Chapter 2
Mereology and Rough Mereology: Rough Mereological Granulation

Ex pluribus unum
[Saint Augustine. Confessions]

In this chapter, we embark on a more specific granulation theory, stemming from the mereological theory of things. This mechanism provides us with tolerance and weak tolerance relations forming a graded similarity in the sense of Chap. 1 and with resulting therefrom granules. To put the necessary notions in a proper order, we are going to discuss mereology, rough mereology and the mereological granulation.

2.1 Mereology

Mereology does address things in terms of parts, cf., [1]. A formal theory of Mereology due to Leśniewski [5] axiomatically defines the notion of a part.

The reader may be aware of the existence of a vast literature on philosophical and ontological aspects of mereology which cannot be mentioned nor discussed here, and, we advise them to consult, e.g., Simons [27], Luschei [11] or Casati and Varzi [2] for discussions of those aspects.

2.1.1 Mereology of Leśniewski

Mereology due to Leśniewski arose from attempts at reconciling antinomies of naïve set theory, see Leśniewski [5, 6, 8], Surma et. al. [9], Sobociński [29, 30]. Leśniewski [5] was the first presentation of the foundations of this attempt as well as the first formally complete exposition of mereology.

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2.1.1.1 On the Notion of Part

The primitive notion of mereology in this formalism is the notion of a *part*. Given some category of things, a relation of a part is a binary relation π which is required to be

M1 *Irreflexive*: *For each thing* x, *it is not true that* $\pi(x, x)$
M2 *Transitive*: *For each triple* x, y, z *of things, if* $\pi(x, y)$ *and* $\pi(y, z)$, *then* $\pi(x, z)$

Remark In the original scheme of Leśniewski, the relation of parts is applied to *individual things* as defined in Ontology of Leśniewski, see Leśniewski [7], Iwanuś [4], Słupecki [28]. Ontology due to Leśniewski is based on the Ontology Axiom:

(AO) $x\varepsilon y$ if and only if $(x\varepsilon x)$ and there is z such that $z\varepsilon y$ and for each z if $z\varepsilon x$

$$\text{then } z\varepsilon y,$$

which singles out x as an *individual* (characterized by the formula $x\varepsilon x$) and y as a *collective thing*, with the copula ε read as *'is'*.

The relation of *part* induces the relation of an *ingredient ingr*, defined as

$$ingr(x, y) \text{ if and only if } \pi(x, y) \ \vee x = y. \tag{2.1}$$

The relation of ingredient is a partial order on things, i.e.,

1. $ingr(x, x)$;
2. If $ingr(x, y)$ and $ingr(y, x)$ then $(x = y)$;
3. If $ingr(x, y)$ and $ingr(y, z)$ then $ingr(x, z)$.

We formulate the third axiom with a help from the notion of an ingredient, see Polkowski [24], Chap. 5,

M3 (*Inference*) For each pair of things x, y, if the property

$$I(x, y):\ \text{if } ingr(t, x),\ \text{then exist } w, z \text{ with } ingr(w, t), ingr(w, z), ingr(z, y)$$

is satisfied, then $ingr(x, y)$.

The predicate of *overlap*, Ov in symbols, is defined by means of

$$Ov(x, y) \text{ if and only if there is } z \text{ such that } ingr(z, x) \text{ and } ingr(z, y). \tag{2.2}$$

Using the overlap predicate, one can write $I(x, y)$ down in the form

$$I_{Ov}(x, y):\ \text{if } ingr(t, x),\ \text{then there is } z \text{ such that } ingr(z, y) \text{ and } Ov(t, z).$$

2.1.1.2 On the Notion of a Class

The notion of a *mereological class* follows; for a non-vacuous property Φ of things, the *class of* Φ, denoted $Cls\Phi$ is defined by the conditions

C1 If $\Phi(x)$, then $ingr(x, Cls\Phi)$
C2 If $ingr(x, Cls\Phi)$, then there is z such that $\Phi(z)$ and $I_{Ov}(x, z)$

In plain language, the class of Φ collects in an individual thing all things satisfying the property Φ. The existence of classes is guaranteed by an axiom

M4 For each non-vacuous property Φ there exists a class $Cls\Phi$

The uniqueness of the class follows by M3. M3 implies also that, for the non-vacuous property Φ, if for each thing z such that $\Phi(z)$ it holds that $ingr(z, x)$, then $ingr(Cls\Phi, x)$.

The notion of an overlap allows for a succinct characterization of a class: for each non-vacuous property Φ and each thing x, it happens that $ingr(x, Cls\Phi)$ if and only if for each ingredient w of x, there exists a thing z such that $Ov(w, z)$ and $\Phi(z)$.

