Chapter 2
Complex Potential and Differentiation

Motivating Problem 2: Extraction of Contaminated Groundwater

Let us revisit Motivating Problem 1. If the contamination is observed at the monitoring well, an extraction well is drilled at the location (1, 0), as shown in Fig. 2.1, to withdraw contaminated groundwater. For the extraction system to be optimized, a precise evaluation of flow behavior of contaminants is essential.

Task 2-1 Draw the flow paths in the flow domain and predict whether the contaminants are detected at the monitoring well.

Task 2-2 Draw the flow paths in the flow domain and evaluate the minimum pumping rate q_e/h at the extraction well to avoid the contamination at the monitoring well.

Task 2-3 Redo Task 2-2 in an analytical manner.

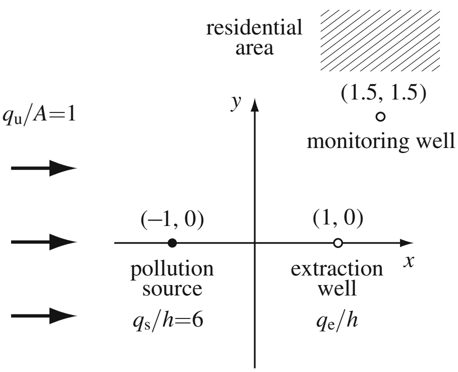

Fig. 2.1 Uniform flow in the x direction with a pollution source at $(-1, 0)$, a monitoring well at (1.5, 1.5), and an extraction well at (1, 0)

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• Solution Strategy to Motivating Problem 2

Although the velocity potential (or its derivative) provides a rough image of flow profiles, as seen in Motivating Problem 1, it does not convey sufficient information to draw exact flow paths. To evaluate the direction of flow, another mathematical function must be acquired.

The strategy, therefore, is to derive a new function that conveys such information. Of course, the new function must be consistent with the physical nature of the velocity potential. This derivation process reveals the reason why complex variables need to be introduced and the usefulness of complex analysis in practical engineering.

2.1 Complex Numbers

To perform complex analysis, the real number system needs to be extended to the complex number system. The algebraic properties and geometric representation of complex numbers are discussed.

2.1.1 Definition

There is no real number x that satisfies the equation $x^2 = -1$. To manipulate these types of equations, the set of complex numbers is introduced. By definition, a complex number z is an ordered pair (x, y) of real numbers x and y, written as

$$z = (x, y) \tag{2.1}$$

The ordered pair (0, 1) is called the imaginary unit and is denoted by i as

$$\mathrm{i} = (0, 1) \tag{2.2}$$

which has the property of

$$\mathrm{i}^2 = -1 \tag{2.3}$$

In practice, a complex number z is expressed in the form

$$z = x + \mathrm{i}y \tag{2.4}$$

The real numbers x and y are referred to as the real part of z and the imaginary part of z, respectively, and are written as

$$\begin{cases} x = \mathrm{Re}\, z \\ y = \mathrm{Im}\, z \end{cases} \tag{2.5}$$

The complex numbers z with $y = 0$ are identified with the real numbers x, and thus the set of complex numbers includes the real numbers as a subset. When $x = 0$, in contrast, the complex numbers $\mathrm{i}y$ are called the pure imaginary numbers.

2.1.2 Algebraic Properties

By definition, two complex numbers are equal if and only if they have the same real parts and the same imaginary parts, that is,

$$x_1 + \mathrm{i}y_1 = x_2 + \mathrm{i}y_2 \quad \text{if and only if} \quad x_1 = x_2 \text{ and } y_1 = y_2 \tag{2.6}$$

Algebraic properties of complex numbers are the same as for real numbers as listed below.

Commutative law of addition $z_1 + z_2 = z_2 + z_1$
Commutative law of multiplication $z_1 z_2 = z_2 z_1$
Associative law of addition $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
Associative law of multiplication $z_1(z_2 z_3) = (z_1 z_2) z_3$
Distributive law $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
Additive identity The complex number $0 = (0, 0)$ satisfies $z_1 + 0 = z_1$
Multiplicative identity The complex number $1 = (1, 0)$ satisfies $z_1 1 = z_1$
Additive inverse $z = -z_1$ if $z + z_1 = 0$
Multiplicative inverse $z = 1/z_1$ if $z z_1 = 1$

Operations with complex numbers can be performed as in the algebra of real numbers and by replacing i^2 by -1 when it occurs. Addition of two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ becomes

$$(x_1 + \mathrm{i}y_1) + (x_2 + \mathrm{i}y_2) = (x_1 + x_2) + \mathrm{i}(y_1 + y_2) \tag{2.7}$$

and subtraction is

$$(x_1 + \mathrm{i}y_1) - (x_2 + \mathrm{i}y_2) = (x_1 - x_2) + \mathrm{i}(y_1 - y_2) \tag{2.8}$$

Multiplication is given by

$$\begin{aligned}(x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) &= x_1x_2 + \mathrm{i}x_1y_2 + \mathrm{i}y_1x_2 + \mathrm{i}^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + \mathrm{i}(x_1y_2 + x_2y_1)\end{aligned} \tag{2.9}$$

where $\mathrm{i}^2 = -1$ is used.

Division of z_1 by z_2 is obtained by multiplying the numerator and denominator by $x_2 - \mathrm{i}y_2$ as

$$\begin{aligned}\frac{x_1 + \mathrm{i}y_1}{x_2 + \mathrm{i}y_2} &= \frac{(x_1 + \mathrm{i}y_1)(x_2 - \mathrm{i}y_2)}{(x_2 + \mathrm{i}y_2)(x_2 - \mathrm{i}y_2)} = \frac{x_1x_2 - \mathrm{i}x_1y_2 + \mathrm{i}y_1x_2 - \mathrm{i}^2y_1y_2}{x_2^2 - \mathrm{i}^2\,y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \mathrm{i}\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\end{aligned} \tag{2.10}$$

where $\mathrm{i}^2 = -1$ is used.

Example 2.1 Algebraic operations result in

$$\begin{aligned}\frac{5+10\mathrm{i}}{3-4\mathrm{i}}+\frac{25}{4+3\mathrm{i}} &= \frac{5+10\mathrm{i}}{3-4\mathrm{i}}\frac{3+4\mathrm{i}}{3+4\mathrm{i}}+\frac{25}{4+3\mathrm{i}}\frac{4-3\mathrm{i}}{4-3\mathrm{i}}\\ &= \frac{15+20\mathrm{i}+30\mathrm{i}+40\mathrm{i}^2}{9-16\mathrm{i}^2}+\frac{100-75\mathrm{i}}{16-9\mathrm{i}^2}\\ &= \frac{-25+50\mathrm{i}}{25}+\frac{100-75\mathrm{i}}{25}=3-\mathrm{i}\end{aligned}$$

which is expressed in standard form.

2.1.3 Complex Plane

Let us consider the geometric representation of complex numbers, which is of great practical importance. As a real number x is often represented by a point on an x line, it is natural to associate a complex number $z = x + \mathrm{i}y$ with a point or a vector in an xy plane, as shown in Fig. 2.2.

The x and y axes are referred to as the real axis and imaginary axis, respectively, and the xy plane is referred to as the complex plane or the z plane. A complex number $z = x + \mathrm{i}y$ is plotted as the point with coordinates (x, y). Each complex number corresponds to one and only one point in the complex plane, and, conversely, each point in the plane corresponds to one and only one complex number.

The absolute value or modulus of a complex number $z = x + \mathrm{i}y$, denoted by $|z|$, is defined by

$$|z| = \sqrt{x^2 + y^2} \tag{2.11}$$

which is the length of the line segment from the origin to z, as shown in Fig. 2.2. If z is real, the modulus is simply the absolute value of z. The distance between two

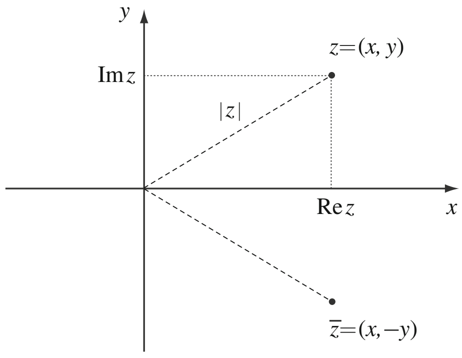


Fig. 2.2 Complex plane representing complex numbers z and $\bar{z}$

points $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ is given by

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{2.12}$$

which is the length of the vector representing $z_1 - z_2$.

Example 2.2 The circle of radius R with its center at z_0 can be expressed as

$$|z - z_0| = R$$

For instance, the equation $|z - 1 + 2\mathrm{i}| = 3$ represents the circle of radius $R = 3$ with its center at $1 - 2\mathrm{i}$.

The complex conjugate of a complex number $z = x + \mathrm{i}y$, denoted by $\bar{z}$, is defined by

$$\bar{z} = x - \mathrm{i}y \tag{2.13}$$

Geometrically, $\bar{z}$ is the point $(x, -y)$, obtained by reflecting $z = (x, y)$ in the real axis as shown in Fig. 2.2.

If $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - \mathrm{i}(y_1 + y_2) = (x_1 - \mathrm{i}y_1) + (x_2 - \mathrm{i}y_2) \tag{2.14}$$

and thus the conjugate of the sum is the sum of the conjugates, that is,

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \tag{2.15}$$

Similarly, the following identities hold:

$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2} \tag{2.16}$$

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2} \tag{2.17}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \tag{2.18}$$

By addition and subtraction, $z + \bar{z} = 2x$ and $z - \bar{z} = 2y\mathrm{i}$, and it follows that

$$\begin{cases} \operatorname{Re} z = \dfrac{z + \bar{z}}{2} \\ \operatorname{Im} z = \dfrac{z - \bar{z}}{2\mathrm{i}} \end{cases} \tag{2.19}$$

By multiplication, $z\bar{z} = x^2 + y^2$, and the modulus of z is given by

$$|z| = \sqrt{z\bar{z}} \tag{2.20}$$

The inverse of z is then given by

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} \tag{2.21}$$

provided $z \neq 0$.

Example 2.3 The circle of radius 1 with its center at $z = 1$, $x^2 + y^2 - 2x = 0$, can be expressed in terms of conjugate coordinates as

$$z\bar{z} - z - \bar{z} = 0$$

where the identities $z\bar{z} = x^2 + y^2$ and $x = (z + \bar{z})/2$ are used.

Example 2.4 The general equation for a circle or line in the xy plane is given by

$$a(x^2 + y^2) + bx + cy + d = 0$$

where a, b, c, and d are real constants, and $a \neq 0$ for a circle and $a = 0$ for a line. Using the aforementioned identities, it can be rewritten as

$$az\bar{z} + b\frac{z + \bar{z}}{2} + c\frac{z - \bar{z}}{2\mathrm{i}} + d = 0$$

or, writing $a = \alpha$, $b/2 + c/(2\mathrm{i}) = \beta$, and $d = \gamma$, it follows that

$$\alpha z\bar{z} + \beta z + \overline{\beta z} + \gamma = 0$$

which is the general equation for a circle or line in terms of conjugate coordinates.

2.1.4 Polar Form of Complex Numbers

A complex number z can also be expressed in terms of polar coordinates r and θ. The positive number r is the length of the vector representing z and θ is the angle that z (as a radius vector) makes with the positive real axis, as shown in Fig. 2.3.

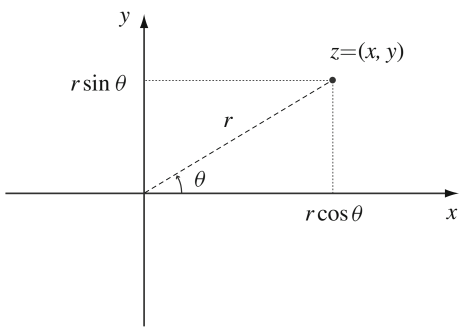


Fig. 2.3 Polar representation of a complex number z

Using the identities

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{2.22}$$

z can be expressed in polar form as

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \tag{2.23}$$

where r is the modulus of z

$$r = |z| = \sqrt{x^2 + y^2} \tag{2.24}$$

and θ is called the argument of z, denoted by $\arg z$, such that

$$\tan\theta = \frac{y}{x} \tag{2.25}$$

For a given $z \neq 0$, θ is determined only up to integer multiples of 2π and has any one of an infinite number of real values, as shown in Fig. 2.4, because of the periodic nature of cosine and sine with period of 2π.

To specify a unique value of $\arg z$, the principal value $\mathrm{Arg}\, z$ is defined such that

$$-\pi < \mathrm{Arg}\, z \leq \pi \tag{2.26}$$

For a positive real number $z = x$, $\mathrm{Arg}\, z = 0$, and for a negative real number, $\mathrm{Arg}\, z = \pi$. The argument is given by

$$\arg z = \mathrm{Arg}\, z + 2n\pi \tag{2.27}$$

where n is an integer.

Example 2.5 For $z = 1 + \mathrm{i}$, its polar form is

$$z = \sqrt{2}\,(\cos(\pi/4) + \mathrm{i}\sin(\pi/4))$$

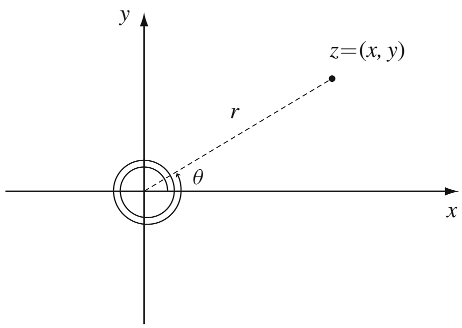


Fig. 2.4 Argument of z given by $\mathrm{Arg}\, z + 2n\pi$

and $|z| = \sqrt{2}$, $\arg z = \pi/4 + 2n\pi$, and $\operatorname{Arg} z = \pi/4$. For $z = -1 - \mathrm{i}$, its polar form is

$$z = \sqrt{2}\,(\cos(-3\pi/4) + \mathrm{i}\sin(-3\pi/4))$$

and $|z| = \sqrt{2}$, $\arg z = -3\pi/4 + 2n\pi$, and $\operatorname{Arg} z = -3\pi/4$.

The inverse of Eq. 2.25, given by

$$\theta = \arctan\frac{y}{x} \tag{2.28}$$

is not always true, and should be used with caution. Since $\tan\theta$ has period π, the arguments of z and $-z$ have the same tangent. The quadrant where z lies must be identified for a proper value.

Example 2.6 For $z = 1 + \mathrm{i}$, it follows that

$$\theta = \arctan\frac{1}{1} = \arctan 1 = \pi/4$$

which is correct. On the other hand, for $z = -1 - \mathrm{i}$, it follows that

$$\theta = \arctan\frac{-1}{-1} = \arctan 1 = \pi/4$$

which is not correct. However, by knowing z lies in the third quadrant and by using $\cos\theta = \sin\theta = -1/\sqrt{2}$, θ is obtained as $-3\pi/4$.

2.1.5 Exponential Form of Complex Numbers

Using the infinite series[1] of expansion $\mathrm{e}^t = 1 + t + t^2/2! + t^3/3! + \cdots$ of elementary calculus with $t = \mathrm{i}\theta$ yields

$$\mathrm{e}^{\mathrm{i}\theta} = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!} + \mathrm{i}\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} \tag{2.29}$$

The first infinite series is $\cos\theta$ and the second is $\sin\theta$, and it follows that

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \tag{2.30}$$

which is known as Euler's formula. Applying this formula to a nonzero complex number z in polar form results in

$$z = r\mathrm{e}^{\mathrm{i}\theta} \tag{2.31}$$

which is the exponential form of z.

[1] Series representations of complex variables are discussed in Chap. 5.

Example 2.7 For $z = 1+\mathrm{i}$ and $z = -1-\mathrm{i}$, the exponential forms are $z = \sqrt{2}\mathrm{e}^{(\pi/4)\mathrm{i}}$ and $z = \sqrt{2}\mathrm{e}^{-(3\pi/4)\mathrm{i}}$, respectively.

2.1.5.1 Multiplication in Exponential Form

Operations in exponential form give us a geometrical understanding of multiplication. Let $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$, as shown in Fig. 2.5; then it follows that

$$z_1 z_2 = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} \tag{2.32}$$

Taking absolute values on both sides reveals that the absolute value of a product is equal to the product of the absolute values of the factors:

$$|z_1 z_2| = |z_1||z_2| \tag{2.33}$$

Taking arguments on both sides reveals that the argument of a product is equal to the sum of the arguments of the factors:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2.34}$$

Figure 2.5 shows these geometrical relations.

