Chapter 5
Stochastic Partial Differential Equations in Hilbert Spaces

In this chapter we study partial differential equations. It is well known [83] that finite dimensional partial differential equations lead to infinite-dimensional ordinary differential equations in the deterministic case involving unbounded operators. The solutions of these can be studied by semigroup methods. However, one has to distinguish between classical solutions and so-called mild solutions. In the stochastic case involving Gaussian noise they are studied in the book [34]. In order to keep our presentation self-contained, we describe in the next section the basic theory of semigroups and how it is used in solving deterministic partial differential equations. This material is taken from [83], where the complete proofs can be found.

5.1 Elements of Semigroup Theory

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be Banach spaces and $\mathbf{L}(E/F)$ be the space of bounded linear operators from E to F. It is known that $\mathbf{L}(E/F)$ is a Banach space, when equipped with the norm

$$\|T\|_{\mathbf{L}(E/F)} = \sup_{\|x\|_E=1} \|Tx\|_F, \quad T \in \mathbf{L}(E/F). \tag{5.1.1}$$

We denote by $\mathbf{L}(F)$= $\mathbf{L}(F/F)$ and by $Id \in \mathbf{L}(F)$ the identity operator.

For $T \in \mathbf{L}(E/F)$, we recall $T^\star \in \mathbf{L}(F^\star, E^\star)$ defined by $\langle x, T^\star y^\star\rangle = \langle Tx, y^\star\rangle$, $x \in E$, $y^\star \in F^\star$, is the adjoint operator of T. If $E = F = H$ is a Hilbert space, the operator T is said to be symmetric if $T = T^\star$, and is non-negative if $\forall h \in H$, $\langle Th, h\rangle \geq 0$.

Definition 5.1.1 A family $\{(S_t), t \geq 0\} \subset \mathbf{L}(E)$ is called a strongly continuous semigroup (C_0-semigroup for short) if the following properties hold:

- $S_0 = \mathrm{Id}$;
- *(semigroup property)* $S_{s+t} = S_s S_t$ *for all* $s, t \geq 0$;
- *(strong continuity property)* $\lim_{t\to 0} S_t x = x$ *for all* $x \in E$.

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Let $\{S_t\}$ be a C_0-semigroup in a Banach space E. Then there exist constants $\alpha \geq 0$ and $M \geq 1$ such that

$$\|S_t\|_{\mathbf{L}(E)} \leq Me^{\alpha t} \quad t \geq 0. \tag{5.1.2}$$

If $M = 1$, then $\{S_t\}$ is called a "pseudo-contraction semigroup". If $\alpha = 0$ then $\{S_t\}$ is said to be "uniformly bounded" and if $\alpha = 0$ and $M = 1$, then $\{S_t\}$ is called a "contraction semigroup". If for every $x \in E$, $t \to S_t x$ is differentiable for $t > 0$, then $\{S_t\}$ is called a "differentiable semigroup".

Note that for a C_0-semigroup, $t \to S_t x$ is continuous for $x \in E$.

Definition 5.1.2 Let $\{S_t\}$ be a C_0-semigroup on E. The linear operator A with domain

$$\mathcal{D}(A) := \{x \in E, \lim_{t \to 0^+} \frac{S_t x - x}{t} \quad exists\}$$

defined by

$$Ax = \lim_{t \to 0^+} \frac{S_t x - x}{t}$$

is called the infinitesimal generator of $\{S_t\}$.

We call $\{S_t\}$ "uniformly continuous" if $\lim_{t \to 0^+} \|S_t - I\|_{\mathbf{L}(E)} = 0$. In this case $\{S_t\}$ is uniformly continuous iff A is bounded and

$$S_t = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

with the convergence in norm for every $t \geq 0$.

Theorem 5.1.3 *Let A be the infinitesimal generator of a C_0-semigroup $\{S_t\}$ on E, then*

(1) *For $x \in E$*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S_s x ds = S_t x.$$

(2) *For $x \in \mathcal{D}(A)$, $S_t x \in \mathcal{D}(A)$ and*

$$\frac{d}{dt} S_t x = AS_t x = S_t Ax.$$

(3) *For* $x \in E$, $\int_0^t S_s x ds \in \mathcal{D}(A)$ *and*

$$A \int_0^t S_s x ds = S_t x - x.$$

(4) *If* $\{S_t\}$ *is differentiable then for* $n = 1, 2, \ldots$, $S_t : E \to \mathcal{D}(A^n)$ *and* $S_t^n := A^n S_t \in \mathbf{L}(E)$.

(5) *For* $x \in \mathcal{D}(A)$

$$S_u x - S_t x = \int_t^u S_s A x ds = \int_t^u A S_t x ds.$$

(6) $\mathcal{D}(A)$ *is dense in* E *and* A *is a closed operator.*

Furthermore $\cap_n \mathcal{D}(A^n)$ is dense in E, and if E is reflexive, then the adjoint semigroup $\{S_t^*\}$ of $\{S_t\}$ is a C_0-semigroup with infinitesimal generator A^*, the adjoint of A.

We shall be dealing with $E = H$, a real separable Hilbert space. In this case, for $h \in H$, we define the graph norm

$$|||h|||_{\mathcal{D}(A)} := (\|h\|_H^2 + \|Ah\|_H^2)^{1/2}.$$

Then $(\mathcal{D}(A), ||| \cdot |||_{\mathcal{D}(A)})$ is a real separable Hilbert space.

Exercise Let A be a closed linear operator on a real separable Hilbert space. Prove that $(\mathcal{D}(A), ||| \cdot |||_{\mathcal{D}(A)})$ is a real separable Hilbert space.

Let $\mathcal{B}(H)$ be the Borel σ-field on H. Then $\mathcal{D}(A) \in \mathcal{B}(H)$ and

$$A : (\mathcal{D}(A), \mathcal{B}(H)|_{\mathcal{D}(A)}) \to (H, \mathcal{B}(H)).$$

Consequently, $\mathcal{B}(H)|_{\mathcal{D}(A)}$ coincides with the Borel σ-field on the Hilbert space $(\mathcal{D}(A), ||| \cdot |||_{\mathcal{D}(A)})$.

Measurability of $\mathcal{D}(A)$-valued functions can be understood with respect to either of the two σ-fields.

Theorem 5.1.4 *Let* $f : [0, T] \to \mathcal{D}(A)$ *be measurable and let* $\int_0^t |||f(s)|||_{\mathcal{D}(A)} ds < \infty$. *Then*

$$\int_0^t f(s) ds \in \mathcal{D}(A) \quad \text{and} \quad \int_0^t A f(s) ds = A \int_0^t f(s) ds.$$

Now we introduce the concept of the resolvent of A, which is needed for Yosida approximation.

Definition 5.1.5 The resolvent set $\rho(A)$ of a closed linear operator A on a Banach space E is the set of all complex numbers λ for which $\lambda I - A$ has a bounded inverse $R(\lambda, A) := (\lambda I - A)^{-1} \in \mathbf{L}(E)$. The family of linear operators $R(\lambda, A)$, $\lambda \in \rho(A)$, is called the resolvent of A.

We note that $R(\lambda, A) : E \to \mathcal{D}(A)$ is one-to-one, i.e.

$$\begin{aligned} & (\lambda I - A)R(\lambda, A)x = x, \quad x \in E \\ \text{and} \quad & R(\lambda, A)(\lambda I - A)x = x, \quad x \in \mathcal{D}(A), \\ \text{giving} \quad & AR(\lambda, A)x = R(\lambda, A)Ax, \quad x \in \mathcal{D}(A). \end{aligned}$$

Exercise Show that $R(\lambda_1, A)R(\lambda_2, A) = R(\lambda_2, A)R(\lambda_1, A)$ for $\lambda_1, \lambda_2 \in \mathcal{D}(A)$.

Lemma 5.1.6 *Let $\{S_t\}$ be a C_0-semigroup with infinitesimal generator A. Let*

$$\alpha_0 := \lim_{t\to\infty} t^{-1} ln(\|S_t\|_{\mathbf{L}(E)}),$$

then any real number $\lambda > \alpha_0$ belongs to the resolvent set $\rho(A)$ and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S_t x dt \quad x \in E.$$

In addition, for $x \in E$

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda, A)x - x\|_E = 0.$$

Theorem 5.1.7 (Hille–Yosida Theorem) *Let $A : \mathcal{D}(A) \subset E \to E$ be a linear operator on a Banach space E. Necessary and sufficient conditions for A to generate a C_0-semigroup are*

(1) *A is closed and $\overline{\mathcal{D}(A)} = E$.*
(2) *There exist $\alpha, M \in \mathbb{R}$ such that for $\lambda > \alpha$, $\lambda \in \rho(A)$*

$$\|R(\lambda, A)^r\|_{\mathbf{L}(E)} \le M(\lambda - \alpha)^{-r}, \quad r = 1, 2, \ldots$$

In this case $\|S_t\|_{\mathbf{L}(E)} \le Me^{\alpha t}$, $t \ge 0$.

