Chapter 2
Preliminaries

Summary. This chapter contains definitions and auxiliary results related to various notions of nowhere differentiability. In particular, in § 2.3, we present a proof of the famous Denjoy–Young–Saks theorem, which may permit the reader to understand better the sense of nowhere differentiability.

2.1 Derivatives

Let $I \subset \mathbb{R}$ be an arbitrary interval containing at least two distinct points.

Definition 2.1.1. For a function $\varphi : I \longrightarrow \mathbb{C}$, set

$$\boldsymbol{\Delta}\varphi(t,u) := \frac{\varphi(u)-\varphi(t)}{u-t}, \quad t,u \in I,\ t \neq u.$$

Recall that φ has a *(finite) derivative* $\varphi'(t)$ at a point $t \in I$ if the limit

$$\varphi'(t) := \lim_{I \ni u \to t} \boldsymbol{\Delta}\varphi(t,u)$$

exists and is finite. In the case $\varphi : I \longrightarrow \mathbb{R}$, we may also consider an *infinite derivative* $\varphi'(t)$ if the limit

$$\varphi'(t) := \lim_{I \ni u \to t} \boldsymbol{\Delta}\varphi(t,u)$$

exists but is infinite, i.e., $\varphi'(t) \in \{-\infty, +\infty\}$.

Remark 2.1.2. If $\varphi : I \longrightarrow \mathbb{C}$, then

$$\boldsymbol{\Delta}\varphi(u_1,u_2) = \frac{u_2-t}{u_2-u_1}\boldsymbol{\Delta}\varphi(t,u_2) + \frac{t-u_1}{u_2-u_1}\boldsymbol{\Delta}\varphi(t,u_1),$$
$$t,u_1,u_2 \in I,\ u_1 < t < u_2.$$

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Consequently:

(a) If a finite derivative $\varphi'(t)$ exists at an interior point $t \in \operatorname{int} I$, then

$$\varphi'(t) = \lim_{\substack{u_1,u_2\to t\\ u_1<t<u_2}} \boldsymbol{\Delta}\varphi(u_1,u_2);$$

note that this fact was already known to T.J. Stieltjes (cf. [Sti14]).

(b) If $\varphi : I \longrightarrow \mathbb{R}$, then

$$\min\{\boldsymbol{\Delta}\varphi(t,u_2), \boldsymbol{\Delta}\varphi(t,u_1)\} \le \boldsymbol{\Delta}\varphi(u_1,u_2) \le \max\{\boldsymbol{\Delta}\varphi(t,u_2), \boldsymbol{\Delta}\varphi(t,u_1)\},$$
$$t, u_1, u_2 \in I,\ u_1 < t < u_2.$$

In particular, if an infinite derivative $\varphi'(t)$ exists at an interior point $t \in \operatorname{int} I$, then

$$\varphi'(t) = \lim_{\substack{u_1,u_2\to t\\ u_1<t<u_2}} \boldsymbol{\Delta}\varphi(u_1,u_2).$$

Definition 2.1.3. Let $\varphi : I \longrightarrow \mathbb{C}$, $t \in I$. We say that φ has a *finite right-* (resp. *left-*) *sided derivative* $\varphi'_+(t)$ (resp. $\varphi'_-(t)$) at t if the limit

$$\varphi'_+(t) := \lim_{\substack{I\ni u\to t\\ u>t}} \boldsymbol{\Delta}\varphi(t,u) = \lim_{I\ni u\to t+} \boldsymbol{\Delta}\varphi(t,u)$$
$$\Big(\text{resp. } \varphi'_-(t) := \lim_{\substack{I\ni u\to t\\ u<t}} \boldsymbol{\Delta}\varphi(t,u) = \lim_{I\ni u\to t-} \boldsymbol{\Delta}\varphi(t,u)\Big)$$

exists and is finite. In the case $\varphi : I \longrightarrow \mathbb{R}$, we allow *infinite one-sided derivatives* $\varphi'_\pm(t) \in \{-\infty, +\infty\}$. Notice that:

- if $t \in I$ is the right endpoint of the interval, then $\varphi'_+(t)$ is not defined and $\varphi'_-(t) = \varphi'(t)$;
- if $t \in I$ is the left endpoint of the interval, then $\varphi'_-(t)$ is not defined and $\varphi'_+(t) = \varphi'(t)$.

