Chapter 2
Basics

Abstract This chapter is devoted to the basic features needed for Cartesian tensors: the components of a position vector with respect to a coordinate system, the scalar product of two vectors, the transformation of the components upon a change of the coordinate system. Special emphasis is put on the orthogonal transformation associated with a rotation of the coordinate system. Then tensors of rank $\ell \geq 0$ are defined via the transformation behavior of their components upon a rotation of the coordinate system, scalars and vectors correspond to the special cases $\ell = 0$ and $\ell = 1$. The importance of tensors of rank $\ell \geq 2$ for physics is pointed out. The parity and time reversal behavior of vectors and tensors are discussed. The differentiation of vectors and tensors with respect to a parameter, in particular the time, is treated.

2.1 Coordinate System and Position Vector

2.1.1 Cartesian Components

Given the origin of a coordinate system, the position of a particle or the center of mass of an extended object is specified by the position vector $\mathbf{r}$, as indicated in Fig. 2.1. In the three-dimensional space we live in, this vector has three components, often referred to as the x-, y- and z-components. We use a (space-fixed) right-handed rectangular coordinate system, also called *Cartesian coordinate system*. It is convenient to label the axes by 1, 2 and 3 and to denote the components of the position vector by r_1, r_2, and r_3. Sometimes, the vector is written as an ordered triple of the form (r_1, r_2, r_3).

For these Cartesian components of the position vector the notation r_μ is preferred, where it is understood that μ, or any other Greek letter used for the subscript, also called indices, can have the value 1, 2 or 3. Of course, the mathematical content is unaffected, when Latin letters are used as subscripts instead of the Greek ones. Here, Latin letters are reserved for the components of two- and four-dimensional vectors or for components in a coordinate system with axes which are not orthogonal.

The components of the sum $\mathbf{S} = \mathbf{r} + \mathbf{s}$ of two vectors $\mathbf{r}$ and $\mathbf{s}$, with the Cartesian components r_1, r_2, r_3 and s_1, s_2, s_3, are given by $r_1 + s_1, r_2 + s_2, r_3 + s_3$. This standard

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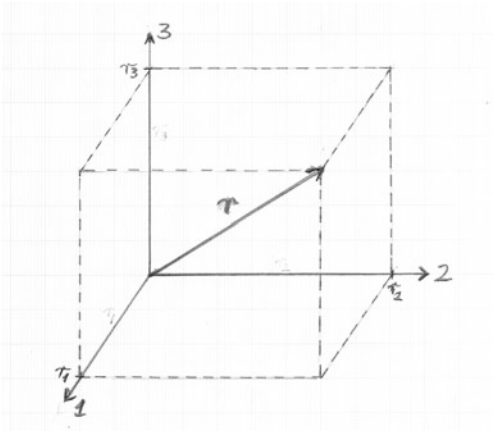

Fig. 2.1 Position vector in a Cartesian coordinate system. The *dashed lines* are guides for the eye

rule for the addition of two vectors can also be written as

$$S_\mu = r_\mu + s_\mu, \tag{2.1}$$

with $\mu = 1, 2, 3$. The multiplication of the vector $\mathbf{r}$ with a real number k, i.e. $\mathbf{R} = k\mathbf{r}$ means, that each component is multiplied by this number, viz.,

$$R_\mu = k\, r_\mu. \tag{2.2}$$

We are still dealing with the same vectors when other Greek letters, like ν, λ, ... or α, β, ... are used as subscripts.

2.1.2 Length of the Position Vector, Unit Vector

For the rectangular coordinate system, the length r of the vector $\mathbf{r}$ is given by the *Euclidian norm*:

$$r^2 = \mathbf{r} \cdot \mathbf{r} = r_1^2 + r_2^2 + r_3^2 := r_\mu\, r_\mu. \tag{2.3}$$

Thus one has

$$r = \sqrt{r_\mu\, r_\mu}. \tag{2.4}$$

The length of the vector is also referred to as its *magnitude* or its *norm*.

Here and in the following, the *summation convention* is used: Greek subscripts which occur twice are summed over. This implies that on one side of an equation, each Greek letter can only show up once or twice as a Cartesian index. Einstein introduced such a summation convention for the components of four-dimensional vectors. For this reason, also the term *Einstein summation convention*, is used.

The vector $\mathbf{r}$, divided by its length r, is the dimensionless *unit vector* $\widehat{\mathbf{r}}$:

$$\widehat{\mathbf{r}} = r^{-1}\mathbf{r}, \tag{2.5}$$

or, in component notation:

$$\widehat{r}_\mu = r^{-1}\, r_\mu. \tag{2.6}$$

The unit vector has magnitude 1:

$$\widehat{r}_\mu\, \widehat{r}_\mu = 1. \tag{2.7}$$

2.1.3 Scalar Product

The scalar product of two position vectors $\mathbf{r}$ and $\mathbf{s}$ with components r_μ and s_μ is

$$\mathbf{r} \cdot \mathbf{s} = r_1\, s_1 + r_2\, s_2 + r_3\, s_3 := r_\mu\, s_\mu. \tag{2.8}$$

Clearly, the length squared (2.3) of the vector $\mathbf{r}$ is its scalar product with itself. Just as in (2.3), the *center dot* "$\cdot$" is essential to indicate the scalar product, when the vectors are written with bold face symbols. The summation convention is used for the component notation. Notice that the "name" of the summation index does not matter, i.e. $r_\mu s_\mu = r_\nu s_\nu = r_\lambda s_\lambda$. What really matters is: a Greek letter occurs twice (and only twice) in a product.

The scalar product has a simple geometric interpretation. In general, the two vectors $\mathbf{r}$ and $\mathbf{s}$ span a plane. We choose the coordinate system such that $\mathbf{r}$ is parallel to the 1-axis and $\mathbf{s}$ is in the 1–2-plane, see Fig. 2.2. Then the components of $\mathbf{r}$ are $(r_1,\ 0,\ 0)$ and those of $\mathbf{s}$ are $(s_1,\ s_2,\ 0)$. The scalar product yields $\mathbf{r} \cdot \mathbf{s} = r_1 s_1$. The lengths of the two vectors are given by $r = r_1$ and $s = \sqrt{s_1^2 + s_2^2}$. The angle between $\mathbf{r}$ and $\mathbf{s}$ is denoted by φ, see Fig. 2.2. One has $s_1 = r\cos\varphi$, and

$$\mathbf{r} \cdot \mathbf{s} = r\, s\, \cos\varphi \tag{2.9}$$

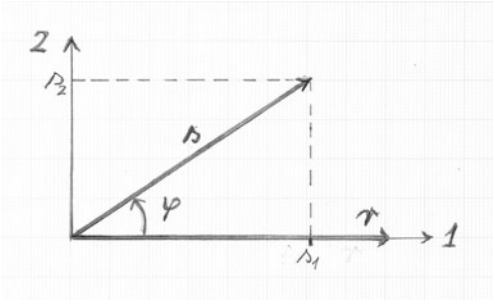

Fig. 2.2 For the geometric interpretation of the scalar product

holds true. Or in words: the scalar product of two vectors is equal to the product of their lengths times the cosine of the angle between them. The scalar product of $\mathbf{s}$ with the unit vector $\widehat{\mathbf{r}}$ is equal to $s \cos\varphi$. The vector $s \cos\varphi \widehat{\mathbf{r}} = (\mathbf{s} \cdot \widehat{\mathbf{r}})\widehat{\mathbf{r}}$ is called the *the projection of* $\mathbf{s}$ *onto the direction of* $\mathbf{r}$.