Remark Uniqueness of the class along with its existence is an axiom in the Leśniewski [5] scheme, from which M3 is derived. Similarly, it is an axiom in the Tarski [32–34] scheme.

Consider two examples,

1. The strict inclusion $\subset$ on sets is a part relation. The corresponding ingredient relation is the inclusion $\subseteq$. The overlap relation is the non-empty intersection. For a non-vacuous family F of sets, the class $ClsF$ is the union $\bigcup F$.
2. For reals in the interval [0, 1], the strict order $<$ is a part relation and the corresponding ingredient relation is the weak order $\leq$. Any two reals overlap; for a set $F \subseteq [0, 1]$, the class of F is $supF$.

2.1.1.3 Notions of Element, Subset

The notion of an element is defined as follows

$$el(x, y) \text{ if and only if for a property } \Phi \ y = Cls\Phi \text{ and } \Phi(x). \tag{2.3}$$

In plain words, $el(x, y)$ means that y is a class of some property and x responds to that property. To establish some properties of the notion of an element, we begin with the property $INGR(x) = \{y : ingr(y, x)\}$, for which the identity $x = ClsINGR(x)$ holds by M3. Hence, $el(x, y)$ is equivalent to $ingr(x, y)$. Thus, each thing x is its own element. This is one of means of expressing the impossibility of the Russell paradox within the mereology, cf., Leśniewski [5], Thms. XXVI, XXVII, see also Sobociński [29].

We observe the extensionality of overlap: *For each pair x, y of things, $x = y$ if and only if for each thing z, the equivalence $Ov(z, x) \Leftrightarrow Ov(z, y)$ holds.* Indeed, assume the equivalence $Ov(z, x) \Leftrightarrow Ov(z, y)$ to hold for each z. If $ingr(t, x)$ then $Ov(t, x)$ and $Ov(t, y)$ hence by axiom M3 $ingr(t, y)$ and with $t = x$ we get $ingr(x, y)$. By symmetry, $ingr(y, x)$, hence $x = y$.

The notion of a subset follows,

$$sub(x, y) \text{ if and only if for each z if } ingr(z, x) \text{ then } ingr(z, y). \tag{2.4}$$

It is manifest that for each pair x, y of things, $sub(x, y)$ holds if and only if $el(x, y)$ holds if and only if $ingr(x, y)$ holds.

For the property $Ind(x) \Leftrightarrow ingr(x, x)$, one calls the class $ClsInd$, *the universe*, in symbols V.

2.1.1.4 The Universe of Things, Things Exterior, Complement

It follows by definition of the universe that

1. The universe V is unique;
2. $ingr(x, V)$ holds for each thing x;
3. For each non-vacuous property Φ, it is true that $ingr(Cls\Phi, V)$.

The notion of an *exterior* thing x to a thing y, $extr(x, y)$, is the following

$$extr(x, y) \text{ if and only if } \neg Ov(x, y). \tag{2.5}$$

In plain words, x is exterior to y when no thing is an ingredient both to x and y.

Clearly, the operator of exterior has properties

1. No thing is exterior to itself;
2. $extr(x, y)$ implies $extr(y, x)$;
3. If for a non-vacuous property Φ, a thing x is exterior to every thing z such that $\Phi(z)$ holds, then $extr(x, Cls\Phi)$.

The notion of a *complement* to a thing, with respect to another thing, is rendered as a ternary predicate $comp(x, y, z)$, cf., Leśniewski [5], par. 14, Def. IX, to be read: 'x is the complement to y with respect to z', and it is defined by means of the following requirements

1. $x = ClsEXTR(y, z)$;
2. $ingr(y, z)$, where $EXTR(y, z)$ is the property which holds for a thing t if and only if $ingr(t, z)$ and $extr(t, y)$ hold.