2.1.5.2 Division in Exponential Form

Operations in exponential form give us a geometrical understanding of division, given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} \tag{2.35}$$

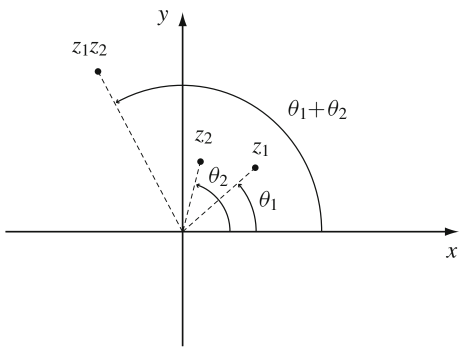


Fig. 2.5 Arguments of z_1, z_2, and $z_1 z_2$

Taking absolute values on both sides reveals that the absolute value of a quotient is equal to the quotient of the absolute values of the factors:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{2.36}$$

Taking arguments on both sides reveals that the argument of a quotient is equal to the difference between the arguments of the factors:

$$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \tag{2.37}$$

It should be noted that these properties are not, in general, valid when arg is replaced by Arg.

Example 2.8 Let us consider $z_1 = -1$ and $z_2 = \mathrm{i}$ and specify $\arg z_1 = -\pi$ and $\arg z_2 = \pi/2$, then $\arg(z_1 z_2) = \arg(-\mathrm{i}) = -\pi/2 = \arg z_1 + \arg z_2$. However, $\operatorname{Arg} z_1 = \pi$ and $\operatorname{Arg} z_2 = \pi/2$; thus, $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = 3\pi/2$, which is not equal to $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-\mathrm{i}) = -\pi/2$.

Example 2.9 Let us consider a generalization of multiplication.

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2 + \cdots \theta_n)}$$

and if $z_1 = z_2 = \cdots = z_n = z$, then

$$z^n = r^n (\cos\theta + \mathrm{i}\sin\theta)^n = r^n \mathrm{e}^{\mathrm{i}n\theta} = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$$

The identity

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta \tag{2.38}$$

is known as De Moivre's theorem.

2.1.6 Roots

Consider a complex number z_0 that satisfies $z^n = z_0$, where n is a positive integer. Then to a given $z_0 \neq 0$ there corresponds n distinct values of z, each of which is called an nth root of z_0. A set of n different roots is denoted by

$$z = z_0^{1/n} \tag{2.39}$$

which is n-valued. The symbol $z_0^{1/n}$ denotes n different roots. If z_0 is a positive real number r, $r^{1/n}$ denotes a set of nth roots, which must be distinguished from $\sqrt[n]{r}$ which is for a single-valued positive root.

Let $z_0 = r(\cos\theta + \mathrm{i}\sin\theta)$ and $z = R(\cos\phi + \mathrm{i}\sin\phi)$; then by De Moivre's theorem, $z^n = z_0$ is rewritten as

$$R^n(\cos n\phi + \mathrm{i}\sin n\phi) = r(\cos\theta + \mathrm{i}\sin\theta) \tag{2.40}$$

Equating the absolute values on both sides yields

$$R = \sqrt[n]{r} \tag{2.41}$$

where $\sqrt[n]{r}$ denotes the positive nth root of r. Equating the arguments on both sides yields

$$\phi = \frac{\theta + 2k\pi}{n} \tag{2.42}$$

where the periodic nature of cosine and sine with period of 2π is considered and $k = 0, 1, 2, \ldots, n-1$.

The n distinct values of the nth roots of $z_0 \neq 0$ are obtained as

$$z_0^{1/n} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + \mathrm{i}\sin\frac{\theta + 2k\pi}{n}\right) \tag{2.43}$$

which lie on a circle of radius $\sqrt[n]{r}$ with its center at the origin and constitute the vertices of a polygon of n sides. In particular, the root with $k = 0$ and $\theta = \operatorname{Arg} z_0$ is called the principal nth root of z_0.

When $z_0 = 1$, it follows that $r = 1$ and $\theta = 0$, and consequently, the nth roots are given by

$$1^{1/n} = \cos\frac{2k\pi}{n} + \mathrm{i}\sin\frac{2k\pi}{n} \tag{2.44}$$

where $k = 0, 1, 2, \ldots, n-1$ and no further roots exist with other values of k. These n values are called the nth roots of unity, which lie on the circle of radius 1 with its center at the origin, called the unit circle.

Let ω denote the value corresponding to $k = 1$ in Eq. 2.44

$$\omega = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n} \tag{2.45}$$

which is called the primitive nth root of unity. Then, according to De Moivre's theorem, it follows that

$$\omega^k = \cos\frac{2k\pi}{n} + \mathrm{i}\sin\frac{2k\pi}{n} \tag{2.46}$$

Hence, the n values of $1^{1/n}$ are given by

$$1, \omega, \omega^2, \ldots, \omega^{n-1}$$

where 1 is the principal nth root of unity.

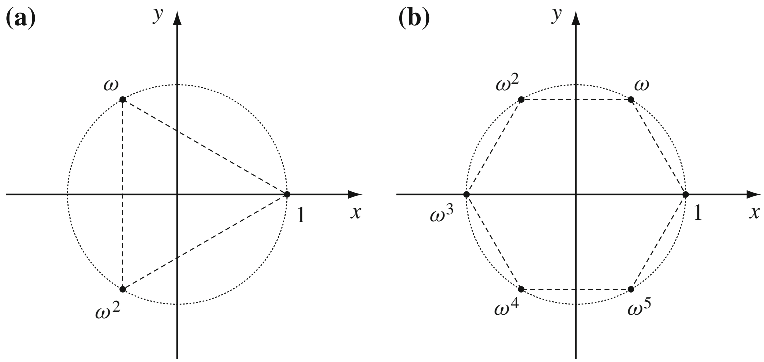


Fig. 2.6 Roots of unity. **a** Three cube roots. **b** Six 6th roots

Figure 2.6 shows the roots of unity for the cases of $n = 3$ and 6. The three cube roots of unity are on the vertices of an equilateral triangle and the six 6th roots of unity are on the vertices of a regular hexagon.

This property can be used to find the nth roots of any nonzero complex number z_0. In general, if α is any particular nth root of z_0 (not necessarily the principal root), then the n distinct values of $z_0^{1/n}$ are obtained as

$$\alpha, \alpha\omega, \alpha\omega^2, \ldots, \alpha\omega^{n-1}$$

since multiplication of α by ω^k corresponds to increasing the argument of α by $2k\pi/n$.

Example 2.10 Let us find all square roots of $-1 = 1(\cos\pi + \mathrm{i}\sin\pi)$. From Eq. 2.43, it follows that

$$(-1)^{1/2} = \sqrt{1}\left(\cos\frac{\pi + 2k\pi}{2} + \mathrm{i}\sin\frac{\pi + 2k\pi}{2}\right)$$

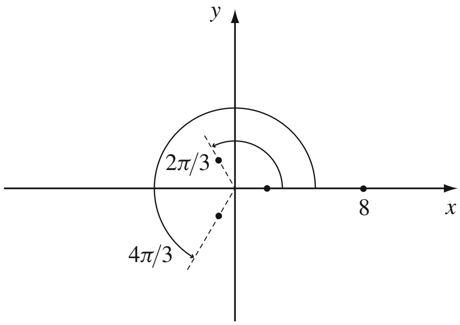
where $k = 0, 1$. The principal square root is i and the other root is $-$i. From Eq. 2.45, $\omega = -1$. Hence, if α is a particular square root, $-\alpha$ is the other root.

Example 2.11 Let us find all cube roots of $8 = 8(\cos 0 + \mathrm{i}\sin 0)$. From Eq. 2.43, it follows that

$$8^{1/3} = \sqrt[3]{8}\left(\cos\frac{2k\pi}{3} + \mathrm{i}\sin\frac{2k\pi}{3}\right)$$

where $k = 0, 1, 2$. The principal cube root is 2 and the other two roots are $-1 + \sqrt{3}\mathrm{i}$ and $-1 - \sqrt{3}\mathrm{i}$, as shown in Fig. 2.7. It is confirmed that three cube roots are on the vertices of an equilateral triangle, the arguments of which are different from each other by $2\pi/3$.

Fig. 2.7 The point $z = 8$ and three cube roots: 2, $-1 + \sqrt{3}\mathrm{i}$, and $-1 - \sqrt{3}\mathrm{i}$



Example 2.12 Let us find all cube roots of $8\mathrm{i} = 8(\cos\pi/2 + \mathrm{i}\sin\pi/2)$. From Eq. 2.43, it follows that

$$(8\mathrm{i})^{1/3} = \sqrt[3]{8}\left(\cos\frac{\pi/2 + 2k\pi}{3} + \mathrm{i}\sin\frac{\pi/2 + 2k\pi}{3}\right)$$

where $k = 0, 1, 2$. The principal cube root is $\sqrt{3} + \mathrm{i}$ and the other two roots are $-\sqrt{3} + \mathrm{i}$ and $-2\mathrm{i}$, as shown in Fig. 2.8.

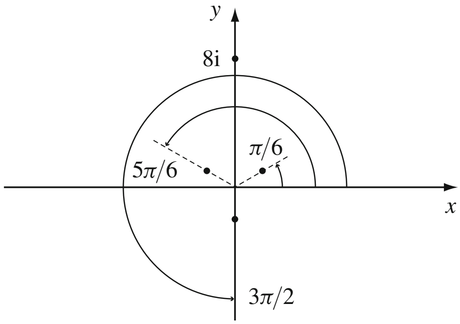
Knowing that a particular cube root of 8i is $\alpha = -2\mathrm{i}$ and

$$\omega = \left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

it follows from Eq. 2.46 that the other two roots are

$$\alpha\omega = -2\mathrm{i}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = \sqrt{3} + \mathrm{i}$$

Fig. 2.8 The point $z = 8\mathrm{i}$ and three cube roots: $\sqrt{3} + \mathrm{i}$, $-\sqrt{3} + \mathrm{i}$, and $-2\mathrm{i}$



and

$$\alpha\omega^2 = -2\mathrm{i}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)^2 = -\sqrt{3} + \mathrm{i}$$

which, of course, are consistent with the solutions from Eq. 2.43.

2.2 Functions of a Complex Variable

Functions of a complex variable can be defined in a similar way to functions of a real variable. However, it should be noted that they operate on a complex variable rather than a real variable, and consequently exhibit interesting properties not shared by their real counterparts.

2.2.1 Definition

A function f defined on S, a set of complex numbers, is a rule that assigns to each value z belonging to S a complex number w. The number w is called the value of f at z, and this correspondence is denoted by

$$w = f(z) \tag{2.47}$$

where z is considered as a complex variable in the set S, called the domain of definition of f (or briefly the domain of f). The set of all values of a function f is called the range of f.

Suppose that $w = u + \mathrm{i}v$ is the value of a function f at $z = x + \mathrm{i}y$:

$$w = u + \mathrm{i}v = f(z) = f(x + \mathrm{i}y) \tag{2.48}$$

This relation implies that each of the real values u and v depends on the real variables x and y, which gives

$$\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases} \tag{2.49}$$

where $u(x, y)$ and $v(x, y)$ are the real functions of the real variables x and y.

Example 2.13 If $f(z) = z^2$, then

$$f(z) = (x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i}$$

and thus

$$\begin{cases} u(x, y) = x^2 - y^2 \\ v(x, y) = 2xy \end{cases}$$

The values of u and v depend on x and y.

It should be noted that a function $f(z)$, depending on the complex variable z, is equivalent to a pair of real functions $u(x, y)$ and $v(x, y)$, each depending on the two real variables x and y. In Example 2.13, a single complex function $f = z^2$ conveys two kinds of real functions, $u = x^2 - y^2$ and $v = 2xy$. This implies that a function of a complex variable contains two kinds of information (u and v), which could satisfy our needs addressed in Solution Strategy.

2.2.2 *Elementary Functions*

Many of the elementary functions appearing in real-variable calculus have natural complex extensions. The functions relevant to the engineering problems considered in this book are reviewed.

2.2.2.1 Polynomial Function

The polynomial function is defined by

$$w = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0 \tag{2.50}$$

where $\alpha_n \neq 0, \alpha_{n-1}, \ldots, \alpha_0$ are the complex constants and a positive integer n is the degree of the polynomial function. The domain of definition is the entire z plane.

2.2.2.2 Rational Function

The rational function is defined by

$$w = \frac{f(z)}{g(z)} \tag{2.51}$$

where $f(z)$ and $g(z)$ are the polynomial functions. The function $f(z)/g(z)$ is defined at each point z except where $g(z) = 0$.

2.2.2.3 Exponential Function

The exponential function is defined by

$$w = \mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) \tag{2.52}$$

where $\mathrm{e} = 2.71828\ldots$ is the natural base of logarithms. The domain of definition is the entire z plane. Complex exponential functions have properties similar to those of real exponential functions, such as $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$ and $\mathrm{e}^{z_1-z_2} = \mathrm{e}^{z_1}/\mathrm{e}^{z_2}$.

2.2.2.4 Logarithmic Function

The inverse of the complex exponential function is the natural logarithmic function and is defined by

$$w = \ln z = \ln r + \mathrm{i}\theta = \ln r + \mathrm{i}(\theta + 2n\pi) \tag{2.53}$$

where $z = re^{\mathrm{i}\theta} = re^{\mathrm{i}(\theta+2n\pi)}$. The function $\ln z$ is defined at each nonzero point z. Since infinitely different values of $\arg z$ are obtained by successively encircling the origin $z = 0$, the complex logarithmic function is infinitely multi-valued. Each of the multiple functions is called a branch of the logarithmic function.

To keep the function single-valued, an artificial barrier that cannot be crossed is introduced. This barrier is called the branch cut, the direction of which is arbitrary. For instance, if the branch cut is set along the negative real half-axis (Fig. 2.9), $\arg z$ becomes single-valued and takes the principal value $-\pi < \mathrm{Arg}\, z \leq \pi$. The value of $\ln z$ corresponding to the principal value is denoted by $\mathrm{Ln}z$:

$$\mathrm{Ln}z = \ln |z| + \mathrm{i}\mathrm{Arg}\, z \tag{2.54}$$

The values of $\arg z$ differ by integer multiples of 2π, and the other values of $\ln z$ is given by

$$\ln z = \mathrm{Ln}z + 2n\pi \mathrm{i} \tag{2.55}$$

where n is an integer. The point common to all branch cuts is called the branch point. The origin is a branch point of the logarithmic function.

Complex logarithmic functions have properties similar to those of real logarithmic functions, such as $\ln(z_1 z_2) = \ln z_1 + \ln z_2$ and $\ln(z_1/z_2) = \ln z_1 - \ln z_2$. It should be noted that these properties are not, in general, valid when ln is replaced by Ln.

Example 2.14 Let us consider $z_1 = z_2 = -1$ and specify $\ln z_1 = \pi \mathrm{i}$ and $\ln z_2 = -\pi \mathrm{i}$, then $\ln(z_1 z_2) = \ln 1 = 0$ and $\ln z_1 + \ln z_2 = 0$, thus the property $\ln(z_1 z_2) = \ln z_1 + \ln z_2$ is satisfied. In contrast, if the principal values are used, $\mathrm{Ln}z_1 = \mathrm{Ln}z_2 = \pi \mathrm{i}$ and $\mathrm{Ln}z_1 + \mathrm{Ln}z_2 = 2\pi \mathrm{i}$, which is not equal to $\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}1 = 0$. Thus, the property $\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}z_1 + \mathrm{Ln}z_2$ is not always valid.

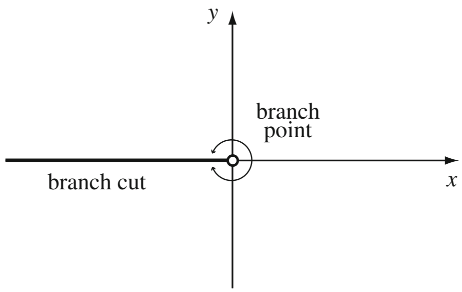


Fig. 2.9 Branch cut and branch point

• **Alternative Solution to Task 1-1**
Using functions of a complex variable, Eq. 1.65 can be rewritten as

$$\begin{aligned}\Phi(z) &= -\frac{q_u}{A}\mathrm{Re}\, z - \frac{q_s}{2\pi h}\mathrm{Re}[\ln(z - z_s)] \\ &= \mathrm{Re}\left[-\frac{q_u}{A}z - \frac{q_s}{2\pi h}\ln(z - z_s)\right]\end{aligned} \tag{2.56}$$

Substituting $q_u/A = 1$, $q_s/h = 6$, and $z_s = -1$ yields

$$\Phi(z) = \mathrm{Re}\left[-z - \frac{3}{\pi}\ln(z + 1)\right]$$

with which the equipotential lines are obtained as shown in Fig. 1.5.

2.3 Complex Differentiation

As is the case with functions of a real variable, the concepts of limit, continuity, and differentiability are important for functions of a complex variable. In developing a theory of differentiation for complex functions, the Cauchy–Riemann equations are introduced, which play a substantial role in complex analysis.