The value of the scalar product reaches its maximum and (negative) minimum when the vectors are parallel ($\varphi = 0$) and anti-parallel ($\varphi = \pi$). The scalar product vanishes for two vectors which are perpendicular to each other, i.e. for $\varphi = \pi/2$. Such vectors are also referred to as *orthogonal* vectors.

2.1 Exercise: Compute Scalar Product for Given Vectors
Compute the length, the scalar products and the angles between the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ which have the components $\{1, 0, 0\}$, $\{1, 1, 0\}$, and $\{1, 1, 1\}$.

2.1.4 Spherical Polar Coordinates

As stated before, the position vector $\mathbf{r}$ has a length, specified by its magnitude $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$, and a direction, determined by the unit vector $\widehat{\mathbf{r}}$, cf. (2.5) and (2.6). These parts of the vector are often referred to as *radial part* and *angular part*. Indeed, the unit vector and thus the direction of $\mathbf{r}$ can be specified by the two *polar angles* ϑ and φ. Conventionally, a particular coordinate system is chosen, the Cartesian coordinates $\{r_1, r_2, r_3\}$ are denoted by $\{x, y, z\}$ which, in turn, are related to the *spherical polar coordinates* r, ϑ, φ by

$$x = r \sin\vartheta \cos\varphi\,, \quad y = r \sin\vartheta \sin\varphi\,, \quad z = r \cos\vartheta. \tag{2.10}$$

Notice, the three numbers for r, ϑ, φ are not components of a vector.

The information given by the Cartesian components of a unit vector corresponds to a point on the unit sphere, identified by the two angles, similar to positions on earth. Notice, however, that the standard choice made for the angle ϑ would correspond to associate $\vartheta = 0$ and $\vartheta = 180°$ with the North Pole and the South Pole, respectively, whereas the equator would be at $\vartheta = 90°$. For positions on earth, one starts counting ϑ from zero on the equator and has to distinguish between North and South, or plus and minus. In any case, the angle spans an interval of 180°, or just π, whereas that of φ is 360°, or 2π.

2.2 Vector as Linear Combination of Basis Vectors

2.2.1 Orthogonal Basis

Examples of orthogonal vectors are the unit vectors $\mathbf{e}^{(i)}$, $i = 1, 2, 3$, which are parallel to the axes 1, 2, 3 of the Cartesian coordinate system. These vectors have the properties $\mathbf{e}^{(1)} \cdot \mathbf{e}^{(1)} = 1$, $\mathbf{e}^{(1)} \cdot \mathbf{e}^{(2)} = 0, \ldots$, in more general terms,

$$\mathbf{e}^{(i)} \cdot \mathbf{e}^{(\mathrm{j})} = \delta_{\mathrm{ij}}. \tag{2.11}$$

Here δ_{ij} is the Kronecker symbol, i.e. $\delta_{\mathrm{ij}} = 1$ for $i = j$ and $\delta_{\mathrm{ij}} = 0$ for $i \neq j$.

The position vector $\mathbf{r}$ can be written as a linear combination of these unit vectors $\mathbf{e}^{(\mathrm{i})}$ according to

$$\mathbf{r} = r_1\, \mathbf{e}^{(1)} + r_2\, \mathbf{e}^{(2)} + r_3\, \mathbf{e}^{(3)}. \tag{2.12}$$

Since the *basis vectors* are not only orthogonal, but also normalized to 1, the Cartesian components are equal to the scalar product of $\mathbf{r}$ with the basis vectors, e.g. $r_1 = \mathbf{e}^{(1)} \cdot \mathbf{r}$.

2.2.2 Non-orthogonal Basis

Three vectors $\mathbf{a}^{(\mathrm{i})}$, with $i = 1, 2, 3$, which are not within one plane, can be used as basis vectors. Then the vector $\mathbf{r}$ can be represented by the linear combination

$$\mathbf{r} = \xi^1\, \mathbf{a}^{(1)} + \xi^2\, \mathbf{a}^{(2)} + \xi^3\, \mathbf{a}^{(3)}, \tag{2.13}$$

with the coefficients ξ^{i}. Scalar multiplication of (2.13) with the basis vectors $\mathbf{a}^{(\mathrm{i})}$ yields

$$\xi_{\mathrm{i}} = \mathbf{a}^{(\mathrm{i})} \cdot \mathbf{r} = \sum_{\mathrm{j}=1}^{3} g_{\mathrm{ij}}\, \xi^{\mathrm{j}}. \tag{2.14}$$

The coefficient matrix

$$g_{\mathrm{ij}} = \mathbf{a}^{(\mathrm{i})} \cdot \mathbf{a}^{(\mathrm{j})} = g_{\mathrm{ji}}, \tag{2.15}$$

is determined by the scalar products of the basis vectors. The coefficients ξ^{i} and ξ_{i} are referred to as *contra-* and *co-variant* components of the vector in a coordinate system with axes specified by the basis vectors $\mathbf{a}^{(\mathrm{i})}$.

In this basis, the square of the length or of the magnitude of the vector is given by

$$\mathbf{r} \cdot \mathbf{r} = \sum_{\mathrm{i}} \sum_{\mathrm{j}} \xi^{\mathrm{i}} \xi^{\mathrm{j}}\, \mathbf{a}^{(\mathrm{i})} \cdot \mathbf{a}^{(\mathrm{j})} = \sum_{\mathrm{i}} \sum_{\mathrm{j}} \xi^{\mathrm{i}} \xi^{\mathrm{j}}\, g_{\mathrm{ji}} = \sum_{\mathrm{i}} \xi^{\mathrm{i}} \xi_{\mathrm{i}}. \tag{2.16}$$

The coefficient matrix g_{ij} characterizes the connection between the co- and the contravariant components and it is essential for the calculation of the norm. Thus it determines the *metric* of the coordinate system.

The geometric meaning of the two different types of components is demonstrated in Fig. 2.3 for the 2-dimensional case.