This definition implies that the notion of a complement is valid only when there exists an ingredient of z exterior to y. Following are basic properties of complement,

1. If $comp(x, y, z)$, then $extr(x, y)$ and $\pi(x, z)$;
2. If $comp(x, y, z)$, then $comp(y, x, z)$.

We let for a thing x, $-x = ClsEXTR(x, V)$. It follows that

1. $-(-x) = x$ *for each thing* x;
2. $-V$ *does not exist.*

We conclude this paragraph with two properties of classes useful in the following.

$$\text{If } \Phi \Rightarrow \Psi \text{ then } ingr(Cls\Phi, Cls\Psi), \tag{2.6}$$

and, a corollary

$$\text{If } \Phi \Leftrightarrow \Psi \text{ then } Cls\Phi = Cls\Psi. \tag{2.7}$$

2.2 Rough Mereology

A scheme of mereology, introduced into a collection of things, sets an exact hierarchy of things of which some are (exact) parts of others; to ascertain whether a thing is an exact part of some other thing is in practical cases often difficult if possible at all, e.g., a robot sensing the environment by means of a camera or a laser range sensor, cannot exactly perceive obstacles or navigation beacons. Such evaluation can be done approximately only and one can discuss such situations up to a degree of certainty only. Thus, one departs from the exact reasoning scheme given by decomposition into parts to a scheme which approximates the exact scheme but does not observe it exactly.

Such a scheme, albeit its conclusions are expressed in an approximate language, can be more reliable, as its users are aware of uncertainty of its statements and can take appropriate measures to fend off possible consequences.

Introducing some measures of overlapping, in other words, the extent to which one thing is a part to the other, would allow for a more precise description of relative position, and would add an expressional power to the language of mereology. Rough mereology answers these demands by introducing the notion of a *part to a degree* with the degree expressed as a real number in the interval $[0, 1]$. Any notion of a part by necessity relates to the general idea of *containment*, and thus the notion of a part to a degree is related to the idea of *partial containment* and it should preserve the essential intuitive postulates about the latter.

The predicate of a part to a degree stems ideologically from and has as one of motivations the predicate of an element to a degree introduced by Zadeh as a basis for fuzzy set theory [36]; in this sense, rough mereology is to mereology as the fuzzy set theory is to the naive set theory. To the rough set theory, owes rough mereology the interest in concepts as things for analysis.

The primitive notion of rough mereology is the notion of a *rough inclusion* which is a ternary predicate $\mu(x, y, r)$ where x, y are *things* and $r \in [0, 1]$, read as '*the thing x is a part to degree at least of r to the thing y*'. Any rough inclusion is associated with a mereological scheme based on the notion of a part by postulating that $\mu(x, y, 1)$ is equivalent to $ingr(x, y)$, where the ingredient relation is defined by the adopted mereological scheme. Other postulates about rough inclusions stem from intuitions about the nature of partial containment; these intuitions can be manifold, a fortiori, postulates about rough inclusions may vary. In our scheme for rough mereology, we begin with some basic postulates which would provide a most general framework. When needed, other postulates, narrowing the variety of possible models, can be introduced.

2.2.1 Rough Inclusions

We have already stated that a rough inclusion is a ternary predicate $\mu(x, y, r)$. We assume that a collection of things is given, on which a part relation π is introduced with the associated ingredient relation *ingr*. We thus apply inference schemes of mereology due to Leśniewski, presented above.

Predicates $\mu(x, y, r)$ were introduced in Polkowski and Skowron [25, 26]; they satisfy the following postulates, relative to a given part relation π and the induced by π relation *ingr* of an ingredient, on a set of things:

RINC1 $\mu(x, y, 1)$ if and only if $ingr(x, y)$

This postulate asserts that parts to degree of 1 are ingredients.

RINC2 If $\mu(x, y, 1)$ then $\mu(z, x, r)$ implies $\mu(z, y, r)$ for every z

This postulate does express a feature of partial containment that a 'bigger' thing contains a given thing 'more' than a 'smaller' thing. It can be called a *monotonicity condition* for rough inclusions.

RINC3 If $\mu(x, y, r)$ and $s < r$ then $\mu(x, y, s)$

This postulate specifies the meaning of the phrase 'a part to a degree at least of r'. From postulates RINC1–RINC3, and known properties of ingredients some consequences follow

1. $\mu(x, x, 1)$;
2. If $\mu(x, y, 1)$ and $\mu(y, z, 1)$ then $\mu(x, z, 1)$;
3. $\mu(x, y, 1)$ and $\mu(y, x, 1)$ if and only if $x = y$;
4. If $x \neq y$ then either $\neg\mu(x, y, 1)$ or $\neg\mu(y, x, 1)$;
5. If, for each z, r, $[\mu(z, x, r)$ if and only if $\mu(z, y, r)]$ then $x = y$.

Property 5 may be regarded as an *extensionality postulate* in rough mereology.

It follows that rough inclusions are in general graded weak tolerance relations in the sense of Chap. 1, par. 2.1.

