

Limit Analysis and Macroscopic Strength of Porous Materials with Coulomb Matrix

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Abstract The paper is devoted to the numerical Limit Analysis of a hollow spheroidal model with a Coulomb solid matrix. In a first part the hollow spheroid model is presented, together with its axisymmetric FEM discretization and its mechanical position. Then, after an adaptation of a previous static code, an original mixed (but fully kinematic) approach dedicated to the axisymmetric problem was elaborated with a specific quadratic velocity field associated to the triangular finite element. Despite the less good conditioning inherent to the axisymmetric modelization, the final conic mixed code appears very efficient, allowing to take into account numerical meshes highly refined. After a first validation in the case of spherical cavities and isotropic loadings, for which the exact solution is known, numerical bounds of the macroscopic strength are provided for both cases of spherical and spheroidal voids. Effects of the friction angle as well as that of the void aspect ratio are fully illustrated.

1 Introduction

Focused to the axisymmetric problem, the present paper aims at responding to two purposes. First it generalizes the linear programming formulations of the limit analysis (LA) in the axisymmetric case for Coulomb materials of [22, 31]: this is partly motivated by recent papers about this subject that only give, in fact, estimates of the resulting bounds, as recently pointed out in [26]. The second purpose deals with the determination of the macroscopic plasticity criterion of the porous Coulomb material with spheroidal voids, never investigated up to now, on the basis of the

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Gurson-like hollow spheroid problem. To overcome the conditioning difficulties of the classic upper bound (or kinematic) approach for solving this problem, a new axisymmetric mixed formulation is presented which preserves the rigorous kinematic character of the resulting solutions. This study was also motivated by the recent results about porous materials with Drucker-Prager matrices in [18]; in this paper the corresponding kinematic mixed approach has allowed—with very much lower computation times—significant improvements of the upper bounds obtained by previous fully 3D numerical methods.

The famous Gurson plasticity criterion [9] is based on the consideration of a hollow von Mises sphere or cylinder in the framework of the Limit Analysis (LA) kinematic approach. Recent studies were devoted to porous materials with a Drucker-Prager matrix and spherical voids ([8, 11, 28], etc.). Even in this relatively simple case of spherical voids, there is no theoretical attempt to micromechanically derive a macroscopic criterion for the Coulomb porous material. This is undoubtedly due to difficulties inherent to the Coulomb criterion which involves all the three stress invariants.

Several extensions of the Gurson model taking into account void shape effects have been also proposed in order to solve various practical cases (see for example [5–7, 15, 16]). Up to our knowledge, similar theoretical studies with pressure-sensitive matrices and non-spherical voids do not exist in the literature: clearly, in this case of spheroidal voids also, there is a strong need of an appropriate research effort to obtain efficient estimates.

On the other hand, using finite element discretizations, both static and kinematic methods of LA have been elaborated for Gurson's problems with cylindrical cavities ([4, 23, 27]). In [30] the Gurson criterion (then with a von Mises matrix) is shown to be relevant for materials with spherical voids by using the same tools and an new three-dimensional numerical model. Using also the hollow sphere model, a recent paper [19] was devoted to spherically porous materials with pressure-sensitive matrices obeying the Drucker-Prager, Mises-Schleicher and Green criteria. On the other hand, in the case of a von Mises matrix, these numerical studies have been extended to take into account the void shape effects by considering a central spheroid void in a matrix with a confocal boundary [21]. They concluded on the relevance of the criteria proposed by the above mentioned studies of Gologanu and Leblond, at least for the investigated porosity cases. Very recently, as mentioned before, these approaches have been extended to the hollow spheroid problem with a Drucker-Prager matrix in [18]. They provided very original bounds which are expected to be used as reference results for validation of future theoretical investigations.

Therefore, the second purpose of the present paper is concerned with providing lower/upper bound results to be used as reference values for forthcoming attempts to determine approximate criteria for porous Coulomb materials in the framework of limit analysis (LA) applied to the model of the hollow sphere or spheroid. Indeed, these numerical approaches provide rigorous bounds (*a posteriori* controllable) to the macroscopic criterion. Moreover, due to the selected projection approach, the direction of the kinematically admissible (macroscopic) strain rates can be easily

obtained as the normal to the kinematic approach in the projection plane, since the homogenized material complies with the associated flow law.

First, we briefly present the hollow spheroid problem and its formulation in terms of LA. Then, we recall the basis of the LA static and mixed methods, and the corresponding expressions for the Coulomb material necessary for obtaining the bounds, or to assess them by post-analysis of the optimal fields. The next step presents the *ad hoc* extended static approach, and details the new formulation of the mixed (but rigorously kinematic) method, in the axisymmetric case for the Coulomb problem. Let us note that it is the first time that the LA mixed approach is applied to an axisymmetric finite element problem with this frictional material. After a validation on the hollow sphere problem under isotropic loading whose the exact solution is known, we finally provide the results of the axisymmetric tests for two values of the friction angle of the Coulomb criterion and three aspect ratios of the cavity. A comparison with Drucker-Prager results of [18] is also given, in order to point out the differences of the two porous material criteria, all porous Coulomb test exhibiting corners, not the Drucker-Prager ones.

2 The Hollow Spheroid Model

The hollow spheroid model is made up of a spheroidal cavity embedded in a confocal spheroidal cell. The solid matrix is a Coulomb material, homogeneous and isotropic. Figure 1 presents the geometric model, where the given aspect ratio a_1/b_1 and the porosity f allow to determine the parameters (a_2 and b_2) of the confocal spheroidal boundary. Let us consider first the three-dimensional point of view, and note Σ and D the macroscopic stress and strain rate tensors. These quantities are related to

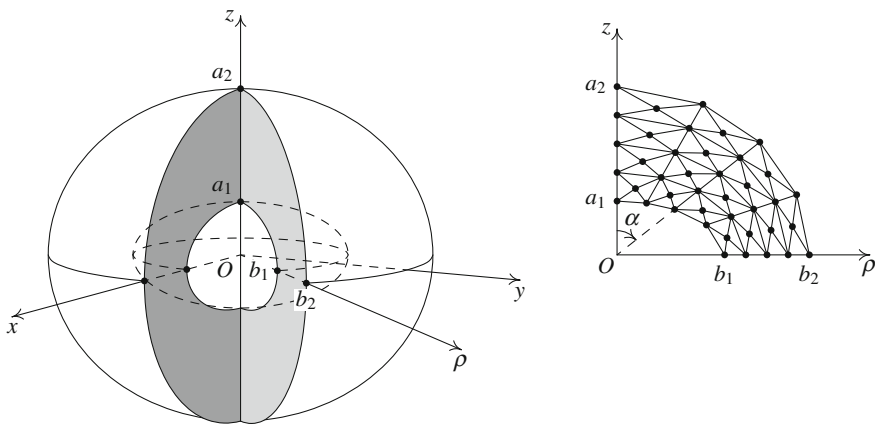


Fig. 1 The hollow spheroid model and its FEM axisymmetric discretization ($a_1/b_1 = 0.5, f = 0.1$)

the microscopic fields by the averages over the model of volume V :

$$\Sigma_{ij} = \frac{1}{V} \int_V \sigma_{ij} dV, \quad D_{ij} = \frac{1}{2V} \int_{\partial V} (u_i n_j + u_j n_i) dS, \quad (1)$$

where u denotes the velocity vector and n the normal vector to the boundary ∂V of the model.