The intersection of the dashed line parallel to the 2-axis with the 1-axis marks the component ξ^1, similarly ξ^2 is found at the intersection of the 2-axis with the dashed line parallel to the 1-axis. The component ξ_1 and ξ_2 are found at the intersections

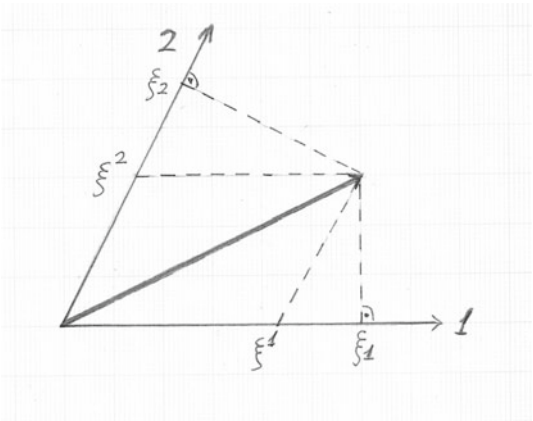

Fig. 2.3 Components of the position vector in a non-orthogonal coordinate system

of the dashed lines perpendicular to the axes. It is understood that the basis vectors along the axes, not shown in Fig. 2.3, are unit vectors.

For basis vectors which are mutually perpendicular and normalized to 1, the matrix g_{ij} reduces to unit matrix δ_{ij}. Consequently co- and the contra-variant components are equal. This is also obvious from Fig. 2.3. The two types of components coincide when the basis vectors are orthogonal. We do not have to distinguish between co- and the contra-variant components when we use the Cartesian coordinate system.

2.3 Linear Transformations of the Coordinate System

The laws of physics do not depend on the choice of a coordinate system. However, in many applications, a specific choice is made. Then it is important to know, how components have to be transformed such that the physics is not changed, when another coordinate system is chosen. Here, we are concerned with *linear transformations* where the coordinates in the new system are linked with those of the original coordinate system by a linear relation. The two types of linear transformations, *translations* and *affine transformations*, also referred to as *linear maps*, are discussed separately. The rotation of a coordinate system is a special case of an affine transformations. Due to its importance, an extra section is devoted to rotations.

2.3.1 Translation

Consider a new coordinate system, that is shifted with respect to the original one by a constant vector **a**. Such a shift is referred to as *translation of the coordinate system*. In Fig. 2.4, a translation within the 1,2-plane is depicted.

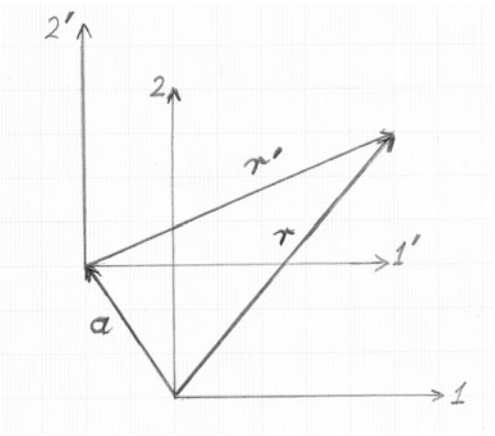

Fig. 2.4 Components of the position vector **r** in shifted coordinate system

The position vector $\mathbf{r}'$ with respect the origin of the shifted coordinate system is related to the original **r** by

$$\mathbf{r}' = \mathbf{r} - \mathbf{a}, \tag{2.17}$$

or in component notation,

$$r'_\mu = r_\mu - a_\mu. \tag{2.18}$$

The inverse transformation, which brings the shifted coordinate system back to the original one, corresponds to a shift by the vector $-\mathbf{a}$.

Notice: the translation of the coordinate system is a passive transformation, which has to be distinguished from the *active translation* of the position of a particle or of an object from **r** to $\mathbf{r} + \mathbf{a}$.

2.3.2 Affine Transformation

For an affine transformation, the components r'_1, r'_2, r'_3 of the position vector $\mathbf{r}'$ in the new coordinate system are linear combinations of the components r_1, r_2, r_3 in the original system. When the components of the vectors are written in columns, the linear mapping can be expressed in the form

$$\begin{pmatrix} r'_1 \\ r'_2 \\ r'_3 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \tag{2.19}$$

The elements T_{11}, T_{12}, ... of the matrix $\mathbf{T}$ characterize the affine transformation. The determinant of $\mathbf{T}$ must not be zero, such that the reciprocal matrix $\mathbf{T}^{-1}$ exists. Standard matrix multiplication is assumed in (2.19). This means, e.g.

$$r_1' = T_{11}\, r_1 + T_{12}\, r_2 + T_{13}\, r_3. \tag{2.20}$$

More general, for $\mu = 1, 2, 3$, one has

$$r_\mu' = T_{\mu 1}\, r_1 + T_{\mu 2}\, r_2 + T_{\mu 3}\, r_3, \tag{2.21}$$

or, with the help of the summation convention

$$r_\mu' = T_{\mu\nu}\, r_\nu. \tag{2.22}$$

Notice: in (2.22), μ is a *free index* which can have any value 1, 2 or 3. The subscript ν, on the other hand, is a summation index, for which any other Greek letter, except μ, could be chosen here.

Sometimes, the relations (2.19) or equivalently (2.22) are expressed in the form

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r}, \tag{2.23}$$

where the matrix-character of $\mathbf{T}$ is indicated by the *bold face sans serif* letter and the center dot "·" implies the summation of products of components.

Notice: in such a notation, the order of factors matters, in contradistinction to the component notation. The equation $\mathbf{r}' = \mathbf{r} \cdot \mathbf{T}$ corresponds to $r_\mu' = r_\nu T_{\nu\mu} = T_{\nu\mu} r_\nu$ which is different from (2.22), unless the transformation matrix $\mathbf{T}$ is symmetric, i.e. unless $T_{\nu\mu} = T_{\mu\nu}$ holds true.

The *inverse transformation*, also called *back-transformation*, links the components of $\mathbf{r}$ with those of $\mathbf{r}'$, according to

$$\mathbf{r} = \mathbf{T}^{-1} \cdot \mathbf{r}', \tag{2.24}$$

with the inverse transformation matrix $\mathbf{T}^{-1}$. Insertion of (2.23) into (2.24) leads to $\mathbf{r} = \mathbf{T}^{-1} \cdot \mathbf{r}' = \mathbf{T}^{-1} \cdot \mathbf{T} \cdot \mathbf{r}$ which implies

$$\mathbf{T}^{-1} \cdot \mathbf{T} = \boldsymbol{\delta}, \tag{2.25}$$

or in component notation,

$$T^{-1}_{\mu\lambda}\, T_{\lambda\nu} = \delta_{\mu\nu}. \tag{2.26}$$

Notice: here μ and ν are free indices, λ is the summation index. The symbol δ indicates the unit matrix, viz.:

$$\boldsymbol{\delta} := \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \tag{2.27}$$

or equivalently, $\delta_{\mu\nu} = 1$ for $\mu = \nu$, and $\delta_{\mu\nu} = 0$ for $\mu \neq \nu$.

Similarly, insertion of (2.24) into (2.23) leads to

$$\mathbf{T} \cdot \mathbf{T}^{-1} = \boldsymbol{\delta}, \tag{2.28}$$

or in component notation

$$T_{\mu\lambda}\, T^{-1}_{\lambda\nu} = \delta_{\mu\nu}. \tag{2.29}$$

For the affine transformation, the left-inverse and the right-inverse matrices are equal.