For $\lambda \in \rho(A)$, consider the family of operators

$$R_\lambda := \lambda R(\lambda, A).$$

Since the range $\mathcal{R}(R(\lambda, A))$ of $R(\lambda, A)$ is such that $\mathcal{R}(R(\lambda, A)) \subset \mathcal{D}(A)$, we define the "Yosida approximation" of A by

$$A_\lambda x = AR_\lambda x, \quad x \in E.$$

Exercise Use $\lambda(\lambda I - A)R(\lambda, A) = \lambda I$ to prove

$$A_\lambda x = \lambda^2 R(\lambda, A) - \lambda I.$$

From the exercise, $A_\lambda \in \mathbf{L}(E)$. Denote by S_t^λ the uniformly continuous semigroup

$$S_t^\lambda x = e^{tA_\lambda} x, \quad x \in E.$$

Using the commutativity of the resolvent, we get $A_{\lambda_1} A_{\lambda_2} = A_{\lambda_2} A_{\lambda_1}$, and clearly

$$A_\lambda S_t^\lambda = S_t^\lambda A_\lambda.$$

Theorem 5.1.8 (Yosida approximation) *Let A be an infinitesimal generator of a C_0-semigroup $\{S_t\}$ on a Banach space E. Then*

(a) $\lim_{\lambda \to \infty} R_\lambda x = x, \quad x \in E.$
(b) $A_\lambda x = Ax, \quad$ *for* $\quad x \in \mathcal{D}(A).$
(c) $\lim_{\lambda \to \infty} S_t^\lambda x = S_t x, \quad x \in E.$

The convergence in (c) is uniform on compact subsets of $\mathbb{R}_+$ and

$$\|S_t^\lambda\|_{\mathbf{L}(E)} \leq M \exp\left(\frac{t \wedge \alpha}{(\lambda - \alpha)}\right)$$

with constants M and α as in the Hille–Yosida Theorem.

We conclude this section by introducing the concept of a "mild" solution. Let us look at the deterministic problem

$$\frac{du(t)}{dt} = Au(t), \ u(0) = x, \quad x \in H.$$

Here H is a real separable Hilbert space and A is an unbounded operator generating a C_0-semigroup.

A classical solution $u : [0, T] \to H$ of the above equation will require a solution to be continuously differentiable and $u(t) \in \mathcal{D}(A)$. However,

$$u^x(t) = S_t x, \quad t \geq 0$$

is considered as a solution to the equation [83, Capt. 4]. For $x \notin \mathcal{D}(A)$, it is not a classical solution. Such a solution is called a "mild solution".

In fact, one can consider the non-homogeneous equation

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t)), \ u(0) = x, \quad x \in H.$$

Then for Bochner integrable $f \in L^1([0, T], H)$, one can consider the integral equation

$$u^x(t) = S_t x + \int_0^t S_{t-s} f(s, u(s)) ds.$$

A solution of this equation is called a "mild solution" if $u \in C([0, T], H)$.

We will consider mild solutions of stochastic partial differential equations (SPDEs) with Poisson noise. Note that the stochastic integral $\int_0^t S_{t-s} f(s, x) q(ds, dx)$, which appears in such SPDEs, is in general not a martingale. However, as for Doob's inequality, the following lemma holds.

Lemma 5.1.9 *Assume $(S_t)_{t \geq 0}$ is pseudo-contractive. Let $q(ds, dx)$ be a compensated Poisson random measure on $\mathbb{R}_+ \times E$ associated to a Poisson random measure N with compensator $dt \otimes \beta(dx)$. For each $T \geq 0$ the following statements are valid:*

1. *There exists a constant $C > 0$ such that for each $f \in \mathcal{L}^2_{T,\beta}(E, H)$ we have*

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \Big\| \int_0^t S_{t-s} f(s, x) q(ds, dx) \Big\|^2\Big] \leq C e^{2\alpha T} \mathbb{E}\Big[\int_0^T \int_E \|f(s, x)\|^2 \beta(dx) ds\Big]. \tag{5.1.3}$$

2. *For all $f \in \mathcal{L}^2_{T,\beta}(E, H_0)$ and all $\epsilon > 0$ we have*

$$\mathbb{P}\Big[\sup_{t \in [0,T]} \Big\| \int_0^t S_{t-s} f(s, x) q(ds, dx) \Big\| > \epsilon\Big] \leq \frac{4e^{2\alpha T}}{\epsilon^2} \mathbb{E}\Big[\int_0^T \int_E \|f(s, x)\|^2 \beta(dx) ds\Big], \tag{5.1.4}$$

where $\int_0^t S_{t-s} f(s, x) q(ds, dx)$ is well defined, if the r.h.s. is finite. $\int_0^t S_{t-s} f(s, x)$ $q(ds, dx)$ is càdlàg.

Proof Let M be the martingale

$$M_t = \int_0^t \int_E f(s, x) q(ds, dx), \quad t \in [0, T].$$

By Theorem 3.6.5 we have

$$\int_0^t S_{t-s} f(s, x) q(ds, dx) = \int_0^t S_{t-s} dM_s, \quad t \in [0, T].$$

Using [44, Theorem 3′.22′] we obtain

$$\begin{aligned}\mathbb{E}\Big[\sup_{t \in [0,T]} \Big\| \int_0^t S_{t-s} f(s, x) q(ds, dx) \Big\|^2\Big] &\leq e^{2\alpha T} (3 + \sqrt{10})^2 \mathbb{E}[\langle M, M \rangle_T] \\ &= e^{2\alpha T} (3 + \sqrt{10})^2 \mathbb{E}\Big[\int_0^T \int_E \|f(s, x)\|^2 \beta(dx) ds\Big],\end{aligned}$$

proving (5.1.3) with $C = (3 + \sqrt{10})^2$, and using [44, Theorem. 5′.16′] we obtain

$$\mathbb{P}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S_{t-s} f(s,x) q(ds,dx) \Big\| > \epsilon\Big] \leq \frac{4e^{2\alpha T}}{\epsilon^2} \mathbb{E}[\langle M, M\rangle_T]$$
$$= \frac{4e^{2\alpha T}}{\epsilon^2} \mathbb{E}\Big[\int_0^T \int_E \|f(s,x)\|^2 \beta(dx) ds\Big],$$

proving (5.1.4).

Let us show that $\int_0^t S_{t-s} f(s,x) q(ds,dx)$ is càdlàg. There is a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \to f$ in $\mathcal{L}_2^\beta(H)$, i.e.

$$\lim_{n\to\infty} \int_0^T \int_H \mathbb{E}[\|f_n(t,u) - f(t,u)\|^2] dt \beta(du) = 0.$$

Let

$$Y_t^n := \int_0^t \int_H S_{t-s} f_n q(ds,du) = \int_0^t S_{t-s} dM_s^n,$$
$$M_s^n := \int_0^t \int_H f_n q(ds,du). \tag{5.1.5}$$

As $S_{t-s} f_n(s,u,\omega)$ belongs to the set $\Sigma(H)$ of simple functions, Y_t^n is a martingale and is càdlàg.

It follows that

$$\mathbb{P}\left(\sup_{0\leq t\leq T} \|Y_t^n - Y_t^m\| > \epsilon\right) \leq 4\frac{e^{2\alpha T}}{\epsilon^2} \mathbb{E}[\langle M_n - M_m\rangle_T]$$
$$\leq 4\frac{e^{2\alpha T}}{\epsilon^2} \int_0^T \int_H \mathbb{E}[\|f_n(t,u)$$
$$- f_m(t,u)\|^2] dt \beta(du).$$

By the Borel–Cantelli Lemma and $f_n \to f$ in $\mathcal{L}_2^\beta(H)$ there is a subsequence $\{Y_t^{n_k}(\omega)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} \|Y_t^{n_k}(\omega) - Y_t^{n_{k+1}}(\omega)\| = 0 \quad \mathbb{P} - a.s.$$

It follows that

$$Y_t(\omega) = \lim_{k\to\infty} Y_t^{n_k}(\omega) \quad \text{uniformly in } [0,T], \quad \mathbb{P} - a.s.$$

We see that Y_t is càdlàg, since $Y_t^{n_k}$ is càdlàg. □

5.2 Existence and Uniqueness of Solutions of SPDEs Under Adapted Lipschitz Conditions

We shall study in this section càdlàg solutions to stochastic partial differential equations with non-Gaussian noise. As stated in Sect. 5.1, we shall treat these equations as ordinary stochastic differential equations in an infinite-dimensional space involving unbounded operators. Let us now assume that H is a separable Hilbert space and let A be a (generally unbounded) linear operator on the domain $\mathcal{D}(A) \subset H$. Assume that A is an infinitesimal generator of a pseudo-contraction semigroup $\{S_t\}_{t\geq 0}$ on H to H.

We want to study the existence and uniqueness of mild solutions of the stochastic differential equation on the interval $[0, T]$

$$\begin{cases} dZ_t = (AZ_t + a(t, Z))dt + \int_H f(t, u, Z)q(dt, du) \\ Z_0 = Z_0(\omega), \end{cases} \tag{5.2.1}$$

where $a(\cdot, z)$, $f(\cdot, u, z)$ are, for fixed $z \in H$, $u \in H$, functions on $D(\mathbb{R}_+, H)$ and Z_t is Z evaluated in t. In other words, we look at the solution of the integral equation

$$Z_t = S_t Z_0 + \int_0^t S_{t-s} a(s, Z)ds + \int_0^t \int_H S_{t-s} f(s, u, Z)q(ds, du), \tag{5.2.2}$$

where the integrals on the r.h.s. are well defined.

As in Chap. 4, we assume that with $\Omega = D(\mathbb{R}_+, H)$, $\mathcal{F}_t$ is a σ-algebra generated by cylinder sets of Ω with base on $[0, t]$. Let us assume throughout that A is an infinitesimal generator of a pseudo-contraction C_0-semigroup. Let

$$a : \mathbb{R}_+ \times D(\mathbb{R}_+, H) \to H, \quad f : \mathbb{R}_+ \times H \times D(\mathbb{R}_+, H) \to H$$

be functions and $\|z\|_\infty := \sup_{0\leq t\leq T} \|z(t)\|_H$, for $T < \infty$.

(a) $f(t, u, z)$ is jointly measurable and, for each $t \in \mathbb{R}_+$, $u \in H$, $f(t, u, \cdot)$ is $\mathcal{F}_t$-adapted.
(b) $a(t, z)$ is jointly measurable and, for each $t \in \mathbb{R}_+$, $a(t, \cdot)$ is $\mathcal{F}_t$-adapted.

If we consider the map $\theta_t : D(\mathbb{R}_+, H) \to D(\mathbb{R}_+, H)$ defined by

$$\begin{aligned} \theta_t(z)(s) &= z_s \quad \text{if} \quad 0 \leq s \leq t \\ &= z_t \quad \text{if} \quad t \leq s \end{aligned}$$

then $f(t, u, z) = f(t, u, \theta_t(z))$ and $a(t, z) = a(t, \theta_t(z))$.