Under the uniform strain rate boundary conditions, i.e. $u_i = D_{ij}x_j$ (where x represents the position vector), on the external boundary, the virtual dissipated power $P_{\text{tot}} = V \Sigma_{ij} D_{ij}$ can be written as follows:

$$P_{\text{tot}} = V Q \cdot q, \quad (2)$$

where the loading vector Q and the generalized velocity q here are defined as:

$$\begin{aligned} Q_1 &= \Sigma_m = \frac{1}{3} (\Sigma_x + \Sigma_y + \Sigma_z), \quad Q_2 = \frac{\Sigma_x + \Sigma_y}{2} - \Sigma_z, \\ Q_3 &= \frac{\sqrt{3}}{2} (\Sigma_x - \Sigma_y), \quad Q_4 = \Sigma_{yz}, \quad Q_5 = \Sigma_{zx}, \quad Q_6 = \Sigma_{xy}; \\ q_1 &= D_x + D_y + D_z, \quad q_2 = \frac{2}{3} \left(\frac{D_x + D_y}{2} - D_z \right), \quad q_3 = \frac{1}{\sqrt{3}} (D_x - D_y), \\ q_4 &= 2D_{yz}, \quad q_5 = 2D_{zx}, \quad q_6 = 2D_{xy}. \end{aligned}$$

From the matrix isotropy and the spheroidal geometry of the model, the resulting material is transversally isotropic around the axis z . Hereafter is investigated the macroscopic criterion $g(\Sigma)$ in the $(Oxyz)$ anisotropy frame of Fig. 1. We search the projection of $g(\Sigma)$ in the (Q_1, Q_2) plane by optimizing Q_2 for fixed $Q_1 = \Sigma_m$, other stress components defined in (2) being free. Consequently, $\frac{\partial g}{\partial \Sigma_{ij}} = 0 = 2D_{ij}$ for $i \neq j$, and $\frac{\partial g}{\partial Q_3} = 0 = q_3$ since the macroscopic material complies with the normality rule. Loadings can then be restricted to principal strain rates D as well as Σ ($(Oxyz)$ is a transverse isotropy frame), with $D_x = D_y$ and $\Sigma_x = \Sigma_y$. Finally, the problem involves only two non-zero loading parameters, Q_1 and Q_2 .

Then, from the symmetries of the whole model, the quarter of the meridian plane of the hollow spheroid is meshed into triangular elements as shown in Fig. 1, right. Note that the macroscopic equivalent stress Σ_{eq} is, in the present case, linked to Q_2 by:

$$\Sigma_{eq} = \sqrt{\frac{3}{2} \Sigma' : \Sigma'} = |Q_2| = |\Sigma_x - \Sigma_z| = |\Sigma_\rho - \Sigma_z|, \quad (3)$$

where Σ' is the deviatoric part of Σ . In this case we can substitute in the above relations Σ_ρ to Σ_x and to Σ_y (Σ_z unchanged), and D_ρ to D_x and D_y . The loading conditions $u_i = D_{ij}x_j$ here become $u_\rho = D_\rho \rho$, $u_z = D_z z$.

To obtain the static approach, we have first extended the axisymmetric finite element code and the process defined in [24]. For the kinematic approach, we have elaborated an original axisymmetric version of the mixed kinematic approach based on specific quadratic velocities in the triangular elements. These *ad hoc* formulations avoids the possible singularities of the axisymmetric equations in order to preserve the rigorous bounding character of the solutions. Hereafter, we present the static process and we detail the new axisymmetric mixed method, and their application to the investigated problem. The final resulting graphs are presented for the usual values of the friction angle $\phi = 10^\circ$ and 20° , and for the aspect factor $a_1/b_1 = 1$, 0.5 and 0.2.

3 LA Methods and Coulomb Criterion

The main goal of limit analysis is the determination of the locus of the limit loads which corresponds to the macroscopic plasticity criterion in the present micro-macro problem. Classically, the limit loads can be determined by using the static (or lower bound) and the kinematic (or upper bound) methods.

3.1 The Static Method

The first one is the static method which is in terms of stresses and leads to a lower bound to the limit loads. A stress field is said admissible if it is statically admissible (SA), and plastically admissible (PA), i.e. verifying the (convex) plasticity criterion $f(\sigma)$; a loading vector $Q(\sigma)$ is admissible if the corresponding σ is admissible. Let us denote K the set of the admissible loading vectors. The final problem reads:

$$Q_{lim} = (Q_1^d, \dots, \lambda_0 Q_i^d, \dots, Q_n^d); \quad (4.i)$$

$$\lambda_0 = \max \{ \lambda, Q(\sigma) = (Q_1^d, \dots, \lambda Q_i^d, \dots, Q_n^d) \} \quad (4.ii)$$

where the stress tensors σ are admissible, and Q_d is a fixed admissible loading vector. In fact, Relation (4.i and 4.ii) holds when all admissible fields σ can be taken into account, which is not the case in general. Then, by varying the direction of Q_d , a set of admissible Q , located near or on ∂K , is obtained: the smallest convex envelope of the corresponding points in K is an inner approach of the boundary ∂K , i.e. a lower bound to the exact macroscopic criterion investigated here.

3.2 The Mixed Kinematic Method

On the other hand, a so-called mixed kinematic formulation was pioneered by Anderheggen and Knopfel [1] for finite element continuous velocity and a linearized von Mises criterion, resulting in linear programming (LP) problems. An extension to the discontinuous velocity case, based on the assumption that the LP duality properties remain valid in non-linear programming, was proposed in [10]. A general extension to discontinuous velocity fields and convex optimization was successfully experienced in [17, 20] for homogeneous von Mises and Gurson materials in plane strain. This mixed formulation is here modified as:

$$\max_{Q, \sigma, T'} F = Vq_d \cdot Q \quad (5.i)$$

$$\text{s.t.} \quad \int_V d : \sigma \, dV + \int_{S_d} [u] \cdot T' \, dS = Vq(u) \cdot Q \quad \forall \text{ KA } u, \quad (5.ii)$$

$$f(\sigma) \leq 0, \quad f_{nt}(T') \leq 0, \quad (5.iii)$$

where d is the strain rate tensor, σ the stress tensor, S_d the union of the velocity discontinuity surfaces, T' the stress vector on these surfaces and $f_{nt}(T')$ the projection of $f(\sigma')$ on the Mohr plane associated to the discontinuity surface of normal n . In (5.i–5.iii) the velocity field u must be kinematically admissible (KA), i.e. piecewise continuous with bounded discontinuities $[u]$ and verifying the boundary conditions and the loading condition $q(u) = q_d$. It can be seen in the above mentioned papers that the optimal velocity field will also be PA (plastically admissible), i.e. there exist everywhere a stress tensor σ or a stress vector T' associated to the strain rate tensor or to the velocity jump by the normality law corresponding to $f(\sigma) = 0$ or $f_{nt}(T') = 0$, respectively.