One-sided derivatives are also called *unilateral derivatives.*

Remark 2.1.4. Let $\varphi : I \longrightarrow \mathbb{C}$.

(a) If a finite $\varphi'_+(t)$ exists, then for every $C > 0$, we have

$$\varphi'_+(t) = \lim_{\substack{I\ni u',u''\to t,\ t<u'<u''\\ |\frac{u''-t}{u''-u'}|\le C}} \boldsymbol{\Delta}\varphi(u',u'').$$

Indeed, we have $\varphi(u) = \varphi(t) + \varphi'_+(t)(u-t) + \alpha(u)(u-t)$, $t < u \in I$, where $\lim_{u\to t+} \alpha(u) = 0$. Hence

$$\begin{aligned}\boldsymbol{\Delta}\varphi(u',u'') &= \frac{\varphi(t) + \varphi'_+(t)(u''-t) + \alpha(u'')(u''-t)}{u''-u'} \\ &\quad - \frac{\varphi(t) + \varphi'_+(t)(u'-t) + \alpha(u')(u'-t)}{u''-u'} \\ &= \varphi'_+(t) + \frac{u''-t}{u''-u'}\alpha(u'') - \frac{u'-t}{u''-u'}\alpha(u') \underset{\substack{I\ni u',u''\to t\\ t<u'<u''}}{\longrightarrow} \varphi'_+(t),\end{aligned}$$

provided $\frac{u''-t}{u''-u'}$ is bounded.

(b) An analogous result may be easily obtained for finite left derivatives.

(c) Notice that (a) is not true for infinite unilateral derivatives.
For example, let $n_1 = 2$, $n_{k+1} = n_k^2$, $k \in \mathbb{N}$. Define $\varphi : \left[0, \frac{1}{4}\right] \longrightarrow \mathbb{R}$, $\varphi(0) := 0$,

$$\varphi(u) := \frac{1}{n_k},\ u \in \left[\tfrac{1}{n_k^3}, \tfrac{1}{n_k^2}\right], \quad \varphi(u) := n_{k+1}u,\ u \in \left[\tfrac{1}{n_{k+1}^2}, \tfrac{1}{n_k^3}\right], \quad k \in \mathbb{N}.$$

Observe that φ is continuous and $\varphi'_+(0) = +\infty$. In fact, for $u \in \left[\frac{1}{n_k^3}, \frac{1}{n_k^2}\right]$, we have $\boldsymbol{\Delta}\varphi(0, u) = \frac{1}{n_k u} \geq n_k$. For $u \in \left[\frac{1}{n_{k+1}^2}, \frac{1}{n_k^3}\right]$, we have $\boldsymbol{\Delta}\varphi(0, u) = n_{k+1}$.
Take $u'_k := \frac{1}{n_k^3}$, $u''_k := \frac{1}{n_k^2}$. Then $\boldsymbol{\Delta}\varphi(u'_k, u''_k) = 0$ and $\frac{u''_k - 0}{u''_k - u'_k} \leq 2$.

(d) A finite derivative $\varphi'(t)$ exists at an interior point $t \in \operatorname{int} I$ iff

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{\substack{t-\delta \leq a_i \leq t \leq b_i \leq t+\delta \\ a_i, b_i \in I,\ a_i < b_i,\ i=1,2}} : |\boldsymbol{\Delta}\varphi(a_1, b_1) - \boldsymbol{\Delta}\varphi(a_2, b_2)| < \varepsilon.$$

Indeed, if the above condition is satisfied, then taking $a_1 = a_2 = t$ (resp. $b_1 = b_2 = t$), we conclude that a finite one-sided derivative $\varphi'_+(t)$ (resp. $\varphi'_-(t)$) exists. Taking $a_1 = b_2 = t$, we get $\varphi'_+(t) = \varphi'_-(t)$. Conversely, if $\varphi'(t) \in \mathbb{R}$ exists, then we use Remark 2.1.2(a).

We will use also the following more general derivatives, introduced, e.g., by U. Dini in [Din92].