(c) There exists a constant $l > 0$ such that, for fixed $t_1, t_2 \in [0, T]$,

$$\int_{t_1}^{t_2} \int_H \|f(t,u,z)\|^2 dt\beta(du) + \int_{t_1}^{t_2} \|a(t,z)\|_H^2 dt \\ \leq l \int_{t_1}^{t_2} (1 + \|\theta_t(z)\|_\infty^2) dt \quad \mathbb{P} - a.s.$$

(d) There exists a constant $K > 0$ such that, for fixed $t_1, t_2 \in [0, T]$ and $z, y \in D(\mathbb{R}_+, H)$,

$$\int_{t_1}^{t_2} \int_H \|f(t,u,z) - f(t,u,y)\|^2 dt\beta(du) + \int_{t_1}^{t_2} \|a(t,z) - a(t,y)\|_H^2 dt \\ \leq K \int_{t_1}^{t_2} \|\theta_t(z) - \theta_t(y)\|_\infty^2 dt \quad P - a.s.$$

Let, for $Z \in D(\mathbb{R}_+; H)$,

$$I(t, Z) := \int_0^t S_{t-s} a(s, Z) ds + \int_0^t \int_{H \setminus \{0\}} S_{t-s} f(s, u, Z) q(ds, du), \quad t \in [0, T]. \tag{5.2.3}$$

Theorem 5.2.1 *Assume (a), (b) and (c). There exists a constant $C_{l,T,\alpha}$ such that for any $\mathcal{F}_t$-stopping time τ*

$$\mathbb{E}[\sup_{0 \leq s \leq t \wedge \tau} \|I(s, Z)\|_H^2] \leq C_{l,T,\alpha}(t + \int_0^t \mathbb{E}[\sup_{0 \leq v \leq s \wedge \tau} \|Z_v\|^2] ds), \quad t \in [0, T]. \tag{5.2.4}$$

Proof

$$\sup_{0 \leq s \leq t \wedge \tau} \|I(s, Z)\|_H^2 \leq 2 \sup_{0 \leq s \leq t \wedge \tau} \| \int_0^s S_{s-v} a(v, Z) dv \|_H^2 \\ + 2 \sup_{0 \leq s \leq t \wedge \tau} \| \int_0^s \int_H S_{t-v} f(v, u, Z) q(dv, du) \|_H^2 \tag{5.2.5}$$

(where we used the inequality $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$, valid for any $x, y \in H$). Using the bound on S_t and condition (c) we obtain

$$\mathbb{E}[\sup_{0 \leq s \leq t \wedge \tau} \| \int_0^s S_{s-v} a(v, Z) dv \|_H^2] \\ \leq \mathbb{E}[\sup_{0 \leq s \leq t} (l e^{\alpha t} \int_0^s (1 + \|\theta_v(Z)\|_\infty) dv)^2] \\ \leq 2e^{2\alpha T} l^2 \{t^2 + t\mathbb{E}[\int_0^{s \wedge \tau} \|\theta_v(Z)\|_\infty^2 dv]\}.$$

Moreover, using Theorem 3 of [44] and (c) we get

$$\mathbb{E}[\sup_{0\le s\le t\wedge\tau}\|\int_0^s\int_H S_{t-v}f(v,u,Z)q(dv,du)\|_H^2] \\ \le 2e^{2\alpha T}l^2(3+\sqrt{10})^2\{t^2+t\mathbb{E}[\int_0^{s\wedge\tau}\|\theta_v(Z)\|_\infty^2 dv]\},$$

$$\mathbb{E}[\sup_{0\le s\le t\wedge\tau}\|I(s,Z)\|_H^2] \le 4e^{2\alpha T}l^2(1+(3+\sqrt{10})^2) \\ \{t^2+t\mathbb{E}[\int_0^{s\wedge\tau}\|\theta_v(Z)\|_\infty^2 dv]\} \\ \le C_{l,T,\alpha}(t+\int_0^t\mathbb{E}[\sup_{0\le v\le s\wedge\tau}\|Z_v\|^2]ds),$$

with $C_{l,T,\alpha} := 4Te^{2\alpha T}l^2(1+(3+\sqrt{10})^2)$. □

Let $T>0$ and

$$\mathcal{H}_2^T := \{\xi := (\xi_s)_{s\in[0,T]} : \xi_s(\omega) \text{ is jointly measurable}, \\ \mathcal{F}_t\text{-adapted}\,;\, \mathbb{E}[\sup_{0\le s\le T}\|\xi_s\|_H^2]<\infty\}.$$

Let us observe that it follows from Theorem 5.2.1 that the map

$$I : \mathcal{H}_2^T \to \mathcal{H}_2^T \\ \xi \to I(\cdot,\xi)$$

is well defined.

Lemma 5.2.2 *Assume (a), (b), (c) and (d). The map $I : \mathcal{H}_2^T \to \mathcal{H}_2^T$ is continuous. There is a constant $C_{\alpha,K,T}$, depending on α, K and T, such that*

$$\mathbb{E}[\sup_{0\le s\le T}\|I(s,Z^1)-I(s,Z^2)\|_H^2] \le C_{\alpha,K,T}\int_0^T\mathbb{E}[\sup_{0\le s\le T}\|Z_s^2-Z_s^1\|_H^2]ds\,. \tag{5.2.6}$$

Exercise Use (d) and the arguments as in Theorem 5.2.1 to prove Lemma 5.2.2.

Theorem 5.2.3 *Let $T>0$, $x\in H$. There is a unique solution $Z := (Z_s)_{s\in[0,T]}$ in $\mathcal{H}_2^T$ which satisfies*

$$Z_t = S_t x + \int_0^t S_{t-s}a(s,Z)ds + \int_0^t\int_H S_{t-s}f(s,u,Z)q(ds,du). \tag{5.2.7}$$

Proof We shall prove that the solution can be approximated in $\mathcal{H}_2^T$ by $Z^n := (Z_s^n)_{s\in[0,T]}$, for $n \to \infty$, $n \in \mathbb{N}$, where

$$Z_s^0(\omega) := S_s x \qquad P - a.s.$$
$$Z_s^{n+1}(\omega) := I(s, Z^n(\omega)) .$$

Note that $(Z_t^n)_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted. Let

$$v_t^n := \mathbb{E}[\sup_{0\le s\le t} \|Z_s^{n+1} - Z_s^n\|_H^2] .$$

Then from Theorem 5.2.1 it follows that there is a constant $V_{\alpha,l,T}(x)$, depending on α, l and T and the initial data x, such that

$$v_t^0 \le \mathbb{E}[\sup_{0\le s\le T} \|Z_s^1 - Z_s^0\|_H^2] \le V_{\alpha,l,T}(x).$$

Similarly as in the proof of Theorem 5.2.1, it can be proven that there is a constant $C_{\alpha,K,T}$ depending on α, K and T, such that

$$v_t^1 \le C_{\alpha,K,T} \int_0^t \mathbb{E}[\sup_{0\le s\le t} \|Z_s^1 - Z_s^0\|_H^2] ds \le \frac{T^2(C_{\alpha,K,T})^2}{2} V_{\alpha,l,T}(x).$$

In a similar way we get by induction that

$$v_t^n \le C_{\alpha,K,T} \int_0^t v_s^{n-1} ds \le \frac{(T C_{\alpha,K,T})^{n+1}}{(n+1)!} V_{\alpha,l,T}(x).$$

Let $\epsilon_n := \left(\frac{(TC_{\alpha,K,T})^{n+1}}{(n+1)!}\right)^{\frac{1}{3}}$. Then:

$$\mathbb{P}(\sup_{0\le t\le T} \|Z_t^{n+1} - Z_t^n\|^2 \ge \epsilon_n) \le \frac{\frac{(TC_{\alpha,K,T})^{n+1}}{(n+1)!} V_{\alpha,l,T}(x)}{(\frac{(TC_{\alpha,K,T})^{n+1}}{(n+1)!})^{\frac{1}{3}}} = \epsilon_n^2 V_{\alpha,l,T}(x).$$

As $\sum_n \epsilon_n^2$ is convergent, we get that $\sum_{n=1}^{\infty} \sup_{0\le t\le T} \|Z_t^{n+1} - Z_t^n\|^2$ converges P-a.s. It follows that there is a process $Z := (Z_t)_{t\in[0,T]}$, $Z \in D([0, T]; H)$, such that Z^n converges to Z, as n goes to infinity, in the space $D([0, T]; H)$ (with the supremum norm), P-a.s. Moreover

$$\mathbb{E}[\sup_{0\le t\le T} \|Z_t - Z_t^n\|^2] = \mathbb{E}[\lim_{m\to\infty} \sup_{0\le t\le T} \| \sum_{k=n}^{n+m-1} (Z_t^{k+1} - Z_t^k)\|^2]$$

$$\leq \mathbb{E}[\lim_{m\to\infty}(\sum_{k=n}^{n+m-1} \sup_{0\leq t\leq T} \|Z_t^{k+1} - Z_t^k\| k\frac{1}{k})^2]$$

$$\leq \sum_{k=n}^{\infty} \mathbb{E}[\sup_{0\leq t\leq T} \|Z_t^{k+1} - Z_t^k\|^2 k^2)] \sum_{k=n}^{\infty} \frac{1}{k^2}$$

$$\leq V_{\alpha,l,T}(x)(\sum_{k=n}^{\infty} \frac{(TC_{\alpha,K,T})^{k+1}k^2}{(k+1)!})$$

$$(\sum_{k=n}^{\infty} \frac{1}{k^2}) \to 0 \quad \text{as} \quad n \to \infty\,,$$