2.3.1 Limit and Continuity

A function $f(z)$ is said to have the limit w_0 at a point $z = z_0$ if

$$\lim_{z \to z_0} f(z) = w_0 \tag{2.57}$$

The point $w = f(z)$ can be made arbitrarily close to w_0 if the point z is chosen close enough to z_0 but distinct from it. In precise terms, for any positive number ε, some positive number δ can be found such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta \tag{2.58}$$

As shown in Fig. 2.10, for each ε neighborhood[2] of w_0, there is a deleted δ neighborhood[3] of z_0 such that every point z in it has an image w lying in the ε neighborhood.

---

[2] An ε neighborhood of a point w_0 is the set of all points w such that $|w - w_0| < \varepsilon$ where ε is any given positive number.

[3] A deleted δ neighborhood of z_0 is a neighborhood of z_0 in which the point z_0 is omitted: $0 < |z - z_0| < \delta$.

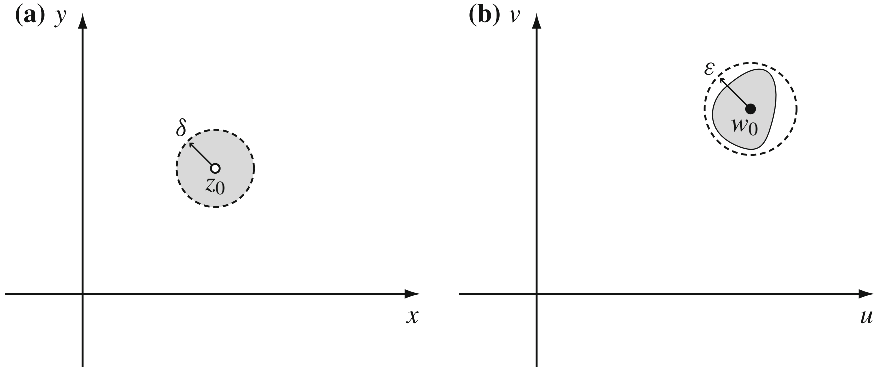


Fig. 2.10 Limit w_0. **a** Deleted δ neighborhood in the z plane. **b** ε neighborhood in the w plane

Because Eq. 2.58 applies to all points in the deleted neighborhood, the symbol $z \to z_0$ in Eq. 2.57 implies that z may approach z_0 from any direction in the complex plane.

A function $f(z)$ is said to be continuous at a point $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{2.59}$$

provided this limit and $f(z_0)$ exist. With the concept of limit, Eq. 2.59 can be rephrased that for any positive number ε, some positive number δ can be found such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta \tag{2.60}$$

If $f(z)$ is continuous at each point in a domain, $f(z)$ is said to be continuous in this domain.

With $z_0 = x_0 + \mathrm{i}y_0$ and $f(z) = u + \mathrm{i}v$, the real and imaginary parts of Eq. 2.59 can be written as

$$\begin{cases} \lim\limits_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u(x_0, y_0) \\ \lim\limits_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v(x_0, y_0) \end{cases} \tag{2.61}$$

which states that the real functions u and v are continuous at (x_0, y_0). A function $f(z)$ is continuous if and only if its real and imaginary parts, u and v, are continuous.

Example 2.15 Let us consider $\lim_{z \to 0}(\bar{z}/z)$. If the limit is to exist, it must be independent of the approaching path of z to 0. Let $z \to 0$ along the x axis ($y = 0$), then it follows

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x}{x} = 1$$

On the other hand, if $z \to 0$ along the y axis ($x = 0$), then it follows that

$$\lim_{z\to 0}\frac{\overline{z}}{z}=\lim_{y\to 0}\frac{-\mathrm{i}y}{\mathrm{i}y}=-1$$

which is different from the previous value, and thus, the limit does not exist.

Example 2.16 Let us examine the continuity of $\overline{z}=x-\mathrm{i}y$. Since $u(x,y)=x$ and $v(x,y)=-y$ are continuous at each point (x,y), $\overline{z}$ is continuous everywhere in the complex plane.

2.3.2 Differentiability

The derivative of a function $f(z)$ at a point z_0, denoted by $f'(z_0)$, is defined by

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \tag{2.62}$$

provided this limit exists. The function $f(z)$ is said to be differentiable at z_0 when its derivative at z_0 exists. As addressed in the definition of the limit, z may approach z_0 from any direction in the complex plane. Hence, differentiability at z_0 implies that the quotient in Eq. 2.62 always approaches a certain value and all these values are equal.

With the increment $\Delta z=z-z_0$, Eq. 2.62 takes the form

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \tag{2.63}$$

Let $w=f(z)$ and $\Delta w=f(z_0+\Delta z)-f(z_0)$, then Eq. 2.63 reduces to

$$f'(z_0)=\frac{\mathrm{d}w}{\mathrm{d}z}=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} \tag{2.64}$$

where $\mathrm{d}w=f'(z_0)\mathrm{d}z$ is the differential of w and $\mathrm{d}z$ is the differential of z. It should be noted that $\mathrm{d}z$ and $\mathrm{d}w$ are not the limits of Δz and Δw as $\Delta z\to 0$; these limits are zero whereas $\mathrm{d}z$ and $\mathrm{d}w$ are not necessarily zero. The differential $\mathrm{d}w$ is a dependent variable determined through $\mathrm{d}w=f'(z_0)\mathrm{d}z$ with the independent variable $\mathrm{d}z$ for a given z_0.

The differentiation rules for functions of a complex variable are the same as in real-variable calculus. For any differentiable functions f and g, the following rules are valid.

Linearity rule If $w=c_1f+c_2g$, where c_1 and c_2 are constants, then

$$w'=(c_1f+c_2g)'=c_1f'+c_2g' \tag{2.65}$$

Product rule If $w = fg$, then

$$w' = (fg)' = f'g + fg' \tag{2.66}$$

Quotient rule If $w = f/g$, where $g \neq 0$, then

$$w' = \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \tag{2.67}$$

Chain rule If $w = f(\chi)$ and $\chi = g(z)$, then

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}w}{\mathrm{d}\chi}\frac{\mathrm{d}\chi}{\mathrm{d}z} \tag{2.68}$$

Example 2.17 Let us prove the chain rule. Consider the increments given by

$$\begin{cases} \Delta w = f(\chi + \Delta\chi) - f(\chi) \\ \Delta\chi = g(z + \Delta z) - g(z) \end{cases}$$

where $\Delta w \to 0$ and $\Delta\chi \to 0$ as $\Delta z \to 0$. If $\Delta\chi \neq 0$, let us define ε as

$$\varepsilon = \frac{\Delta w}{\Delta\chi} - \frac{\mathrm{d}w}{\mathrm{d}\chi} \tag{2.69}$$

so that $\varepsilon \to 0$ as $\Delta\chi \to 0$. Rearranging Eq. 2.69 for Δw gives

$$\Delta w = \frac{\mathrm{d}w}{\mathrm{d}\chi}\Delta\chi + \varepsilon\Delta\chi \tag{2.70}$$

If $\Delta\chi = 0$, according to the definition of Δw, $\Delta w = 0$, and ε is set as zero. Hence, for any $\Delta\chi$ ($\Delta\chi \neq 0$ or $\Delta\chi = 0$), Eq. 2.70 holds.

Dividing Eq. 2.70 by $\Delta z \neq 0$ and in the limit of $\Delta z \to 0$

$$\begin{aligned} \lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = \frac{\mathrm{d}w}{\mathrm{d}z} &= \lim_{\Delta z\to 0}\left(\frac{\mathrm{d}w}{\mathrm{d}\chi}\frac{\Delta\chi}{\Delta z} + \varepsilon\frac{\Delta\chi}{\Delta z}\right) \\ &= \frac{\mathrm{d}w}{\mathrm{d}\chi}\lim_{\Delta z\to 0}\frac{\Delta\chi}{\Delta z} + \lim_{\Delta z\to 0}\varepsilon\lim_{\Delta z\to 0}\frac{\Delta\chi}{\Delta z} \\ &= \frac{\mathrm{d}w}{\mathrm{d}\chi}\frac{\mathrm{d}\chi}{\mathrm{d}z} + \lim_{\Delta z\to 0}\varepsilon\frac{\mathrm{d}\chi}{\mathrm{d}z} \\ &= \frac{\mathrm{d}w}{\mathrm{d}\chi}\frac{\mathrm{d}\chi}{\mathrm{d}z} \end{aligned}$$

where $\varepsilon \to 0$ in the limit of $\Delta z \to 0$ is used, and the chain rule is proved.

Example 2.18 Let us find the derivative of $w = z^2$. The increment Δw is

$$\begin{aligned}\Delta w &= (z + \Delta z)^2 - z^2 = z^2 + 2z\Delta z + \Delta z^2 - z^2 \\ &= 2z\Delta z + \Delta z^2\end{aligned}$$

From Eq. 2.64, it follows that

$$w' = \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z} = 2z$$

which is identical to the differentiation in real-variable calculus. The power rule of differentiation

$$(z^n)' = nz^{n-1} \tag{2.71}$$

generally holds for complex functions.

Example 2.19 Let us find the derivative of $w = 1/z$. The increment Δw is

$$\begin{aligned}\Delta w &= \frac{1}{z + \Delta z} - \frac{1}{z} = \frac{z - z - \Delta z}{(z + \Delta z)z} \\ &= -\frac{\Delta z}{(z + \Delta z)z}\end{aligned}$$

From Eq. 2.64, it follows that

$$w' = -\lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z(z + \Delta z)z} = -\frac{1}{z^2}$$

which is identical to the differentiation in real-variable calculus. The power rule of differentiation

$$(z^{-n})' = -nz^{-n-1} \tag{2.72}$$

generally holds for complex functions.

2.3.3 Analytic Functions

A function $f(z)$ is said to be analytic at a point z_0 when there exists a neighborhood $|z - z_0| < \delta$ for all points of which $f'(z)$ exists. That is to say, if $f(z)$ is analytic at z_0, $f(z)$ must be differentiable not only at z_0, but also at all points in some δ neighborhood of z_0. This concept is raised because differentiability of $f(z)$ merely at a single point z_0 is of no practical interest.

When the derivative $f'(z)$ exists at all points z in a domain, a function $f(z)$ is said to be analytic in this domain and is referred to as an analytic function. In particular, if $f(z)$ is analytic at all points in the entire complex plane, then $f(z)$ is referred to as an entire function.

Example 2.20 Let us examine the analyticity of $w = z^2$. In Example 2.18, the derivative is obtained as $w' = 2z$ and it is obvious that z^2 is analytic at all points in the entire plane. The function z^2 is entire.

Example 2.21 Let us examine the analyticity of $w = |z|^2$. The increment Δw is

$$\begin{aligned}\Delta w &= |z + \Delta z|^2 - |z|^2 = (z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z} \\ &= z\overline{\Delta z} + \bar{z}\Delta z + \overline{\Delta z}\Delta z\end{aligned}$$

From Eq. 2.64, it follows that

$$w' = \lim_{\Delta z \to 0} \frac{z\overline{\Delta z} + \bar{z}\Delta z + \overline{\Delta z}\Delta z}{\Delta z} = \bar{z} + \lim_{\Delta z \to 0}\left(z\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}\right)$$

If the limit exists, it can be found by letting $\Delta z = (\Delta x, \Delta y)$ approach 0 in any manner. Setting $\Delta y = 0$ and letting $\Delta z = (\Delta x, 0)$ approach 0 yields $\overline{\Delta z} = \Delta z$, which gives the value of derivative $w' = \bar{z} + z$. Similarly, setting $\Delta x = 0$ and letting $\Delta z = (0, \Delta y)$ approach 0 yields $\overline{\Delta z} = -\Delta z$, which gives the value of derivative $w' = \bar{z} - z$. Since limits are unique, $\bar{z} + z$ must be equal to $\bar{z} - z$, and it follows that $z = 0$ and $w' = 0$. The function $|z|^2$ has the derivative only at the point $z = 0$. It is not differentiable at any point in a neighborhood of $z = 0$, and thus, $|z|^2$ is not analytic.

Example 2.22 Let us examine the analyticity of $w = 1/z$. In Example 2.19, the derivative is obtained as $w' = -1/z^2$ and it is obvious that $1/z$ is analytic at all nonzero points in the complex plane.

If a function $f(z)$ is not analytic at a point z_0, but every neighborhood of z_0 contains at least one point where $f(z)$ is analytic, then the point z_0 is called the singular point or singularity of $f(z)$. For instance, the function $1/z$ has a singular point at $z = 0$, since $1/z$ is analytic at all nonzero points in the complex plane. On the other hand, the function $|z|^2$ is nowhere analytic and has no singularity.

2.3.4 L'Hospital's Rule

If $f(z)$ is analytic, Eq. 2.62 can be rewritten as

$$\frac{f(z) - f(z_0)}{z - z_0} - f'(z) = \varepsilon \tag{2.73}$$

where $\varepsilon \to 0$ as $z \to z_0$. It follows that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z - z_0) \tag{2.74}$$

Now, let us consider analytic functions $f(z)$ and $g(z)$ in a domain containing z_0 and suppose that $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$. Then, using Eq. 2.74 and the fact that $f(z_0) = g(z_0) = 0$, it follows that

$$\begin{cases} f(z) = f'(z_0)(z - z_0) + \varepsilon_1(z - z_0) \\ g(z) = g'(z_0)(z - z_0) + \varepsilon_2(z - z_0) \end{cases} \tag{2.75}$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $z \to z_0$. Then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)} \tag{2.76}$$

which is known as L'Hospital's rule which often converts the quotient to a determinate form and allows the limit to be evaluated.

Example 2.23 Let us evaluate

$$\lim_{z\to \mathrm{i}} \frac{f(z)}{g(z)} = \lim_{z\to \mathrm{i}} \frac{z^7 + \mathrm{i}}{z^3 + \mathrm{i}}$$

Since $f(\mathrm{i}) = g(\mathrm{i}) = 0$ and $f(z)$ and $g(z)$ are analytic at $z = \mathrm{i}$, L'Hospital's rule can be applied as

$$\lim_{z\to \mathrm{i}} \frac{f'(z)}{g'(z)} = \frac{7\mathrm{i}^6}{3\mathrm{i}^2} = \frac{7}{3}\mathrm{i}^4 = \frac{7}{3}$$

and the limit is obtained as $7/3$.

Example 2.24 In the case $f'(z_0) = g'(z_0) = 0$, L'Hospital's rule can be extended. Let us evaluate

$$\lim_{z\to 0} \frac{f(z)}{g(z)} = \lim_{z\to 0} \frac{1 - \cos z}{z^2}$$

Since $f(0) = g(0) = 0$ and $f(z)$ and $g(z)$ are analytic at $z = 0$, L'Hospital's rule can be applied as

$$\lim_{z\to 0} \frac{f'(z)}{g'(z)} = \lim_{z\to 0} \frac{\sin z}{2z}$$

where $\sin z$ and $2z$ are analytic and equal to 0 when $z = 0$. By applying L'Hospital's rule again, it follows that

$$\lim_{z\to 0} \frac{f''(z)}{g''(z)} = \frac{\cos 0}{2} = \frac{1}{2}$$

and the limit is obtained as $1/2$.

2.3.5 Cauchy–Riemann Equations in Cartesian Form

To perform complex analysis on any function, the analyticity of the function is essential. Let us assume the derivative $f'(z)$ at z exists as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{2.77}$$

By the definition of the limit, Δz can approach zero along any path in a neighborhood of z. If the approaching path is set horizontally, $\Delta z = \Delta x$, Eq. 2.77 can be written as

$$\begin{aligned}
f'(z) &= \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + \mathrm{i}v(x+\Delta x, y) - u(x, y) - \mathrm{i}v(x, y)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + \mathrm{i}\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right] \\
&= \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}
\end{aligned} \tag{2.78}$$

provided that the partial derivatives exist.

Similarly, if the approaching path is set vertically, $\Delta z = \mathrm{i}\Delta y$, Eq. 2.77 can be written as

$$\begin{aligned}
f'(z) &= \lim_{\Delta y \to 0} \frac{f(z+\mathrm{i}\Delta y) - f(z)}{\mathrm{i}\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + \mathrm{i}v(x, y+\Delta y) - u(x, y) - \mathrm{i}v(x, y)}{\mathrm{i}\Delta y} \\
&= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x, y)}{\mathrm{i}\Delta y} + \mathrm{i}\frac{v(x, y+\Delta y) - v(x, y)}{\mathrm{i}\Delta y} \right] \\
&= -\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\end{aligned} \tag{2.79}$$

where the identity $1/\mathrm{i} = -\mathrm{i}$ is used.