Definition 2.1.5. Let $\varphi : I \longrightarrow \mathbb{R}$, $t \in I$. The *lower* (resp. *upper*) *right Dini derivative* $D_+\varphi(t)$ (resp. $D^+\varphi(t)$) of φ at t is defined as

$$D_+\varphi(t) := \liminf_{I \ni u \to t+} \boldsymbol{\Delta}\varphi(t, u) \in \overline{\mathbb{R}}$$
$$\Big(\text{resp. } D^+\varphi(t) := \limsup_{I \ni u \to t+} \boldsymbol{\Delta}\varphi(t, u) \in \overline{\mathbb{R}}\Big).$$

Analogously, the *lower* (resp. *upper*) *left Dini derivative* $D_-\varphi(t)$ (resp. $D^-\varphi(t)$) of φ at t is defined as

$$D_-\varphi(t) := \liminf_{I \ni u \to t-} \boldsymbol{\Delta}\varphi(t, u) \in \overline{\mathbb{R}}$$
$$\Big(\text{resp. } D^-\varphi(t) := \limsup_{I \ni u \to t-} \boldsymbol{\Delta}\varphi(t, u) \in \overline{\mathbb{R}}\Big).$$

Similarly to the above, $D^+\varphi(t)$ and $D_+\varphi(t)$ (resp. $D^-\varphi(t)$ and $D_-\varphi(t)$) are not defined if $t \in I$ is the right (resp. left) endpoint of the interval.

Remark 2.1.6. (a) $\varphi'_+(t)$ exists iff $D^+\varphi(t) = D_+\varphi(t)$; $\varphi'_-(t)$ exists iff $D^-\varphi(t) = D_-\varphi(t)$.
(b) $D^-\varphi = -D_-(-\varphi)$, $D_+\varphi = -D^+(-\varphi)$.
(c) $D^-\overset{\vee}{\varphi}(t) = -D_+\varphi(-t)$, $D_-\overset{\vee}{\varphi}(t) = -D^+\varphi(-t)$, where $\overset{\vee}{\varphi}(t) := \varphi(-t)$ (provided that $-I = I$).

Remark 2.1.7. If $\varphi : I \longrightarrow \mathbb{R}$ is continuous, then the functions $D^+\varphi$, $D_+\varphi$, $D^-\varphi$, $D_-\varphi$ are Borel measurable.

We will prove that $D^+\varphi$ is Borel measurable (the remaining cases are left to the reader as an Exercise). We may assume that the right endpoint of I does not belong to I. It suffices to show that for every $C \in \mathbb{R}$, the set $A_C := \{t \in I : D^+\varphi(t) < C\}$ is Borel measurable. Fix a $C \in \mathbb{R}$. Let $N \in \mathbb{N}$ be such that $I_n := \{t \in I : t + \frac{1}{n} \in I\} \neq \varnothing$ for $n \geq N$. Now we need only observe that in view of the continuity of φ, we have

$$A_C = \bigcup_{n\in\mathbb{N}_N,\ k\in\mathbb{N}} \bigcap_{h\in\mathbb{Q}\cap(0,\frac{1}{n})} \Big\{t \in I_n : \frac{\varphi(t+h)-\varphi(t)}{h} \le C - \frac{1}{k}\Big\}.$$

Notice that the result remains true for arbitrary Borel-measurable functions $\varphi : I \longrightarrow \mathbb{R}$ (cf. [Ban22]).

2.2 Families of Continuous Nowhere Differentiable Functions

Recall that our principal aim is to discuss *continuous* nowhere differentiable functions. To simplify notation related to nowhere differentiability, we define the following classes of continuous nowhere differentiable functions.