where we used Schwarz's inequality. It follows that, as n goes to infinity, Z^n also converges to Z in the space $\mathcal{H}_2^T$. From Lemma 5.2.2 it follows that $(Z_t)_{0\leq t\leq T}$ solves (5.2.7). We shall prove that the solution is unique. Suppose that $(Z_t)_{0\leq t\leq T}$ and $(Y_t)_{0\leq t\leq T}$ are two solutions of (5.2.7). Let

$$\mathcal{V}_t := \mathbb{E}[\sup_{0\leq s\leq t} \|Z_s - Y_s\|_H^2]\,.$$

Then similarly as before we get

$$\mathcal{V}_t \leq C_{\alpha,K,T} \int_0^t \mathcal{V}_s$$

and by induction

$$\mathcal{V}_t \leq \frac{(C_{\alpha,K,T}t)^n}{n!} \mathbb{E}[\sup_{0\leq s\leq T} \|Z_s - Y_s\|_H^2] \to 0 \quad \text{as} \quad n \to \infty$$

i.e. $\mathcal{V}_t = 0 \ \forall t \in [0, T]$. □

5.3 Existence and Uniqueness of Solutions of SPDEs Under Markovian Lipschitz Conditions

Let us assume that we are given

$$a : \mathbb{R}_+ \times H \to H,$$

$$f : \mathbb{R}_+ \times H \times H \to H.$$

Assume

(A) $f(t,u,z)$ is jointly measurable,
(B) $a(t,z)$ is jointly measurable,
and for fixed $T > 0$
(C) there is a constant $L > 0$ such that

$$T\|a(t,z) - a(t,z')\|^2 + \int_H \|f(t,u,z) - f(t,u,z')\|^2 \beta(du) \le L\|z - z'\|^2$$
$$\text{for all } t \in [0,T], \quad z, z' \in F,$$

(D) there is a constant $K > 0$ such that

$$T\|a(t,z)\|^2 + \int_H \|f(t,u,z)\|^2 \beta(du) \le K(\|z\|^2 + 1)$$
$$\text{for all } t \in [0,T], \quad z \in F.$$

We assume again that A is the infinitesimal generator of a pseudo-contraction semigroup $(S_t)_{t\in[0,T]}$. If we consider functions $a(t,z) = a(t,z_t)$ and $f(t,u,z) = f(t,u,z_t)$ on $D(\mathbb{R}_+, H)$ then our previous theorem tells us that the equation

$$\begin{aligned} Z_t = S_t Z_0 &+ \int_0^t S_{t-s} a(s, Z_s) ds \\ &+ \int_0^t \int_H S_{t-s} f(s,u,Z_s) q(ds,du) \quad \mathbb{P} - a.s. \quad \forall t \in [0,T] \end{aligned} \tag{5.3.1}$$

has a unique solution on $\mathcal{H}_2^T$.

However, we shall now consider the SPDE on any filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ (satisfying the usual conditions) and show that it has a unique càdlàg solution in S_T^2 as defined in Chap. 4.

Theorem 5.3.1 *Suppose assumptions (A)–(D) are satisfied. Then for* $Z_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$*, there exists a unique mild solution in* S_T^2 *to* (5.3.1)*, with initial condition* Z_0*, such that* Z_t *is* $\mathcal{F}_t$*-measurable.*

Proof Define the process

$$\begin{aligned} (\mathcal{S}Z)_t := S_t Z_0 &+ \int_0^t S_{t-s} a(s, Z_s) ds \\ &+ \int_0^t \int_H S_{t-s} f(s,u,Z_s) q(ds,du) \quad t \in [0,T]. \end{aligned} \tag{5.3.2}$$

Consider

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T}\|(\mathcal{S}Z)_t\|^2] \le 3\mathbb{E}[\sup_{0\le t\le T}\|S_t Z_0\|^2] \\ + 3\mathbb{E}[\sup_{0\le t\le T}\|\int_0^t\int_H S_{t-s}a(s,Z_s)ds\|^2] \\ + 3\mathbb{E}[\sup_{0\le t\le T}\|\int_0^t S_{t-s}f(s,u,Z_s)q(ds,du)\|^2].\end{aligned}$$

Using the fact that $\|S_t\| \le e^{\alpha t}$ for $t \ge 0$, inequality (5.1.3) and (D) we get

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T}\|(\mathcal{S}Z)_t\|^2] &\le 3e^{2\alpha T}\mathbb{E}[\|Z_0\|^2] \\ &\quad + 3e^{2\alpha T}\mathbb{E}[\|\int_0^T a(s,Z_s)ds\|^2] \\ &\quad + 3e^{2\alpha T}\mathbb{E}[\|\int_0^T\int_H f(s,u,Z_s)q(ds,du)\|^2] \\ &\le 3e^{2\alpha T}\mathbb{E}[\|Z_0\|^2] + 3Te^{2\alpha T}\mathbb{E}[\int_0^T\|a(s,Z_s)\|^2 ds] \\ &\quad + 3e^{2\alpha T}C\mathbb{E}[\int_0^T\int_H\|f(s,u,Z_s)\|^2 ds\beta(du)] \\ &\le 3e^{2\alpha T}\mathbb{E}[\|Z_0\|^2] + 3e^{2\alpha T}C(K\mathbb{E}[\int_0^T\|Z_s\|^2 ds] + K)\end{aligned}$$

where C is any fixed constant such that (5.1.3) holds and such that $C > 1$.

This shows that $\mathcal{S}$ maps S_T^2 into itself.

For $Y, Z \in S_T^2$ using again (5.1.3)

$$\begin{aligned}\mathbb{E}[\sup_{0\le s\le t}\|(\mathcal{S}Y)_s - (\mathcal{S}Z)_s\|^2] &\le 2\mathbb{E}[\sup_{0\le s\le t}\|\int_0^s\int_H S_{s-v}(a(v,Y_v) \\ &\quad - a(v,Z_v))dv\|^2] \\ &\quad + 2\mathbb{E}[\sup_{0\le s\le t}\|\int_0^s S_{s-v}(f(v,u,Y_v) \\ &\quad - f(v,u,Z_v))q(dv,du)\|^2] \\ &\le 2Te^{2\alpha T}\mathbb{E}[\int_0^t\|a(s,Y_s) - a(s,Z_s)\|^2 ds] \\ &\quad + 2e^{2\alpha T}C\mathbb{E}[\int_0^T\int_H\|f(s,u,Y_s) \\ &\quad - f(s,u,Z_s)\|^2 ds\beta(du)].\end{aligned}$$

Using (C) we get

$$\mathbb{E}[\sup_{0\le s\le t}\|(\mathcal{S}Y)_s-(\mathcal{S}Z)_s\|^2]\le 2Le^{2\alpha T}C\mathbb{E}[\int_0^t\|Y_s-Z_s\|^2ds]$$
$$\le 2Le^{2\alpha T}C\mathbb{E}[\int_0^t\sup_{0\le v<s}\|Y_v-Z_v\|^2ds].$$

By induction we get

$$\mathbb{E}[\sup_{0\le s\le t}\|(\mathcal{S}Y)_s-(\mathcal{S}Z)_s\|^2]\le\frac{2^n(CL)^ne^{2\alpha nT}}{n!}\mathbb{E}[\sup_{0\le t<T}\|Y_t-Z_t\|^2ds].$$

Hence for some $n\in\mathbb{N}$, $\mathcal{S}$ is a contraction, yielding the conclusion by the fixed point theorem. □

Corollary 5.3.2 *Let $0<T<\infty$, and assume (A), (B), (C) and (D). Let $(Z_t^\xi)_{t\in[0,T]}$ (resp. $(Z_t^\eta)_{t\in[0,T]}$) be the solution to (5.3.1) with initial condition ξ (resp. η), then*

$$\mathbb{E}[\|Z_t^\xi-Z_t^\eta\|^2]\le C_{t,\alpha}\|\xi-\eta\|^2,$$

with constant $C_{t,\alpha}$ depending on t and α.

Exercise Prove the corollary by computing $\mathbb{E}[\|Z_t^\xi-Z_t^\eta\|^2]$.

We assume again that A is the infinitesimal generator of a pseudo-contraction semigroup $(S_t)_{t\in[0,T]}$ and conditions (A), (B), (C), (D) hold.
Let

$$Z_0(\omega)=\xi\quad\mathbb{P}-a.s.$$

and let $(Z_t)_{t\in[0,T]}$ be the unique càdlàg process solving $\mathbb{P}$-a.s. (5.3.1) for every $t\in[0,T]$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be the Yosida approximation to A (see Sect. 5.1). For every fixed $T>0$, there exists a unique càdlàg process $(Z_t^n)_{t\in[0,T]}$ such that $\int_0^T\mathbb{E}[\|Z_s^n\|^2]ds<\infty$ and such that $(Z_t^n)_{t\in[0,T]}$ is a strong solution of

$$dZ_t^n=A_nZ_t^ndt+a(t,Z_t^n)dt+\int_H f(s,u,Z_s^n)q(ds,du)$$

with initial condition ξ (see Chap. 4 or [65]). Moreover, $(Z_t^n)_{t\in[0,T]}$ is also a mild solution, i.e. $\mathcal{P}$-a.s.

$$Z_t^n=S_t^n\xi+\int_0^tS_{t-s}^na(s,Z_s^n)ds+\int_0^t\int_HS_{t-s}^nf(s,u,Z_s^n)q(ds,du)\qquad(5.3.3)$$

for every $t \in [0, T]$ and such that conclusions the conditions in Theorem 5.3.1 are satisfied. We shall prove the following result:

Theorem 5.3.3

$$\lim_{n\to\infty} \mathbb{E}[\|Z_t - Z_t^n\|^2] = 0$$

uniformly in $[0, T]$.

