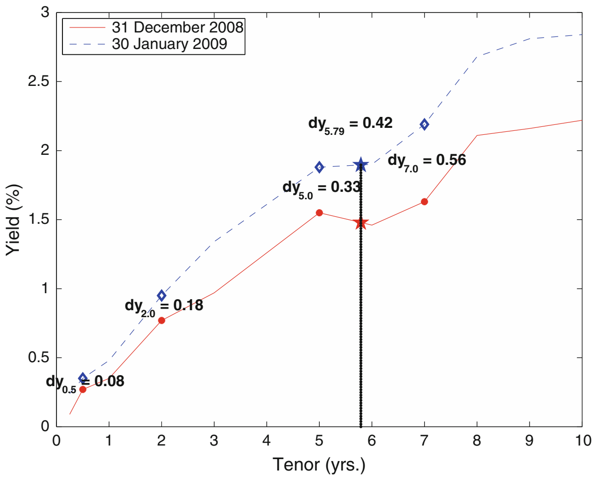


Fig. 2.6 Key-rate and modified durations. Here we examine the change in five key rates (i.e., 6-months, 2-years, 5-years, and 7 years) for a given bond along with a 5.79 year maturity

Table 2.3 Key-rate and modified-duration example

Key-rate tenor	D_{t_k}	Δy_{t_k}	$-D_{t_k}\,\Delta y_{t_k}$	Δy	$\boldsymbol{-D_{t_k}\,\Delta y}$
6 months	0.02	8	−0.2	42	**−0.8**
2 years	0.14	18	−2.5	42	**−5.9**
5 years	4.16	33	−137.3	42	**−174.7**
7 years	0.75	56	−42.0	42	**−31.5**
Total/weighted average	5.07	36	−182.0	42	**−212.9**

This table describes the key-rate durations and key-rate movements for a UST bond with a tenor of 5.79 years. The yield changes are also graphically represented in Fig. 2.6. All yield values are represented in basis points.

36 basis points of widening. The difference between the actual 42 basis points and the approximated 36 basis points accounts for the 30 basis points of difference in the approximation.

A more robust, albeit perhaps less satisfying, way to perform the computation, would be to use the key-rate durations, but restrict the yield changes at each key-rate point to be equal to the 42 basis-point widening experienced by the actual bond yield. This amounts to adjusting Eq. (2.32) as follows,

$$-\sum_{k=1}^{n} D_{t_k} \boxed{\Delta y} \approx -D_M \Delta y. \tag{2.33}$$

This essentially distributes the total curve return—without any under- or overstatement—across the pre-determined key tenors in the sovereign yield curve.

While this is clearly a rather extreme example, it does highlight the fact that the change in the yield of a given bond need not be equal to the key-rate weighted average of key-rate yield movements.[13] We will return to this point again in the following chapters when we discuss performance attribution and the estimation of portfolio risk.

2.6 Spread Exposure

Thus far, we have focused on bonds issued by so-called risk-free borrowers. Risk-free, in this context, implies that these bonds are subject to no, or at least very little, credit risk. It would be tempting to classify these risk-free borrowers as governments, but in fact, some government bonds do have a relatively high prob-

[13]For a portfolio with numerous bonds across the curve, the effect is likely to be relatively small. Nevertheless, for computations that require a high degree of accuracy, such as performance attribution, this is likely to remain an unacceptable source of error.

ability of default or losses associated with a deterioration of their credit quality.[14] A good example would be the Eurozone, where some government issuers such as Germany and France could be considered risk-free borrowers, while others—such as Greece, Portugal, Ireland, or Spain at the time this document was written—could be considered to have considerable credit risk. Simply put, there are many fixed-income instruments including bonds issued by some governments, supranational entities, government agencies, and corporations that are exposed to credit risk. Credit risk is clearly a risk factor. Moreover, it is probably a collection of risk factors that might be organized in varying degrees of granularity.[15] Our analysis thus far does not permit us to compute the exposure of a fixed-income instrument to changes in its credit quality. In this section, we will address this shortcoming.

Determination of the exposure of a fixed-income instrument to changes in its underlying credit quality requires the identification, within the context of our bond-price equation, of some element that relates to credit risk. At first glance, the price of an agency, supranational, or even a corporate bond has the same form as government bond,

$$V(t, y) = \sum_{i=1}^{n} \frac{c_{t_i}}{(1 + y)^{t_i - t}}. \tag{2.34}$$

It remains the sum of the individual cash-flows discounted by its yield. This form is unfortunately *not* very helpful.