- $\mathfrak{ND}(I)$:= the set of all $\varphi \in \mathcal{C}(I,\mathbb{C})$ that are nowhere differentiable in the finite sense;
- $\mathfrak{ND}^\infty(I)$:= the set of all $\varphi \in \mathcal{C}(I)$ that are nowhere differentiable in the finite or infinite sense;
- $\mathfrak{ND}_\pm(I)$:= the set of all $\varphi \in \mathcal{C}(I,\mathbb{C})$ such that for every $t \in I$, there is neither a finite right nor a finite left derivative at t;
- $\mathfrak{ND}_\pm^\infty(I) = \mathfrak{B}(I)$:= the set of all *Besicovitch functions*, i.e., the set of all $\varphi \in \mathcal{C}(I)$ such that for every $t \in I$, there is neither a finite or infinite right nor a finite or infinite left derivative at t (cf. § 7.5);
- $\mathfrak{M}(I)$:= the set of all *Morse functions*, i.e., the set of all $\varphi \in \mathcal{C}(I)$ such that

$$\max\{|D^+\varphi(t)|, |D_+\varphi(t)|\} = \max\{|D^-\varphi(t)|, |D_-\varphi(t)|\} = +\infty, \quad t \in I;$$

  we skip the left (resp. right) $\max\{\dots\}$ if t is the right (resp. left) endpoint of the interval;
- $\mathfrak{BM}(I) = \mathfrak{B}(I) \cap \mathfrak{M}(I)$:= the set of all *Besicovitch–Morse functions* (cf. § 11.1).

Notice that

$$\mathfrak{BM}(I) \subset \mathfrak{M}(I) \subset \mathfrak{ND}_\pm(I) \subset \mathfrak{ND}(I),$$
$$\mathfrak{BM}(I) \subset \mathfrak{B}(I) = \mathfrak{ND}_\pm^\infty(I) \subset \mathfrak{ND}^\infty(I).$$

Remark 2.2.1. Observe that if I is an open interval, then there exists a real-analytic increasing diffeomorphism $\sigma : \mathbb{R} \longrightarrow I$. In particular, if a continuous function $\varphi : I \longrightarrow \mathbb{C}$ belongs to one of the above classes of nowhere differentiable functions on I, then the function $\varphi \circ \sigma$ belongs to the corresponding class on $\mathbb{R}$.

The above remark permits us to transport many results from I to $\mathbb{R}$ and vice versa.

2.3 The Denjoy–Young–Saks Theorem

The following result may give some feelings for the general behavior of functions with respect to their differentiability. *On a first reading, the reader may skip the proof.*

Theorem 2.3.1 (Denjoy–Young–Saks)**.** *Let $I \subset \mathbb{R}$ be an arbitrary nontrivial interval. Let $f : I \longrightarrow \mathbb{R}$. Then there exists a set $E \subset I$ of Lebesgue measure zero such that for every $x \in I \setminus E$, either*

- *a finite $f'(x)$ exists, or*
- *$D^+f(x) = D_-f(x) \in \mathbb{R}$ and $D_+f(x) = -\infty$, $D^-f(x) = +\infty$, or*

- *$D^- f(x) = D_+ f(x) \in \mathbb{R}$ and $D^+ f(x) = +\infty$, $D_- f(x) = -\infty$, or*
- *$D^- f(x) = D^+ f(x) = +\infty$ and $D_- f(x) = D_+ f(x) = -\infty$.*

Remark 2.3.2. Symbolically, for $x \in I \setminus E$ we have the following four possibilities:

$$\begin{array}{c|c} & \\ * & * \\ & \end{array} \qquad \begin{array}{c|c} +\infty & \\ * & * \\ & -\infty \end{array} \qquad \begin{array}{c|c} & +\infty \\ * & * \\ -\infty & \end{array} \qquad \begin{array}{c|c} +\infty & +\infty \\ & \\ -\infty & -\infty \end{array}$$

If f is continuous, the result was first proved by A. Denjoy in [Den15]. The case in which f is measurable was solved by G.C. Young in [You16b]. Finally, the general case was proved by S. Saks in [Sak24]. Our elementary proof is due to E.H. Hanson [Han34].

Corollary 2.3.3. *Let $f : I \longrightarrow \mathbb{R}$, $f \in \mathfrak{ND}(I)$. Then at almost all points of I, the function f has no one-sided (finite or infinite) derivatives.*

The following two classical results from measure theory will be important for the proof.