Proof We have

$$\begin{aligned}\mathbb{E}[\|Z_t - Z_t^n\|^2] \le\ & 2^3\|S_t^n\xi - S_t\xi\|^2 \\ & + 2^3\mathbb{E}[\|\int_0^t (S_{t-s}a(s, Z_s) - S_{t-s}^n a(s, Z_s^n)ds\|^2] \\ & + 2^3\mathbb{E}[\|\int_0^t \int_H (S_{t-s}f(s, u, Z_s) - S_{t-s}^n f(s, u, Z_s^n)q(ds, du)\|^2].\end{aligned} \tag{5.3.4}$$

We shall analyze separately the three terms on the right-hand side of inequality (5.3.4). As for the first term, we remark that

$$\lim_{n\to\infty} \|S_t^n\xi - S_t\xi\| = 0.$$

By Sect. 5.1 (Yosida approximation) we get that the convergence is uniform in $[0, T]$.

Let us consider the second term on the right-hand side of (5.3.4). We have:

$$\begin{aligned}&\mathbb{E}[\|\int_0^t (S_{t-s}a(s, Z_s) - S_{t-s}^n a(s, Z_s^n)ds\|^2] \\ &\le 2T\int_0^t \mathbb{E}[\|S_{t-s}a(s, Z_s) - S_{t-s}^n a(s, Z_s)\|^2]ds \\ &\quad + 2T\int_0^t \mathbb{E}[\|S_{t-s}^n a(s, Z_s) - S_{t-s}^n a(s, Z_s^n)\|^2]ds\end{aligned} \tag{5.3.5}$$

$$\lim_{n\to\infty} \|S_{t-s}a(s, Z_s(\omega)) - S_{t-s}^n a(s, Z_s(\omega))\| = 0 \quad P - a.s. \tag{5.3.6}$$

and

$$\begin{aligned}\|S_{t-s}a(s, Z_s(\omega)) - S_{t-s}^n a(s, Z_s(\omega))\|^2 &\le C_T\|a(s, Z_s(\omega))\|^2 \\ &\le C_T K(\|Z_s(\omega)\|^2 + 1)\,.\end{aligned} \tag{5.3.7}$$

This is a consequence of uniform convergence and condition (D). By the Lebesgue dominated convergence theorem it follows that the first term on the r.h.s. of (5.3.5) converges to zero.

Let us consider the second term on the r.h.s. of (5.3.5). We observe that from uniform convergence and the Lipschitz condition (C) it follows that

$$T\|S^n_{t-s}a(s, Z_s(\omega)) - S^n_{t-s}a(s, Z^n_s(\omega))\|^2 \leq C_T L\|Z_s(\omega) - Z^n_s(\omega)\|^2$$

so that

$$2T\int_0^t \mathbb{E}[\|S^n_{t-s}a(s, Z_s) - S^n_{t-s}a(s, Z^n_s)\|^2\, ds \leq 2C_T L\int_0^t \mathbb{E}[\|Z_s - Z^n_s\|^2]\, ds.$$

It follows that for all $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mathbb{E}[\|\int_0^t (S_{t-s}a(s, Z_s) - S^n_{t-s}a(s, Z^n_s)ds\|^2] \leq \epsilon + 2C_T L\int_0^t \mathbb{E}[\|Z_s - Z^n_s\|^2]\, ds.$$

Let us consider the third term in (5.3.4). By similar arguments as above, it can be proved that

$$\mathbb{E}[\|\int_0^t\int_H (S_{t-s}f(s, u, Z_s) - S^n_{t-s}f(s, u, Z^n_s)q(ds, du)\|^2]$$
$$\leq \epsilon + 2C_T L\int_0^t \mathbb{E}[\|Z_s - Z^n_s\|^2]\, ds.$$

It follows that

$$\mathbb{E}[\|Z_t - Z^n_t\|^2] \leq 2^3\|S^n_t\xi - S_t\xi\|^2 + 2^4\epsilon 2^4 C_T L\int_0^t \mathbb{E}[\|Z_s - Z^n_s\|^2]\, ds.$$

Using Gronwall's Lemma we get

$$\mathbb{E}[\|Z_t - Z^n_t\|^2] \leq (2^3\|S^n_t\xi - S_t\xi\|^2 + 2^4\epsilon)\exp(2^4 TLC_T)$$

so that (5.3.4) gives the result. □

5.4 The Markov Property of the Solution of SPDEs

Let $B_b(H)$ denote the set of bounded real valued functions on H. We first prove that the Markov property holds for the semigroup associated to the mild solutions of (5.3.1):

Let $0 < v < T$ and $\xi \in H$. Let $(Z(t, v, \xi))_{t\in[v,T]}$ denote the solution of the following integral equation

$$Z_t = S_{t-v}\xi + \int_v^t S_{t-s}a(s, Z_s)ds + \int_v^t \int_H S_{t-s} f(s, u, Z_s)q(ds, du) \quad (5.4.1)$$

(in the sense of Theorem 5.3.1). Let $\mathcal{F}_t^Z$ denote the σ-algebra generated by $Z(\tau, v, \xi)$, with $\tau \le t, \tau \ge v$. Let $v \le s \le t \le T$ and $P_{s,t}$ be the linear operator on $B_b(H)$, defined by

$$(P_{s,t})(\phi)(x) = \mathbb{E}[\phi(Z(t, s; x))] \quad \text{for} \quad \phi \in B_b(H) \quad x \in H\,. \quad (5.4.2)$$

Then the Markov property holds, i.e.

Theorem 5.4.1 *Let* $0 \le v \le s \le t \le T$. *Then*

$$\mathbb{E}[\phi(Z(t, v; \xi))/\mathcal{F}_s^Z] = (P_{s,t})(\phi)(Z(s, v; \xi)) \text{ for any } \phi \in B_b(H).$$

Proof As $\mathcal{F}_s^Z \subset \mathcal{F}_s$, it is sufficient to prove that

$$\mathbb{E}[\phi(Z(t, v; \xi))/\mathcal{F}_s] = (P_{s,t})(\phi)(Z(s, v; \xi))\,. \quad (5.4.3)$$

From the uniqueness of the solution we get

$$Z(t, v; \xi)(\omega) = Z(t, s; Z(s, v; \xi)(\omega))(\omega) \quad \mathbb{P} - a.s. \quad (5.4.4)$$

Let

$$\eta(\omega) := Z(s, v; \xi)(\omega). \quad (5.4.5)$$

Then from (5.4.4) it follows that (5.4.3) can be written as

$$\mathbb{E}[\phi(Z(t, s; \eta))/\mathcal{F}_s] = (P_{s,t})(\phi)(Z(s, v; \eta))\,. \quad (5.4.6)$$

It is enough to show that (5.4.6) holds for every $\phi \in C_b(H)$, with $C_b(H)$ denoting the set of continuous real-valued bounded functions on H. We first assume that ϕ is linear and bounded.

Moreover, let us first consider the case where

$$\eta(\omega) = x \in H \quad \mathbb{P} - a.s.$$

As x is constant and because of the independent increment property of the cPrm, $Z(t, s; \eta(\omega))$ is independent of $\mathcal{F}_s$. In fact $\mathcal{F}_s$ is the σ-algebra generated by the pure jump Lévy process with compensator $ds\beta(dx)$. See Sect. 2.4 and Sect. 3.3, or [3, Sect. 2].

$$\mathbb{E}[\phi(Z(t, s; \eta))/\mathcal{F}_s] = \mathbb{E}[\phi(Z(t, s, x))] = P_{s,t}(\phi(x))$$

so that (5.4.6) holds for this particular case.

Now we prove (5.4.6) for the case where

$$\eta(\omega) := \sum_1^n a_j \mathbf{1}_{A_j}(Z(s, v; \xi)) \tag{5.4.7}$$

with $\{A_j, j = 1, \dots, n\}$ a partition of H and $a_1, \dots, a_n \in H$. In this case

$$Z(t, s; \eta(\omega))(\omega) = \sum_1^n Z(t, s; a_j)\mathbf{1}_{A_j}(Z(s, v; \xi)) \quad \mathbb{P} - a.s.,$$

$$\phi(Z(t, s; \eta(\omega))(\omega)) = \sum_1^n \phi(Z(t, s; a_j))\mathbf{1}_{A_j}(Z(s, v; \xi)) \quad \mathbb{P} - a.s.,$$

and

$$\begin{aligned} \mathbb{E}[\phi(Z(t, s; \eta)/\mathcal{F}_s] &= \mathbb{E}[\sum_1^n \phi(Z(t, s; a_j))\mathbf{1}_{A_j}(Z(s, v; \xi))/\mathcal{F}_s] \\ &= \sum_1^n P_{s,t}(\phi)(a_j)\mathbf{1}_{A_j}(Z(s, v, \xi) = P_{s,t}(\phi)(\eta), \end{aligned} \tag{5.4.8}$$

where in (5.4.8) we used that $\phi(Z(t, s; a_j))$ are independent of $\mathcal{F}_s$ and $\mathbf{1}_{A_j}(Z(s, v; \xi))$ are $\mathcal{F}_s$-measurable.