2.4 Rotation of the Coordinate System

2.4.1 Orthogonal Transformation

Affine transformations, which conserve the rule for the computation of the length or the norm of the position vector, and likewise the scalar product of two vectors, are of special importance. Coordinate transformations with this property are called *orthogonal transformations*. Proper rotations and rotations combined with a mirroring of the coordinate system are special cases to be discussed in detail.

The orthogonal transformations are defined by the requirement that

$$r'_\mu\, r'_\mu = r_\mu\, r_\mu, \tag{2.30}$$

where it is understood, that a relation of the form (2.23) holds true. This then is a condition on the properties of the transformation matrix $\mathbf{T}$. Here and in the following, the symbol $\mathbf{U}$ is used for the norm-conserving orthogonal transformation matrices. The letter "U" is reminiscent of "unitarian".

The property of the orthogonal matrix is inferred as follows. Use of $r'_\lambda = U_{\lambda\mu} r_\mu$ and $r'_\lambda = U_{\lambda\nu} r_\nu$ yields $r'_\lambda r'_\lambda = U_{\lambda\mu} U_{\lambda\nu} r_\mu r_\nu$. On the other hand (2.30) requires this expression to be equal to $r_\mu r_\mu = \delta_{\mu\nu} r_\mu r_\nu$. Thus one has

$$U_{\lambda\mu} U_{\lambda\nu} = \delta_{\mu\nu}. \tag{2.31}$$

Notice: here the summation index λ is the front index for both matrices $\mathbf{U}$. Reversal of the order of the subscripts yields the corresponding component of the *transposed matrix*, labelled with the superscript "T". Thus one has $U_{\lambda\mu} = U^{\mathrm{T}}_{\mu\lambda}$, and (2.31) is equivalent to

$$U^{\mathrm{T}}_{\mu\lambda}\, U_{\lambda\nu} = \delta_{\mu\nu}. \tag{2.32}$$

This orthogonality relation for the transformation matrix is equivalent to

$$\mathbf{U}^{\mathrm{T}} \cdot \mathbf{U} = 1, \tag{2.33}$$

where it is understood that the 1 on the right hand side stands for the unit matrix. Comparison of (2.32) and (2.33) with (2.26) and (2.25) reveals: the inverse $\mathbf{U}^{-1}$ of the orthogonal matrix $\mathbf{U}$ is just its transposed $\mathbf{U}^{\mathrm{T}}$:

$$\mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}, \tag{2.34}$$

or

$$U_{\mu\nu}^{-1} = U_{\nu\mu}. \tag{2.35}$$

Use of the inverse transformation in considerations similar to those which lead to (2.31) and of (2.35) yield the orthogonality relation with the summation index at the back,

$$U_{\mu\lambda} U_{\nu\lambda} = \delta_{\mu\nu}, \tag{2.36}$$

or, equivalently,

$$\mathbf{U} \cdot \mathbf{U}^{\mathrm{T}} = 1. \tag{2.37}$$

Summary

The coordinate transformation

$$r'_{\mu} = U_{\mu\nu}\, r_{\nu}, \tag{2.38}$$

where the matrix $U_{\mu\nu}$ has the property

$$U_{\mu\lambda}\, U_{\nu\lambda} = U_{\lambda\mu}\, U_{\lambda\nu} = \delta_{\mu\nu} \tag{2.39}$$

guarantees that the scalar product of two vectors (2.8) and consequently, the expression (2.4) for the length of a vector are invariant under this transformation. Furthermore, the relation (2.39) means that the reciprocal $\mathbf{U}^{-1}$ of $\mathbf{U}$ is equal to the transposed matrix $\mathbf{U}^{\mathrm{T}}$ which, in turn is defined by $U_{\mu\nu}^{\mathrm{T}} = U_{\nu\mu}$.

Simple Examples

The simplest examples for transformation matrices which obey (2.39) are $U_{\mu\nu} = \delta_{\mu\nu}$ and $U_{\mu\nu} = -\delta_{\mu\nu}$, or in matrix notation:

$$\mathbf{U} = \boldsymbol{\delta} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{U} = -\boldsymbol{\delta} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.40}$$

which, respectively, induce the identity transformation and a reversal of the directions of the coordinate axes. The latter case means a transformation to the 'mirrored' coordinate system.

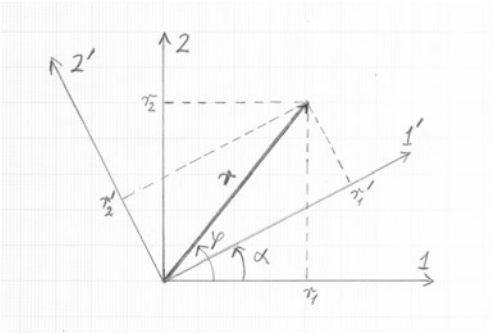

Fig. 2.5 The components of the position vector **r** in the original coordinate system and in one rotated about the 3-axis by the angle α are given by the projections of **r** on the coordinate axes 1, 2 and 1′, 2′, respectively

2.4.2 Proper Rotation

In general, an orthogonal transformation is either a proper rotation or a rotation combined with mirrored axes. The relation (2.39) implies $(\det \mathbf{U})^2 = 1$, thus $\det \mathbf{U} = \pm 1$. In the case of a *proper rotation*, the determinant "det" of the transformation matrix is equal to 1. Check the sign of the determinant for the simple matrices shown in (2.40).

An instructive nontrivial special case is the rotation of the coordinate system about one of its axes, e.g. the 3-axis as in Fig. 2.5 by an angle α. Let **r** be a vector located in the 1–2-plane, the angle between **r** and the 1-axis is denoted by φ. Then one has $r_1 = r\cos\varphi, r_2 = r\cos\varphi, r_3 = 0$, where r is the length of the vector. From the figure one infers: $r_1' = r\cos(\varphi-\alpha) = r(\cos\varphi\cos\alpha+\sin\varphi\sin\alpha) = r_1\cos\alpha+r_2\sin\alpha$ and $r_2' = r\sin(\varphi-\alpha) = r(\sin\varphi\cos\alpha-\cos\varphi\sin\alpha) = -r_1\sin\alpha+r_2\cos\alpha$; furthermore $r_3' = 0$. Thus the rotation matrix $U_{\mu\nu} = U_{\mu\nu}(3|\,\alpha)$, also denoted by $\mathbf{U}(3|\,\alpha)$, reads:

$$\mathbf{U} = \mathbf{U}(3|\,\alpha) := \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.41}$$

A glance at (2.41) shows $\mathbf{U}(3|\,-\alpha) = \mathbf{U}^{\mathrm{T}}(3|\,\alpha)$. This is expected on account of (2.39) equivalent to $\mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}$, since the rotation by the angle $-\alpha$ corresponds to the inverse transformation.