By a *model* for rough mereology, we mean a quadruplex

$$M = (V_M, \pi_M, ingr_M, \mu_M)$$

where V_M is a set with a part relation $\pi_M \subseteq V_M \times V_M$, the associated ingredient relation $ingr_M \subseteq V_M \times V_M$, and a relation $\mu_M \subseteq V_M \times V_M \times [0, 1]$ which satisfies RINC1–RINC3.

We now describe some models for rough mereology which at the same time give us methods by which we can define rough inclusions, see Polkowski [14–17, 20–22], a detailed discussion may be found in Polkowski [24].

2.2.1.1 Rough Inclusions from T-norms

We resort to *continuous t-norms* which are continuous functions $T : [0, 1]^2 \to [0, 1]$ which are 1. symmetric; 2. associative; 3. increasing in each coordinate; 4. satisfying boundary conditions $T(x, 0) = 0, T(x, 1) = x$, cf., Polkowski [24], Chaps. 4, 6, Hájek [3], Chap. 2. Classical examples of continuous t-norms are

1. $L(x, y) = max\{0, x + y - 1\}$ (the *Łukasiewicz t-norm*);
2. $P(x, y) = x \cdot y$ (the *product t-norm*);
3. $M(x, y) = min\{x, y\}$ (the *minimum t-norm*).

The *residual implication* $\Rightarrow_T$ induced by a continuous t-norm T is defined as

$$x \Rightarrow_T y = max\{z : T(x, z) \leq y\}. \tag{2.8}$$

One proves that $\mu_T(x, y, r) \Leftrightarrow x \Rightarrow_T y \geq r$ is a rough inclusion; particular cases are

1. $\mu_L(x, y, r) \Leftrightarrow min\{1, 1 - x + y \geq r\}$ (the *Łukasiewicz implication*);
2. $\mu_P(x, y, r) \Leftrightarrow \frac{y}{x} \geq r$ when $x > 0$, $\mu_P(x, y, 1)$ when $x = 0$ (the *Goguen implication*);
3. $\mu_M(x, y, r) \Leftrightarrow y \geq r$ when $x > 0$, $\mu_M(x, y, 1)$ when $x = 0$ (the *Gödel implication*).

A particular case of continuous t-norms are *Archimedean t-norms* which satisfy the inequality $T(x, x) < x$ for each $x \in (0, 1)$. It is well–known, see Ling [10], that each archimedean t-norm T admits a representation

$$T(x, y) = g_T(f_T(x) + f_T(y)), \tag{2.9}$$

where, the function $f_T : [0, 1] \to R$ is continuous decreasing with $f_T(1) = 0$, and $g_T : R \to [0, 1]$ is the *pseudo–inverse* to f_T, i.e., $g \circ f = id$. It is known, cf., e.g., Hájek [3], that up to an isomorphism there are two Archimedean t-norms: L and P. Their representations are

$$f_L(x) = 1 - x;\;\; g_L(y) = 1 - y, \tag{2.10}$$

and,

$$f_P(x) = exp(-x);\;\; g_P(y) = -ln\, y. \tag{2.11}$$

For an Archimedean t-norm T, we define the rough inclusion μ^T on the interval $[0, 1]$ by means of

$$(ari)\; \mu^T(x, y, r) \Leftrightarrow g_T(|x - y|) \geq r, \tag{2.12}$$

equivalently,

$$\mu^T(x, y, r) \Leftrightarrow |x - y| \leq f_T(r). \tag{2.13}$$

It follows from (2.13), that μ^T satisfies conditions RINC1–RINC3 with *ingr* as identity $=$.

To give a hint of proof: for RINC1: $\mu^T(x, y, 1)$ if and only if $|x - y| \leq f_T(1) = 0$, hence, if and only if $x = y$. This implies RINC2. In case $s < r$, and $|x - y| \leq f_T(r)$, one has $f_T(r) \leq f_T(s)$ and $|x - y| \leq f_T(s)$.

Specific recipes are

$$\mu^L(x, y, r) \Leftrightarrow |x - y| \leq 1 - r, \tag{2.14}$$

and,

$$\mu^P(x, y, r) \Leftrightarrow |x - y| \leq -ln\, r. \tag{2.15}$$

Both residual and archimedean rough inclusions satisfy the *transitivity condition*, cf., Polkowski [24], Chap. 6,

$$\text{(Trans) } \mathit{If}\ \mu(x, y, r)\ \mathit{and}\ \mu(y, z, s),\ \mathit{then}\ \mu(x, z, T(r, s)).$$

To recall the proof, assume, e.g., $\mu^T(x, y, r)$ and $\mu^T(y, z, s)$, i.e., $|x - y| \leq f_T(r)$ and $|y - z| \leq f_T(s)$. Hence, $|x - z| \leq |x - y| + |y - z| \leq f_T(r) + f_T(s)$, hence, $g_T(|x - z|) \geq g_T(f_T(r) + f_T(s)) = T(r, s)$, i.e., $\mu^T(x, z, T(r, s))$. Other cases go along the same lines. Let us observe that rough inclusions of the form (ari) are also *symmetric*, hence they are graded tolerance relations, whereas residual rough inclusions are graded weak tolerance relations needing not be symmetric.