The previous formulation gives the exact solution if all velocity and stress fields could be taken into account. In general this is not the case when we consider a discretization of the mechanical system in finite elements. However, the following axisymmetric formulation is formulated for preserving the fully kinematic character of the final result by using convexity properties concerning the set of the PA strain rates (and of the PA velocity jumps) and the unit dissipated powers.

3.3 The Coulomb Material

The original Coulomb criterion reads:

$$f(\sigma) = |\sigma_i - \sigma_j| - 2c \cos \phi + (\sigma_i + \sigma_j) \sin \phi \leq 0 \quad (6)$$

where σ_i and σ_j refer to the principal stresses without any *a priori* order ($i, j = 1, 2, 3$, $i \neq j$), c is the cohesion of the material and ϕ the internal friction angle. In [22], this

criterion is written in the axisymmetric case in the form of three conic inequalities that are linearized to obtain a linear programming problem. From [24] the Coulomb criterion finally results in one conic inequation and three linear inequations:

$$\begin{cases} \sqrt{(\sigma_\rho - \sigma_z)^2 + 4\tau_{\rho z}^2} \leq U, \\ U \leq -\alpha \sin \phi + 2c \cos \phi, \\ U \leq -\left(\alpha - 2\sigma_\theta \frac{1-\sin \phi}{1+\sin \phi}\right) + 4c \frac{\cos \phi}{1+\sin \phi}, \\ U \leq \left(\alpha - 2\sigma_\theta \frac{1+\sin \phi}{1-\sin \phi}\right) + 4c \frac{\cos \phi}{1-\sin \phi}, \end{cases} \quad (7)$$

with $\alpha = \sigma_\rho + \sigma_z$.

In the Mohr plane related to a velocity discontinuity facet of normal n , the Coulomb criterion classically reads:

$$f_{nt}(T) = |\sigma_{nt}| + \sigma_n \tan \phi - c \leq 0, \quad (8)$$

Let us recall the PA (plastic admissibility) condition for the strain rates and the volumic dissipated power [25]:

$$(|d_1| + |d_2| + |d_3|) \sin \phi \leq tr(d); \quad \pi_{vol}(d) = \frac{c}{\tan \phi} tr d \quad (9)$$

where d_1 , d_2 and d_3 are the principal strain rates. From (9), the domain of the PA strain rates is a convex cone whose apex corresponds to the null tensor.

The corresponding relations about the velocity discontinuities read:

$$[u_n] \geq |[u_t]| \tan \phi; \quad \pi_{disc}([u]) = c \frac{[u_n]}{\tan \phi} = c |[u_t]| \quad (10)$$

The expressions (9) and (10) here are only used in the post-analysis of the optimal solution of the mixed method. As mentioned before, two numerical implementation methods were carried out for this study. We first begin with the static method, more precisely the static iterative process which will deliver a lower bound for the solution of the Limit Analysis problem.

4 Numerical Implementation of the Axisymmetric Static Method

4.1 The Stress Field

Since the affine formulation is too “poor” to give satisfactory results, the FEM discontinuous stress field is expressed with *quadratic* functions, as follows:

$$\begin{cases} \sigma_\rho = A + B\rho + Cz + H\rho z + I\rho^2 + Jz^2, \\ \sigma_\theta = A + B_\theta\rho + Cz + H_\theta\rho z + I_\theta\rho^2 + Jz^2, \\ \sigma_z = A_z + B_z\rho + C_zz + H_z\rho z + I_z\rho^2 + J_zz^2, \\ \tau_{\rho z} = B_\tau\rho + H_\tau\rho z + I_\tau\rho^2. \end{cases} \quad (11)$$

It is worth noting that the formulation has been adjusted so as to eliminate ρ from the fraction denominators containing ρ , in the equilibrium equations.

4.2 The SA Conditions

From (11), the SA conditions give rise to linear relations satisfying the definition of the macroscopic stresses, the equilibrium equations inside the triangles, the stress vector continuity across the element sides, the boundary and symmetry stress conditions in the horizontal plane of the (hemispherical) mesh.

4.3 The PA Conditions

The conic inequation of (7) was first used as such, and enhanced with the conic code MOSEK. Unfortunately, this conic formulation did not appear robust enough with significative meshing sizes. In a second attempt the cone was “PieceWise Linearized” (as in [24]), using the classic “PWL” method generating a polyhedric cone and a system of m linear inequations; the resulting problem was solved with the (interior point) linear programming code XA [3]. Due to the necessity of a m value sufficient to obtain precise results, the conditioning problems persisted for the spheroidal case.

Finally, only the so-called “BTN” linearization (based on a projection of cones) of Ben Tal and Nemirovski [2] (see [23]) induced a sufficiently robust formulation for the oblate cavity tests. This method gives an equilibrated number of additional rows ($3m + 1$) and auxiliary columns ($2m$) so that using $m = 6$ is equivalent to the PWL approach with a polyhedron of $2^m = 64$ sides, i.e. a value inducing an unreachable size of the final problem when using the PWL approach.

Moreover, because of the non-affine character of the stress field, the criterion cannot be verified everywhere by imposing it at the three apexes of the triangle only. Hence, the criterion is imposed at fifteen points (or more) regularly located in the element. Fortunately, the quadratic variation and the discontinuities of the stress field made that only moderately sizes of the mesh were necessary in the tests, due also to efficiency of the iterative process below.

4.4 The Post-analysis Process

After the optimization, a rigorous post-analysis is carried out:

- verification of the stress vector continuity across every boundary between adjacent elements; in fact, the stress vector jump is always smaller than 10^{-5} ;
- subdivision of each element into a large number of “subtriangles” (more than 200); accurate computation of Q_1 and Q_2 values, by integral calculation on each subtriangle, followed by a summation on the whole domain;
- verification of the Coulomb criterion inside each subtriangle of each element; this verification is performed with the three original inequations (6), since the stress field is known at this stage.