Now we prove (5.4.6) for the case where $\eta(\omega)$ is given according to (5.4.5). (From the proof it follows in particular that the r.h.s. of (5.4.3) is $\mathcal{F}_s^Z$-measurable.) There is a sequence of simple functions $\eta_n(\omega)$ of the form (5.4.7) such that, if for a given natural number M we denote $\eta_n^M := \eta_n \wedge M$, then

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathbb{E}[\|\eta_n^M - \eta\|^2] = 0. \tag{5.4.9}$$

Similar to the proof of Corollary 5.3.2 it follows that

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathbb{E}[\|Z(t, s; \eta_n^M) - Z(t, s; \eta)\|^2] = 0\,.$$

There is a subsequence (by abuse of notation we denote it in the same way as the original sequence), for which

$$\lim_{M\to\infty} \lim_{n\to\infty} Z(t, s; \eta_n^M)(\omega) = Z(t, s; \eta)(\omega) \quad \mathbb{P} - a.s.$$

As ϕ is continuous and bounded, it follows from (5.4.8) that

$$\begin{aligned} \mathbb{E}[\phi(Z(t, s; \eta)/\mathcal{F}_s)] &= \lim_{M\to\infty} \lim_{n\to\infty} \mathbb{E}[\phi(Z(t, s, \eta_n^M)/\mathcal{F}_s)] \\ &= \lim_{M\to\infty} \lim_{n\to\infty} P_{s,t}(\phi)(\eta_n^M) = P_{s,t}(\phi)(\eta). \end{aligned}$$

Given $\phi \in C_b(H)$ there exists a sequence of linear bounded functions ϕ_n converging, up to a set of Borel measure zero, to ϕ (see e.g. [103], Chap. V.5). It follows that $\phi_n(Z(t,s;\eta) \to \phi(Z(t,s;\eta)$ $\mathbb{P}$ -a.s., when $n \to \infty$. ϕ_n can be chosen to be uniformly bounded, so that

$$\lim_{n\to\infty} \mathbb{E}[\phi_n(Z(t,s;\eta)/\mathcal{F}_s)] = \mathbb{E}[\phi(Z(t,s;\eta)/\mathcal{F}_s)]. \qquad \square$$

Theorem 5.4.2 *Let $T > 0$, $f(s,u,z)) = f(u,z)$, $a(s,z) = a(z)$ and $x \in H$, then $(Z(t,0;x)(\omega))_{t\in[0,T]}$ is a homogenous Markov process.*

Proof It is sufficient to prove that

$$P_{s,t} = P_{0,t-s} \quad \text{for} \quad \text{all} \quad 0 \le s \le t \le T \tag{5.4.10}$$

together with the Markov property in Theorem 5.4.1 implies that the Chapman–Kolmogorov equation holds for the transition probabilities associated to $P_{s,t}$, $0 \le s \le t \le T$ and $(Z(t,0;x)(\omega))_{t\in[0,T]}$ is a Markov process.

Let us remark that the compensated Lévy random measure $q(ds,du)(\omega)$ is translation invariant in time, i.e. if $t > 0$ and $\tilde{q}(ds,du)(\omega)$ denotes the unique σ-finite measure on $\mathcal{B}(\mathbb{R}_+ \times H)$ which extends the pre -measure $\tilde{q}(ds,du)(\omega)$ on $S(\mathbb{R}_+) \times \mathcal{B}(H)$, such that $\tilde{q}((s,\tau],\Lambda) := q((s+t,\tau+t],\Lambda)$, for $(s,\tau] \times \Lambda \in S(\mathbb{R}_+) \times \mathcal{B}(H)$, then $\tilde{q}(B)$ and $q(B)$ are equally distributed for all $B \in \mathcal{B}(\mathbb{R}_+ \times H)$.

It follows that

$$\begin{aligned}
&Z(t+h,t;x)\\
&= S_h x + \int_t^{t+h} S_{t+h-s} a(Z(s,t;x))ds + \int_t^{t+h}\int_H S_{t+h-s} f(Z(s,t;x))q(ds,du)\\
&= S_h x + \int_0^h S_{h-s} a(Z(t+s,t;x))ds + \int_0^h \int_H S_{h-s} f(u, Z(t+s,t;x))\tilde{q}(ds,du)\\
&= S_h x + \int_0^h S_{h-s} a(Z(t+s,t;x))ds + \int_0^h \int_H S_{h-s} f(u, Z(t+s,t;x))q(ds,du).
\end{aligned}$$

By uniqueness (Theorem 5.3.1) it follows that $Z(t+h,t;x)(\omega)$ and $Z(h,0;x)(\omega)$ have the same distribution, completing the proof. $\square$

5.5 Existence of Solutions for Random Coefficients

Let $L_2^T := L_2^T([0,T] \times \Omega, (\mathcal{F}_t)_{t\in[0,T]})$ be the space of processes $(Z_t(\omega))_{t\in[0,T]}$ which are jointly measurable and

(i) Z_t is $\mathcal{F}_t$-measurable,
(ii) $\int_0^T \mathbb{E}[\|Z_s\|^2]ds < \infty$.

Definition 5.5.1 We say that two processes $Z_t^i(\omega) \in L_2^T$, $i = 1, 2$, are $dt \otimes P$-equivalent if they coincide for all $(t, \omega) \in \Gamma$, with $\Gamma \in \mathcal{B}([0, T]) \otimes \mathcal{F}_T$, and $dt \otimes P(\Gamma^c) = 0$. We denote by $\mathcal{L}_2^T$ the set of $dt \otimes P$-equivalence classes.

Remark 5.5.2 $\mathcal{L}_2^T$, with norm

$$\|Z_t\|_{\mathcal{L}_2^T} := (\int_0^T \mathbb{E}[\|Z_s\|^2]ds)^{1/2},$$

is a Hilbert space.

In this section we assume that the coefficients are random and adapted to the filtration and prove the existence of a solution in $\mathcal{L}_2^T$. We assume here the growth and Lipschitz conditions of the coefficients independent of ω, but depending on the points in H. We assume in fact that we are given

$$a : \mathbb{R}_+ \times H \times \Omega \to H,$$

$$f : \mathbb{R}_+ \times H \times H \times \Omega \to H,$$

such that
(A$'$) $f(t, u, z, \omega)$ is jointly measurable such that for all $t \in [0, T]$, $u \in E$ and fixed $z \in H$, $f(t, u, z, \cdot)$ is $\mathcal{F}_t$-adapted,
(B$'$) $a(t, z, \omega)$ is jointly measurable such that for all $t \in [0, T]$, and fixed $z \in H$, $a(t, z, \cdot)$ is $\mathcal{F}_t$-adapted, and for fixed $T > 0$
(C$'$) there is a constant $L > 0$ such that

$$T\|a(t, z, \omega) - a(t, z', \omega)\|^2 + \int_H \|f(t, u, z, \omega) - f(t, u, z', \omega)\|^2\beta(du) \le L\|z - z'\|^2$$
$$\text{for all} \quad t \in [0, T], \qquad z, z' \in H, \quad \text{and} \quad \mathbb{P} - a.e.\ \omega \in \Omega,$$

(D$'$) there is a constant $K > 0$ such that

$$T\|a(t, z, \omega)\|^2 + \int_H \|f(t, u, z, \omega)\|^2\beta(du) \le K(\|z\|^2 + 1)$$
$$\text{for all} \quad t \in ([0, T], \quad z \in H, \quad \text{and} \quad \mathbb{P} - a.e.\ \omega \in \Omega.$$

Theorem 5.5.3 *Let $0 < T < \infty$ and suppose that (A$'$), (B$'$), (C$'$), (D$'$) are satisfied. Let $x \in H$. Then there is a unique process $(Z_t)_{0 \le t \le T} \in \mathcal{L}_2^T$ which satisfies*

$$Z_t(\omega) = S_t x + \int_0^t S_{t-s}a(s, Z_s(\omega), \omega)ds + \int_0^t \int_H S_{t-s}f(s, u, Z_s(\omega), \omega)q(dsdu) \quad \forall t \in [0, T]. \tag{5.5.1}$$

As a consequence of Theorem 5.5.3 we have:

Corollary 5.5.4 *Let* $0 < T < \infty$ *and suppose that (A′), (B′), (C′), (D′) are satisfied. Then there is up to stochastic equivalence a unique process* $(Z_t)_{0\leq t\leq T} \in L_2^T$ *which satisfies* (5.5.1).

Remark 5.5.5 As a consequence of Lemma 5.1.9 we have that $(Z_t)_{0\leq t\leq T}$ is càdlàg.

Before proving Theorem 5.5.3 we prove some properties of the following function

$$\begin{aligned} K_t(x,\xi)(\omega) := S_t x + \int_0^t S_{t-s} a(s,\xi_s(\omega),\omega) ds \\ + \int_0^t \int_H S_{t-s} f(s,u,\xi_s(\omega),\omega) q(ds,du) \end{aligned}$$

with $x \in H$ and $\xi := (\xi_s)_{s\in[0,T]} \in \mathcal{L}_2^T$.

Lemma 5.5.6 *For any* $T > 0$ *there is a constant* C_T^1 *such that*

$$\int_0^T \mathbb{E}[\|K_t(x,\xi) - K_t(x,\eta)\|^2] dt \ \leq\ C_T^1 \int_0^T \mathbb{E}[\|\xi_t - \eta_t\|^2] dt.$$

Proof

$$\begin{aligned} &\int_0^T \mathbb{E}[\|K_t(x,\xi) - K_t(x,\xi)\|^2] dt \\ &\leq 2e^{2\alpha T} T \int_0^T \mathbb{E}\left[\| \int_0^t (a(s,\xi_s) - a(s,\eta_s) ds\|^2\right] dt \\ &\quad + 2e^{2\alpha T} \int_0^T \int_0^t \int_H \mathbb{E}[\|(f(s,u,\xi_s) - f(s,u,\eta_s)\|^2 ds \beta(du)] dt \\ &\leq 2LTe^{2\alpha T} \int_0^T \mathbb{E}[\|\xi_s - \eta_s\|^2] dt \ < \infty\,, \end{aligned}$$

where we applied the bounds on S_t. This proves the lemma. □

Let

$$K(x,\xi):\ H \times \mathcal{L}_2^T \to \mathcal{L}_2^T \tag{5.5.2}$$

be such that its projection at time $t \in [0,T]$ is given by $K_t(x,\xi)$.