With a bit of reflection, this drawback can be resolved. The credit element is embedded in the yield. Supranational bonds, for example, trade at a higher yield (i.e., lower price) than a treasury security with an equivalent tenor to account for this incremental credit risk. We can exploit this fact and decompose a security's yield into *two* distinct components:

1. the equivalent treasury yield; and
2. the credit spread.

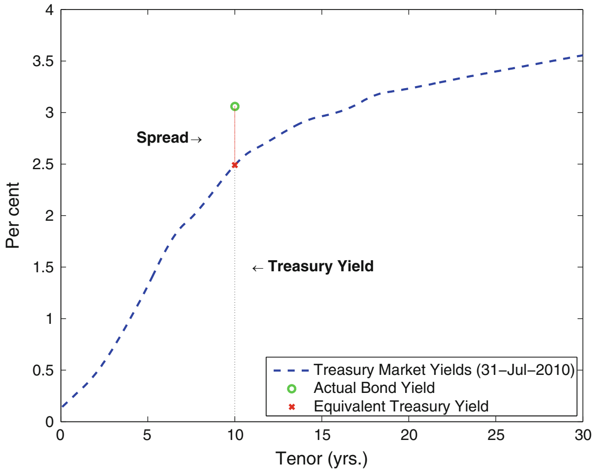
What does this mean? The *equivalent treasury yield* represents the yield that a bond subject to credit risk would have if it was issued by the risk-free issuer in that currency. An example would be a US agency bond with a tenor of 4.76 years–its equivalent treasury yield would be the yield of a US Treasury bond with the same tenor of 4.76 years. To perform this computation, one requires a mathematical model of the underlying risk-free yield curve. Such a model permits one to easily determine the yield of a US Treasury bond with a 4.76 year bond, as it is unlikely that such a bond would exist in the market.

---

[14] This latter situation is also termed credit migration.

[15] One could, for example, use categories such as sovereign, agency, supranational, or corporate. It would also be possible to have subcategories with each of these groups or even, in the limit, focus on individual names.

Fig. 2.7 Decomposing the bond yield. This figure provides a graphical representation of how, armed with a risk-free yield curve, one can decompose the yield of a credit bond into the sum of its equivalent treasury yield and its credit spread



The difference between the bond's actual yield and this equivalent treasury yield is particularly interesting. It is called the *credit spread*. This credit spread, or simply spread, is essentially the additional yield demanded by the market—over and above the yield required for the risk-free borrower—to compensate for the incremental credit risk associated with that particular bond.[16] Thus, the larger the credit spread of a specific bond, the less optimistic is the market's assessment of the general credit quality of the underlying issuer. Changes in the spread over time consequently represent adjustments in the overall credit quality of the underlying issuer. Should, for example, the spread widen (narrow), then this would typically imply that the market has downgraded (upgraded) its view of the bond's credit quality.

Figure 2.7 demonstrates graphically how, with a yield-curve model for the necessary risk-free yield curve, one can decompose the yield into the equivalent treasury yield and a credit spread. Such a decomposition is quite practical as it breaks out a bond's yield into the risk-free component—common to all fixed-income instruments in that market—and an idiosyncratic credit spread component. Using this decomposition, our bond-price equation becomes,

$$V(t, \hat{y}, s) = \sum_{i=1}^{n} \frac{c_{t_i}}{(1 + \underbrace{\hat{y} + s}_{y})^{t_i - t}}, \tag{2.35}$$

[16]To be precise, the spread may also include additional yield demanded by lenders to compensate for lower liquidity of the credit bond relative to the underlying government curve. Decomposition of the credit and liquidity aspects of the credit spread is not trivial and we will, in our development ignore such effects. It is nonetheless important to be aware that part of the credit spread may be attributable to liquidity and, in some cases, its contribution can be important.

where $\hat{y}$ denotes the equivalent treasury yield and s represents the associated credit spread.[17] This simple decomposition separates *two* risk factors—the treasury yield curve and the credit spread—that were previously linked through the security's yield.