To set an example, let us verify the following inequation in a subtriangle:

$$|\sigma_1 - \sigma_2| - 2c \cos \phi + (\sigma_1 + \sigma_2) \sin \phi \leq 0;$$

One has to calculate the k ratio: $k = \frac{|\sigma_1 - \sigma_2| + (\sigma_1 + \sigma_2) \sin \phi}{2c \cos \phi}$ and verify the inequality: $k \leq 1$;

if $k > 1$, the current “faulty” element’s number is stored and, if k is greater than the value previously stored during the current post-analysis step, k replaces this previous smaller value;

- if at least one element is non-PA, the optimization is reiterated, after modification of the PA conditions in each “faulty” element: the original cohesion, c , or its previously modified value, c_c is replaced with a new smaller fictitious c_c value: $c_c = c_c/k$, in order to make the PA conditions in this element more severe during the next optimization step;
- the whole process is reiterated, as many times as necessary, as long as the solution is not found admissible *everywhere* in the domain, with respect to the original Coulomb criterion.

5 Numerical Implementation of the Axisymmetric Mixed Method

The axisymmetric mesh in the frame (ρ, z) is given on the right of Fig. 1 with 4 sectors ($n_s = 4$) and 4 layers ($n_\rho = 4$). For each aspect ratio a_1/b_1 of the cavity and for a given porosity, the matrix boundaries of the mesh are adapted to obtain their confocal forms. Since these boundaries are not homothetic, the porosity of the resulting mesh is not exactly the same as the input one. Therefore, in a first step for each case of porosity and aspect ratio, the distribution of the angle α is optimized to obtain the desired porosity by progressively concentrating this distribution towards the more curved zone.

5.1 The Virtual Velocity Field

In each triangle, the FEM discontinuous displacement velocity field is expressed as follows:

$$\begin{cases} u_\rho = \rho (A + B \rho + C z) \\ u_\theta = 0 \\ u_z = D + E \rho + F z + H \rho z + I \rho^2 + J z^2 \end{cases} \quad (12)$$

$$d_\rho = \frac{\partial u_\rho}{\partial \rho}; d_\theta = \frac{u_\rho}{\rho}; d_z = \frac{\partial u_z}{\partial z}; d_{\rho z} = \frac{1}{2} \left(\frac{\partial u_\rho}{\partial z} + \frac{\partial u_z}{\partial \rho} \right) \Rightarrow \{d\} = [B]\{X\} \quad (13)$$

Thus, each triangle generates nine constants ($X_i = A, B, \dots, J$) which are the final virtual variables of the discretized model. The resulting strain rate field inside the triangle can be easily cast into the form $\{d(\rho, z)\} = [B(\rho, z)]\{X\}$ where the components of X are the (virtual) variables of the triangle. From this choice, the resulting strain rate tensor d cannot become singular and it varies linearly in the element: this linearity will be used later to (strictly) upper bound the (convex) dissipated power on the triangle.

Along a inter-element side the velocity jump $[u]$ is quadratic. To upper bound as above the corresponding dissipated power, the jump is linearized by enforcing the jump at the side middle to be equal to half the sum of its values at the ends of the discontinuity. Note that another interesting methodology for maintaining the PA character of a quadratic $[u]$ along the discontinuity is given in [14], but it is not easily applicable when using the present mixed method.

Note also that, in the recent LA numerical literature for axisymmetric problems, the drawback of the singularity at the origin is avoided by enforcing the PA conditions only at the centroid of the element, giving by this way only an estimate of the real solution.

5.2 Formulation of the Mixed Kinematic Method

5.2.1 Contribution of the Element Velocity Fields

The first part of the integral in (5.ii) becomes here, for a triangle of volume V :

$$P_V = \int_V d : \sigma dV = 2\pi \int_V \{d\}^t \{\sigma\} \rho d\rho dz \quad (14)$$

As detailed in [17, 19], from the Karush-Kuhn-Tucker optimality conditions, the product $\{d\}^t \{\sigma\}$ becomes the convex unit dissipated power $\pi_V(d)$ in the optimal solution. Then, we substitute to this product its linear interpolation $\mathcal{L}(\{d\}^t \{\sigma\})$ between its values at each vertex of the triangular element. Since the product $\rho \mathcal{L}$

is quadratic, the integral $\int \rho \mathcal{L} d\rho dz$ is classically calculated from its values at the medium of the sides of the triangle. Since each term of the product is linear, the final result depends only on the product values at the element vertices. Consequently, a stress tensor $\{\sigma\} = (\sigma_\rho, \sigma_z, \sigma_\theta, \sigma_{\rho z})^T$ is assigned at each vertex of the triangle, *without any variation assumption over the triangle*.

We finally obtain for the triangle with vertices $i = 1$ to 3 and area A :

$$P_V \leq \frac{\pi A}{6} \sum_{i=1,3} C_i \{d\}_i^t \{\sigma\}_i, \quad (15)$$

where $C_i = 2\rho_i + \rho_{i+1} + \rho_{i+2}$, with $\rho_4 = \rho_1$ and $\rho_5 = \rho_2$.

It is important to note that the strain rate d of the optimal solution will be PA at the vertices of the triangle; from the convexity of the set of PA strain rates associated to the present (convex) criterion, and from the linear variation of d in (13), it can be deduced (as first noticed in [13]) that the strain rate will be PA all over the element, a *sine qua non* condition for preserving the upper bound character of the result.

5.2.2 Contribution of the Velocity Discontinuities

The second part of the integral in (5.ii) is the sum of the power contribution of each discontinuity surface L_{1-2} (of ends noted 1 and 2):

$$P_d = \int_{L_{1-2}} [u] \cdot T' dS = 2\pi \int_{L_{1-2}} \{[u]\}^t \{T'\} \rho d\rho dz \quad (16)$$

According to [25], the product $[u] \cdot T'$ becomes the dissipated power $\pi_d([u])$ when the stress vector T' and the velocity jump vector $[u]$ are associated (by the normality law) relatively to the $f_{nt}(T')$ criterion. Here also, we can use the convexity of $\pi_d([u])$ since $[u]$ is constrained to vary linearly along the discontinuity side: to the product $\{[u]\}^t \{T'\}$ we substitute its linear interpolation $\mathcal{L}(\{[u]\}^t \{T'\})$ between its values at each end of the discontinuity side of normal n . Then, we can upper bound P_d by calculating the quadratic expression with Simpson formula, and by using the linearity of each term of the product $\rho \mathcal{L}$. By allocating an auxiliary stress vector $T' = (\sigma_{nn}, \sigma_{nt})$ (expressed in the orthonormal (n, t) frame of the side) at each of the two apices of the discontinuity side L_{1-2} , we finally obtain:

$$P_d \leq \frac{2\pi l}{6} \sum_{i=1,2} C_i \left(\{[u]_i\}^t \{T'_i\} \right), \quad (17)$$

where l is the length of L_{1-2} , $C_i = 2\rho_i + \rho_{i+1}$ with $\rho_3 = \rho_1$. It can be noted that the optimal jump $[u]$ will be PA all along L_{1-2} from the convexity of the PA jump set associated to the criterion f_{nt} .