Lemma 5.5.7 *There exists a constant* α_T*, depending on* T*, such that* $\alpha_T \in (0,1)$ *and*

$$\|K(x,\xi)(\omega) - K(x,\eta)(\omega)\|_{\mathcal{L}_2^T} \leq \alpha_T \|\xi - \eta\|_{\mathcal{L}_2^T}. \tag{5.5.3}$$

Proof Let $\mathbf{S}\xi := K_t(x, \xi)$. We shall prove that $\mathbf{S}^n$ is a contraction operator on $\mathcal{L}_2^T$, for sufficiently large values of $n \in \mathbb{N}$. By Lemma 5.5.6 it follows by induction that

$$\int_0^T \mathbb{E}[\, \|\mathbf{S}^n \xi_t - \mathbf{S}^n \eta_t\|^2\,]dt \leq C_T^{1\,n} \int_0^T dt \int_0^T ds_1 \int_0^T ds_2, \dots, \int_0^T \mathbb{E}[\|\xi_{s_n} - \eta_{s_n}\|^2]ds_n$$
$$\leq C_T^{1\,n} \frac{T^n}{n!} \int_0^T \mathbb{E}[\|\xi_s - \eta_s\|^2]ds.$$

From this we get that, for sufficiently large values of $n \in \mathbb{N}$, the operator $\mathbf{S}^n$ is a contraction operator on $\mathcal{L}_2^T$ and therefore has a unique fixed point. Suppose that $\mathbf{S}^{n_0}$ is a contraction operator on $\mathcal{L}_2^T$. We get

$$\begin{aligned}\int_0^T dt\mathbb{E}[\|\mathbf{S}\xi_t - \mathbf{S}\eta_t\|^2] &= \int_0^T dt\mathbb{E}[\|\mathbf{S}^{kn_0+1}\xi_t - \mathbf{S}^{kn_0+1}\eta_t\|^2] \\ &\leq \frac{C_T^{1\,kn_0} T^{kn_0}}{kn_0!} \int_0^T dt\mathbb{E}[\|\mathbf{S}\xi_t - \mathbf{S}\eta_t\|^2] \\ &\leq \frac{C_T^{1\,kn_0+1} T^{kn_0}}{kn_0 + 1!} \int_0^T dt\mathbb{E}[\|\xi_t - \eta_t\|^2] \to 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

□

Proof of Theorem 5.5.3 From (5.5.3) it follows that $K(x, \xi)$ is a contraction on $\mathcal{L}_2^T$ for every fixed $x \in H$. We get by the contraction principle that there exists a $\phi \in C(H, \mathcal{L}_2^T)$ such that

$$K(x, \phi(x)) = \phi(x)$$

for every fixed $x \in H$. $\phi(x) := (Z_t^x(\omega))_{t\in[0,T]}$ is the solution of (5.5.1). □

5.6 Continuous Dependence on Initial Data, Drift and Noise Coefficients

Let $T > 0$. Let us assume that (A), (B), (C), (D) or (A′), (B′), (C′), (D′) are satisfied for $f_0(t, u, z, \omega) := f(t, u, z, \omega)$ and $a_0(t, z, \omega) := a(t, z, \omega)$. Moreover, we assume that this also holds for $f_n(t, u, z, \omega)$ and $a_n(t, z, \omega)$, for any $n \in \mathbb{N}$. Let $(Z_t)_{t\in[0,T]}$ be a solution of (5.5.1) (in the sense of the previous theorems, depending on the hypothesis). We denote by $(Z_t^n(\omega))_{[0,T]}$ the unique solution of

$$Z_t^n(\omega) = S_t Z_0^n(\omega) + \int_0^t S_{t-s} a_n(s, Z_s^n(\omega), \omega)ds$$
$$+ \int_0^t \int_H S_{t-s} f_n(s, u, Z_s^n(\omega), \omega) q(ds, du)$$

(in the sense of the previous theorems). We prove the following result:

Theorem 5.6.1 *Assume that there is a constant $K > 0$ such that for all $n \in \mathbb{N}_0$, $t \in [0, T]$ and $z \in H$*

$$\|a_n(t, z, \omega)\|^2 + \int_H \|f_n(t, u, z, \omega)\|^2 \beta(du) \leq K(\|z\|^2 + 1) \quad \mathbb{P} - a.s. \tag{5.6.1}$$

Assume that there is a constant L such that for all $n \in \mathbb{N}_0$, $t \in [0, T]$ and z, $z' \in H$:

$$T\|a_n(t, z, \omega) - a_n(t, z', \omega)\|^2 + \int_H \|f_n(t, u, z, \omega) - f_n(t, u, z', \omega)\|^2 \beta(du) \\ \leq L\|z - z'\|^2 \quad \mathbb{P} - a.s. \tag{5.6.2}$$

Moreover, assume that

$$\sup_{n \in \mathbb{N}_0} \mathbb{E}[\|Z_0^n)\|^2] < \infty, \tag{5.6.3}$$

$$\lim_{n \to \infty} \mathbb{E}[\|Z_0^n - Z_0\|^2] = 0 \tag{5.6.4}$$

(where $Z_0^0(\omega) := Z_0(\omega)$) and assume that for every $t \in [0, T]$ and fixed $z \in H$

$$\lim_{n \to \infty} \{T\|a_n(t, z, \omega) - a(t, z, \omega)\|^2 + \int_H \|f_n(t, u, z, \omega) - f(t, u, z, \omega)\|^2 \beta(du)\} \\ = 0 \quad \mathbb{P} - a.s. \tag{5.6.5}$$

Then

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E}[\|Z_t^n - Z_t\|^2] = 0.$$

Proof Let $t \leq T$, then:

$$\begin{aligned}\mathbb{E}[\|Z_t^n - Z_t\|^2] \leq 2^5 e^{2\alpha T} \{ &\mathbb{E}[\|Z_0^n - Z_0\|^2] + 2L \int_0^t \mathbb{E}[\|Z_t^n - Z_t\|^2] ds \\ &+ 2T \int_0^t \mathbb{E}[\|a_n(s, Z_s) - a(s, Z_s)\|^2] ds\} \\ &+ 2^5 e^{2\alpha T} \{2 \int_0^t \int_H \mathbb{E}[\|f_n(s, u, Z_s) - f(s, u, Z_s)\|^2] \beta(du) ds\},\end{aligned}$$

where the latter inequality is proved by using a bound on $\|S_t\|$ and inequality (5.6.2).

Let

$$\gamma_t^n := T \int_0^t \mathbb{E}[\|a_n(s, Z_s) - a(s, Z_s)\|^2]ds,$$

$$\delta_t^n := \int_0^t \int_H \mathbb{E}[\| f_n(s, u, Z_s) - f(s, u, Z_s)\|^2]\beta(du)ds.$$

As

$$\lim_{n\to\infty} \|a_n(s, Z_s, \omega) - a(s, Z_s, \omega)\|^2 \\ + \int_H \| f_n(s, u, Z_s, \omega) - f(s, u, Z_s, \omega)\|^2 \beta(du) = 0\,, \quad \mathbb{P} - a.s.$$

and (5.6.1) implies

$$\|a_n(t, Z_s(\omega), \omega)\|^2 + \int_H \| f_n(t, u, Z_s(\omega), \omega)\|^2 \beta(du) \\ \le K(\|Z_s(\omega)\|^2 + 1) \quad \mathbb{P} - a.s.\,,$$

it follows that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \delta_t^n + \lim_{n\to\infty} \sup_{t\in[0,T]} \gamma_t^n = 0\,.$$

The conclusion then follows by using Gronwall's inequality. □

5.7 Differential Dependence of the Solutions on the Initial Data

In this section we continue to assume, as before, that the coefficients a and f satisfy the conditions (A), (B), (C) and (D) and we shall prove the differential dependence of the solution of (5.3.1) with respect to the initial data. Let

$$K_t(x, \xi) := S_t x + \int_0^t S_{t-s} a(s, \xi_s)ds + \int_0^t \int_H S_{t-s} f(s, u, \xi_s) q(ds, du)$$

with $x \in H$ and $\xi := (\xi_s)_{s\in[0,T]} \in \mathcal{L}_2^T$.

Lemma 5.7.1 *For any $T > 0$ there is a constant C_T^1, resp. C_T^2, such that*

$$\int_0^T \mathbb{E}[\| K_t(x, \xi) - K_t(x, \eta)\|^2]dt \le C_T^1 \int_0^T \mathbb{E}[\|\xi_t - \eta_t\|^2]dt, \tag{5.7.1}$$

$$\int_0^T \mathbb{E}[\|K_t(x,\xi) - K_t(y,\xi)\|^2]dt \leq C_T^2\|x-y\|^2. \tag{5.7.2}$$

Proof Note that (5.7.1) is a special case of Lemma 5.5.6. The proof of (5.7.2) is similar to that of Lemma 5.2.2. □

Let

$$K(x,\xi) : H \times \mathcal{L}_2^T \to \mathcal{L}_2^T$$

be such that its projection at time $t \in [0, T]$ is given by $K_t(x, \xi)$.

Remark 5.7.2 From Theorem 5.3.1 we know that there is a unique solution $(Z_t^x(\omega))_{t\in[0,T]}$ of (5.3.1). Hence, from Theorem 5.3.1 we know that for every fixed $x \in H$

$$K(x, Z_t^x(\omega)) = Z_t^x(\omega) \quad \mathbb{P} - a.s. \tag{5.7.3}$$

We shall now prove some facts about the map K.

Theorem 5.7.3 *Let $\xi \in \mathcal{L}_2^T$ be fixed. The map*

$$K(\cdot,\xi) : H \to \mathcal{L}_2^T$$

is Fréchét differentiable and its derivative $\frac{\partial K}{\partial x}$ along the direction $h \in H$ is such that

$$\frac{\partial K_t(x,\xi)}{\partial x}(h) = S_t h.$$

The proof of Theorem 5.7.3 is easy and follows from the Frechét differentiability of S_t.

Remark 5.7.4 It follows in particular that $\frac{\partial K}{\partial x}$ is in $\mathcal{L}(H;\mathcal{L}_2^T)$.

Let us denote by $\frac{\partial}{\partial z}$ the Fréchét derivative in H. Starting from here we assume that the coefficients a and f in the SPDE also satisfy the following conditions

(E) $\frac{\partial}{\partial z} f(t,u,z)$ exists for all $t \in (0, T]$ and fixed $u \in H$,

(F) $\frac{\partial}{\partial z} a(t,z)$ exists for all $t \in (0, T]$.