Theorem 2.3.4 (Vitali Covering Theorem; Cf. [KK96], Theorem 0.3.2)**.** *Let $S \subset \mathbb{R}$ be bounded and let $\mathcal{F}$ be a family of bounded closed intervals, none consisting of a single point, such that for every $x \in S$ and $\varepsilon > 0$, there exists a $P \in \mathcal{F}$ such that $x \in P$ and $\operatorname{diam}(P) \le \varepsilon$. Then there exists an at most countable subfamily $\mathcal{F}^0 \subset \mathcal{F}$, consisting of pairwise disjoint intervals, such that*

$$\mathcal{L}\Big(S \setminus \bigcup_{P \in \mathcal{F}^0} P\Big) = 0,$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$.

Theorem 2.3.5 (Lebesgue Density Theorem; Cf. [KK96], Theorem 2.2.1)**.** *Let $A \subset \mathbb{R}$. Then for almost all $x \in A$ and for every sequence $(P_s)_{s=1}^{\infty}$ of bounded intervals with $x \in P_s$ and $0 < \operatorname{diam}(P_s) \longrightarrow 0$, we have*

$$\lim_{s \to +\infty} \frac{\mathcal{L}^*(A \cap P_s)}{\mathcal{L}(P_s)} = 1,$$

where $\mathcal{L}^*$ stands for the outer Lebesgue measure on $\mathbb{R}$.

Proof of Theorem 2.3.1. Using Remark 2.2.1, we may assume that $I = \mathbb{R}$.

Step 1^o. It suffices to prove that there exists a zero-measure set $E_0 = E_0(f)$ such that for every $x \in \mathbb{R} \setminus E_0$, either

- $D^+ f(x) = D_- f(x) \in \mathbb{R}$, or
- $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$.

Indeed, then we put $E := E_0(f) \cup E_0(-f)$.

Step 2^o. The main idea of the proof is to show that:

(a) the set $E_1 := \{x \in \mathbb{R} : D^+ f(x) = +\infty,\ D_- f(x) \ne -\infty\}$ is of measure zero,
(b) the set $E_2 := \{x \in \mathbb{R} : D_- f(x) = -\infty,\ D^+ f(x) \ne +\infty\}$ is of measure zero,
(c) the set $E_3 := \{x \in \mathbb{R} : D^+ f(x) < D_- f(x)$ or $D^- f(x) < D_+ f(x)\}$ is at most countable,
(d) the set $E_4 := \{x \in \mathbb{R} : D^+ f(x), D_- f(x) \in \mathbb{R},\ D^+ f(x) \ne D_- f(x)\}$ is of measure zero.

Observe that (b) follows from (a) applied to the function $-f$.

Suppose for a moment that the above properties are already proven. Put $E_0 := E_1 \cup E_2 \cup E_3 \cup E_4$ and fix an $x \in \mathbb{R} \setminus E_0$. By (d), we need to check only that if $D^+ f(x)$ or $D_- f(x)$ is infinite, then $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$. The configurations from (a) and (b) are excluded. Thus, their remains the case $D^+ f(x) = -\infty$ (resp. $D_- f(x) = +\infty$), but then, in view of (c), $D_- f(x) = -\infty$ (resp. $D^+ f(x) = +\infty$), which contradicts (b) (resp. (a)).

Step 3^o. Proof of (a).

We have

$$E_1 = \bigcup_{r\in\mathbb{Q},\ n\in\mathbb{N}} A_{r,n},$$

where

$$A_{r,n} := \{x \in \mathbb{R} : D^+f(x) = +\infty,\ \forall_{x'\in(x-\frac{1}{n},x)} :\ \boldsymbol{\Delta} f(x,x') > r\}.$$

We need to prove only that each set $A_{r,n}$ is of measure zero. Fix r, $n \in \mathbb{N}$, and $b \in A_{r,n}$. Let $a \in \mathbb{R}$ be such that $0 < b - a < \frac{1}{n}$. Put $S := A_{r,n} \cap (a,b)$. Take an arbitrary $t \in \mathbb{R}$ and let

$$\mathcal{F}_t := \{[p,q] : q > p,\ [p,q] \subset (a,b),\ p \in S,\ \boldsymbol{\Delta} f(p,q) > t\}.$$