5.2.3 Expression of the External Power

To the global vector of the virtual u -velocities are added the generalized velocities q_1, q_2, q_3 in order to form the final virtual vector $\{u\}$. Thus, from (2), the external power can be written as:

$$P_{ext} = V(q \cdot Q) = V\{q\}^T \{Q\} = \{u\}^T V[\beta] \{Q\} \quad (18)$$

where $\{q\} = [\beta]^T \{u\}$.

5.2.4 The PA Stress Conditions

In the global (x, y, z) reference frame, the Coulomb yield condition is written as in (7), where the cone can be easily cast into the Lorentz form required by the conic optimizer MOSEK [12]:

$$\sqrt{x_1^2 + x_2^2} \leq U \quad (19)$$

together with the three inequalities for each vertex of the triangular elements.

The criterion (8) for the stress vector T' gives rise to the following inequalities:

$$\sigma'_{nt} + \sigma'_n \tan \phi \leq c, \quad -\sigma'_{nt} + \sigma'_n \tan \phi \leq c, \quad (20)$$

which results in two linear constraints in terms of the real variables σ'_{nn} and σ'_{nt} for each ends of the discontinuity sides.

5.2.5 The Final Mixed Problem and the KA Conditions

Finally, the numerical form of the variational mechanical problem (5.i–5.iii) is as following:

$$\begin{aligned} & \text{Max } V\{q^d\}^T \{Q\} \\ & \text{s. t. } -[\alpha]\{\sigma\} - [\alpha']\{T'\} + V[\beta]\{Q\} = 0, \\ & f(\sigma) \leq 0 \quad \forall \sigma ; f_n(T') \leq 0 \quad \forall T', \\ & +\text{KA velocity conditions.} \end{aligned} \quad (21)$$

Indeed, a systematic change of d and u in terms of the final $\{X\}$ virtual variables is performed through specific subroutines of the Fortran code which generates the final problem in the MPS format required by MOSEK.

As shown in the detailed analysis of [17, 19], we can identify the dual variables of the solution of this optimization problem with the $\{X\}$ components. This analysis also details how the resulting velocity field is plastically admissible and how, by adding

auxiliary columns, the kinematically admissible character of the optimal velocity field can be ensured, as in the following.

- We previously defined two supplementary rows (constraints) whose associated virtual variables are q_1, q_2 and two new columns for the corresponding macroscopic stresses Q_1, Q_2 . At each apex and at the middle of the boundary triangle sides, we impose the loading conditions $u_\rho = D_\rho \rho, u_z = D_z z$. This is done by adding one additional column (i.e., an additional variable) for each of these conditions. Let us note the chosen kinematic parameters as $\{q\} = [A_D]\{D\}$ with $\{D\}^T = (D_\rho, D_z)$; then, for example, the terms of a condition $u_\rho - \rho D_\rho = 0 \implies u_\rho - \rho [A_D]^{-1}\{q\} = 0$ are dispatched on the (X, q) components (corresponding to u_ρ and q) of the additional column.
- A similar technique is used to impose the null symmetry value to the u_z components on the ρ axis, and to make that the jump $[u]$ of the middle of the discontinuity side is equal to half the sum of its end values.

6 Application to Sphere and Spheroid Models

In the following tests, the porosity f is taken equal to 0.1, and the friction angles to 10° and 20° , for the sake of homogeneity.

6.1 Comparison with Exact Results for Spherical Voids

Table 1 gives the values obtained with the static and mixed codes for isotropic loadings (i.e. $Q_2 = 0$) in tension and compression, together with the exact values given in [29]. It can be seen that the exact solution was always located between the numerical bounds that are very close; this also points out that the linearization of the matrix boundaries has no real influence with the selected n_s values. All the calculations were made on an Apple Mac Book Pro with a 2.7 GHz Core i7 and 16 gigabytes of RAM.

In the case of the static approach, the final numerical LP problem involved 1,024 triangles giving rise to 384,000 constraints and 249,000 free variables. For one point of the searched macroscopic criterion, 5–6 iterations, of approximately one minute each, were needed to obtain the convergence of the process described in Sect. 4.4.

Table 1 Comparison to Σ_m exact values [29]—Coulomb matrix—spherical void—porosity $f = 0.1$ —cohesion $c = 1$ —(n_s, n_ρ) : (16, 16)/static, (48, 48)/kinematic

	Friction angle ϕ	10°	20°
Compression	Present-static	−5.0970	−10.705
	<i>Exact value</i>	−5.1397	−10.805
	Present-kinematic	−5.1531	−10.844
Tension	Present-static	2.0543	1.4851
	<i>Exact value</i>	2.0704	1.4911
	Present-kinematic	2.0717	1.4916

The final mixed mesh of 9,200 triangles gives a conic problem with a constraint matrix involving 276,100 rows, 276,350 columns, and 27,650 Lorentz cones; the problem is solved in about 20 s by the version 5 of Mosek, using the four cores of the Intel Core i7 processor. This remarkable performance is mainly due to the quadratic variation of the velocity field and to the relatively small size of the final problem even for a very refined meshing of the problem. It can be also noted that the linearization of the velocity discontinuities, in order to preserve the kinematic character of the method, has not a significative influence on the performance in terms of upper bound solutions.

6.2 Numerical Results for Axisymmetric Loadings in the Spherical Void Case

In the reference [24], the hollow sphere problem with a Coulomb matrix was investigated, unfortunately only for non-negative Q_2 only. The corresponding results are recalled (JCAM dashed lines) in Figs. 2 and 3, for the friction angles $\phi = 10^\circ$ and 20° , respectively.

The JCAM static approach (in blue color) was obtained with a PWL version of the static code: this explains the better results (continuous lines) by using the present (conic projection) algorithm of Ben Tal and Nemirovski of Sect. 4.3. It can be seen

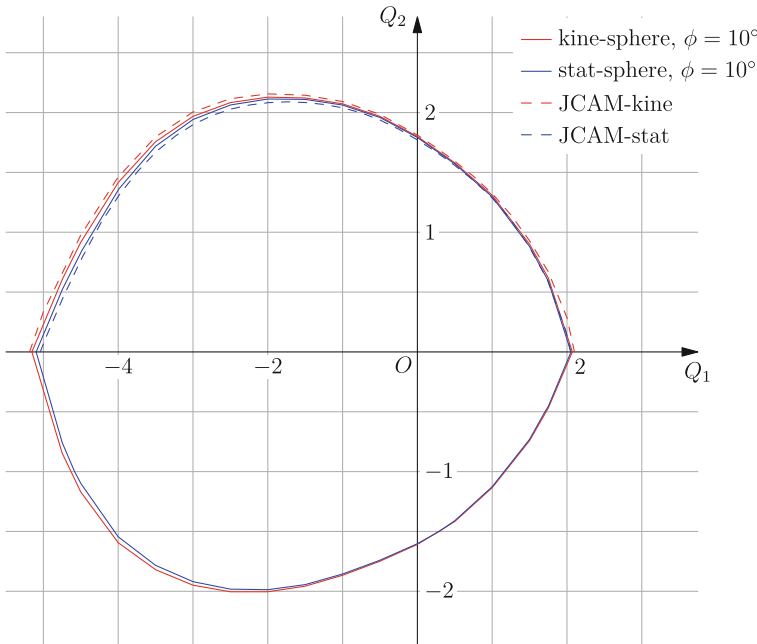


Fig. 2 Present results—Coulomb matrix— $f = 0.1$ —spherical cavity— $\phi = 10^\circ$; $c = 1$

