

Preface

This set of notes corresponds to the lectures of a postgraduate short course given by the author at the BCAM—Basque Centre for Applied Mathematics at Bilbao of Spain in early May 2013.

The purpose of these notes is to give a quick and elementary, yet rigorous, presentation of the rudiments of the so-called theory of Viscosity Solutions which applies to fully nonlinear first- and second- order Partial Differential Equations (PDE). For such equations, particularly for second- order ones, solutions generally are nonsmooth and standard approaches in order to define a “weak solution” do not apply: classical, strong almost everywhere (a.e.), weak, measure-valued and distributional solutions either do not exist or may not even be defined. The main reason for the latter failure is that, the standard idea of using “integration-by-parts” in order to pass derivatives to smooth test functions by duality, is not available for non-divergence structure PDE.

The name of this theory originates from the “vanishing viscosity method” developed first for first- order fully nonlinear PDE (Hamilton-Jacobi PDE). Today, though, it comprises an independent theory of “weak” solutions which applies to fully nonlinear elliptic and parabolic PDE and in most cases has no or little relation to the idea of adding a viscosity term. The formal notions have been introduced by P.L. Lions and M.G. Crandall in the early 1980s for first- order PDE, following preceding contributions of L.C. Evans. The extension to the case of second- order PDEs came around the 1990s by H. Ishii and P.L. Lions.

Interesting PDE to which the theory applies arise in Geometry and Geometric Evolution (Monge-Ampère PDE, Equations of Motion by Mean Curvature), Optimal Control and Game Theory (Hamilton-Jacobi-Bellman PDE, Isaacs PDE, Differential Games) and Calculus of Variations in L^p and L^∞ (Euler-Lagrange PDE, p -Laplacian, Aronsson PDE, ∞ -Laplacian).

Due to the vastness of the subject, a drastic choice of material is required for a brief and elementary introduction to the subject of Viscosity Solutions. In the case at hand, our main criterion has been nothing more but personal taste. Hence, herein, we shall restrict ourselves to the second- order degenerate elliptic case, focusing in particular on applications in the modern field of Calculus of Variations in L^∞ .

An inspection of the Table of Contents gives an idea about the organisation of the material. A rather immediate observation of the expert is that, unlike most standard texts on Viscosity Solutions where Uniqueness and Comparison form the centre of gravity of the exposition, herein they appear rather late in the presentation and are not overemphasised. This shift of viewpoint owes to that the author is directed mostly towards extensions of the theory to the vector case of systems. In this realm, the primary focus switches to existence methods, while comparison and uniqueness are not true in general, not even in the most ideal cases.

Throughout these notes, no previous knowledge is assumed on the reader's behalf. Basic graduate-level mathematical maturity suffices for the reading of the first six chapters, which is the general theory. For the next two chapters which concern applications to Calculus of Variations, some familiarity with weak derivatives and functionals is assumed, which does not go much deeper than the definitions. The last chapter collects, mostly without proofs, extensions and perhaps more advanced related topics which we were not able to cover in detail in this introduction to the subject.

There exist several excellent expository texts in the literature on Viscosity Solutions and Calculus of Variations in L^∞ . In particular, we would like to point out [CIL, C2, L1, Ko, C1, B, G, MR, Dr]. However, all the sources we are aware of are either of more advanced level, or have a different viewpoint. We believe that the main contribution of the present text is *elementary and is mostly addressed to students and non-experts*. This “textbook style” is reflected also in the fact that references do not appear in the main text. Most of the results are not optimal, but instead there has been a huge, and hopefully successful effort for the main ideas to be illustrated as clearly as possible. We hope that these notes serve as a suitable *first reading* on the theory of Viscosity Solutions.

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