For the existence of the derivative $f'(z)$, the values of Eqs. 2.78 and 2.79 must be equal. By equating the real and imaginary parts on the right-hand sides of these equations, necessary conditions for the existence of $f'(z)$ can be derived as

$$\begin{cases}
\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\
\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}
\end{cases} \tag{2.80}$$

which are known as the Cauchy–Riemann equations.

These observations imply the necessity of the Cauchy–Riemann equations for a function to be analytic and can be summarized as follows.

Theorem 2.1 (Cauchy–Riemann equations: Necessity) *If $f(z) = u + \mathrm{i}v$ is analytic in a domain, the first-order partial derivatives of u and v with respect to x and y exist and satisfy the Cauchy–Riemann equations at all points in this domain.*

Example 2.25 Let us consider the analytic function $f(z) = z^2$. Since

$$\begin{cases} u(x, y) = x^2 - y^2 \\ v(x, y) = 2xy \end{cases}$$

the partial derivatives are

$$\begin{cases} \partial u/\partial x = 2x \\ \partial u/\partial y = -2y \end{cases}$$

and

$$\begin{cases} \partial v/\partial y = 2x \\ -\partial v/\partial x = -2y \end{cases}$$

which satisfy the Cauchy–Riemann equations. The derivative is

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = 2x + 2y\mathrm{i} = 2(x + \mathrm{i}y) = 2z$$

which is identical to the differentiation in real-variable calculus. As also noted in Example 2.18, the power rule of differentiation generally holds for complex functions.

Example 2.26 Let us consider the function $f(z) = |z|^2$. Since

$$\begin{cases} u(x, y) = x^2 + y^2 \\ v(x, y) = 0 \end{cases}$$

the partial derivatives are

$$\begin{cases} \partial u/\partial x = 2x \\ \partial u/\partial y = 2y \end{cases}$$

and

$$\begin{cases} \partial v/\partial y = 0 \\ -\partial v/\partial x = 0 \end{cases}$$

which do not satisfy the Cauchy–Riemann equations except at 0, and thus, $f(z) = |z|^2$ is not analytic. The derivative is

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = 0$$

only at $z = 0$.

In real-variable analysis, the functions x^2 and $|x|^2$ are identical and have the same derivative of $2x$. The very different conclusions of Examples 2.25 and 2.26 exemplify an interesting feature of differentiability of functions of a complex variable.

It should be noted that differentiability of $f(z) = u + \mathrm{i}v$ is not ensured by individual differentiability of u and v. Functions u and v need to be related through the Cauchy–Riemann equations. This is different from continuity of $f(z)$, which is equivalent to individual continuity of u and v.

Example 2.27 Let us consider real functions $u(x, y) = x$ and $v(x, y) = -y$, which are differentiable. The partial derivatives are

$$\begin{cases} \partial u/\partial x = 1 \\ \partial u/\partial y = 0 \end{cases}$$

and

$$\begin{cases} \partial v/\partial y = -1 \\ -\partial v/\partial x = 0 \end{cases}$$

which do not satisfy the Cauchy–Riemann equations, and $\bar{z} = x - \mathrm{i}y$ is not analytic.

It should be noted that $\bar{z}$ is not differentiable anywhere even though its real and imaginary parts (u and v) are continuous (and therefore $\bar{z}$ is continuous as shown in Example 2.16) and differentiable.

Theorem 2.1 states that the Cauchy–Riemann equations are necessary for the existence of the derivative $f'(z)$ but does not cover the sufficiency. However, if the first partial derivatives in Eq. 2.80 are continuous in a domain, then it can be shown that the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in this domain.

Theorem 2.2 (Cauchy–Riemann equations: Sufficiency) *If u and v have continuous first-order partial derivatives with respect to x and y that satisfy the Cauchy–Riemann equations in a domain, the function $f(z) = u + \mathrm{i}v$ is analytic in this domain.*

Proof In view of the continuity of the first-order partial derivatives of u with respect to x and y, the increment Δu is given by

$$\begin{aligned} \Delta u &= u(x + \Delta x, y + \Delta y) - u(x, y) \\ &= u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) + u(x, y + \Delta y) - u(x, y) \\ &= \left(\frac{\partial u}{\partial x} + \varepsilon_u\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_u\right)\Delta y \\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_u \Delta x + \eta_u \Delta y \end{aligned}$$

where the identity Eq. 2.74 is used with $\varepsilon_u \to 0$ as $\Delta x \to 0$ and $\eta_u \to 0$ as $\Delta y \to 0$.

Similarly, in view of the continuity of the first-order partial derivatives of v with respect to x and y, the increment Δv is given by

$$\begin{aligned}\Delta v &= v(x+\Delta x, y+\Delta y) - v(x,y)\\ &= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_v \Delta x + \eta_v \Delta y\end{aligned}$$

where $\varepsilon_v \to 0$ as $\Delta x \to 0$ and $\eta_v \to 0$ as $\Delta y \to 0$.

The increment Δw is then given by

$$\begin{aligned}\Delta w &= \Delta u + \mathrm{i}\Delta v\\ &= \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y\end{aligned}$$

where $\varepsilon = \varepsilon_u + \mathrm{i}\varepsilon_v \to 0$ and $\eta = \eta_u + \mathrm{i}\eta_v \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Assuming that the Cauchy–Riemann equations are satisfied, $\partial u/\partial y$ is replaced by $-\partial v/\partial x$ and $\partial v/\partial y$ by $\partial u/\partial x$, and then

$$\begin{aligned}\Delta w &= \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{-\partial v}{\partial x} + \mathrm{i}\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y\\ &= \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)(\Delta x + \mathrm{i}\Delta y) + \varepsilon\Delta x + \eta\Delta y\\ &= \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)\Delta z + \varepsilon\Delta x + \eta\Delta y\end{aligned}$$

Dividing by Δz and in the limit of $\Delta z \to 0$

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}$$

which indicates that the derivative exists and the function $f(z)$ is analytic. □

Example 2.28 Let us consider the property of a function $f(z)$, when $f(z)$ is analytic in a domain and $|f(z)|$ is constant in this domain. It follows that

$$|f(z)|^2 = |u + \mathrm{i}v|^2 = u^2 + v^2 = \text{constant}$$

Taking the partial derivatives with respect to x and y yields

$$\begin{cases} u(\partial u/\partial x) + v(\partial v/\partial x) = 0\\ u(\partial u/\partial y) + v(\partial v/\partial y) = 0\end{cases}$$

From the Cauchy–Riemann equations, it follows that

$$\begin{cases} u(\partial u/\partial x) - v(\partial u/\partial y) = 0\\ u(\partial u/\partial y) + v(\partial u/\partial x) = 0\end{cases}$$

which gives

$$\begin{cases} (u^2+v^2)(\partial u/\partial x) = 0 \\ (u^2+v^2)(\partial u/\partial y) = 0 \end{cases}$$

This implies that $u^2+v^2 = 0$ or $\partial u/\partial x = \partial u/\partial y = 0$. If $u^2+v^2 = 0$, then $u = v = 0$ and $f(z) = 0 =$ constant. If $u^2+v^2 \neq 0$, from the Cauchy–Riemann equations, then also $\partial v/\partial x = \partial v/\partial y = 0$. Hence, u is constant and v is constant; consequently, $f(z)$ is constant. In either case, if $|f(z)|$ is constant, $f(z)$ is constant.

2.3.6 Cauchy–Riemann Equations in Polar Form

When complex functions are expressed in polar coordinates, the Cauchy–Riemann equations in polar form are of practical use. Let us assume the derivative $f'(z)$ at $z = re^{i\theta}$ exists, which is given by Eq. 2.77. By the definition of the limit, Δz can approach zero along any path in a neighborhood of z. If the approaching path is set along the ray θ, $\Delta z = \Delta r e^{i\theta}$, Eq. 2.77 can be written as

$$\begin{aligned} f'(z) &= \lim_{\Delta r \to 0} \frac{f((r+\Delta r)e^{i\theta}) - f(re^{i\theta})}{\Delta r e^{i\theta}} \\ &= \lim_{\Delta r \to 0} \frac{u(r+\Delta r, \theta) + iv(r+\Delta r, \theta) - u(r,\theta) - iv(r,\theta)}{\Delta r e^{i\theta}} \\ &= \lim_{\Delta r \to 0} \left[\frac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r e^{i\theta}} + i\frac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r e^{i\theta}} \right] \\ &= e^{-i\theta} \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \right) \end{aligned} \tag{2.81}$$

provided that the partial derivatives exist.

Similarly, if the approaching path is set along the circle r, $\Delta z = r(e^{i(\theta+\Delta\theta)} - e^{i\theta})$, Eq. 2.77 can be written as

$$\begin{aligned} f'(z) &= \lim_{\Delta\theta \to 0} \frac{f(re^{i(\theta+\Delta\theta)}) - f(re^{i\theta})}{r(e^{i(\theta+\Delta\theta)} - e^{i\theta})} \\ &= \lim_{\Delta\theta \to 0} \frac{u(r,\theta+\Delta\theta) + iv(r,\theta+\Delta\theta) - u(r,\theta) - iv(r,\theta)}{re^{i\theta}\Delta\theta} \frac{\Delta\theta}{e^{i\Delta\theta} - 1} \\ &= \lim_{\Delta\theta \to 0} \left[\frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{re^{i\theta}\Delta\theta} + i\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{re^{i\theta}\Delta\theta} \right] \frac{\Delta\theta}{e^{i\Delta\theta} - 1} \\ &= e^{-i\theta} \left(\frac{1}{r}\frac{\partial u}{\partial \theta} + i\frac{1}{r}\frac{\partial v}{\partial \theta} \right) \frac{1}{i} = e^{-i\theta} \left(-i\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} \right) \end{aligned} \tag{2.82}$$

where L'Hospital's rule is used to obtain $\lim_{\Delta\theta \to 0} \Delta\theta/(e^{i\Delta\theta} - 1) = 1/i$.

For the existence of the derivative $f'(z)$, the values of Eqs. 2.81 and 2.82 must be equal. Equating the real and imaginary parts on the right-hand sides of these equations yields

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r} \end{cases} \tag{2.83}$$

which are the Cauchy–Riemann equations in polar form.

Example 2.29 Let us consider the function $f(z) = \ln z = \ln r + \mathrm{i}\theta$. Since

$$\begin{cases} u(r,\theta) = \ln r \\ v(r,\theta) = \theta \end{cases}$$

the partial derivatives are

$$\begin{cases} \partial u/\partial r = 1/r \\ (1/r)(\partial u/\partial \theta) = 0 \end{cases}$$

and

$$\begin{cases} (1/r)(\partial v/\partial \theta) = 1/r \\ -\partial v/\partial r = 0 \end{cases}$$

which satisfy the Cauchy–Riemann equations, and $f(z) = \ln z$ is analytic. The derivative is

$$f'(z) = \mathrm{e}^{-\mathrm{i}\theta}\,(\partial u/\partial r + \mathrm{i}\partial v/\partial r) = \mathrm{e}^{-\mathrm{i}\theta}(1/r) = 1/(r\mathrm{e}^{\mathrm{i}\theta}) = 1/z$$

which is identical to the differentiation in real-variable calculus.

Example 2.30 Let us consider the function $f(z) = 1/z = r^{-1}\mathrm{e}^{-\mathrm{i}\theta}$. Since

$$\begin{cases} u(r,\theta) = \cos\theta/r \\ v(r,\theta) = -\sin\theta/r \end{cases}$$

the partial derivatives are

$$\begin{cases} \partial u/\partial r = -\cos\theta/r^2 \\ (1/r)(\partial u/\partial \theta) = -\sin\theta/r^2 \end{cases}$$

and

$$\begin{cases} (1/r)(\partial v/\partial \theta) = -\cos\theta/r^2 \\ -\partial v/\partial r = -\sin\theta/r^2 \end{cases}$$

which satisfy the Cauchy–Riemann equations, and $f(z) = 1/z$ is analytic. The derivative is

$$\begin{aligned} f'(z) &= \mathrm{e}^{-\mathrm{i}\theta} (\partial u/\partial r + \mathrm{i}\partial v/\partial r) = \mathrm{e}^{-\mathrm{i}\theta} \left(-\cos\theta/r^2 + \mathrm{i}\sin\theta/r^2\right) \\ &= -\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta}/r^2 = -1/(r\mathrm{e}^{\mathrm{i}\theta})^2 = -1/z^2 \end{aligned}$$

which is identical to the differentiation in real-variable calculus.

2.4 Harmonic Functions

As discussed in Chap. 1, Laplace's equation occurs in many engineering problems and plays a central role in potential theory. Functions that fulfill Laplace's equation are called harmonic functions, which are discussed in this section. In particular, analyticity of complex functions and the existence of a harmonic conjugate are of great practical importance.

2.4.1 Analyticity of Elementary Functions

Analyticity of the elementary functions reviewed in Sect. 2.2.2 is examined.

2.4.1.1 Polynomial Function

The power rule of differentiation (identical to that in real-variable calculus) holds for functions of a complex variable, and thus

$$\begin{aligned} &\frac{\mathrm{d}(\alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0)}{\mathrm{d}z} \\ &\quad = n\alpha_n z^{n-1} + (n-1)\alpha_{n-1} z^{n-2} + \cdots + \alpha_1 \end{aligned} \tag{2.84}$$

and the polynomial function is analytic in the entire complex plane.

2.4.1.2 Rational Function

The quotient rule of differentiation (identical to that in real-variable calculus) holds for functions of a complex variable, and thus

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{f(z)}{g(z)} = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} \tag{2.85}$$

and the rational function is analytic except at the points where $g(z) = 0$.

2.4.1.3 Exponential Function

By definition (Eq. 2.52), $w = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = u + \mathrm{i}v$, then

$$\begin{cases} \dfrac{\partial u}{\partial x} = \mathrm{e}^x \cos y = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\mathrm{e}^x \sin y = -\dfrac{\partial v}{\partial x} \end{cases} \tag{2.86}$$

indicating the Cauchy–Riemann equations are satisfied. Hence, the required derivative exists and is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \mathrm{e}^x \cos y + \mathrm{i}\mathrm{e}^x \sin y = \mathrm{e}^z \tag{2.87}$$

which is identical to the result in real-variable calculus and the exponential function is analytic in the entire complex plane.

2.4.1.4 Logarithmic Function

Let $w = \ln z$, then $z = \mathrm{e}^w$ and $\mathrm{d}z/\mathrm{d}w = \mathrm{e}^w = z$, thus

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln z = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1}{\mathrm{d}z/\mathrm{d}w} = \frac{1}{z} \tag{2.88}$$

which is identical to the result in real-variable calculus and consistent with Example 2.29. The logarithmic function is analytic except at the branch point, $z = 0$, and on the branch cut, the negative real half-axis. The origin and each point on the negative real half-axis are the singular points of $\ln z$.

2.4.2 Laplace's Equation in Cartesian Form

If a function $f(z) = u + \mathrm{i}v$ is analytic in a domain, the Cauchy–Riemann equations are satisfied in this domain. As is to be proved in Chap. 4 (Corollary 4.3), it is possible to assume that u and v have continuous second partial derivatives.

Differentiating both sides of the first Cauchy–Riemann equation in Cartesian form (Eq. 2.80) with respect to x and both sides of the second with respect to y yields

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y} \\ \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y} \end{cases} \tag{2.89}$$

Adding these two equations gives Laplace's equation in terms of u:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.90}$$

Similarly, differentiating both sides of the first Cauchy–Riemann equation with respect to y and both sides of the second with respect to x yields

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y^2} \\ \dfrac{\partial^2 u}{\partial x \partial y} = -\dfrac{\partial^2 v}{\partial x^2} \end{cases} \tag{2.91}$$

Subtracting the second equation from the first gives Laplace's equation in terms of v:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{2.92}$$

Solutions of Laplace's equation having continuous first and second partial derivatives are called harmonic functions. Hence, the real and imaginary parts of an analytic function are harmonic functions, as seen with Eqs. 2.90 and 2.92.

Theorem 2.3 (Harmonic function) *If a function $f(z) = u + \mathrm{i}v$ is analytic in a domain, its component functions u and v are harmonic in this domain.*

2.4.3 Laplace's Equation in Polar Form

For analytic functions expressed in polar coordinates, Laplace's equation in polar form (rather than Cartesian form) is of practical use. Differentiating both sides of the first Cauchy–Riemann equation in polar form (Eq. 2.83) with respect to r and both sides of the second with respect to θ yields

$$\begin{cases} r\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\partial u}{\partial r} = \dfrac{\partial^2 v}{\partial r \partial \theta} \\ \dfrac{1}{r}\dfrac{\partial^2 u}{\partial \theta^2} = -\dfrac{\partial^2 v}{\partial r \partial \theta} \end{cases} \tag{2.93}$$

Adding these two equations gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2.94}$$

which is Laplace's equation in terms of u in polar form.