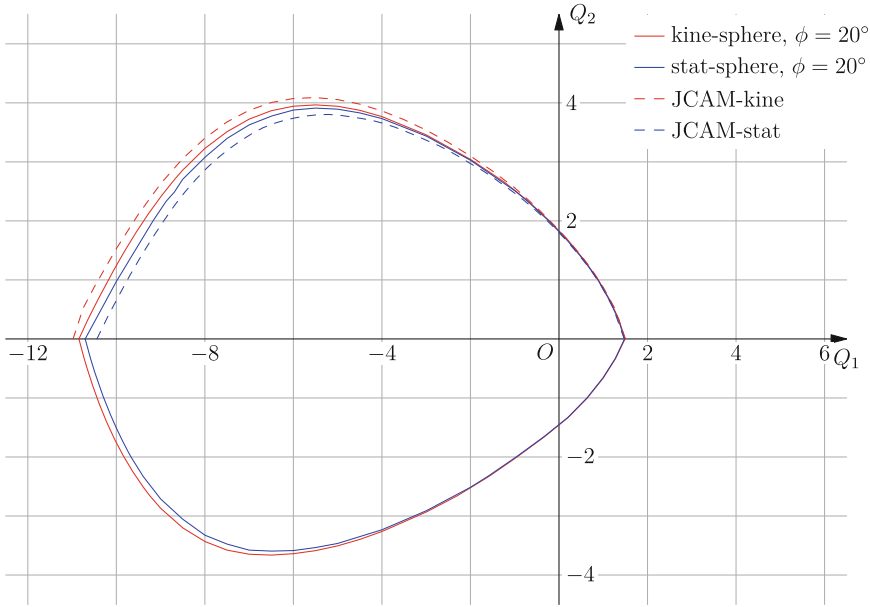


Fig. 3 Present results—Coulomb matrix— $f = 0.1$ —spherical cavity— $\phi = 20^\circ$; $c = 1$

also that the proposed mixed kinematic code has allowed similar improving of the previous results (dashed red lines) obtained with a classic kinematic approach that cannot run the refined meshes allowed by the mixed code.

The final kinematic and static criterions are very close, and always in the good order, even when they are almost indistinguishable. The presence of the singular points on the average stress axis are confirmed. The influence of the third stress invariant is also shown (recall that the second invariant is here $\Sigma_{eq} = |Q_2|$), since, owing to the isotropy of the macroscopic material, the plasticity criterion only depends on the three stress invariants. Indeed, in the Figs. 2 and 3, the Σ_m axis would be a symmetry axis of the plasticity criterion if this one does not depend on the third stress invariant. It must be emphasized that in the present case, the influence of third stress invariant comes from a combination of an effect that exists even if the matrix is not sensitive to the this third invariant, and of that which comes from the Coulomb matrix. In particular, the apexes on figure are signatures of this sensitivity of the Coulomb matrix. These points will be commented again in the following when comparing Coulomb and Drucker-Prager matrices.

6.3 Porous Coulomb Material with Oblate Voids

We consider first the case of the 0.5 aspect ratio a_1/b_1 where both axisymmetric codes are used. The case of the aspect ratio 0.2 is then investigated by using the

same static and mixed code. Recalling that this problem has not been studied in the literature up to now, it should be useful to remember that the present axisymmetric bounds are also strict bounds when considering the full 3D problem, in the Q_i space.

6.3.1 Case $a_1/b_1 = 0.5$

In Figs. 4 and 5 are shown the results for an oblate cavity having an aspect ratio $a_1/b_1 = 0.5$.

First, the kinematic and static bounds, here also are very close, always remaining correctly ordered as in the case of spherical cavities. It can be seen that the graphs presents a small rotation around the frame origin, the singular point being not on the average stress axis. As expected in this case of transverse isotropy, taking into account that the axis of the anisotropy frame (x, y, z) are also loading axis, the influence of the third stress invariant is here more important.

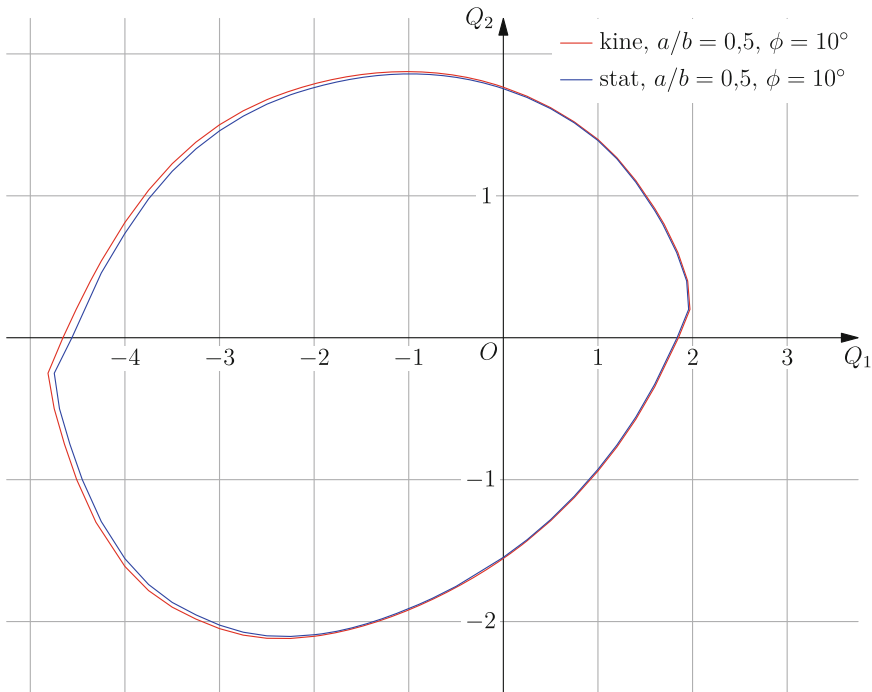


Fig. 4 Present results—Coulomb matrix— $f = 0.1$; $a_1/b_1 = 0.5$ — $\phi = 10^\circ$; $c = 1$