2.2.1.2 Rough Inclusions in Information Systems (Data Tables)

An important domain where rough inclusions will play a dominant role in our analysis of reasoning by means of parts is the realm of *information systems* of Pawlak [13], cf., Polkowski [24], Chap. 6. We will define information rough inclusions denoted with a generic symbol μ^I.

We recall that an *information system* (a *data table*) is represented as a pair (U, A) where U is a finite set of things and A is a finite set of *attributes*; each attribute $a : U \to V$ maps the set U into the *value set* V. For an attribute a and a thing v, $a(v)$ is the value of a on v.

For things u, v the *discernibility set* $DIS(u, v)$ is defined as

$$DIS(u, v) = \{a \in A : a(u) \neq a(v)\}. \tag{2.16}$$

For an (ari) μ_T, we define a rough inclusion μ_T^I by means of

$$(airi)\ \mu_T^I(u, v, r) \Leftrightarrow g_T(\frac{|DIS(u, v)|}{|A|}) \geq r. \tag{2.17}$$

Then, μ_T^I is a rough inclusion with the associated ingredient relation of identity and the part relation empty. These relations are graded tolerance relations.

For the Łukasiewicz t-norm, the *airi* μ_L^I is given by means of the formula

$$\mu_L^I(u, v, r) \Leftrightarrow 1 - \frac{|DIS(u, v)|}{|A|} \geq r. \tag{2.18}$$

We introduce the set $IND(u, v) = A \setminus DIS(u, v)$. With its help, we obtain a new form of (2.18)

$$\mu_L^I(u, v, r) \Leftrightarrow \frac{|IND(u, v)|}{|A|} \geq r. \tag{2.19}$$

The formula (2.19) witnesses that the reasoning based on the rough inclusion μ_L^I is the probabilistic one which goes back to Łukasiewicz [12]. Each (airi)–type rough inclusion μ_T^I satisfies the transitivity condition (Trans) and is symmetric.

2.2.1.3 Rough Inclusions on Sets and Measurable Sets

Formula (2.19) can be abstracted for set and geometric domains. For finite sets A, B,

$$\mu^S(A, B, r) \Leftrightarrow \frac{|A \cap B|}{|A|} \geq r \tag{2.20}$$

defines a rough inclusion μ^S. For bounded measurable sets X, Y in an Euclidean space E^n,

$$\mu^G(A, B, r) \Leftrightarrow \frac{||A \cap B||}{||A||} \geq r, \tag{2.21}$$

where, $||A||$ denotes the area (the Lebesgue measure) of the region A, defines a rough inclusion μ^G. Both μ^S, μ^G are neither symmetric nor transitive, hence, they are graded weak tolerance relations.

Other rough inclusions and their weaker variants will be defined in later chapters.

2.3 Granules from Rough Inclusions

The idea of mereological granulation of knowledge, see Polkowski [16–19], cf., surveys Polkowski [21, 22], presented here finds an effective application in problems of synthesis of classifiers from data tables. This application consists in granulation of data at preprocessing stage in the process of synthesis: after granulation, a new data set is constructed, called a *granular reflection*, to which various strategies for rule synthesis can be applied. This application can be regarded as a process of *filtration* of data, aimed at reducing noise immanent to data. Application of rough inclusions leads to a formal theory of granules of various *radii* allowing for various choices of coarseness degree in data.

Granules are formed here as simple granules in the sense of Chap. 1, with tolerance or weak tolerance induced by a rough inclusion. Assume that a rough inclusion μ is given along with the associated ingredient relation *ingr*, as in postulate RINC1.

The *granule* $g_\mu(u, r)$ of the radius r about the center u is defined as the class of property $\Phi^{\mu}_{u,r}$, i.e.,

$$\Phi^{\mu}_{u,r}(v) \text{ if and only if } \mu(v, u, r). \tag{2.22}$$

The granule $g_\mu(u, r)$ is defined by means of

$$g_\mu(u, r) = Cls\Phi^{\mu}_{u,r}. \tag{2.23}$$

Properties of granules depend, obviously, on the type of rough inclusion used in their definitions. We consider separate cases, as some features revealed by granules differ from a rough inclusion to a rough inclusion. The reader is asked to refer to the axiom M3 for the tool for mereological reasoning, which is going to be used in what follows.