It is clear that $(S, \mathcal{F}_t)$ satisfies the assumptions of the Vitali covering theorem. Thus there exists an at most countable subfamily $\mathcal{F}_t^0 \subset \mathcal{F}_t$, consisting of pairwise disjoint intervals, such that $\mathcal{L}(S \setminus \bigcup_{P\in\mathcal{F}_t^0} P) = 0$. Take $P_1, \dots, P_N \in \mathcal{F}_t^0$, $P_i = [p_i, q_i]$. Then $(a,b) \setminus \bigcup_{i=1}^N P_i = \bigcup_{j=1}^M (\alpha_j, \beta_j)$, where the intervals $(\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)$ are pairwise disjoint and $\beta_j \in A_{r,n}$, $j = 1, \dots, M$. In particular, $\boldsymbol{\Delta} f(\alpha_j, \beta_j) > r$. Consequently,

$$\begin{aligned} f(b) - f(a) &= \sum_{j=1}^M (f(\beta_j) - f(\alpha_j)) + \sum_{i=1}^N (f(q_i) - f(p_i)) \\ &> r\sum_{j=1}^M (\beta_j - \alpha_j) + t\sum_{i=1}^N (q_i - p_i) = (t-r)\sum_{i=1}^N \mathcal{L}(P_i) + r(b-a). \end{aligned}$$

Thus

$$f(b) - f(a) \geq (t-r) \sum_{P\in\mathcal{F}_t^0} \mathcal{L}(P) + r(b-a).$$

Observe that

$$\sum_{P\in\mathcal{F}_t^0} \mathcal{L}(P) = \mathcal{L}(\bigcup_{P\in\mathcal{F}_t^0} P) \geq \mathcal{L}^*(S).$$

Consequently, for $t > r$, we get

$$f(b) - f(a) \geq (t-r)\mathcal{L}^*(S) + r(b-a).$$

Letting $t \longrightarrow +\infty$, we conclude that $\mathcal{L}^*(S) = \mathcal{L}(A_{r,n} \cap (a,b)) = 0$. Hence, $\mathcal{L}(A_{r,n}) = 0$.

Step 4^o. Proof of (c).

It suffices to prove that the set $A := \{x \in \mathbb{R} : D^+f(x) < D_-f(x)\}$ is of measure zero (and then apply this result to $-f$). Observe that

$$A = \bigcup_{r\in\mathbb{Q},\ n\in\mathbb{N}} A_{r,n},$$

where

$$A_{r,n} := \{x \in \mathbb{R} : \forall_{x'\in(x-\frac{1}{n},x),\ x''\in(x,x+\frac{1}{n})} :\ \boldsymbol{\Delta} f(x,x') < r < \boldsymbol{\Delta} f(x,x'')\}.$$

It is clear that if $x, y \in A_{r,n}$, then $|x - y| \geq \frac{1}{n}$. Consequently, $A_{r,n}$ is at most countable.

Step 5^o. Proof of (d).

We have

$$E_4 \setminus E_3 = \bigcup_{\substack{r_1,r_2,r_3,r_4 \in \mathbb{Q} \\ r_1>r_2>r_3>r_4,\ n\in\mathbb{N}}} A_{r_1,r_2,r_3,r_4,n},$$

where

$$\begin{aligned} A_{r_1,r_2,r_3,r_4,n} := \{x \in \mathbb{R} : r_4 < D_- f(x) < r_3 < r_2 < D^+ f(x) < r_1, \\ \forall_{x' \in (x-\frac{1}{n}, x)} : \boldsymbol{\Delta} f(x,x') > r_4,\ \forall_{x'' \in (x, x+\frac{1}{n})} : \boldsymbol{\Delta} f(x,x'') < r_1\}. \end{aligned}$$

Fix $r_1 > r_2 > r_3 > r_4$, $n \in \mathbb{N}$, and $a, b \in A_{r_1,r_2,r_3,r_4,n}$ such that $0 < b - a < \frac{1}{n}$. Put $S := A_{r_1,r_2,r_3,r_4,n} \cap (a,b)$. In view of the proof of Step 3^o with $(r,t) = (r_4, r_2)$, we get

$$f(b) - f(a) \ge (r_2 - r_4)\mathcal{L}^*(S) + r_4(b-a).$$

Let

$$\mathcal{F} := \{[p,q] : q > p,\ [p,q] \subset (a,b),\ q \in S,\ \boldsymbol{\Delta} f(p,q) < r_3\}.$$

It is clear that $(S, \mathcal{F})$ satisfies the assumptions of the Vitali covering theorem. Thus there exists an at most countable subfamily $\mathcal{F}^0 \subset \mathcal{F}$, consisting of pairwise disjoint intervals, such that $\mathcal{L}^*\Big(S \setminus \bigcup_{P\in\mathcal{F}^0} P\Big) = 0$.