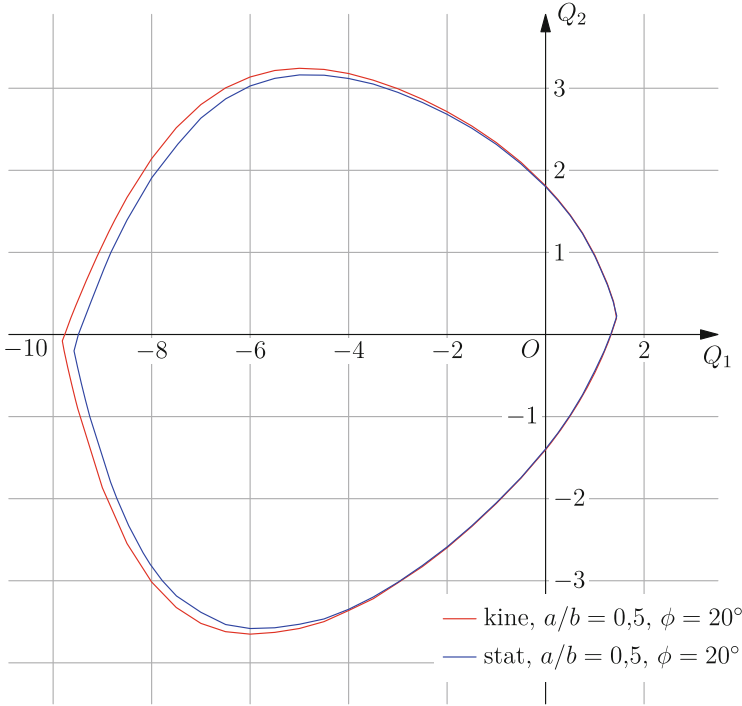


Fig. 5 Present results—Coulomb matrix— $f = 0.1$; $a_1/b_1 = 0.5$ — $\phi = 20^\circ$; $c = 1$

6.3.2 Case $a_1/b_1 = 0.2$

Figures 6 and 7 illustrate the results for an oblate cavity having this time the aspect ratio $a_1/b_1 = 0.2$. Here, the bounds are close, however in a lesser extent than above for mainly compressive loadings, and the dissymmetry with respect to the average stress axis is noticeably more important.

For comparison between the two cases with the two usual pressure-dependent matrices, we report also the graphs obtained for a Drucker-Prager matrix in the reference [18]; both graphs are reported to the cohesion c , after identification of the two criteria in plane strain from [25], in such a manner that the boundary of the domain of plastically admissible stresses for Drucker-Prager is innerly tangent to the corresponding Coulomb boundary. As expected, the criterion of the porous Drucker-Prager material remains innerly tangent to the porous Coulomb criterion, without any angular point in the Drucker-Prager case. Again, existence of such angular points is a signature of the sensitivity of the Coulomb matrix to the third invariant of the stress deviator. Since angular points might induce strong effects in strain localization phenomena, it can be seen that replacing the Coulomb criterion with an *ad hoc*

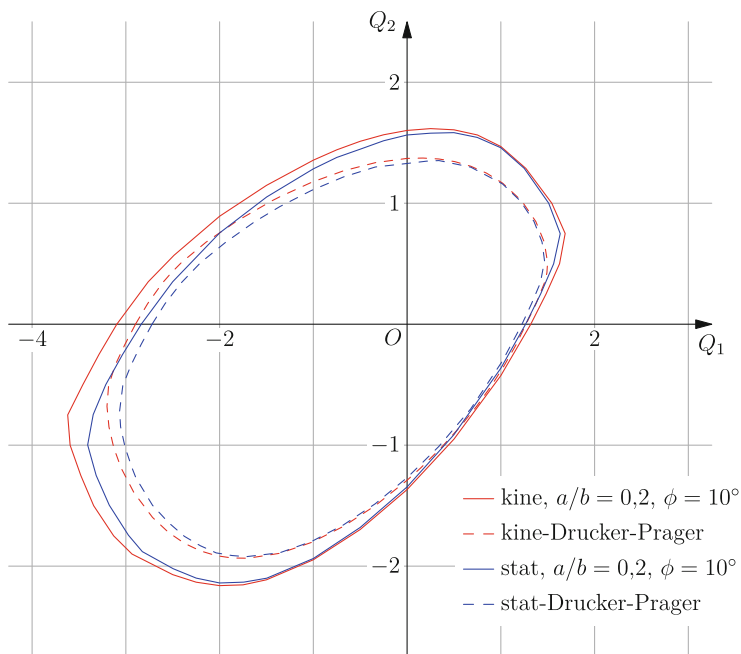


Fig. 6 Present results—Drucker-Prager matrix— $f = 0.1$; $a_1/b_1 = 0.2$ — $\phi = 10^\circ$; $c = 1$

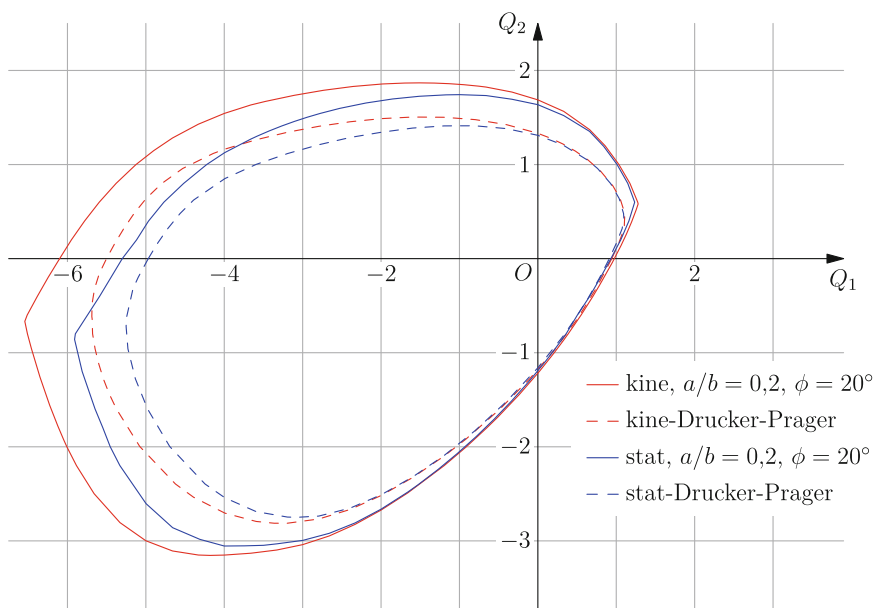


Fig. 7 Present results—Coulomb matrix— $f = 0.1$; $a_1/b_1 = 0.2$ — $\phi = 20^\circ$; $c = 1$

Drucker-Prager for elastoplastic 3D computations can be problematic, even for the present porous materials.