Similarly, differentiating both sides of the first Cauchy–Riemann equation with respect to θ and both sides of the second with respect to r yields

$$\begin{cases} \dfrac{\partial^2 u}{\partial r \partial \theta} = \dfrac{1}{r}\dfrac{\partial^2 v}{\partial \theta^2} \\ \dfrac{\partial^2 u}{\partial r \partial \theta} = -\dfrac{\partial v}{\partial r} - r\dfrac{\partial^2 v}{\partial r^2} \end{cases} \tag{2.95}$$

Subtracting the second equation from the first gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \tag{2.96}$$

which is Laplace's equation in terms of v in polar form.

2.4.4 Harmonic Conjugate

When two harmonic functions u and v satisfy the Cauchy–Riemann equations (Eq. 2.80 or 2.83) in a domain, v is said to be a harmonic conjugate of u in this domain. It is evident that if a function $f(z) = u + \mathrm{i}v$ is analytic in a domain, v is a harmonic conjugate of u, and conversely, that if v is a harmonic conjugate of u in a domain, the function $f(z) = u + \mathrm{i}v$ is analytic in this domain.

Theorem 2.4 (Harmonic conjugate) *A function $f(z) = u + \mathrm{i}v$ is analytic in a domain if and only if v is a harmonic conjugate of u.*

Example 2.31 Let us consider real functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Since these functions are respectively the real and imaginary parts of the analytic function $f(z) = z^2$, v is a harmonic conjugate of u.

In contrast, the function $f(z) = v + \mathrm{i}u$ is not analytic, since the partial derivatives

$$\begin{cases} \partial v/\partial x = 2y \\ \partial v/\partial y = 2x \end{cases}$$

and

$$\begin{cases} \partial u/\partial y = -2y \\ -\partial u/\partial x = -2x \end{cases}$$

do not satisfy the Cauchy–Riemann equations, and thus, u is not a harmonic conjugate of v.

If v is a harmonic conjugate of u in some domain, it is not, in general, true that u is a harmonic conjugate of v in this domain. Instead, it is true that if v is a harmonic conjugate of u, $-u$ is a harmonic conjugate of v, that is, $f(z) = v - \mathrm{i}u$ is analytic.

This can be understood by noting that $f(z) = u + \mathrm{i}v$ is rewritten as $-\mathrm{i}f(z) = v - \mathrm{i}u$ and that $f(z)$ is analytic if and only if $-\mathrm{i}f(z)$ is analytic.

Proposition 2.1 (Existence of a harmonic conjugate) *If a function u is harmonic, there exists a harmonic conjugate v such that $f = u + \mathrm{i}v$ is an analytic function.*

Proof A harmonic conjugate v satisfies the Cauchy–Riemann equations $\partial v/\partial y = \partial u/\partial x$ and $\partial v/\partial x = -\partial u/\partial y$. Integrating $\partial v/\partial y = \partial u/\partial x$ with respect to y while keeping x constant gives

$$v = \int \frac{\partial u}{\partial x} \mathrm{d}y + t(x)$$

where $t(x)$ is a real function of x. Substituting this into the second Cauchy–Riemann equation yields

$$\frac{\partial}{\partial x} \int \frac{\partial u}{\partial x} \mathrm{d}y + t'(x) = -\frac{\partial u}{\partial y}$$

Since u is harmonic, differentiating this equation with respect to y results in Laplace's equation, revealing that a formula for $t'(x)$ involves x alone and $t(x)$ can be obtained by integration of $t'(x)$. □

Example 2.32 Let us consider a real function $u(x, y) = y^3 - 3x^2y$. The second partial derivatives are

$$\begin{cases} \partial^2 u/\partial x^2 = -6y \\ \partial^2 u/\partial y^2 = 6y \end{cases}$$

which satisfy Laplace's equation $\partial u^2/\partial x^2 + \partial u^2/\partial y^2 = 0$ and the function u is harmonic. From the Cauchy–Riemann equations, it follows that

$$\begin{cases} \partial v/\partial y = \partial u/\partial x = -6xy \\ \partial v/\partial x = -\partial u/\partial y = -3y^2 + 3x^2 \end{cases}$$

Integrating the first equation with respect to y while keeping x constant gives

$$v = -3xy^2 + t(x)$$

where $t(x)$ is an arbitrary real function of x. Substituting this into the second equation above yields

$$-3y^2 + t'(x) = -3y^2 + 3x^2$$

which results in $t'(x) = 3x^2$, and thus, $t(x) = x^3 + \text{constant}$. Hence, the harmonic conjugate of u is found as

$$v = -3xy^2 + x^3 + \text{constant}$$

and it follows that

$$\begin{aligned} f(z) &= y^3 - 3x^2y + \mathrm{i}(-3xy^2 + x^3 + \text{constant}) \\ &= \mathrm{i}z^3 + \text{constant} \end{aligned}$$

which is analytic.

2.5 Stream Function and Complex Potential

Finally, Solution Strategy is accomplished in this section. By virtue of Proposition 2.1, another mathematical function in addition to the velocity potential is constructed. It is shown that the function so created has the properties required to solve Motivating Problem 2.

2.5.1 Definition

The velocity potential Φ satisfies Laplace's equation, as shown in Chap. 1, and therefore, is harmonic. From Proposition 2.1, it follows that there must exist a harmonic conjugate, denoted by Ψ, such that

$$\Omega = \Phi + \mathrm{i}\Psi \tag{2.97}$$

is analytic. The function Ψ is called the stream function, since Ψ is related to fluid streams as revealed in the later section. The analytic function Ω is called the complex potential.

Here, the properties of the velocity potential and stream function are summarized. As the vector function $\nabla\Phi$ is conservative, the vector function $\nabla\Psi$ is also conservative; thus, the properties associated with conservative fields, such as path independence of line integrals and the exactness of the differential form (Appendix C.2), hold for the stream function Ψ.

The complex potential Ω is analytic, and its component functions Φ and Ψ are harmonic, satisfying Laplace's equation:

$$\begin{cases} \nabla^2\Phi = 0 \\ \nabla^2\Psi = 0 \end{cases} \tag{2.98}$$

Laplace's equation in terms of Φ is physically derived in Chap. 1. Although a physical derivation is also possible, Laplace's equation in terms of Ψ has immediately been derived through a mathematical manipulation.

In addition, Φ and Ψ are related through the Cauchy–Riemann equations

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = \dfrac{\partial \Psi}{\partial y} \\ \dfrac{\partial \Phi}{\partial y} = -\dfrac{\partial \Psi}{\partial x} \end{cases} \tag{2.99}$$

in Cartesian form or

$$\begin{cases} \dfrac{\partial \Phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \Psi}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial \Phi}{\partial \theta} = -\dfrac{\partial \Psi}{\partial r} \end{cases} \tag{2.100}$$

in polar form.

As illustrated in Example 2.32, given a harmonic function (velocity potential Φ), the corresponding harmonic conjugate (stream function Ψ) can be obtained through the Cauchy–Riemann equations. In Chap. 1, the velocity potential for uniform flow and that for a source or sink are derived, to which the corresponding stream functions can be derived through the property of harmonic functions.

2.5.2 *Uniform Flow*

The velocity potential for uniform flow in the x direction is given by Eq. 1.55 as

$$\Phi(x) = -\frac{q_{\mathrm{u}}}{A}x + \Phi_0 \tag{2.101}$$

From the Cauchy–Riemann equations in Cartesian form (Eq. 2.99), it follows that

$$\begin{cases} \dfrac{\partial \Psi}{\partial y} = \dfrac{\partial \Phi}{\partial x} = -\dfrac{q_{\mathrm{u}}}{A} \\ \dfrac{\partial \Psi}{\partial x} = -\dfrac{\partial \Phi}{\partial y} = 0 \end{cases} \tag{2.102}$$

Integrating the first equation with respect to y while keeping x constant gives

$$\Psi = -\frac{q_{\mathrm{u}}}{A}y + t(x) \tag{2.103}$$

where $t(x)$ is an arbitrary real function of x. Substituting this equation into the second equation of Eq. 2.102 gives $t'(x) = 0$ or $t(x) =$ constant. Hence, the stream function for uniform flow is obtained as

$$\Psi = -\frac{q_{\mathrm{u}}}{A}y + \Psi_0 \tag{2.104}$$

where Ψ_0 is an arbitrary additive constant.

Substituting Eqs. 2.101 and 2.104 into Eq. 2.97 yields the complex potential for uniform flow

$$\Omega = -\frac{q_{\mathrm{u}}}{A} z \tag{2.105}$$

where for simplicity a constant is omitted without essential loss of generality. This immediately shows that Ω is analytic in the entire complex plane.

2.5.3 Sources and Sinks

The velocity potential for sources and sinks is given by Eq. 1.58 as

$$\Phi(r) = -\frac{q_{\mathrm{w}}}{2\pi h} \ln r + \Phi_0 \tag{2.106}$$

From the Cauchy–Riemann equations in polar form (Eq. 2.100), it follows that

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial \Psi}{\partial \theta} = \dfrac{\partial \Phi}{\partial r} = -\dfrac{q_{\mathrm{w}}}{2\pi h}\dfrac{1}{r} \\ \dfrac{\partial \Psi}{\partial r} = -\dfrac{1}{r}\dfrac{\partial \Phi}{\partial \theta} = 0 \end{cases} \tag{2.107}$$

Integrating the first equation with respect to θ while keeping r constant gives

$$\Psi = -\frac{q_{\mathrm{w}}}{2\pi h}\theta + t(r) \tag{2.108}$$

where $t(r)$ is an arbitrary real function of r. Substituting this equation into the second equation of Eq. 2.107 gives $t'(r) = 0$ or $t(r) =$ constant. Hence, the stream function for sources and sinks is obtained as

$$\Psi = -\frac{q_{\mathrm{w}}}{2\pi h}\theta + \Psi_0 \tag{2.109}$$

where Ψ_0 is an arbitrary additive constant.

Substituting Eqs. 2.106 and 2.109 into Eq. 2.97 yields

$$\Omega = -\frac{q_{\mathrm{w}}}{2\pi h}(\ln r + \mathrm{i}\theta) = -\frac{q_{\mathrm{w}}}{2\pi h} \ln z \tag{2.110}$$

where for simplicity a constant is omitted without essential loss of generality. When a source or sink is located at $z_{\mathrm{w}} = (x_{\mathrm{w}}, y_{\mathrm{w}})$, the radius r is the distance between z and the source or sink, $r = |z - z_{\mathrm{w}}|$ and the argument θ is the angle that $z - z_{\mathrm{w}}$ (as a radius vector) makes with the positive real axis, $\theta = \arg(z - z_{\mathrm{w}})$. Hence, the complex potential for a source or sink at z_{w} can be written as

$$\Omega = -\frac{q_{\mathrm{w}}}{2\pi h} \ln(z - z_{\mathrm{w}}) \tag{2.111}$$

where a positive q_w corresponds to emanating flow from a source and a negative q_w corresponds to converging flow toward a sink. The complex potential Ω is analytic except at $z = z_\mathrm{w}$ and on the branch cut.

Example 2.33 Let us confirm that the stream functions satisfy Laplace's equation. For uniform flow

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial y^2} = -\frac{\partial}{\partial y}\frac{q_\mathrm{u}}{A} = 0$$

For a source or sink, using Eq. A.12

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} = -\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{q_\mathrm{w}}{2\pi h} = 0$$

The stream function for uniform flow and that for a source or sink indeed satisfy Laplace's equation.

2.5.4 Streamlines

Streamlines represent the paths of imaginary fluid particles in a flow domain and are defined as the instantaneous curves that are at every point tangent to the direction of the velocity at that point. Along a streamline, an element of arc $\mathrm{d}\mathbf{r} = (\mathrm{d}x, \mathrm{d}y)$ is parallel to the velocity $\mathbf{V} = (V_x, V_y)$, as shown in Fig. 2.11.

The mathematical expression defining a streamline is therefore given by

$$\frac{\mathrm{d}x}{V_x} = \frac{\mathrm{d}y}{V_y} \tag{2.112}$$

or equivalently

$$V_y\mathrm{d}x - V_x\mathrm{d}y = 0 \tag{2.113}$$

where V_x and V_y are the velocity components in the x and y directions, such that a function $V = V_x + \mathrm{i}V_y$ corresponds to the velocity vector $\mathbf{V}$.

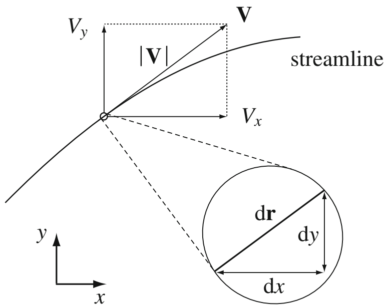


Fig. 2.11 An element of arc $\mathrm{d}\mathbf{r}$ along a streamline and the velocity vector $\mathbf{V}$

According to Darcy's law, Eq. 1.33 (or Eq. 1.24 in general), the velocity components are given as minus the gradient of Φ:

$$\begin{cases} V_x = -\dfrac{\partial \Phi}{\partial x} \\ V_y = -\dfrac{\partial \Phi}{\partial y} \end{cases} \tag{2.114}$$

Applying the Cauchy–Riemann equations (Eq. 2.99) to Eq. 2.114 yields another expression for the velocity components in terms of Ψ:

$$\begin{cases} V_x = -\dfrac{\partial \Psi}{\partial y} \\ V_y = \dfrac{\partial \Psi}{\partial x} \end{cases} \tag{2.115}$$

Now, substituting Eq. 2.115 into 2.113 gives

$$\frac{\partial \Psi}{\partial x}\mathrm{d}x + \frac{\partial \Psi}{\partial y}\mathrm{d}y = 0 \tag{2.116}$$

Since the exactness of the differential form (Eq. C.9 in Appendix C.2) holds for the stream function, the left-hand side of Eq. 2.116 is the exact differential form of Ψ; thus, along a streamline, it follows that

$$\mathrm{d}\Psi = 0 \tag{2.117}$$

Equation 2.117 implies that, in the same way as equipotential lines are obtained as curves of constant velocity potential Φ (Sect. 1.3.3), streamlines are obtained by setting Ψ equal to a constant in the equation

$$\Psi = \Psi(x, y) = \text{constant} \tag{2.118}$$

which describes a family of curves, for various values of the constant. That is, the level curves of Ψ are the streamlines.

Example 2.34 Figure 2.12 shows the streamlines of uniform flow given by Eq. 2.104 (or the imaginary part of Eq. 2.105) and a source or sink at z_w given by Eq. 2.109 (or the imaginary part of Eq. 2.111). For uniform flow, the streamlines are straight lines parallel to the flow direction which are all the same distance apart. For a source or sink, the streamlines are rays of constant θ emanating from or converging toward z_w.

Note that these figures are respectively drawn by contouring the values of Ψ computed within the domain. In Fig. 2.12b, a streamline is overlapped by the thick line from the source or sink in the negative x direction, which is caused by the jump on the contour plot and corresponds to the branch cut.

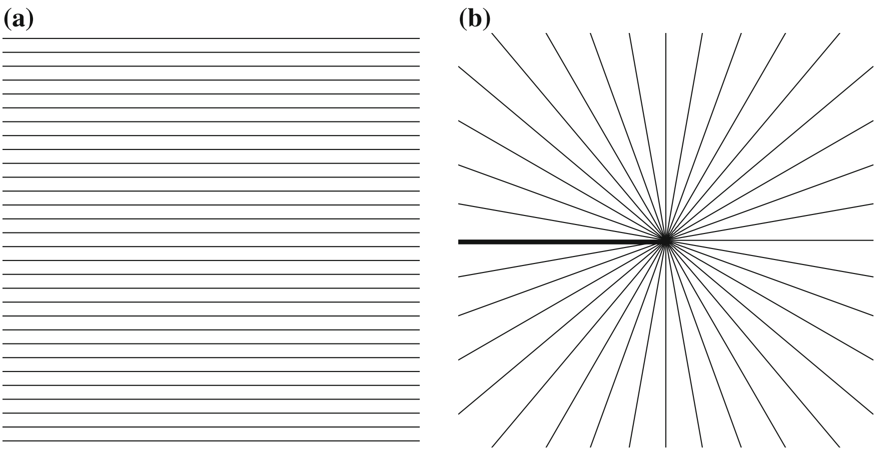


Fig. 2.12 Streamlines. **a** Uniform flow. **b** Source or sink

2.5.5 Complex Velocity

The velocity components V_x and V_y can be obtained by complex differentiation. From Eq. 2.78, the derivative of Ω with respect to z is given by

$$\frac{\mathrm{d}\Omega}{\mathrm{d}z} = \frac{\partial \Phi}{\partial x} + \mathrm{i}\frac{\partial \Psi}{\partial x} = -V_x + \mathrm{i}V_y \tag{2.119}$$

where Eqs. 2.114 and 2.115 are used. The complex velocity W is defined by

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = V_x - \mathrm{i}V_y \tag{2.120}$$

from which the velocity components V_x and V_y are directly obtained. The magnitude of velocity $|\mathbf{V}|$ is also obtained as

$$|\mathbf{V}| = \sqrt{V_x^2 + V_y^2} = |W| \tag{2.121}$$

The velocity components in Cartesian and polar coordinates are related as

$$\begin{cases} V_x = V_r \cos\theta - V_\theta \sin\theta \\ V_y = V_r \sin\theta + V_\theta \cos\theta \end{cases} \tag{2.122}$$

where V_r and V_θ are the velocity components in the r and θ directions, respectively. Hence, the complex velocity in polar coordinates is given by

$$\begin{aligned} W &= V_r \cos\theta - V_\theta \sin\theta - \mathrm{i}(V_r \sin\theta + V_\theta \cos\theta) \\ &= (V_r - \mathrm{i}V_\theta)\mathrm{e}^{-\mathrm{i}\theta} \end{aligned} \tag{2.123}$$

A point where both V_x and V_y vanish is called a stagnation point. From Eqs. 2.78 and 2.79, it follows that

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = -\frac{\partial \Phi}{\partial x} - \mathrm{i}\frac{\partial \Psi}{\partial x} = \mathrm{i}\frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial y} = 0 \qquad (2.124)$$

at the stagnation point. Since $\partial\Psi/\partial x = 0$ and $\partial\Psi/\partial y = 0$, streamlines can intersect each other or abruptly change direction at the stagnation point.