Take $P_1, \dots, P_N \in \mathcal{F}_t^0$, $P_i = [p_i, q_i]$. Then $(a,b) \setminus \bigcup_{i=1}^N P_i = \bigcup_{j=1}^M (\alpha_j, \beta_j)$, where the intervals $(\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)$ are pairwise disjoint and $\alpha_j \in A_{r_1,r_2,r_3,r_4,n}$, $j = 1, \dots, M$. In particular, $\boldsymbol{\Delta} f(\alpha_j, \beta_j) < r_1$. Consequently,

$$f(b) - f(a) \le (r_3 - r_1) \sum_{P \in \mathcal{F}_t^0} \mathcal{L}(P) + r_1(b-a) \le (r_3 - r_1)\mathcal{L}^*(S) + r_1(b-a).$$

Hence

$$\frac{\mathcal{L}^*(S)}{b-a} = \frac{\mathcal{L}^*(A_{r_1,r_2,r_3,r_4,n} \cap [a,b])}{\mathcal{L}([a,b])} \le \frac{r_1 - r_4}{r_1 - r_4 + r_2 - r_3} < 1. \tag{2.3.1}$$

Suppose that $\mathcal{L}^*(A_{r_1,r_2,r_3,r_4,n}) > 0$. Then by the Lebesgue density theorem, there exists a point $b \in A_{r_1,r_2,r_3,r_4,n}$ such that

$$\lim_{a \to b-} \frac{\mathcal{L}^*(A_{r_1,r_2,r_3,r_4,n} \cap [a,b])}{\mathcal{L}([a,b])} = 1. \tag{2.3.2}$$

In particular, in view of (2.3.1), there are no sequences $(a_s)_{s=1}^\infty \subset A_{r_1,r_2,r_3,r_4,n}$ such that $0 < b - a_s < \frac{1}{n}$ and $a_s \longrightarrow b$. Thus $A_{r_1,r_2,r_3,r_4,n} \cap (b, b - \frac{1}{s}) = \varnothing$ for $s \gg 1$, which contradicts (2.3.2). $\square$

2.4 Series of Continuous Functions

Many of the functions discussed in this book will be of the form

$$\varphi(t) := \sum_{n=0}^{\infty} \varphi_n(t), \quad t \in I,$$

where $\varphi_n : I \longrightarrow \mathbb{C}$ is continuous, $n \in \mathbb{N}_0$, and the series is *normally convergent*, i.e.,

$$A := \sum_{n=0}^{\infty} (\sup_{t \in I} |\varphi_n(t)|) < +\infty.$$

In particular, such a series is *uniformly convergent*, and therefore, the function φ is continuous. Obviously, φ is bounded and $|\varphi(x)| \leq A$, $x \in I$.

Remark 2.4.1. It is well known that if, moreover, each function $\varphi_n : I \longrightarrow \mathbb{C}$ is differentiable and the series $\sum_{n=0}^{\infty} \varphi_n'$ is uniformly convergent (e.g., normally convergent) in I, then φ is differentiable and $\varphi'(t) = \sum_{n=0}^{\infty} \varphi_n'(t)$, $t \in I$.