Moreover we assume that

$$|||\frac{\partial}{\partial z}a(s,z)|||^2 + \int_H |||\frac{\partial}{\partial z} f(s,z,u)|||^2\beta(du) < \infty \quad \text{uniformly in} \quad z \in H, \quad \text{and} \quad s \in [0,T], \tag{5.7.4}$$

where $|||\cdot|||$ denotes the operator norm of the Fréchét derivative in H.

Theorem 5.7.5 *Let $x \in H$ be fixed.*

$$K(x, \cdot) : \mathcal{L}_2^T \to \mathcal{L}_2^T \tag{5.7.5}$$

is Gateaux differentiable and its derivative $\frac{\partial K}{\partial \xi}$ along the direction $\xi \in \mathcal{L}_2^T$ satisfies

$$\frac{\partial K_t(x,\xi)}{\partial \xi}(\eta_t) = \int_0^t S_{t-s} \frac{\partial}{\partial z} a(s, \xi_s)(\eta_s) ds + \int_0^t \int_H S_{t-s} \frac{\partial}{\partial z} f(s, u, \xi_s)(\eta_s) q(ds, du)$$

(with the notation $\frac{\partial}{\partial z} a(s, \xi_s(\omega))$ (resp. $\frac{\partial}{\partial z} f(s, u, \xi_s(\omega))$) for $\frac{\partial}{\partial z} a(s, z)$ (resp. $\frac{\partial}{\partial z}$ $f(s, u, z)$), at $z = \xi_s(\omega)$).

Proof For any fixed $x \in H$, and $\xi, \eta \in \mathcal{L}_2^T$ we consider the map $r \to K(x, \xi + r\eta)$ from $\mathbb{R}$ to $\mathcal{L}_2^T$. We have

$$K_t(x, \xi + r\eta) = S_t x + \int_0^t S_{t-s} a(s, \xi_s + r\eta_s) ds + \int_0^t \int_H S_{t-s} f(s, u, \xi_s + r\eta_s) q(ds, du).$$

It follows that

$$\frac{1}{r}(K_t(x, \xi + r\eta) - K(x, \xi) = \int_0^t S_{t-s} \frac{(a(s, \xi_s + r\eta_s) - a(s, \xi_s))}{r} ds + \int_0^t \int_H S_{t-s} \frac{(f(s, u, \xi_s + r\eta_s) - f(s, u, \xi_s))}{r} q(ds, du).$$

Let us fix $z \in H$ and define for any $r \neq 0$:

$$a_r(t, z, y) := \frac{a(t, z + ry) - a(t, z)}{r}$$

$$f_r(t, u, z, y) := \frac{f(t, u, z + ry) - f(t, u, z)}{r}$$

where $t \in [0, T]$, $y \in H$. $a_r(t, y, \xi_s(\omega))$ and $f_r(t, u, y, \xi_s(\omega))$ satisfy the conditions (5.6.1) and (5.6.2) with r instead of n (and y instead of z). Moreover, $\frac{\partial}{\partial z} a(s, \xi_s(\omega)) y$ and $\frac{\partial}{\partial z} f(s, u, \xi_s(\omega)) y$ satisfy the same conditions, by condition (5.7.4).

Analogous to (5.6.5), we have (also using the Lipschitz conditions) that

$$\lim_{r\to 0}\{T\|a_r(t,y,\xi_t(\omega))-\frac{\partial}{\partial z}a(t,\xi_t(\omega))y\|^2+\int_{E\setminus\{0\}}\|f_r(t,u,y,\xi_t(\omega))\\ -\frac{\partial}{\partial z}f(t,u,\xi_t(\omega))y\|^2\beta(du)\}=0 \quad \mathbb{P}-a.s.$$

Defining similarly as before

$$\gamma_t^r := T\int_0^t \mathbb{E}[\|a_r(s,\eta_s,\xi_s,)-\frac{\partial}{\partial z}a(s,\xi_s)\eta_s\|^2]ds$$

$$\delta_t^r := \int_0^t\int_H \mathbb{E}[\|f_r(s,u,\eta_s,\xi_s)-\frac{\partial}{\partial z}f(s,u,\xi_s)\eta_s\|^2]\beta(du)ds,$$

and operating in a similar way as in the proof of Theorem 5.6.1, we obtain the desired result. □

We also assume
(G) $\frac{\partial}{\partial z}a(s,z)$ is continuous in z ds-a.s.
(H) $\frac{\partial}{\partial z}f(s,u,z)$ is continuous ds-a.s. in the norm $\|\cdot\|_{\mathcal{L}^2(d\beta)}$ of $\mathcal{L}^2(d\beta)$.

Theorem 5.7.6 *For any fixed $\eta\in\mathcal{L}_2^T$ the function*

$$\frac{\delta}{\delta\xi}K(x,\xi)\eta : H\times\mathcal{L}_2^T\to\mathcal{L}_2^T \tag{5.7.6}$$

is continuous.

Proof of Theorem 5.7.6 Let (x^n,ξ^n) converge to (x,ξ) in $H\times\mathcal{L}_2^T$. For any $n\in\mathbb{N}$ we have that

$$\frac{\partial}{\partial\xi}K(x^n,\xi^n)\eta_t-\frac{\partial}{\partial\xi}K(x,\xi)\eta_t=\int_0^t S_{t-s}(\frac{\partial}{\partial z}a(s,\xi_s^n)\eta_s-\frac{\partial}{\partial z}a(s,\xi_s)\eta_s)\\ +\int_0^t\int_H S_{t-s}(\frac{\partial}{\partial z}f(s,u,\xi_s^n)\eta_s\\ -\frac{\partial}{\partial z}f(s,u,\xi_s)\eta_s)q(ds,dx).$$

From $\|S_t\|\le e^{\alpha t}$ it follows that

$$\int_0^T\mathbb{E}[\|\frac{\partial}{\partial\xi}K(x^n,\xi^n)\eta_t-\frac{\partial}{\partial\xi}K(x,\xi)\eta_t\|^2]dt\\ \le 2Te^{2\alpha T}\int_0^T\mathbb{E}[\|\frac{\partial}{\partial z}a(s,\xi_s^n)\eta_s-\frac{\partial}{\partial z}a(s,\xi_s)\eta_s\|^2ds$$

$$+ 2Te^{2\alpha T} \int_0^T \int_H \mathbb{E}[\| \frac{\partial}{\partial z} f(s, u, \xi_s^n)\eta_s - \frac{\partial}{\partial z} f(s, u, \xi_s)\eta_s \|^2] ds \beta(du). \tag{5.7.7}$$

$\xi^n \to \xi$ in $\mathcal{L}_2^T$ as $n \to \infty$ implies that there is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\xi_s^{n_k} \to \xi_s$ $ds \otimes d\mathbb{P}$ -a.s. in $[0, T] \times \Omega$, as $k \to \infty$. Hence we have

$$\||\frac{\partial}{\partial z} a(s, \xi_s^{n_k}(\omega))\eta_s - \frac{\partial}{\partial z} a(s, \xi_s(\omega))\eta_s|\| \to 0 \quad ds \otimes d\mathbb{P} - a.e.$$
$$\text{in} \quad [0, T] \times \Omega \quad \text{as} \quad k \to \infty$$

and

$$\int_H \||\frac{\partial}{\partial z} f(s, u, \xi_s^n(\omega))\eta_s - \frac{\partial}{\partial z} f(s, u, \xi_s(\omega))\eta_s|\|^2 \beta(du) \to 0$$
$$a.e. \quad ds \otimes d\mathbb{P} \quad in \quad [0, T] \times \Omega. \tag{5.7.8}$$

We get by the Lebesgue dominated convergence theorem that $\frac{\partial}{\partial \xi} K(x, \xi)\eta$ is continuous. □

Corollary 5.7.7 *Let us assume that all the hypotheses of Theorem 5.7.6 hold. Let $(Z_t^x)_{t\in[0,T]}$ denote the solution of (5.3.1) with initial condition*

$$Z_0(\omega) = x \quad \mathbb{P} - a.s.$$

Then $(\frac{\partial}{\partial x} Z_t^x)_{t\in[0,T]}$ is a solution of

$$\frac{\partial}{\partial x} Z_t^x = \int_0^t (S_{t-s} \frac{\partial}{\partial z} a(s, Z_s^x) \frac{\partial}{\partial x} Z_s^x)\, ds$$
$$+ \int_0^t \int_H (S_{t-s} \frac{\partial}{\partial z} f(s, u, Z_s^x) \frac{\partial}{\partial x} Z_s^x)\, q(ds, dx). \tag{5.7.9}$$

Proof The statement of Corollary 5.7.7 is a consequence of Theorems 5.7.3–5.7.6, Remark 5.7.4 and Proposition C.0.3 in Appendix C of [15] (see also Appendix C of [19], where the Gaussian case is considered). □

5.8 Remarks and Related Literature

In this chapter, we have studied Hilbert space valued SPDEs. A special case of SPDEs in Banach spaces with certain restrictions on the partial differential operator was considered in [38].

Our presentation is based on [2]. The technique is a generalization of that used in [35] (see also [34]) and is generalized from [36].

The material on Gateaux differentiability with respect to the initial value was generalized in [72].

This work has found applications to financial models in [32].

For our work in Chap. 7 on stability theory we provide the Yosida approximations for mild solutions. As the approximating solutions are strong solutions, we can apply Itô's formula for these. The general case of non-anticipating coefficients is of interest in view of the applications presented in [24].

We refer the reader to [33] where Sz.-Nagy's dilation theorem is used to study uniqueness by relating mild solutions to strong solutions. However, in this form, one does not know how to study the asymptotic behaviour of the equation in Sect. 6.1 to obtain the result on invariant measure in [71], which is done in Chap. 7.

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