By analogy to (2.41), the transformation matrix for a rotation by the angle β about the 2-axis is

$$\mathbf{U} = \mathbf{U}(2|\,\beta) := \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}. \tag{2.42}$$

The two rotation matrices $\mathbf{U}(3|\,\alpha)$ and $\mathbf{U}(2|\,\beta)$ do not commute, i.e. one has

$$U(3|\,\alpha)_{\mu\lambda}U(2|\,\beta)_{\lambda\nu} \neq U(2|\,\beta)_{\mu\lambda}U(3|\,\alpha)_{\lambda\nu}.$$

This is explained as follows. When first a rotation $\mathbf{U}(3|\,\alpha)$ about the 3-axis is performed, the subsequent rotation induced by $\mathbf{U}(2|\,\beta)$ is about the new coordinate axis $2'$. On the other hand, the rotation $\mathbf{U}(3|\,\alpha)$, performed after the $2'$ rotation, is about the new $3'$-axis.

A general rotation about an arbitrary axis can be expressed by three successive rotations of the type (2.41) by 3 *Euler angles* about the 3-axis, the new 2-axis, and the new 3-axis, viz.: $U_{\mu\nu} = U_{\mu\lambda}(3|\,\gamma)U_{\lambda\kappa}(2|\,\beta)U_{\kappa\nu}(3|\,\alpha)$.

In most applications, it is not necessary to compute or to perform rotations explicitly. However, the behavior of the components of the position vector is essential for the definition of a vector and of a tensor, as used in physics.

2.5 Definitions of Vectors and Tensors in Physics

2.5.1 Vectors

A quantity $\mathbf{a}$ with Cartesian components a_μ, $\mu = 1, 2, 3$ is called a *vector* when, upon a rotation of the coordinate system, its components are transformed just like the components of the position vector, cf. (2.38). This means, the components a'_μ, in the rotated coordinate system, are linked with the components in the original system by

$$a'_\mu = U_{\mu\nu}\, a_\nu. \tag{2.43}$$

Here $U_{\mu\nu}$ are the elements of a transformation matrix for a proper rotation of the coordinate system.

Differentiation with respect to time t does not affect the vector character of a physical quantity. Thus the velocity $v_\mu = \mathrm{d}r_\mu(t)/\mathrm{d}t$ and the acceleration $\mathrm{d}v_\mu(t)/\mathrm{d}t$ are vectors. The linear momentum $\mathbf{p}$, being equal to the mass of a particle times its velocity, and the force $\mathbf{F}$ are vectors. This guarantees that Newton's equation of motion $\mathrm{d}\mathbf{p}/\mathrm{d}t = \mathbf{F}$, or in components

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}t} = F_\mu, \tag{2.44}$$

is form-invariant against a rotation of the coordinate system.

Warning

A rotated coordinate system must not be confused with a *rotating coordinate system*. A rotating coordinate system is an accelerated system where additional forces, like the Coriolis force and a centrifugal force, have to be taken into account in the equation of motion.

2.5.2 What is a Tensor?

Tensors are important "tools" for the characterization of anisotropies; but what is meant by the notion *tensor*? Here mainly Cartesian tensors of rank ℓ, $\ell = 0, 1, 2, \ldots$ are treated. These are quantities with ℓ indices which change in a specific way, when the coordinate system is rotated. More specifically: a Cartesian tensor of rank ℓ is a quantity with ℓ indices, e.g. $A_{\mu_1\mu_2\ldots\mu_\ell}$, whose Cartesian components $A'_{\mu_1\mu_2\ldots\mu_\ell}$ in a rotated coordinate system are obtained from the original ones by the application of ℓ rotation matrices $\mathbf{U}$ to each one of the indices, viz.:

$$A'_{\mu_1\mu_2\ldots\mu_\ell} = U_{\mu_1\nu_1} U_{\mu_2\nu_2} \ldots U_{\mu_\ell\nu_\ell}\, A_{\nu_1\nu_2\ldots\nu_\ell}. \tag{2.45}$$

In this sense, *scalars* and *vectors* are tensors of rank $\ell = 0$ and $\ell = 1$. Examples for vectors are the position vector $\mathbf{r}$ of a particle, its velocity $\mathbf{v}$, its linear momentum $\mathbf{p}$, as already mentioned before, but also its orbital angular momentum $\mathbf{L}$, its spin $\mathbf{s}$, as well as an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$.

Tensors of rank $\ell = 2$ are frequently referred to as *tensors* without indicating their rank. Examples are the moment of inertia tensor, the pressure tensor or the stress tensor. Applications will be discussed later.

A second rank tensor can also be written as a matrix. However, it is distinguished from an arbitrary 3×3-matrix by the transformation properties of its components, just as not any 3-tuple is a vector in the sense described above. Of course, the matrix notation does not work for tensors of rank 3 or of higher rank.

2.5.3 Multiplication by Numbers and Addition of Tensors

The multiplication of a tensor by real number k means the multiplication of all its elements by this number, which is almost trivial in component notation:

$$k\,(\mathbf{A})_{\mu_1\mu_2\ldots\mu_\ell} = k\, A_{\mu_1\mu_2\ldots\mu_\ell}. \tag{2.46}$$

The addition of two tensors of the same rank implies that the corresponding components are added. When a tensor $\mathbf{C}$ is said to be the sum of the tensors $\mathbf{A}$ and $\mathbf{B}$, this means:

$$C_{\mu_1\mu_2\ldots\mu_\ell} = A_{\mu_1\mu_2\ldots\mu_\ell} + B_{\mu_1\mu_2\ldots\mu_\ell}. \tag{2.47}$$

The addition of two tensors makes sense only when both have the same rank ℓ. Of course, the rank of the resulting sum is also ℓ.

Notice, though it may sound somewhat confusing, tensors of a fixed rank ℓ (with $\ell = 0, 1, 2, \ldots$) are elements of a vector space.

2.5.4 Remarks on Notation

The Cartesian components of tensors are unambiguously specified by Greek subscripts, e.g. a_μ and $a_{\mu\nu}$. As practiced already above, it is sometimes more convenient to use *boldface* and *boldface sans serif* letters, e.g. $\mathbf{a}$ and a to indicate that a quantity is a vector or (second rank) tensor. An alternative "invariant" notation for tensors of rank ℓ (which is preferred in hand writing) is to underline a letter ℓ times, e.g. $\underline{a}$ and $\underline{\underline{a}}$ for a vector and a tensor of rank 2. When Cartesian components are not written explicitly, a *center dot* $\cdot$ must be used to indicate a "contraction", i.e. a summation over indices. The scalar product $\mathbf{a}\cdot\mathbf{b} = a_\mu b_\mu$ has to be distinguished from the *dyadic product* $\mathbf{a}\,\mathbf{b}$, equivalent to $a_\mu b_\nu$, which is a second rank tensor. The scalar product of a second rank tensor with a vector, e.g. $\mathsf{C}\cdot\mathbf{b}$, equivalent to $C_{\mu\nu}b_\nu$, is a vector whose components are computed by analogy to the multiplication of a matrix with a "column vector". The quantity $\mathsf{C}\,\mathbf{b}$, on the other hand, stands for the third rank tensor $C_{\mu\nu}b_\lambda$.