In case of Archimedean t-norm–induced rough inclusions (ari), or (airi)–type rough inclusions, by their transitivity, and symmetry, the important property holds, see Polkowski [21, 24].

Proposition 6 *In case of a symmetric and transitive rough inclusion μ, for each pair u, v of objects , and $r \in [0, 1]$, $ingr(v, g_\mu(u, r))$ if and only if $\mu(v, u, r)$ holds. In effect, the granule $g_\mu(u, r)$ can be represented as the set $\{v : \mu(v, u, r)\}$.*

Proof (op.cit., op.cit.) Assume that $ingr(v, g_\mu(u, r))$ holds. Thus, there exists z such that $Ov(z, v)$ and $\mu(z, u, r)$. There is x with $ingr(x, v)$, $ingr(x, z)$, hence, by transitivity of μ, also $\mu(x, u, r)$ holds. By symmetry of μ, $ingr(v, x)$, hence, $\mu(v, x, r)$ holds also $\square$

In case of rough inclusions in information systems, induced by residual implications generated by continuous t-norms, we have a positive case, for the minimum t-norm M, see Polkowski [24].

Proposition 7 *For the rough inclusion μ induced by the residual implication $\Rightarrow_M$, due to the minimum t-norm M, and $r < 1$, the relation $ingr(v, g_\mu(u, r))$ holds if and only if $\mu(v, u, r)$ holds.*

Proof (loc.cit.) We recall the proof. The rough inclusion μ has the form $\mu(v, u, r)$ if and only if $\frac{|IND(v,s)|}{|A|} \Rightarrow_M \frac{|IND(u,s)|}{|A|} \geq r$. If $ingr(v, g_\mu(u, r))$ holds, then by the class definition, there exists z such that $Ov(v, z)$ and $\mu(z, u, r)$ hold. Thus, we have

w with $ingr(w, v)$ and $\mu(w, u, r)$ by transitivity of μ and the fact that $ingr(w, z)$. By definition of μ, $ingr(w, v)$ means that $|IND(w, s)| \leq |IND(v, s)|$. As $\mu(w, u, r)$ with $r < 1$ means that $|IND(u, s)| \geq r$ because of $|IND(w, s)| \geq |IND(u, s)|$, the condition $|IND(w, s)| \leq |IND(v, s)|$ implies that $\mu(v, u, r)$ holds as well □

The case of the rough inclusion μ induced either by the product t-norm $P(x, y) = x \cdot y$, or by the Łukasiewicz t-norm L, is a bit more intricate. To obtain in this case some positive result, we exploit the averaged t-norm $\vartheta(\mu)$ defined for the rough inclusion μ, induced by a t-norm T, by means of the formula, see Polkowski [24], Chap. 7, par. 7.3, from which this result is taken,

$$\vartheta(\mu)(v, u, r) \Leftrightarrow \forall z.\exists a, b.\mu(z, v, a), \mu(z, u, b), a \Rightarrow_T b \geq r. \tag{2.24}$$

Our proposition for the case of the t-norm P is, op.cit.

Proposition 8 *For $r < 1$, $ingr(v, g_{\vartheta(\mu)}(u, r))$ holds if $\mu(v, u, a \cdot r)$, where $\mu(v, t, a)$ holds for t which obeys conditions $ingr(t, v)$ and $\vartheta(\mu)(t, u, r)$.*

Proof $ingr(v, g_{\vartheta(\mu)}(u, r))$ implies that there is w such that $Ov(v, w)$ and $\vartheta(\mu)(w, u, r)$, so we can find t with properties, $ingr(t, w)$, $ingr(t, v)$, hence, by transitivity of $\vartheta(\mu)$ also $\vartheta(\mu)(t, u, r)$.

By definition of $\vartheta(\mu)$, there are a, b such that $\mu(v, t, a)$, $\mu(v, u, b)$, and $a \Rightarrow_P b \geq r$, i.e., $\frac{b}{a} \geq r$. Thus, $\mu(v, u, b)$ implies $\mu(v, u, a \cdot r)$ □

An analogous reasoning brings forth in case of the rough inclusion μ induced by residual implication due to the Łukasiewicz implication L, the result that, op.cit.