2.5 Hölder Continuity

Definition 2.5.1. Let $\alpha \in (0, 1]$. We say that a continuous function $\varphi : I \longrightarrow \mathbb{C}$ is:

- *α-Hölder continuous at a point* $t \in I$ ($\varphi \in \mathfrak{H}^\alpha(I; t)$) if

$$\exists_{c,\,\delta>0}\ \forall_{h \in (-\delta,\delta)\cap(I-t)} :\ |\varphi(t+h) - \varphi(t)| \leq c|h|^\alpha;$$

- *Lipschitz at a point* $t \in I$ if $\varphi \in \mathfrak{H}^1(I; t)$;
- *α-Hölder continuous* ($\varphi \in \mathfrak{H}^\alpha(I)$) if

$$\exists_{C>0}\ \forall_{t,u \in I} :\ |\varphi(u) - \varphi(t)| \leq C|u - t|^\alpha;$$

- *Lipschitz continuous* if φ is 1-Hölder continuous;
- *M-Lipschitz at a point* $t \in I$ (where $M > 0$) if

$$\forall_{u \in I} :\ |\varphi(u) - \varphi(t)| \leq M|u - t|.$$

Remark 2.5.2. (a) Observe that if $\varphi : I \longrightarrow \mathbb{C}$ is a bounded continuous function, then φ is α-Hölder continuous at t iff

$$\exists_{c>0}\ \forall_{u \in I} :\ |\varphi(u) - \varphi(t)| \leq c|u - t|^\alpha \text{ (EXERCISE)};$$

in particular, φ is 1-Hölder continuous at t iff φ is M-Lipschitz at t for some $M > 0$.

(b) If a finite derivative $\varphi'(t)$ exists, then φ is Lipschitz at t.

(c) It is known (cf. [KK96], Theorems 1.2.8, 6.1.5, 6.1.15) that if $\varphi : I \longrightarrow \mathbb{C}$ is Lipschitz continuous, then there exists a zero-measure set $S \subset I$ such that $\varphi'(t)$ exists for all $t \in I \setminus S$.

(d) Assume that I is a bounded closed interval and let T_M denote the set of all $\varphi \in \mathcal{C}(I, \mathbb{C})$ such that for every $t \in I$, the function φ is not M-Lipschitz at t. Consider $\mathcal{C}(I, \mathbb{C})$ as a metric space endowed with the distance $d(\varphi, \psi) := \max_I |\varphi - \psi|$. Then T_M is open in $\mathcal{C}(I, \mathbb{C})$[1] (EXERCISE). Consequently, the set $T := \bigcap_{M \in \mathbb{Q}_{>0}} T_M$ of all functions that are nowhere Lipschitz on I is a Borel set. Observe that $T \subset \mathfrak{ND}(I)$.

[1] Recall that a pair (X, d) is a *metric space* if $d : X \times X \longrightarrow \mathbb{R}_+$, $(d(x, y) = 0 \Longleftrightarrow x = y)$, $d(x, y) = d(y, x)$, and $d(x, y) \leq d(x, z) + d(z, y)$. A set $A \subset X$ is called *open* if for each $a \in A$, there exists an $r > 0$ such that $\{x \in X : d(x, a) < r\} \subset A$.

Definition 2.5.3. For $\alpha > 0$, we say that a continuous function $\varphi : I \longrightarrow \mathbb{C}$ is:

- *nowhere α-Hölder continuous* ($\varphi \in \mathcal{NH}^\alpha(I)$) if $\forall_{t\in I} : \ \varphi \notin \mathcal{H}^\alpha(I;t)$;
- *α-anti-Hölder continuous* if

$$\exists_{\varepsilon>0} \ \forall_{t\in I,\ \delta\in(0,1)} \ \exists_{\substack{h_\pm\in(0,\delta)\\ t\pm h_\pm\in I}} : |\varphi(t \pm h_\pm) - \varphi(t)| > \varepsilon\delta^\alpha;$$

we skip h_+ (resp. h_-) if t is the right (resp. left) endpoint of the interval;

- *weakly α-anti-Hölder continuous* if

$$\exists_{\varepsilon>0} \ \forall_{t\in I,\ \delta\in(0,1)} \ \exists_{h\in(-\delta,\delta)\cap(I-t)} : |\varphi(t + h) - \varphi(t)| > \varepsilon\delta^\alpha.$$

Remark 2.5.4. Let $\alpha \in (0, 1)$.

(a) If φ is α-anti-Hölder continuous, then $\varphi \in \mathcal{M}(I) \subset \mathcal{ND}_\pm(I)$.

(b) If φ is weakly α-anti-Hölder continuous, then φ is nowhere 1-Hölder continuous, and hence $\varphi \in \mathcal{ND}(I)$.

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