The invariant notation appears to be "simpler" than the component notation. Here both notations are used. The components of Cartesian tensors are specified explicitly when new relations are introduced and when ambiguities in the order of subscripts could arise as, e.g., in the products $a_{\mu\nu}b_{\nu\mu}$ and $a_{\mu\nu}b_{\mu\nu}$ of two tensors a and b. The invariant notation is preferred only when it can be translated uniquely into the component form.

2.5.5 Why the Emphasis on Tensors?

The physical content of equations must be invariant under a rotation of the coordinate system. For the linear relation

$$b_\mu = C_{\mu\nu}\, a_\nu, \tag{2.48}$$

between two vectors $\mathbf{a}$ and $\mathbf{b}$, this implies that the components of the coefficient matrix C have to transform under a rotation like the components of a tensor of rank 2. In short, C *is* a second rank tensor. The proof is as follows. We assume that $\mathbf{a}$ and $\mathbf{b}$ are vectors. This means, in the rotated coordinate system, the components of $\mathbf{b}$ are related to the original ones by $b'_\mu = U_{\mu\lambda}b_\lambda$. Use of (2.48) leads to

$$b'_\mu = U_{\mu\lambda}\, C_{\lambda\kappa}\, a_\kappa .$$

The components of $\mathbf{a}$ are related to those of $\mathbf{a}'$ by $a_\kappa = U^{-1}_{\kappa\nu}a'_\nu = U_{\nu\kappa}a'_\nu$. In the last equality it has been used that the inverse and the transposed of the transformation matrix U, cf. (2.35), are equal. Insertion into the previous equation leads to $b'_\mu = U_{\mu\lambda}\, C_{\lambda\kappa}U_{\nu\kappa}a'_\nu$, which is equivalent to

$$b'_\mu = C'_{\mu\nu}\, a'_\nu, \tag{2.49}$$

with the quantity $\mathbf{C}'$ linked with $\mathbf{C}$ by

$$C'_{\mu\nu} = U_{\mu\lambda} U_{\nu\kappa}\, C_{\lambda\kappa}. \tag{2.50}$$

The relation (2.50) proves: $\mathbf{C}$ is a second rank tensor.

Examples for linear relations like (2.48) are those between the angular momentum and the angular velocity of a solid body, where the moment of inertia tensor occurs, and between the electric polarization and the electric field in a "linear medium". Here, the susceptibility tensor plays the role of $\mathbf{C}$.

Similarly, the linear relation $b_{\mu\nu} = C_{\mu\nu\lambda\kappa} a_{\lambda\kappa}$ between two second rank tensors $\mathbf{a}$ and $\mathbf{b}$ implies that, in this case, $\mathbf{C}$ is a tensor of rank 4. The elasticity and the viscosity tensors linking the stress tensor or the pressure tensor with the gradient of the displacement and of the velocity field, respectively, are of this type.

The generalization of (2.48) is a linear relation between a tensor $\mathbf{b}$ of rank ℓ with tensor $\mathbf{a}$ of rank k of the form

$$b_{\mu_1\mu_2\ldots\mu_\ell} = C_{\mu_1\mu_2\ldots\mu_\ell\,\nu_1\nu_2\ldots\nu_k}\, a_{\nu_1\nu_2\ldots\nu_k}. \tag{2.51}$$

Here $\mathbf{C}$ is a tensor of rank $\ell + k$.

In physics, examples for linear relations linking tensors of rank 1 with tensors of rank 1, 2, 3 and of tensors of 2 with tensors of rank 1, 2, 3 of tensors, and so on, were already discussed over hundred years ago in the book *Lehrbuch der Kristallphysik* where Woldemar Voigt introduced the notion *tensor*.

The relation (2.51) is a *linear mapping* of $\mathbf{a}$ on $\mathbf{b}$. Nevertheless, the physical content may describe non-linear effects, when the tensor $\mathbf{a}$ stands for a product of tensors. Examples occur in non-linear optics. For strong electric fields, the induced electric polarization contains not only the standard term linear in the field, but also contributions bilinear and of third order in the electric field. The material coefficient characterizing these effects, called higher order susceptibilities, are tensors of rank 3 and 4.

2.6 Parity

2.6.1 Parity Operation

In addition to their rank, tensors are classified by their *parity*. The parity is either equal to 1 or -1 when the physical quantity considered is an eigenfunction of the *parity operator* $\mathscr{P}$. The *parity operation* is an active transformation where the position vector $\mathbf{r}$ is replaced by $-\mathbf{r}$, cf. Fig. 2.6. This active 'mirroring' should not be confused with the mirroring of the coordinate system as described by the transformation matrix $U_{\mu\nu} = -\delta_{\mu\nu}$.

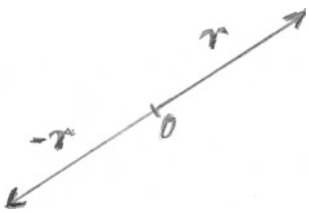

Fig. 2.6 Parity operation: $\mathbf{r} \rightarrow -\mathbf{r}$

The parity operator $\mathscr{P}$, when applied on any function $f(\mathbf{r})$, yields $f(-\mathbf{r})$:

$$\mathscr{P} f(\mathbf{r}) = f(-\mathbf{r}). \tag{2.52}$$

Clearly, one has $\mathscr{P} f(-\mathbf{r}) = f(\mathbf{r})$ and consequently

$$\mathscr{P}^2 = 1, \tag{2.53}$$

or $(\mathscr{P} - 1)(\mathscr{P} + 1) = 0$. Thus the eigenvalues of the parity operator are

$$P = \pm 1. \tag{2.54}$$

Usually eigenfunctions are referred to as having *positive* or *negative* parity, when $P = 1$ and $P = -1$, respectively, applies.

2.6.2 Parity of Vectors and Tensors

In most applications tensors, and this includes vectors, are eigenfunction of the parity operator. Tensors of rank ℓ with

$$\mathscr{P} = (-1)^{\ell} \tag{2.55}$$

are called *proper tensors*, those with

$$\mathscr{P} = -(-1)^{\ell} = (-1)^{\ell+1} \tag{2.56}$$

are referred to as *pseudo tensors*.

For vectors ($\ell = 1$), also the terms *polar vector* and *axial vector* are used to distinguish between proper and pseudo vectors. Examples for polar vectors are the linear momentum $\mathbf{p}$ of a particle and the electric field, whereas the angular momentum and the magnetic field are axial vectors, as will be discussed later.