Proposition 9 *For $r < 1$, $ingr(v, g_{\vartheta(\mu)}(u, r))$ holds if and only if $\mu(v, u, r + a - 1)$ holds, where $\mu(v, t, a)$ holds for t such that $ingr(t, v)$ and $\vartheta(\mu)(t, u, r)$.*

The two last propositions can be recorded jointly in the form

Proposition 10 *For $r < 1$, and μ induced by residual implications either $\Rightarrow_P$ or $\Rightarrow_L$, $ingr(v, g_{\vartheta(\mu)}(u, r))$ holds if and only if $\mu(v, u, T(r, a))$ holds, where $\mu(v, t, a)$ holds for t such that $ingr(t, v)$ and $\vartheta(\mu)(t, u, r)$.*

Granules as collective concepts can be objects for rough mereological calculi.

2.3.1 Rough Inclusions on Granules

Due to the feature of mereology that it operates (due to the class operator) only on level of individuals, one can extend rough inclusions from objects to granules; the formula for extending a rough inclusion μ to a rough inclusion $\overline{\mu}$ on granules is a modification of mereological axiom M3, see Polkowski [24], Chap. 7, par. 7.4:

$\overline{\mu}(g, h, r)$ if and only if for each z if $ingr(z, g)$ then there is w such that

$$ingr(w, h) \text{ and } \mu(z, w, r).$$

Proposition 11 *The predicate $\overline{\mu}(g, h, r)$ is a rough inclusion on granules.*

Proof To recall the proof, see that $\mu(g, h, 1)$ means that for each object z with $ingr(z, g)$ there exists an object w with $ingr(w, h)$ such that $\mu(z, w, 1)$, i.e., $ingr(z, w)$, which, by the inference rule implies that $ingr(g, h)$. This proves RINC1. For RINC2, assume that $\mu(g, h, 1)$ and $\mu(k, g, r)$ so for each $ingr(x, k)$ there is $ingr(y, g)$ with $\mu(x, y, r)$. For y there is z such that $ingr(z, h)$ and $\mu(y, z, 1)$, hence, $\mu(x, z, r)$ by property RINC2 of μ. Thus, $\mu(k, h, r)$. RINC2 follows and RINC3 is obviously satisfied. □

We now examine rough mereological granules with respect to their properties.

2.4 General Properties of Rough Mereological Granules

In this exposition, we follow the results presented in Polkowski [24] with references given therein. The basic properties are collected in

Proposition 12 *The following constitute a set of basic properties of rough mereological granules*

1. *If $ingr(y, x)$ then $ingr(y, g_\mu(x, r))$;*
2. *If $ingr(y, g_\mu(x, r))$ and $ingr(z, y)$ then $ingr(z, g_\mu(x, r))$;*
3. *If $\mu(y, x, r)$ then $ingr(y, g_\mu(x, r))$;*
4. *If $s < r$ then $ingr(g_\mu(x, r), g_\mu(x, s))$,*

which follow straightforwardly from properties RINC1–RINC3 of rough inclusions and the fact that *ingr* is a partial order, in particular it is transitive, regardless of the type of the rough inclusion μ.

For T–transitive rough inclusions, we can be more specific, and prove

Proposition 13 *For each T-transitive rough inclusion μ,*

1. *If $ingr(y, g_\mu(x, r))$ then, for each s, $ingr(g_\mu(y, s), g_\mu(x, T(r, s)))$;*
2. *If $\mu(y, x, s)$ with $1 > s > r$, then there exists $\alpha < 1$ with the property that $ingr(g_\mu(y, \alpha), g_\mu(x, r))$.*

Proof Property 1 follows by transitivity of μ with the t-norm T. Property 2 results from the fact that the inequality $T(s, \alpha) \geq r$ has a solution in α, e.g., for $T = P$, $\alpha \geq \frac{r}{s}$, and, for $T = L$, $\alpha \geq 1 - s + r$ □

It is natural to regard granule system $\{g_r^{\mu_t}(x) : x \in U; r \in (0, 1)\}$ as a neighborhood system for a topology on U that may be called the *granular topology*.