We are now in familiar territory. As with modified duration for nominal and inflation-linked bonds and the notion of key-rate durations, it would seem that the derivative is a good place to start to determine the sensitivity of the bond's price to a change in the spread. The derivative of Eq. (2.35) with respect to the credit spread, s, is

$$\begin{aligned} \frac{\partial V(t,\hat{y},s)}{\partial s} &= \frac{\partial}{\partial s}\left(\sum_{i=1}^{n}\frac{c_{t_i}}{(1+\hat{y}+s)^{t_i-t}}\right), \\ &= -\frac{1}{(1+\hat{y}+s)}\sum_{i=1}^{n}\frac{(t_i-t)\,c_{t_i}}{(1+\hat{y}+s)^{t_i-t}}. \end{aligned} \tag{2.36}$$

If we recall, however, that $y = \hat{y} + s$ we can make a number of substitutions to simplify Eq. (2.36) as follows,

$$\begin{aligned} \frac{\partial V(t,\hat{y},s)}{\partial s} &= -\frac{1}{(1+\underbrace{\hat{y}+s}_{y})}\sum_{i=1}^{n}\frac{(t_i-t)\,c_{t_i}}{(1+\underbrace{\hat{y}+s}_{y})^{t_i-t}}, \\ &= -\frac{1}{(1+y)}\sum_{i=1}^{n}\frac{(t_i-t)\,c_{t_i}}{(1+y)^{t_i-t}}, \\ &= \frac{\partial V(t,y)}{\partial y}. \end{aligned} \tag{2.37}$$

Ultimately, there is nothing different about the sensitivity of the bond price whether the yield change comes from a movement in the equivalent treasury yield ($\hat{y}$), the credit spread (s), or the overall yield (y). While this is hardly a surprise, given that the terms enter additively into the bond-price equation, it is nonetheless a useful result.

If we proceed to divide both sides of Eq. (2.37) by $\frac{\partial V(\hat{y},s)}{\partial s}$, we arrive at the equivalent of modified duration, but for spread movements. It is termed, quite naturally, the spread duration and has the following form,

$$D_S = \frac{1}{V(t,\hat{y},s)}\frac{\partial V(t,\hat{y},s)}{\partial s}, \tag{2.38}$$

[17]The explicit introduction of the credit spread into the bond price equation permits us to include the credit risk factor into our mathematical framework.

$$= -\frac{1}{V(1+y)} \sum_{i=1}^{n} \frac{(t_i - t)\, c_{t_i}}{(1+y)^{t_i - t}},$$
$$= D_m.$$

We see again that spread duration and modified duration are equivalent for a plain-vanilla bond.

The spread duration is used in practice by replacing the partial derivative in the spread duration with its linear approximation $\frac{\Delta V}{\Delta s}$ as follows,

$$D_S \approx -\frac{1}{V(t,\hat{y},s)} \frac{\overbrace{V(t,\hat{y},s+\Delta s) - V(t,\hat{y},s)}^{\Delta V(t,\hat{y},s)}}{\Delta s}, \tag{2.39}$$
$$-D_s \Delta s \approx \frac{\Delta V(t,\hat{y},s)}{\Delta V(t,\hat{y},s)(t,\hat{y},s)}.$$

The spread duration, therefore, is basically the *exposure* of the bond to a small change in its credit spread—or, more generally, a small change in the credit quality of the bond issuer. It should be noted that for some fixed-income instruments, most notably floating-rate notes, the spread and modified durations do *not* coincide. In these cases, it is typical to compute the actual spread duration using an alternative approach. Floating-rate notes are addressed in Chap. 4.

In many commercial applications, the modified and spread durations, while close, do not usually perfectly coincide. For complex securities, the reason is obvious. The cash flows also depend on the spread, which leads to the following bond-price equation,

$$V(t,\hat{y},s) = \sum_{i=1}^{n} \frac{c_{t_i}(s)}{(1+\underbrace{\hat{y}+s}_{y})^{t_i - t}}, \tag{2.40}$$

Clearly, differentiating Eq. (2.40) with respect to s will not reduce to the modified duration. For plain-vanilla instruments, the bond-price equation is also often written as follows,

$$V(t,\hat{y},s) = \sum_{i=1}^{n} \frac{c_{t_i}}{(1+\underbrace{z(t,t_i)+s_z}_{y})^{t_i - t}}, \tag{2.41}$$