Example 2.35 For uniform flow $\Omega = -(q_\mathrm{u}/A)z$, the complex velocity is

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = \frac{q_\mathrm{u}}{A}$$

Thus, $V_x = q_\mathrm{u}/A$ and $V_y = 0$. There is no stagnation point in uniform flow. For a source at the origin $\Omega = -(q_\mathrm{w}/2\pi h)\ln z$, the complex velocity is

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = \frac{q_\mathrm{w}}{2\pi h}\frac{1}{z}$$

By using the exponential form $z = r\mathrm{e}^{\mathrm{i}\theta}$, W becomes

$$W = \frac{q_\mathrm{w}}{2\pi h}\frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta} = (V_r - \mathrm{i}V_\theta)\mathrm{e}^{-\mathrm{i}\theta}$$

Thus, $V_r = (q_\mathrm{w}/2\pi h)/r$ and $V_\theta = 0$. The origin is a singular point, where W is indeterminate.

• Solution to Task 2-1

The complex potential for uniform flow with a source can be obtained by applying the principle of superposition to Eqs. 2.105 and 2.111. When the source with a flow rate q_s is located at z_s, the solution is

$$\Omega(z) = -\frac{q_\mathrm{u}}{A}z - \frac{q_\mathrm{s}}{2\pi h}\ln(z - z_\mathrm{s}) \qquad (2.125)$$

Substituting $q_\mathrm{u}/A = 1$, $q_\mathrm{s}/h = 6$, and $z_\mathrm{s} = -1$ into the solution yields

$$\Omega(z) = -z - \frac{3}{\pi}\ln(z + 1)$$

with which the corresponding complex potentials are computed. The real and imaginary parts of $\Omega(z)$ yield the velocity potential and stream function, respectively, and the equipotential lines and streamlines are obtained as shown in Fig. 2.13.

Since Eq. 1.65 is the real part of Eq. 2.125, the equipotential lines shown in Fig. 2.13a coincide with those in Fig. 1.5, which are of little use to trace the contaminants. On the other hand, the streamlines shown in Fig. 2.13b visually reveal

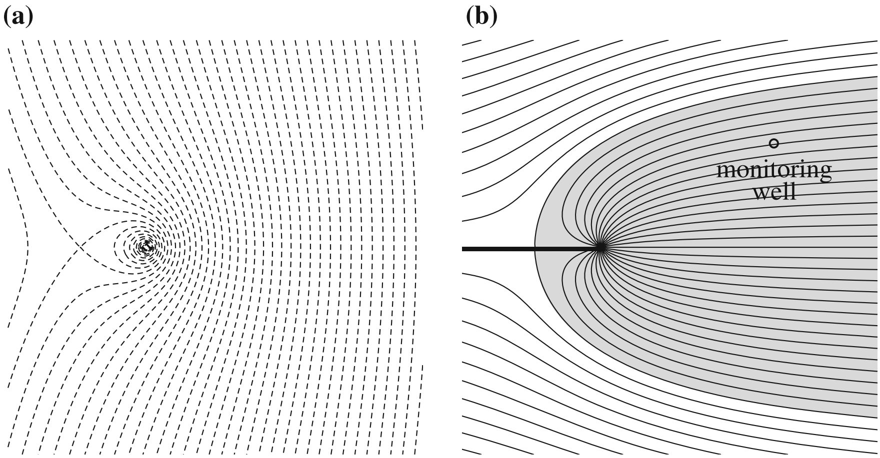


Fig. 2.13 Uniform flow with a source ($q_s/h = 6$). **a** Equipotential lines. **b** Streamlines. (As is the case with Example 2.34, the thick line emanating from the pollution source in the negative x direction corresponds to the branch cut and overlaps a streamline)

that the monitoring well is invaded by the contaminants from the pollution source. The stream function conveys useful information to draw exact flow paths.

The complex velocity is given by

$$W = -\frac{d\Omega}{dz} = 1 + \frac{3}{\pi}\frac{1}{z+1}$$

and from the solution of $W = 0$, a stagnation point is found at $z = -1 - 3/\pi = -1.955$, where the streamlines intersect each other and abruptly change direction, as seen in Fig. 2.13b.

The streamline passing through the stagnation point is called the dividing streamline (or water divide), which separates the fluid from uniform flow and the fluid from the pollution source. The monitoring well is inside the dividing streamline, and thus, the contaminants from the pollution source are detected at the monitoring point.

Let us further consider the dividing streamline. From Eq. 2.125, the stream function is given by

$$\Psi(z) = -\frac{q_u}{A}y - \frac{q_s}{2\pi h}\theta_s \tag{2.126}$$

where $\theta_s = \arg(z - z_s)$. For the streamlines in the upper half plane, the stagnation point is given by $y = 0$ and $\theta_s = \pi$; thus, the stream function at the stagnation point becomes $\Psi = -q_s/2h$. The equation for the dividing streamline is given by

$$-\frac{q_s}{2h} = -\frac{q_u}{A}y - \frac{q_s}{2\pi h}\theta_s \tag{2.127}$$

As x approaches ∞, θ_s approaches 0, and Eq. 2.127 gives

$$y = \frac{q_s/2h}{q_u/A} \tag{2.128}$$

which is the asymptote of the dividing streamline.

In a similar way, for the streamlines in the lower half plane, the stagnation point is given by $y = 0$ and $\theta_s = -\pi$; thus, the stream function becomes $\Psi = q_s/2h$. The equation for the dividing streamline is given by

$$\frac{q_s}{2h} = -\frac{q_u}{A}y - \frac{q_s}{2\pi h}\theta_s \tag{2.129}$$

and the asymptote of the dividing streamline is obtained as

$$y = -\frac{q_s/2h}{q_u/A} \tag{2.130}$$

The two asymptotes of the dividing streamline are given by

$$y = \pm\frac{q_s/2h}{q_u/A} \tag{2.131}$$

Substituting $q_u/A = 1$ and $q_s/h = 6$ into the equation yields $y = \pm 3$, the area between which at $x = \infty$ is contaminated.

• Solution to Task 2-2

The complex potential for uniform flow with a source and a sink can be obtained by applying the principle of superposition to Eqs. 2.105 and 2.111. When the source with a flow rate q_s is located at z_s and the sink with a flow rate q_e is at z_e, the solution is

$$\Omega(z) = -\frac{q_u}{A}z - \frac{q_s}{2\pi h}\ln(z - z_s) - \frac{q_e}{2\pi h}\ln(z - z_e) \tag{2.132}$$

Substituting $q_u/A = 1$, $q_s/h = 6$, $z_s = -1$, and $z_e = 1$ into the solution yields

$$\Omega(z) = -z - \frac{3}{\pi}\ln(z + 1) - \frac{q_e}{2\pi h}\ln(z - 1) \tag{2.133}$$

with which the corresponding complex potentials are computed. After a process of trial and error, it is found that $q_e/h = -3.5$ gives a possible solution to the extraction system, as shown in Fig. 2.14.

The complex velocity is given by

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = 1 + \frac{3}{\pi}\frac{1}{z+1} - \frac{3.5}{2\pi}\frac{1}{z-1}$$

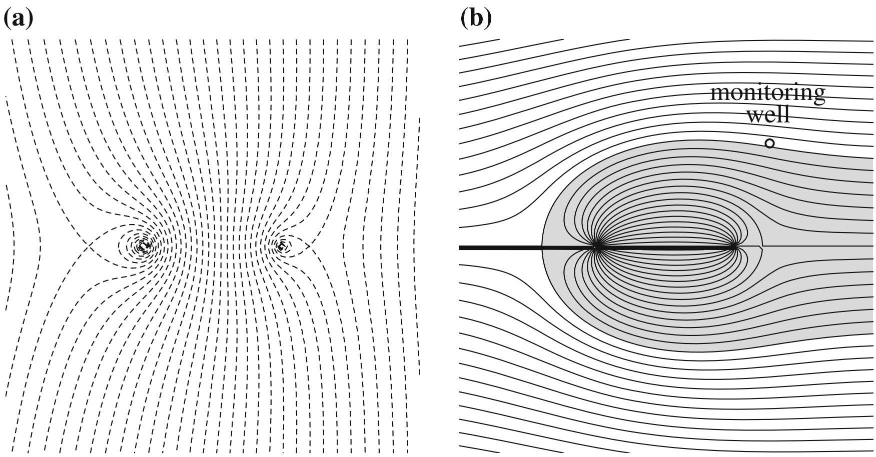


Fig. 2.14 Uniform flow with a source ($q_s/h = 6$) and a sink ($q_e/h = -3.5$). **a** Equipotential lines. **b** Streamlines. (The thick line emanating from the pollution source and that from the extraction well in the negative x direction correspond to the individual branch cuts and overlap streamlines)

and from the solution of $W = 0$, stagnation points are found at $z = -1.796$ and $z = 1.398$, as seen in Fig. 2.14b. The monitoring well is outside the dividing streamline and the streamlines emanating from the pollution source do not flow through the monitoring well. Hence, the contaminants are not detected at the monitoring point. Note that this is an approximate visual evaluation.

To evaluate analytically the minimum pumping rate to avoid the contamination at the monitoring well, the dividing streamline is of great use. As can be seen from Fig. 2.14, the dividing streamline originally emanates from the pollution source, then passes through and abruptly changes direction at the stagnation point. It is obvious that the area surrounded by the dividing streamline depends on the pumping rate of the extraction well; as the rate increases, the area shrinks. Hence, the dividing streamline that passes through the monitoring well corresponds to the required minimum pumping rate.

• Solution to Task 2-3

Taking the imaginary part of Eq. 2.133 yields the stream function

$$\Psi(z) = -y - \frac{3}{\pi}\theta_s - \frac{q_e}{2\pi h}\theta_e$$

where $\theta_s = \arg(z - z_s) = \arg(z + 1)$ and $\theta_e = \arg(z - z_e) = \arg(z - 1)$. For the streamlines in the upper half plane, the stagnation point is given by $y = 0$, $\theta_s = \pi$, and $\theta_e = \pi$; thus, the stream function at the stagnation point becomes

$$\Psi = 0 - \frac{3}{\pi}\pi - \frac{q_e}{2\pi h}\pi = -3 - \frac{1}{2}\frac{q_e}{h}$$

The minimum pumping rate can be obtained by equating this value and the stream function at the location of the monitoring well.

For the monitoring well, $y = 1.5, \theta_s = \arctan(1.5/2.5)$, and $\theta_e = \arctan(1.5/0.5)$; thus, the stream function becomes

$$-3 - \frac{1}{2}\frac{q_e}{h} = -1.5 - \frac{3}{\pi}\arctan(3/5) - \frac{q_e}{2\pi h}\arctan 3$$

which results in the analytical solution of $q_e/h = -3.267$. Figure 2.15 shows the equipotential lines and streamlines with $q_e/h = -3.267$. It is confirmed that the dividing streamline indeed passes through the monitoring well.

The complex velocity is given by

$$W = -\frac{d\Omega}{dz} = 1 + \frac{3}{\pi}\frac{1}{z+1} - \frac{3.267}{2\pi}\frac{1}{z-1}$$

and from the solution of $W = 0$, stagnation points are found at $z = -1.806$ and $z = 1.371$, as seen in Fig. 2.15b. The monitoring well is exactly on the streamline that emanates from the pollution source, and thus, $q_e/h = -3.267$ is the minimum value for the required flow rate.

From Eq. 2.132, the stream function for uniform flow with a source and a sink is given by

$$\Psi(z) = -\frac{q_u}{A}y - \frac{q_s}{2\pi h}\theta_s - \frac{q_e}{2\pi h}\theta_e \tag{2.134}$$

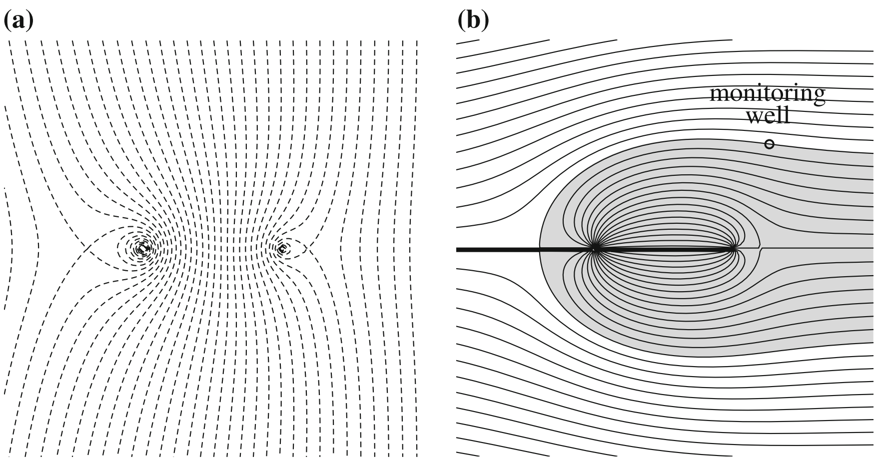


Fig. 2.15 Uniform flow with a source ($q_s/h = 6$) and a sink ($q_e/h = -3.267$). **a** Equipotential lines. **b** Streamlines

With the same argument as before, it can be shown that the equations for the dividing streamlines are given by

$$\mp\left(\frac{q_s}{2h}+\frac{q_e}{2h}\right)=-\frac{q_u}{A}y-\frac{q_s}{2\pi h}\theta_s-\frac{q_e}{2\pi h}\theta_e \tag{2.135}$$

As x approaches ∞, θ_s and θ_e approach 0, and the two asymptotes of the dividing streamline are obtained as

$$y=\pm\frac{q_s/2h+q_e/2h}{q_u/A} \tag{2.136}$$

Substituting $q_u/A = 1$, $q_s/h = 6$, and $q_e/h = -3.267$ into the equation yields $y = \pm 1.367$, the area between which at $x = \infty$ is contaminated.

Motivating Problem 3: Groundwater Flow Over a Circular Pillar

Groundwater flows in the x direction with a uniform flow velocity $q_u/A = 1$. For some construction purpose, an impermeable circular pillar of radius $R = 1$ is installed through the flow medium, as shown in Fig. 2.16.

Task 3-1 Draw the flow profile and discuss the effect of the circular pillar on the groundwater flow.

Task 3-2 Evaluate the discharge profile along the y axis.

- **Solution Strategy to Motivating Problem 3**

At first glance, Motivating Problem 3 may appear totally different from Motivating Problem 2, where uniform flow with a source and a sink is considered and no obstacle to flow is dealt with. This is not necessarily so.

Further insights into the properties of complex potential broaden the interpretation of streamlines and the usage of stream function. Indeed, a slight modification of the solution to Motivating Problem 2 gives the solution to the current problem.

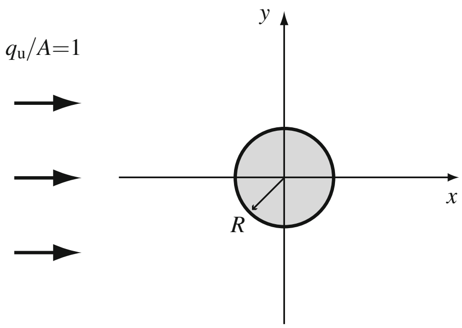


Fig. 2.16 Uniform flow over a circular pillar

2.6 Further Topics in Complex Potential

To broaden its practical applicability, additional aspects of complex potential are discussed in this section, including the orthogonal families of curves, streamlines as impermeable boundaries, and the discharge evaluation with the stream function.