7 Conclusion

The first purpose of the present paper was to generalize the formulations of the limit analysis (LA) in the axisymmetric case of [24], in order to obtain robust conic formulations without losing the rigorous bound character of the solutions. The second purpose was to provide numerical, but rigorous bounds to the macroscopic criterion of a “porous Coulomb material” with spherical and oblate voids, a problem not studied in the literature up to now, at least to our knowledge. These bounds, not only allow to characterize the macroscopic plastic properties of this class of materials, but are also expected to serve as a reference results for forthcoming theoretical investigations.

To obtain accurate results for the non-spherical voids, the static approach was modified to use the conic projection algorithm of [2], and an original mixed axisymmetric approach has been elaborated which results in a second order conic programming problem. Both resulting codes have allowed remarkable performances by using specific quadratic fields together with FEM refinements of the problem unreachable before. After a validation in the case of a spherical cavity and isotropic loadings, by comparison to the known exact solution, we provide illustrations of the macroscopic plasticity criterion. This has been done for usual values of the friction angle and of the void aspect ratio. The obtained results allow to highlight, among others, the shape and the size of the plasticity criterion depending on the aspect ratio of the cavity. A comparison with the porous Drucker-Prager material is finally given in order to illustrate the noticeable differences with the present porous Coulomb material, the same geometric and loading conditions being considered. In view of all the above results, it is clear that the paper provides interesting bounding data which call for a great effort in the theoretical side in order to formulate new macroscopic criterion corresponding to the porous material with a Coulomb matrix.

References

1. Anderheggen E, Knopf H (1972) Finite element limit analysis using linear programming. *Int J Solids Struct* 8:1413–1431
2. Ben-Tal A, Nemirovskii A (2001) Lectures on modern convex optimization. Analysis, algorithms and engineering applications. Series on Optimization. SIAM-MPS
3. Byer J (2000) XA, optimisation library. Sunset Software Technology. San Marino
4. Francescato P, Pastor J, Riveill-Reydet B (2004) Ductile failure of cylindrically porous materials. Part I: plane stress problem and experimental results. *Eur J Mech/A Solids* 23:181–190
5. Garajeu M, Suquet P (1997) Effective properties of porous ideally plastic or viscoplastic materials containing rigid particles. *J Mech Phys Solids* 45:873–902
6. Gologanu M, Leblond JB (1993) Approximate models for ductile metals containing non-spherical voids—case of axisymmetric prolate ellipsoidal cavities. *J Mech Phys Solids* 41(11):1723–1754

7. Gologanu M, Leblond JB, Perrin G, Devaux J (1994) Approximate models for ductile metals containing non-spherical voids—case of axisymmetric oblate ellipsoidal cavities. *J Eng Mater Technol* 116:290–297
8. Guo TF, Faleskog J, Shih CF (2008) Continuum modeling of a porous solid with pressure sensitive dilatant matrix. *J Mech Phys Solids* 56:2188–2212
9. Gurson AL (1977) Continuum theory of ductile rupture by void nucleation and growth—part I: yield criteria and flow rules for porous ductile media. *J Eng Mater Technol* 99:2–15
10. Krabbenhoft K, Lyamin AV, Hijaj M, Sloan SW (2005) A new discontinuous upper bound limit analysis formulation. *Int J Numer Methods Eng* 63:1069–1088
11. Lee JH, Oung J (2000) Yield functions and flow rules for porous pressure-dependent strain-hardening polymeric materials. *J Appl Mech* 67:288–297
12. MOSEK ApS. (2010) C/O Symbion Science Park, Fruebjergvej 3, Box 16, 2100 Copenhagen ϕ , Denmark, <http://www.mosek.com>
13. Makrodimopoulos A, Martin CM (2007) Upper bound limit analysis using simplex strain elements and second-order cone programming. *Int J Numer Anal Methods Geomech* 31(6):835–865
14. Makrodimopoulos A, Martin CM (2008) Upper bound limit analysis using discontinuous quadratic displacement fields. *Commun Numer Methods Eng* 24:911–927
15. Monchiet V, Charkaluk E, Kondo D (2007) An improvement of Gurson-type models of porous materials by using Eshelby-like trial velocity fields. *C R Mécanique* 335:32–41
16. Monchiet V, Charkaluk E, Kondo D (2013) Macroscopic yield criteria for ductile materials containing spheroidal voids: an Eshelby-like velocity fields approach. *Mech Mater* <http://dx.doi.org/10.1016/j.mechmat.2013.05.006>
17. Pastor F (2007) Résolution par des méthodes de point intérieur de problèmes de programmation convexe posés par l'analyse limite. Thèse de doctorat, Facultés universitaires Notre-Dame de la Paix, Namur
18. Pastor F, Kondo D (2014) Limit analysis and lower/upper bounds to the macroscopic criterion of Drucker-Prager materials with spheroidal voids. *C R Mécanique* 342:96–105
19. Pastor F, Kondo D, Pastor J (2013) 3D-FEM formulations of limit analysis methods for porous pressure-sensitive materials. *Int J Numer Methods Eng* 95:847–870
20. Pastor F, Loute E, Pastor J, Trillat M (2009) Mixed method and convex optimization for limit analysis of homogeneous Gurson materials: a kinematical approach. *Eur J Mech-A/Solids* 28:25–35
21. Pastor F, Kondo D (2013) Assessment of hollow spheroid models for ductile failure prediction by limit analysis and conic programming. *Eur J Mech A/Solids* 38:100–114
22. Pastor J, Turgeman S (1982) Limit analysis in axisymmetrical problems: numerical determination of complete statical solutions. *Int J Mech Sci* 24:95–117
23. Pastor J, Francescato P, Trillat M, Loute E, Rousselier G (2004) Ductile failure of cylindrically porous materials. Part II: other cases of symmetry. *Eur J Mech/A Solids* 23:191–201
24. Pastor J, Thoré Ph, Pastor F (2010) Limit analysis and numerical modeling of spherically porous solids with Coulomb and Drucker-Prager matrices. *J Comput Appl Math* 234:2162–2174
25. Salençon J (1983) Calcul à la rupture et analyse limite. Presses des Ponts et Chaussées, Paris
26. Sloan SW (2013) Geotechnical stability analysis. *Géotechnique* 63:531–572
27. Thai-The H, Francescato P, Pastor J (1998) Limit analysis of unidirectional porous media. *Mech Res Commun* 25:535–542
28. Thoré P, Pastor F, Pastor J (2011) Hollow sphere models, conic programming and third stress invariant. *Eur J Mech/A Solids* 30:63–71
29. Thoré Ph, Pastor F, Pastor J, Kondo D (2009) Closed form solutions for the hollow sphere model with Coulomb and Drucker-Prager materials under isotropic loadings. *C R Mécanique Acad Sci Paris* 337:260–267
30. Trillat M, Pastor J (2005) Limit analysis and Gurson's model. *Eur J Mech A/Solids* 24:800–819
31. Turgeman S, Pastor J (1982) Limit analysis: a linear formulation of the kinematic approach for axisymmetric mechanic problems. *Int J Numer Anal Methods Geomech* 6:109–128

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