2.6.3 Consequences for Linear Relations

The electromagnetic interaction underlying all relevant interactions encountered in every days life, i.e. in gases, liquids and solids, is invariant under the parity operation. The equations governing physical properties and phenomena must not violate this parity invariance. This means, for example, when the vector **b** in the relation $b_\mu = C_{\mu\nu} a_\nu$ has the parity -1 (polar vector), the vector **a** and the tensor **C** must have the parities -1 and 1 (polar vector and proper tensor) or 1 and -1 (axial vector and pseudo tensor). More general, let P_a, P_b, P_C the values of the parities of the tensors **a**, **b**, **C** in the linear relation (2.51). Parity invariance requires

$$P_\mathrm{b} = P_\mathrm{C}\, P_\mathrm{a}. \tag{2.57}$$

Likewise, when the parities of **a** and **b** are given by their physical meaning, the coefficient tensor **C** must have the parity

$$P_\mathrm{C} = P_\mathrm{a}\, P_\mathrm{b}, \tag{2.58}$$

in order that the linear relation (2.51) does not violate parity.

2.6.4 Application: Linear and Nonlinear Susceptibility Tensors

The electric field **E**, the electric displacement field **D** and the electric polarization **P** used in *electrodynamics* are polar vectors. They are linked by the general relation

$$\mathbf{D} = \varepsilon_0\, \mathbf{E} + \mathbf{P},$$

where ε_0 is the electric permeability coefficient of the vacuum. In a material, called *linear medium*, the electric polarization is linearly related to the electric field, according to

$$P_\mu = \varepsilon_0\, \chi_{\mu\nu}\, E_\nu,$$

where $\chi_{\mu\nu}$ is the *linear susceptibility tensor*. In the special case of a linear medium, one has

$$D_\mu = \varepsilon_0\, \varepsilon_{\mu\nu}\, E_\nu, \quad \varepsilon_{\mu\nu} = \varepsilon_0\, (\delta_{\mu\nu} + \chi_{\mu\nu}),$$

with the dimensionless dielectric tensor $\varepsilon_{\mu\nu}$. In general, in particular for strong electric fields as, e.g. encountered in a (focussed) laser beam, terms nonlinear in the electric field give significant contributions to the electric polarization. Up to third order in the electric field, the electric polarization is given by

$$P_\mu = \varepsilon_0 \left(\chi^{(1)}_{\mu\nu} E_\nu + \chi^{(2)}_{\mu\nu\lambda} E_\nu E_\lambda + \chi^{(3)}_{\mu\nu\lambda\kappa} E_\nu E_\lambda E_\kappa + \ldots\right). \tag{2.59}$$

Here $\chi^{(1)}_{\mu\nu} \equiv \chi_{\mu\nu}$ is the linear susceptibility tensor. The third and fourth rank tensors $\chi^{(2)}_{\mu\nu\lambda}$ and $\chi^{(3)}_{\mu\nu\lambda\kappa}$ characterize the second and third order susceptibilities. In optics, these terms are responsible for the second and third harmonics generation, where a part of the incident light with frequency ω is converted into light with the frequencies 2ω and 3ω, respectively.

Both the electric field and the electric polarization have negative parity. Conservation of parity enforces that the linear and the third order susceptibility tensors must have positive parity, i.e. they are proper tensors of rank 2 and 4, respectively. In the simple case of an *isotropic medium*, these tensors reduce to $\chi^{(1)}_{\mu\nu} = \chi^1 \delta_{\mu\nu}$ and $\chi^{(3)}_{\mu\nu\lambda\kappa} = \chi^3 \delta_{\mu\nu}\delta_{\lambda\kappa}$, with (proper) scalar coefficients χ^1 and χ^3. The second order susceptibility, underlying the second harmonic generation (and also the generation of a zero frequency field), must have negative parity. This can be provided by a polar vector $\mathbf{d}$ in the medium, such as dipole moment or internal electric field, or even by the vector normal to a surface. Then the second order susceptibility tensor $\chi^{(2)}_{\mu\nu\lambda}$ will contain contributions proportional to $\mathrm{d}_\mu \delta_{\nu\lambda}$ and to $\delta_{\mu\nu} d_\lambda$.

Notice: as far as the tensor algebra is concerned, the terms nonlinear in the electric field in (2.59) still are "linear relations" between P_μ and the tensors $E_\nu E_\lambda$ and $E_\nu E_\lambda E_\kappa$, which are of second and third order in the components of the electric field vector.

2.7 Differentiation of Vectors and Tensors with Respect to a Parameter

2.7.1 Time Derivatives

Just like scalars, vectors and tensors can dependent on parameters. In most applications in physics, one deals with functions of the time t. The time derivative of a tensor $\mathbf{A}$ is a tensor again. It is defined as the time derivatives of all its components, viz.,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}\right)_{\mu\nu\ldots} \equiv (\dot{\mathbf{A}})_{\mu\nu\ldots} = \frac{\mathrm{d}}{\mathrm{d}t} A_{\mu\nu\ldots}. \tag{2.60}$$

It is recalled that the tensor character of a quantity is intimately linked with the transformation behavior of its components under a rotation of the coordinate system, cf. (2.45). Since the transformation matrix $\mathbf{U}$ is "timeless", the differentiation with respect to time and the rotation of the coordinate system commute. Thus the time derivative of a tensor of rank ℓ obeys the same transformation rules, it is also a tensor of rank ℓ.

The parity operation is also timeless. Thus it commutes with the differentiation with respect to time. Consequently, the time derivative $\dot{\mathbf{A}}$ of a tensor has the same parity as the original tensor $\mathbf{A}$.

In short: neither the property of a physical quantity being a tensor, nor its parity behavior are affected by differentiating it with respect to time.

2.7.2 Trajectory and Velocity

The trajectory of a mass point or of the center of mass of any solid object is described by the time dependence of its position vector $\mathbf{r} = \mathbf{r}(t)$, or equivalently, $r_\mu = r_\mu(t)$, $\mu = 1, 2, 3$. The velocity $\mathbf{v}$ is defined by

$$v_\mu = \frac{\mathrm{d}}{\mathrm{d}t} r_\mu \equiv \dot{r}_\mu. \tag{2.61}$$

The velocity is a polar vector, just as the position vector.

The unit vector

$$\widehat{v}_\mu = v^{-1}\, v_\mu = (\dot{r}_\nu \dot{r}_\nu)^{-1/2}\, \dot{r}_\mu, \tag{2.62}$$

points in the direction of the tangent of the curve describing the trajectory. It is referred to as *tangential vector*.

Two simple types of motion are considered next.

1. **Motion along a straight line**. The trajectory is determined by

   $$r_\mu(t) = r_\mu^0 + f(t)\, e_\mu,$$

   where r_μ^0 and the unit vector e_μ are constant. The differentiable function $f(t)$ is assumed to be equal to zero for $t = 0$, then $r_\mu(0) = r_\mu^0$. For $r_\mu^0 = 0$, the line runs through the origin. The resulting velocity is

   $$v_\mu(t) = \frac{\mathrm{d}f}{\mathrm{d}t}\, e_\mu.$$

   Here, one has $\widehat{v}_\mu = e_\mu = \text{const.}$ and $v = \dot{f}$. For a *straight uniform motion*, not only the direction of the velocity, but also the speed v is constant. Then $f(t) = v\, t$ hold true.
2. **Motion on a circle**. The motion on a circle with the radius R and the angular velocity w is described by

   $$r_1 = R\cos(wt), \quad r_2 = R\sin(wt), \quad r_3 = 0,$$

   where, obviously, the circle lies in the 1–2-plane. The origin of the coordinate system is the center of the circle. At time $t = 0$, the position vector points in the

1-direction. For $w > 0$, the motion runs counterclockwise, i.e. in the mathematically positive sense. Assuming $R = \text{const.}$ and $w = \text{const.}$, the components of the velocity are

$$v_1 = -Rw\,\sin(wt), \quad v_2 = Rw\,\cos(wt), \quad v_3 = 0.$$

In this case, the velocity is perpendicular to the position vector, it is purely tangential.