2.6.1 Orthogonal Families

Let us consider two families of curves

$$\begin{cases} u(x, y) = p \\ v(x, y) = s \end{cases} \tag{2.137}$$

where u and v are the real and imaginary parts of an analytic function $f(z)$ and p and s are any constants.

Let $u(x, y) = p_1$ and $v(x, y) = s_1$, where p_1 and s_1 are particular constants, be any two members of the respective families. Differentiating $u(x, y) = p_1$ with respect to x gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{2.138}$$

Then the slope of $u(x, y) = p_1$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial u}{\partial x}\bigg/\frac{\partial u}{\partial y} \tag{2.139}$$

In a similar way, the slope of $v(x, y) = s_1$ is obtained as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial v}{\partial x}\bigg/\frac{\partial v}{\partial y} \tag{2.140}$$

The product of the slopes becomes

$$\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right)\bigg/\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) = -\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)\bigg/\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) = -1 \tag{2.141}$$

where the Cauchy–Riemann equations are used. This indicates that, if a function $f(z) = u + \mathrm{i}v$ is analytic, the families of curves given by Eq. 2.137 are orthogonal. This is true only if the partial derivatives in Eq. 2.141 are not equal to zero.

Since the complex potential $\Omega = \Phi + \mathrm{i}\Psi$ is analytic, it follows that two families of curves, equipotential lines and streamlines, defined by

$$\begin{cases} \Phi(x, y) = p \\ \Psi(x, y) = s \end{cases} \tag{2.142}$$

are orthogonal. Each curve of equipotential lines is perpendicular to each curve of streamlines at the point of intersection. This is true only if the partial derivatives of Φ and Ψ with respect to x and y are not equal to zero, that is, the complex velocity W is not equal to zero.

As discussed in Sect. 2.5.5, when $W = 0$, the point is a stagnation point. If the point of intersection is a stagnation point, the equipotential line and the streamline may not be mutually orthogonal. Instead, streamlines may intersect each other or abruptly change direction.

Example 2.36 The complex potential for uniform flow given by Eq. 2.105 has two families of curves, equipotential lines and streamlines

$$\begin{cases} \Phi(x, y) = -\dfrac{q_\mathrm{u}}{A} x = p \\ \Psi(x, y) = -\dfrac{q_\mathrm{u}}{A} y = s \end{cases}$$

which are respectively parallel to y and x axes, and thus obviously orthogonal, as shown in Fig. 2.17a.

Similarly, the complex potential for a source or sink given by Eq. 2.111 has the following two families:

$$\begin{cases} \Phi(x, y) = -\dfrac{q_\mathrm{w}}{2\pi h} \ln r = -\dfrac{q_\mathrm{w}}{4\pi h} \ln\left(x^2 + y^2\right) = p \\ \Psi(x, y) = -\dfrac{q_\mathrm{w}}{2\pi h} \theta = -\dfrac{q_\mathrm{w}}{2\pi h} \arctan \dfrac{y}{x} = s \end{cases}$$

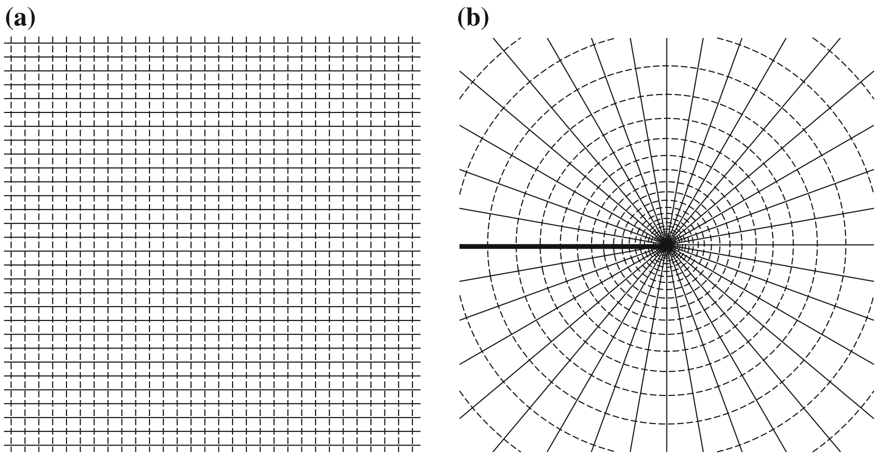


Fig. 2.17 Orthogonality between equipotential lines and streamlines. **a** Uniform flow. **b** Source or sink

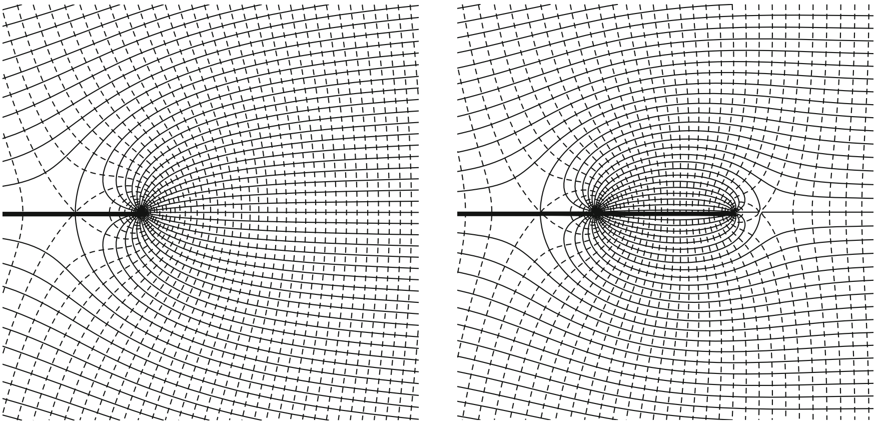


Fig. 2.18 Orthogonality between equipotential lines and streamlines of Motivating Problem 2. **a** Contamination. **b** Extraction

and the slopes $\mathrm{d}y/\mathrm{d}x$ of $\Phi = p_1$ and $\Psi = s_1$ are respectively obtained as $-x/y$ and y/x, which implies the two families are orthogonal (Fig. 2.17b).

Example 2.37 The orthogonality also holds for superposition of different types of flow as long as the resultant flow is defined by analytic functions. Figure 2.18 shows the orthogonality between the equipotential lines and streamlines for uniform flow with a source and a sink (Motivating Problem 2). Note that at stagnation points, $z = -1.955$ in Fig. 2.18a and $z = -1.806$ and 1.371 in Fig. 2.18b, the equipotential lines and streamlines are not mutually orthogonal and the streamlines abruptly change direction.

As discussed in Sect. 1.2, many physical problems are governed by Laplace's equation, and thus, the solutions are given by a harmonic function Φ and its harmonic conjugate Ψ. For individual physical processes, the orthogonality between Φ and Ψ holds. Table 2.1 summarizes the orthogonal families for different physical phenomena.

Table 2.1 Orthogonal families for physical processes

Physical process	$\Phi(x, y)$	$\Psi(x, y)$
Fluid flow	Equipotential lines	Streamlines
Fickian diffusion	Concentration	Lines of solute flow
Heat conduction	Isotherms	Heat flow lines
Gravitational fields	Gravitational potential	Lines of force
Electrostatic fields	Equipotential lines	Lines of electrical force

2.6.2 Streamlines as Impermeable Boundaries

It follows from the orthogonality that the velocity potential is constant in the direction normal to the streamlines, across which no flow occurs. Hence, the streamlines can be interpreted as impermeable boundaries. In Figs. 2.13, 2.14, and 2.15, for instance, the shaded areas may be interpreted as $\subset$-shaped obstacles instead of contaminated areas.

For uniform flow with a source and a sink given by Eq. 2.132, if the flow rates are balanced, $q_\mathrm{s} = -q_\mathrm{e}$ and $z_\mathrm{s} = -z_\mathrm{e} = -d$, then

$$\Omega(z) = -\frac{q_\mathrm{u}}{A}z - \frac{q_\mathrm{s}}{2\pi h}\ln\frac{z+d}{z-d} \tag{2.143}$$

Figure 2.19 shows the streamlines given by Eq. 2.143. It is seen that the dividing streamline forms an oval contour.

Although Eq. 2.143 is derived for uniform flow around a source-sink pair, the resultant flow can be interpreted as the flow over an oval object, which is known as Rankine's oval. Inside the oval contour, the fluid emanates from the source and flows into the sink, whereas the fluid outside the oval is attributed to uniform flow.

The stagnation points are obtained from the solution of

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = \frac{q_\mathrm{u}}{A} + \frac{q_\mathrm{s}}{2\pi h}\left(\frac{1}{z+d} - \frac{1}{z-d}\right) = 0 \tag{2.144}$$

which yields

$$z = \pm\sqrt{d^2 + \frac{A}{q_\mathrm{u}}\frac{q_\mathrm{s}}{\pi h}d} \tag{2.145}$$

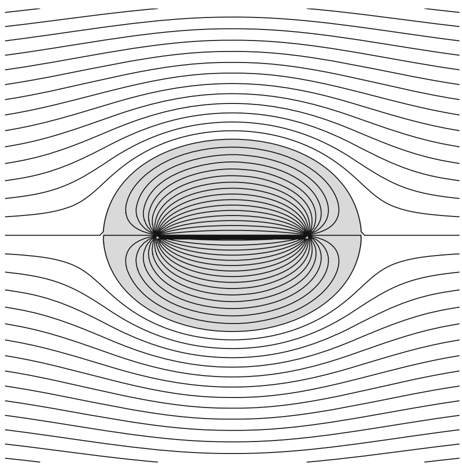


Fig. 2.19 Uniform flow over Rankine's oval

There exist two stagnation points, one to the left of the source and the other to the right of the sink, both on the x axis (Fig. 2.19).

It is deduced that the shape of Rankine's oval depends on the distance between the source and the sink, $2d$, and the ratio between the flow rates, $q_\mathrm{s}/q_\mathrm{u}$. As the source and the sink get closer to each other (d approaches zero), while $(q_\mathrm{s}/q_\mathrm{u})d$ is held constant, the oval shape becomes more and more circular.

The coalescence of the source and the sink, with the flow rates equal in magnitude but opposite in sign, at a single point forms a dipole. By letting d approach zero and $m = q_\mathrm{s}/(2\pi h)$ approach ∞ in such a way that $2dm = \sigma$ is finite, the second term on the right-hand side of Eq. 2.143 yields

$$\begin{aligned} -\lim_{d\to 0} \frac{q_\mathrm{s}}{2\pi h} \ln \frac{z+d}{z-d} &= -\lim_{d\to 0} m\,[\ln(z+d) - \ln(z-d)] \\ &= -\lim_{d\to 0} 2dm \frac{\ln(z+d) - \ln(z-d)}{2d} \\ &= -\sigma \frac{\mathrm{d}}{\mathrm{d}z} \ln z = -\frac{\sigma}{z} \end{aligned} \tag{2.146}$$

which represents flow caused by a dipole. Then, the complex potential Eq. 2.143 becomes

$$\Omega(z) = -\frac{q_\mathrm{u}}{A} z - \frac{\sigma}{z} \tag{2.147}$$

which expresses flow over a circular object.

By using the exponential form $z = r\mathrm{e}^{\mathrm{i}\theta}$, the stream function is found to be

$$\Psi = \mathrm{Im}\,\Omega(r,\theta) = -\mathrm{Im}\left[\frac{q_\mathrm{u}}{A} r\mathrm{e}^{\mathrm{i}\theta} + \frac{\sigma}{r}\mathrm{e}^{-\mathrm{i}\theta}\right] = -\left(\frac{q_\mathrm{u}}{A} r - \frac{\sigma}{r}\right)\sin\theta \tag{2.148}$$

which becomes zero when $r = \sqrt{\sigma/(q_\mathrm{u}/A)}$ or $\sin\theta = 0$. Hence, the streamline $\Psi = 0$ consists of the circle

$$|z| = \sqrt{\frac{\sigma}{q_\mathrm{u}/A}} \tag{2.149}$$

and the x axis ($\theta = 0$ and $\theta = \pi$). This implies that the streamline on the x axis (beyond the stagnation points) passes along the circumference of the circle and confirms that Rankine's oval becomes a circle in the limit of $d \to 0$.

It should be noted that the stagnation points given by Eq. 2.145 approach

$$\begin{aligned} z &= \pm \lim_{d\to 0} \sqrt{d^2 + \frac{A}{q_\mathrm{u}} \frac{q_\mathrm{s}}{\pi h} d} = \pm \lim_{d\to 0} \sqrt{d^2 + \frac{2dm}{q_\mathrm{u}/A}} \\ &= \pm \sqrt{\frac{\sigma}{q_\mathrm{u}/A}} \end{aligned} \tag{2.150}$$

which are on the circle. The same result follows from the solution of $W = 0$. From Eq. 2.147, the corresponding complex velocity is

$$W = \frac{q_u}{A} - \frac{\sigma}{z^2} \tag{2.151}$$

and stagnation points are found on the circular object at

$$z = \pm\sqrt{\frac{\sigma}{q_u/A}} \tag{2.152}$$

The distance between the two is the diameter of the circle, and thus the radius is $\sqrt{\sigma/(q_u/A)}$. Let R be the radius of the circle, then $\sigma = (q_u/A)R^2$ and Eq. 2.147 becomes

$$\Omega(z) = -\frac{q_u}{A}\left(z + \frac{R^2}{z}\right) \tag{2.153}$$

for flow over a circular object of radius R.

• **Solution to Task 3-1**

The complex potential for uniform flow over a circular pillar is given by Eq. 2.153. Substituting $q_u/A = 1$ and $R = 1$ into the solution yields

$$\Omega(z) = -z - \frac{1}{z} \tag{2.154}$$

with which the corresponding complex potentials are computed. Figure 2.20 shows the equipotential lines and streamlines.

The complex velocity is given by

$$W = -\frac{d\Omega}{dz} = 1 - \frac{1}{z^2} \tag{2.155}$$

and from the solution of $W = 0$, stagnation points are found at $z = \pm 1$, as expected. Flow outside the circumference of the pillar is caused by uniform flow only. Away from the object, flow is essentially uniform as can be seen by evenly spaced horizontal streamlines. When approaching the pillar, flow is gradually distorted and the streamlines detour and become denser near the circumference of the pillar.

2.6.3 Discharge and Stream Function

Let us consider a discharge $\Delta q/h$ (in terms of volume per unit thickness normal to the xy plane per unit time) flowing in the channel between two streamlines with stream functions of Ψ_1 and Ψ_2, respectively, as shown in Fig. 2.21. This discharge is constant in the channel because flow cannot leave across the streamlines.

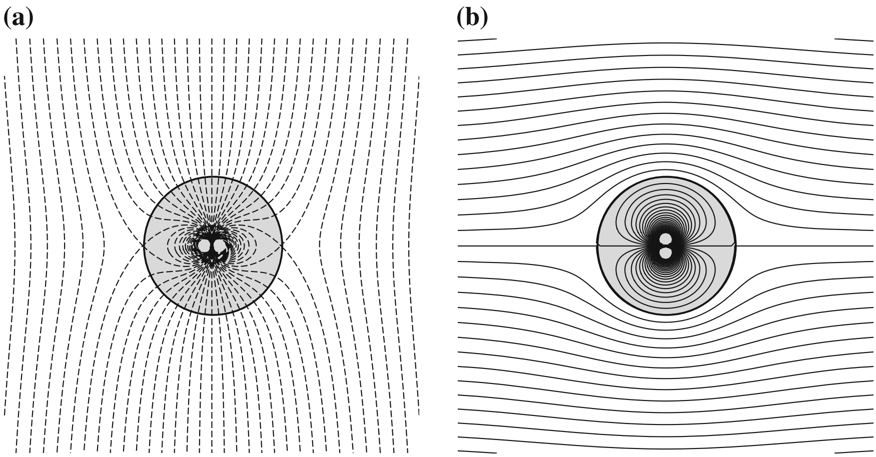


Fig. 2.20 Uniform flow over an impermeable circular object of radius $R = 1$. **a** Equipotential lines. **b** Streamlines. (In the vicinity of the center of the object, equipotential lines and streamlines become too dense and some of them are not plotted)

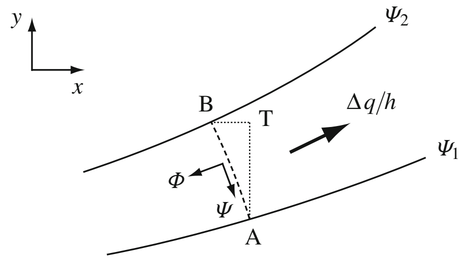


Fig. 2.21 Discharge $\Delta q/h$ between two streamlines with stream functions of Ψ_1 and Ψ_2

A temporary point T is set so that the segment AT is parallel to the y axis and the segment TB is parallel to the x axis. By continuity of flow, the discharge must pass through ATB, and thus

$$\frac{\Delta q}{h} = \int_{\mathrm{A}}^{\mathrm{T}} V_x \mathrm{d}y + \int_{\mathrm{B}}^{\mathrm{T}} V_y \mathrm{d}x \tag{2.156}$$

where V_x and V_y are given by Eq. 2.115; thus

$$\begin{aligned}\frac{\Delta q}{h} &= -\int_{\mathrm{A}}^{\mathrm{T}} \frac{\partial \Psi}{\partial y}\mathrm{d}y + \int_{\mathrm{B}}^{\mathrm{T}} \frac{\partial \Psi}{\partial x}\mathrm{d}x \\ &= -\Psi_{\mathrm{T}} + \Psi_{\mathrm{A}} + \Psi_{\mathrm{T}} - \Psi_{\mathrm{B}} = \Psi_{\mathrm{A}} - \Psi_{\mathrm{B}} \\ &= \Psi_1 - \Psi_2\end{aligned} \tag{2.157}$$

where Ψ_{A}, Ψ_{B}, and Ψ_{T} are the stream functions at A, B, and T, respectively.