In order to make this idea explicit, we define classes of the form

$$N^T(x, r) = Cls(\psi_{r,x}^{\mu_T}), \tag{2.25}$$

where,

$$\psi_{r,x}^{\mu_T}(y) \text{ if and only if there is } s > r \text{ such that } \mu_T(y, x, s). \tag{2.26}$$

We declare the system $\{N^T(x, r) : x \in U; r \in (0, 1)\}$ to be a neighborhood basis for a topology θ_μ. This is justified by the following

Proposition 14 *Properties of the system* $\{N^T(x, r) : x \in U,\ r \in (0, 1)\}$ *are as follows:*

1. *If* $ingr(y, N^T(x, r))$ *then there is* $\delta > 0$ *such that* $ingr(N^T(y, \delta), N^T(x, r))$*;*
2. *If* $s > r$ *then* $ingr(N^T(x, s), N^T(x, r))$*;*
3. *If* $ingr(z, N^T(x, r))$ *and* $ingr(z, N^T(y, s))$ *then there is* $\delta > 0$ *such that* $ingr(N^T(z, \delta), N^T(x, r))$ *and* $ingr(N^T(z, \delta), N^T(y, s))$.

Proof For Property 1, $ingr(y, N^t(x, r))$ implies that there exists an $s > r$ such that $\mu_t(y, x, s)$. Let $\delta < 1$ be such that $t(u, s) > r$ whenever $u > \delta$; δ exists by continuity of t and the identity $t(1, s) = s$. Thus, if $ingr(z, N^t(y, \delta))$, then $\mu_t(z, y, \eta)$ with $\eta > \delta$ and $\mu_t(z, x, t(\eta, s))$ hence $ingr(z, N^t(x, r))$.

Property 2 follows by RINC3 and Property 3 is a corollary to properties 1 and 2. This concludes the argument. □

2.5 Ramifications of Rough Inclusions

In problems of classification, it turns out important to be able to characterize locally the distribution of values in data. The idea that metrics used in classifier construction should depend locally on the training set is, e.g., present in classifiers based on the idea of nearest neighbor, see, e.g., a survey in Polkowski [23]: for nominal values, the metric Value Difference Metric (VDM) in Stanfill and Waltz [31] takes into account conditional probabilities $P(d = v|a_i = v_i)$ of decision value given the attribute value, estimated over the training set *Trn*, and on this basis constructs in the value set V_i of the attribute a_i a metric $\rho_i(v_i, v_i') = \sum_{v \in V_d} |P(d = v|a_i = v_i) - P(d = v|a_i = v_i')|$. The global metric is obtained by combining metrics ρ_i for all attributes $a_i \in A$ according to one of many-dimensional metrics.

This idea was also applied to numerical attributes in Wilson and Martinez [35] in metrics *IVDM* (Interpolated VDM) and *WVDM* (Windowed VDM). A modification of the *WVDM* metric based again on the idea of using probability densities in determining the window size was proposed as *DBVDM* metric.

In order to construct a measure of similarity based on distribution of attribute values among objects, we resort to residual implications, of the form μ_T; this rough inclusion can be transferred to the universe U of an information system; to this end, first, for given objects u, v, and $\varepsilon \in [0, 1]$, factors

$$dis_\varepsilon(u, v) = \frac{|\{a \in A : |a(u) - a(v)| \geq \varepsilon\}|}{|A|}, \tag{2.27}$$

and,

$$ind_{\varepsilon}(u, v) = \frac{|\{a \in A : |a(u) - a(v)| < \varepsilon\}|}{|A|}, \tag{2.28}$$

are introduced. The weak variant of rough inclusion $\mu_{\rightarrow_T}$ is defined, see Polkowski [20], as

$$\mu^*_T(u, v, r) \text{ if and only if } dis_{\varepsilon}(u, v) \rightarrow_T \ ind_{\varepsilon}(u, v) \geq r. \tag{2.29}$$

Particular cases of this similarity measure induced by, respectively, t-norm *min*, t-norm $P(x, y)$, and t-norm L are,

1. For $T = M(x, y) = min(x, y)$, $x \Rightarrow_{min} y$ *is* y in case $x > y$ and 1 otherwise, hence, $\mu^*_{min}(u, v, r)$ if and only if $dis_{\varepsilon}(u, v) > ind_{\varepsilon}(u, v) \geq r$ with $r < 1$ and 1 otherwise.
2. For $t = P$, with $P(x, y) = x \cdot y$, $x \Rightarrow_P y = \frac{y}{x}$ when $x > y$ and 1 when $x \leq y$, hence, $\mu^*_P(u, v, r)$ if and only if $\frac{ind_{\varepsilon}(u,v)}{dis_{\varepsilon}(u,v)} \geq r$ *with* $r < 1$ and 1 otherwise.
3. For $t = L$, $x \Rightarrow_L y = min\{1, 1 - x + y\}$, hence, $\mu^*_L(u, v, r)$ if and only if $1 - dis_{\varepsilon}(u, v) + ind_{\varepsilon}(u, v) \geq r$ with $r < 1$ and 1 otherwise.

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