2.7.3 Radial and Azimuthal Components of the Velocity

The position vector $r_\mu(t)$ can be written as a product of its magnitude $r = (r_\nu r_\nu)^{1/2}$ and the unit vector $\widehat{r}_\mu(t)$, according to $r_\mu = r\widehat{r}_\mu$. Then one has

$$\frac{\mathrm{d}}{\mathrm{d}t} r_\mu = \frac{\mathrm{d}r}{\mathrm{d}t}\widehat{r}_\mu + r\frac{\mathrm{d}}{\mathrm{d}t}\widehat{r}_\mu. \tag{2.63}$$

The *radial* component of the velocity is the first term on the right hand side of (2.63), which is parallel to the position vector $\mathbf{r}$. It describes the change of the length of $\mathbf{r}$. The second term, associated with the change of the direction of $\mathbf{r}$, is called the *azimuthal component*, sometimes also the *tangential component* of the velocity, because it is perpendicular to $\mathbf{r}$. This can be seen quickly as follows. Notice that $\widehat{r}_\nu\widehat{r}_\nu = 1$. The time derivative of this equation yields $2\widehat{r}_\nu \frac{\mathrm{d}}{\mathrm{d}t}\widehat{r}_\nu = 0$, which implies that the derivative of the radial unit vector $\widehat{\mathbf{r}}$ is perpendicular to $\widehat{\mathbf{r}}$. Alternatively, the definition of the unit vector, viz., $\widehat{r}_\mu = r_\mu\, r^{-1} = r_\mu (r_\nu r_\nu)^{-1/2}$ and the chain rule can be used to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{r}_\mu = r^{-1}\,\frac{\mathrm{d}}{\mathrm{d}t} r_\mu - r^{-3} r_\mu r_\nu\,\frac{\mathrm{d}}{\mathrm{d}t} r_\nu = r^{-1}\,(\delta_{\mu\nu} - \widehat{r}_\mu\widehat{r}_\nu)\,v_\nu. \tag{2.64}$$

The projection tensor $\delta_{\mu\nu} - \widehat{r}_\mu\widehat{r}_\nu$ guarantees that $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{r}_\mu$ is perpendicular to $\widehat{r}_\mu$.

Notice that the word *tangential* is used with two slightly different meanings, which only coincide for the motion on a circle. In one case it refers to the tangent of a trajectory which points in the direction of the velocity. In the second case, just discussed here, where the word "azimuthal" is more appropriate, it means the direction perpendicular to the position vector.

2.8 Time Reversal

The trajectory of a particle or of the center of mass of an extended object is described by the time dependence of the position vector $\mathbf{r} = \mathbf{r}(t)$. One may ask the question: does the trajectory $\mathbf{r}(-t)$ also describe a physically possible motion? In other words,

does physics allow the backward motion just as well as the forward motion. If the answer to this question is "yes", the motion is called *reversible*, otherwise it is referred to as *irreversible*. In movies and in computer simulations, one can let the time run backwards. In real physics, just as in real life, this is not possible. On the other hand, physics deals both with reversible processes, like the celestial motion of a planet around the sun and with irreversible processes, like an earthly motion, damped by friction. It is desirable to know, whether the equations governing the dynamics, describe a reversible or an irreversible behavior, even before these equations are solved. This can be found out by inspecting the *time reversal behavior* of all terms in the relevant equations.

The time reversal behavior of a physical quantity is called *even* or *odd*, or also denoted by plus + or minus −, depending on whether the *time reversal operator*, applied on this quantity, leaves it unchanged or changes its sign. The time reversal operator does not change the position vector $\mathbf{r}$. Application to the velocity $\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}$ yields $-\mathbf{v}$. More generally, the first derivative of a physical variable has a time reversible behavior, which is just opposite to that of the original variable. Clearly, the acceleration $\mathbf{a} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}$ is even under time reversal.

The idea behind these considerations is as follows: observe a process, e.g. the trajectory of a particle, from time $t = 0$ to the time t_{obs}, then change the sign of the velocity and of all relevant variables, which are odd under the time reversal operation and let the time run forward till $2t_{\mathrm{obs}}$. When the process comes back to the original state, e.g. a particle runs back to its initial position, the process is called *reversible*. If the process does not return to its original state, it is called *irreversible*. When all physical variables in an equation governing the dynamics of a process have the same time reversal behavior, *time reversal invariance* is obeyed, otherwise the time reversal invariance is violated. A simple example is Newton's equation of motion for a single particle. Mass times acceleration is even under time reversal. When the force is just a function of the position vector, it is also even and, as a consequence, the equation describes a reversible dynamics. When, on the other hand, the force has a frictional contribution proportional to the velocity, the equation of motion involves terms with different time reversal behavior, the motion is irreversible. The motion is damped provided that the friction coefficient has the correct sign.

To distinguish in the theoretical description between reversible and irreversible phenomena, it is important to know the time reversal behavior of vectors and tensors used in physics. As already mentioned, the position vector $\mathbf{r}$ is not affected by the time reversal operator, the velocity $\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}$, however, changes sign, when t is replaced by $-t$. Likewise, the linear momentum $\mathbf{p} = m\mathbf{v}$, and also the orbital angular momentum, as discussed later, are odd under time reversal. The acceleration, being the second derivative of $\mathbf{r}$ with respect to time, is even under time reversal.

In Table 2.1, parity and the time reversible behavior of some vectors are indicated by plus or minus. The parity of all these vectors is uniquely determined. This is also true for the time reversal behavior of $\mathbf{r}$, $\mathbf{v}$, $\mathbf{p}$ and of the acceleration $\mathbf{a}$, of the angular velocity $\mathbf{w}$, and of the orbital angular momentum $\mathbf{L}$. As will be discussed later, this also applies for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. When the time

Table 2.1 The parity and time-reversal behavior of some vectors

Physical quantity	**r**	**v**	**p**	**a**	**F**	**w**	**L**	**T**	**E**	**B**
Parity	−	−	−	−	−	+	+	+	−	+
Time reversal	+	−	−	+	±	−	−	±	+	−

reversal behavior of the force **F** and of the torque **T** are positive, the dynamics is reversible. Forces and torques, however, contain terms with the other time reversal behavior, when friction plays a role. Then the dynamics is irreversible. The time reversal behavior of tensors occurring in applications will be discussed later.

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