(continued)

implying that each cash flow is discounted at its own individual spot rate, $z(t, t_i)$ for $i = 1, \ldots, n$. In this case, the credit spread, which is denoted as s_z, will not be the same as the s computed using our simple decomposition. That is, $s_z \neq s$. In this case, when the spread duration is calculated—using a numerical approach—the result is not equal to the modified duration. For normal plain-vanilla instruments, however, the difference is minimal and, for portfolio analytic purposes, we simply assume that spread duration and modified duration are equal. In the case of complex securities, as described in Eq. (2.40), this may be a poor assumption. In this case, it is safer to employ a numerical approximation.

2.7 Foreign-Exchange Exposure

Fixed-income securities are not always denoted in one's base currency. Often, they are denominated in foreign currency. This implies that a movement in the exchange rate between the foreign currency and your base currency—all else being equal—leads to changes in the value of your investment. As such, the foreign-exchange rate is a risk factor.

Since the foreign-exchange rate is a risk factor, it would be useful to understand the exposure of a generic fixed-income security to the exchange rate. To do this, we return as usual to the bond-price equation. Denote E_t as the exchange rate between the security and your base currency. Consequently, your security's value may be written as,

$$
\begin{aligned}
V(t, y, E_t) &= E_t \underbrace{\sum_{i=1}^{I} c_{t_i} \delta(t, t_i)}_{\substack{\text{Base currency} \\ \text{value: } V(t,y)}}, \\
&= E_t V(t, y).
\end{aligned} \tag{2.42}
$$

If we differentiate V with respect to E_t, we have

$$
\begin{aligned}
\frac{\partial V(t, y, E_t)}{\partial E_t} &= \frac{\partial E_t V(t, y)}{\partial E_t}, \\
&= V(t, y).
\end{aligned} \tag{2.43}
$$

The result is quite intuitive. A security's exposure to a foreign-exchange rate is the entire investment.[18]

[18] A possible exception would be a dual-currency bond, where the coupon and notional amounts are denominated in different currencies. In this case, the security would need to be split into two

Table 2.4 Summarizing exposures

Factor	Exposure	Definition	Value
Yield	Modified duration	$\frac{1}{V}\frac{\partial V}{\partial \mathbf{y}}$	D_M
	Convexity	$\frac{1}{V}\frac{\partial^2 V}{\partial \mathbf{y}^2}$	C
	kth key-rate duration	$\frac{1}{V}\frac{\partial V}{\partial \mathbf{y}_{\mathbf{t_k}}}$	D_{t_k}
Time	Carry	$\frac{1}{V}\frac{\partial V}{\partial \mathbf{t}}$	y
Credit spread	Spread duration	$\frac{1}{V}\frac{\partial V}{\partial \mathbf{s}}$	D_s
FX	FX exposure	$\frac{\partial V}{\partial \mathbf{E_t}}$	V

This table summarizes the exposures computed to the various risk factors identified throughout the course of this chapter

2.8 Concluding Thoughts

This has been a detailed chapter with numerous mathematical digressions. The common thread among these computations and derivations was the determination of the exposure of a fixed-income security to a collection of different risk factors: yield, time, spread, key rates and foreign-exchange rates. In short, this chapter has been about computing the exposure, or sensitivity, of a fixed-income security to various risk factors. Table 2.4 summarizes all of the operations that we performed on the bond-price expression in terms of factors and exposures.

These exposures provide an insight into the change in the value of a generic fixed-income security associated with a given movement in the underlying risk factor. Note, however, that they consider each risk factor in isolation. We naturally wish to examine them in a joint fashion—this requires additional effort. In the next chapter, we will make ample use of these exposures and employ them to link our set of risk factors to the return of our fixed-income security.

Reference

1. W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edn. (Cambridge University Press, Cambridge, 1992)

separate instruments: an annuity for the coupon stream and a zero-coupon bond for the notional value. Each would then require a separate currency definition.

Springer

http://www.springer.com/978-3-319-12666-1

Fixed-Income Portfolio Analytics
A Practical Guide to Implementing, Monitoring and Understanding Fixed-Income Portfolios
Bolder, D.J.
2015, XXVII, 544 p. 170 illus., 15 illus. in color., Hardcover
ISBN: 978-3-319-12666-1