It follows that the discharge $\Delta q/h$ between two streamlines is given as the difference in the values of stream function corresponding to those streamlines. It should be noted that the path used for the integration is immaterial, since path independence of line integrals holds for the stream function Ψ. That is, the result depends only on Ψ_1 and Ψ_2.

Since Φ increases in the direction against the flow direction and Φ and Ψ form a Cartesian coordinate system, Eq. 2.157 can be written as

$$\frac{\Delta q}{h} = -\Delta\Psi \tag{2.158}$$

The dimension of stream function is that of discharge, volume per unit thickness per unit time.

The same result can be derived through the definition of the discharge, Eq. 1.15, across a curve C. Applying Eq. 2.115 to Eq. 1.15, it follows that

$$\begin{aligned} Q &= \int_{\mathrm{C}} \left(V_x \mathrm{d}y - V_y \mathrm{d}x\right) = \int_{\mathrm{C}} \left(-\frac{\partial\Psi}{\partial y}\mathrm{d}y - \frac{\partial\Psi}{\partial x}\mathrm{d}x\right) \\ &= -\int_{\mathrm{C}} \mathrm{d}\Psi \end{aligned} \tag{2.159}$$

which states that the discharge Q can be obtained by using the integral of the stream function along the curve C. For the points A and B in Fig. 2.21, consider the connecting curve C; then it follows that

$$Q = -\int_{\mathrm{C}} \mathrm{d}\Psi = -\int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}\Psi = \Psi_{\mathrm{A}} - \Psi_{\mathrm{B}} = \Psi_1 - \Psi_2 \tag{2.160}$$

which is consistent with Eq. 2.157.

When C is a closed curve (A = B) and there is no singularity interior to C, the discharge becomes zero, since path independence of line integrals holds for Ψ and Eq. 2.159 yields $Q = \Psi_{\mathrm{A}} - \Psi_{\mathrm{B}} = 0$. Conversely, if there exist singularities interior to C, the discharge across C may not be zero.

Example 2.38 Let us consider a source with a flow rate per unit thickness q_{w}/h at the origin, the complex potential of which is given by

$$\Omega = -\frac{q_{\mathrm{w}}}{2\pi h}\ln z = -\frac{q_{\mathrm{w}}}{2\pi h}\ln r - \mathrm{i}\frac{q_{\mathrm{w}}}{2\pi h}\theta \tag{2.161}$$

For the stream function $\Psi = -(q_{\mathrm{w}}/2\pi h)\theta$, it follows that

$$\frac{\partial\Psi}{\partial\theta} = -\frac{q_{\mathrm{w}}}{2\pi h}$$

From Eq. 2.159, the discharge across a circle C of radius r_0 with its center at the origin, $|z| = r_0$, is given by

$$Q = -\oint_{C} \mathrm{d}\Psi = \int_{0}^{2\pi} \frac{q_w}{2\pi h}\mathrm{d}\theta = \frac{q_w}{h}$$

which is independent of r_0 and equal to the flow rate per unit thickness of the source.

It should be noted that the nonzero discharge across the circle enclosing the source does not contradict the condition of incompressibility, Eq. 1.17. The source is the singularity at which a fluid of discharge q_w/h is introduced. If a closed curve does not enclose the source, the discharge becomes zero.

2.6.4 Circulation and Velocity Potential

Let us consider a circulation Γ along a curve C connecting points A and B. Applying Eq. 2.114 to the definition of the circulation, Eq. 1.19, it follows that

$$\begin{aligned}\Gamma &= \int_{C} \left(V_x \mathrm{d}x + V_y \mathrm{d}y\right) = \int_{C} \left(-\frac{\partial \Phi}{\partial x}\mathrm{d}x - \frac{\partial \Phi}{\partial y}\mathrm{d}y\right) \\ &= -\int_{C} \mathrm{d}\Phi \end{aligned} \tag{2.162}$$

which states that the circulation Γ can be obtained by using the integral of the velocity potential along the curve C.

When C is a closed curve (A = B) and there is no singularity interior to C, the circulation becomes zero, since path independence of line integrals holds for Φ and Eq. 2.162 yields $\Gamma = \Phi_A - \Phi_B = 0$. Conversely, if there exist singularities interior to C, the circulation along C may not be zero.

Example 2.39 Let us consider a source with a flow rate per unit thickness q_w/h at the origin, expressed by Eq. 2.161. For the velocity potential $\Phi = -(q_w/2\pi h)\ln r$, it follows $\partial\Phi/\partial\theta = 0$. From Eq. 2.162, the circulation along a circle C is given by

$$\Gamma = -\oint_{C} \mathrm{d}\Phi = -\int_{0}^{2\pi} 0\mathrm{d}\theta = 0$$

which satisfies the condition of irrotationality.

• Solution to Task 3-2

The stream function for uniform flow over a circular pillar of radius R is the imaginary part of Eq. 2.153, given by

$$\Psi(x, y) = -\frac{q_u}{A}\left(y - \frac{R^2 y}{x^2 + y^2}\right)$$

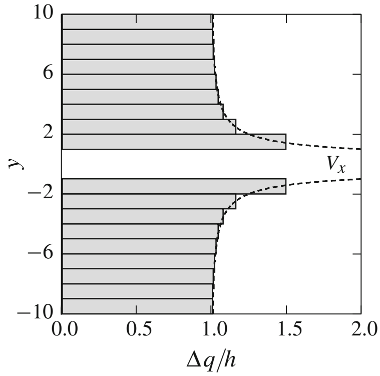


Fig. 2.22 Discharge (*bars*) and velocity (*dashed lines*) of uniform flow over an impermeable circular object of radius $R = 1$

Substituting $q_\mathrm{u}/A = 1$, $R = 1$, and $x = 0$ into the solution yields the stream function along the y axis:

$$\Psi(0, y) = -y + \frac{1}{y} \tag{2.163}$$

From Eq. 2.158, the discharge $\Delta q/h$ flowing through an interval Δy can be evaluated by using the difference in the values of stream function. Figure 2.22 shows the values of $\Delta q/h$ for $\Delta y = 1$ along the y axis.

In the limit of $\Delta y \to 0$, the discharge $\Delta q/h$ passing through Δy reduces to

$$\lim_{\Delta y \to 0} \frac{\Delta q/h}{\Delta y} = -\lim_{\Delta y \to 0} \frac{\Delta \Psi}{\Delta y} = -\frac{\partial \Psi}{\partial y}$$

which is equal to V_x, and from Eq. 2.163, it follows that

$$V_x = 1 + \frac{1}{y^2} \tag{2.164}$$

along the y axis.

The same result can be directly obtained from the complex velocity, Eq. 2.155, which gives

$$\begin{cases} V_x = 1 - \dfrac{x^2 - y^2}{(x^2 + y^2)^2} \\ V_y = -\dfrac{2xy}{(x^2 + y^2)^2} \end{cases}$$

and it follows that

$$\begin{cases} V_x = 1 + \dfrac{1}{y^2} \\ V_y = 0 \end{cases}$$

along the y axis.

The profile of V_x given by Eq. 2.164 is shown in Fig. 2.22. The velocity takes its maximum $V_x = 2$ on the circumference of the object at $y = \pm 1$ and decreases as $|y|$ increases. In the limit of $|y| \to \infty$, V_x approaches 1, which is equal to the uniform flow velocity $q_{\mathrm{u}}/A = 1$.

2.6.5 Dipoles

The dipole derived in Eq. 2.146 is oriented in the x direction. The orientation of the dipole depends on the direction from which the source approaches the sink. Let us consider the source approaching the sink in the direction making an angle δ with the x axis. Substituting $d\mathrm{e}^{\mathrm{i}\delta}$ for d into Eq. 2.146 yields the complex potential

$$\Omega = -\frac{\sigma \mathrm{e}^{\mathrm{i}\delta}}{z} \tag{2.165}$$

for a dipole oriented at an angle δ to the x direction.

Figure 2.23 shows the equipotential lines and streamlines for a dipole with $\delta = \pi/6$. The curves are families of circles through the dipole with their centers on mutually orthogonal lines.

Example 2.40 In the solution to Task 3-1, let us consider a dipole oriented against uniform flow, that is, $\delta = \pi$ in Eq. 2.165. Then the complex potential for uniform flow with a dipole oriented against it becomes

$$\Omega = -z + \frac{1}{z} \tag{2.166}$$

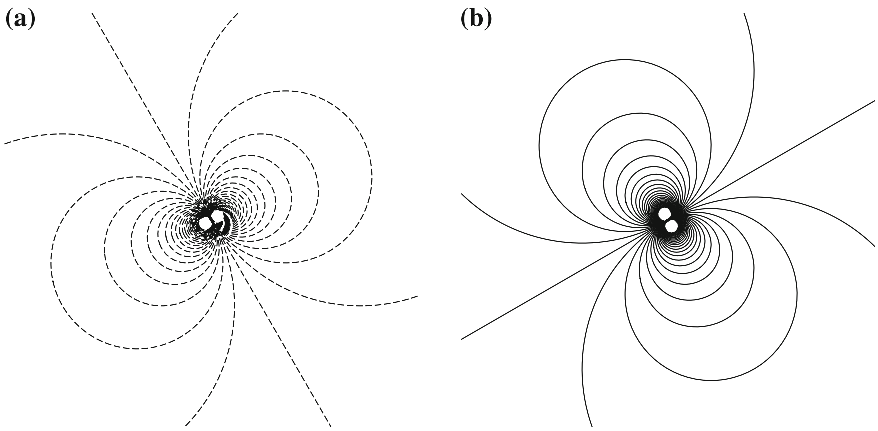


Fig. 2.23 A dipole in the direction making an angle $\pi/6$ with the x direction. **a** Equipotential lines. **b** Streamlines. (In the vicinity of the origin, equipotential lines and streamlines become too dense and some of them are not plotted)

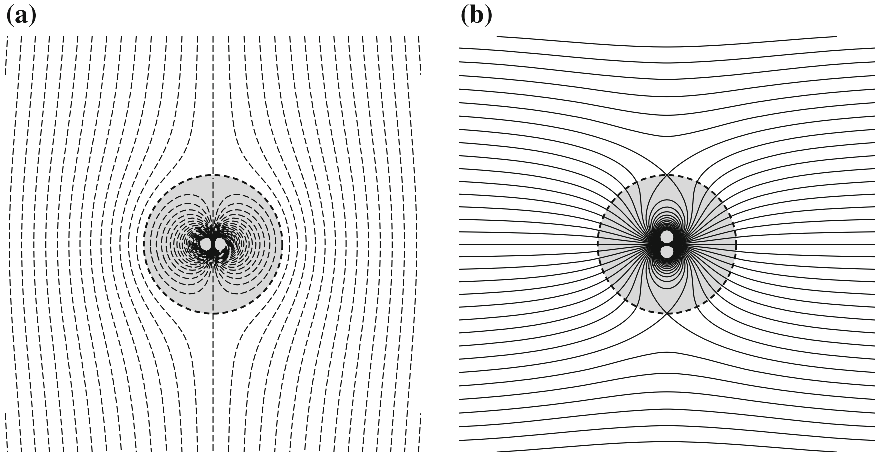


Fig. 2.24 Uniform flow with a dipole oriented against the flow. **a** Equipotential lines. **b** Streamlines

with which the corresponding equipotential lines and streamlines are obtained as shown in Fig. 2.24.

The circumference of the circular object coincides with an equipotential line, which indicates that the circular pillar becomes an equipotential object rather than an obstacle to flow. Consequently, the streamlines are perpendicular to the circular object. Inflow occurs along the upstream (left-hand) side of the circle and outflow along the downstream (right-hand) side.

The complex velocity is given by

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = 1 + \frac{1}{z^2}$$

and from the solution of $W = 0$, stagnation points are found at $z = \pm\mathrm{i}$, where the streamlines intersect each other, as seen in Fig. 2.24b. Away from the equipotential object, flow is essentially uniform as can be seen by evenly spaced horizontal streamlines. When approaching the circular object, flow is gradually distorted and the streamlines become perpendicular to the circular equipotential line, except at the stagnation points.

2.6.6 Vortices

A vortex is a point around which a fluid flows along concentric circles. The corresponding complex potential is given by interchanging the velocity potential and stream function for a source or sink; mathematically, the complex potential for a vortex at z_v can be obtained by dividing that for a source or sink by an imaginary unit i

$$\Omega = -\frac{\gamma}{2\pi\mathrm{i}}\ln(z - z_\mathrm{v}) \tag{2.167}$$

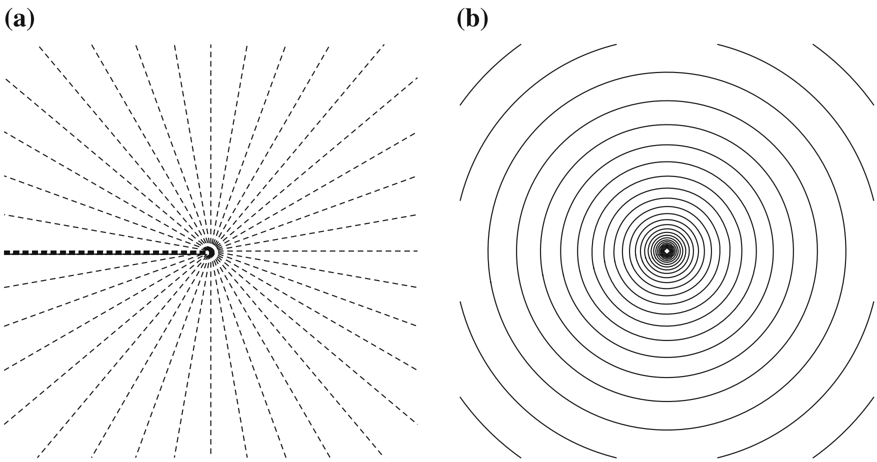


Fig. 2.25 Flow caused by a vortex. **a** Equipotential lines. (The thick line emanating from the vortex in the negative x direction corresponds to the branch cut and overlaps an equipotential line). **b** Streamlines

where γ is the strength of the vortex. The resultant flow caused by the vortex is shown in Fig. 2.25. The equipotential lines are rays emanating from the vortex and the streamlines are concentric circles centered at the vortex.

By using the exponential form $z - z_\mathrm{v} = r\mathrm{e}^{\mathrm{i}\theta}$, it follows that

$$\Omega = -\frac{\gamma}{2\pi\mathrm{i}}(\ln r + \mathrm{i}\theta) = -\frac{\gamma}{2\pi}\theta + \mathrm{i}\frac{\gamma}{2\pi}\ln r \tag{2.168}$$

For the stream function $\Psi = (\gamma/2\pi)\ln r$, it follows that $\partial\Psi/\partial\theta = 0$. From Eq. 2.159, the discharge across a circle C with its center at z_v is given by

$$Q = -\oint_{\mathrm{C}} \mathrm{d}\Psi = \int_0^{2\pi} 0\mathrm{d}\theta = 0 \tag{2.169}$$

which satisfies the condition of incompressibility.

For the velocity potential $\Phi = -(\gamma/2\pi)\theta$, it follows that

$$\frac{\partial\Phi}{\partial\theta} = -\frac{\gamma}{2\pi} \tag{2.170}$$

From Eq. 2.162, the circulation along a circle C enclosing $z = z_\mathrm{v}$ is given by

$$\Gamma = -\oint_{\mathrm{C}} \mathrm{d}\Phi = \int_0^{2\pi} \frac{\gamma}{2\pi}\mathrm{d}\theta = \gamma \tag{2.171}$$

which is equal to the strength of the vortex.

The nonzero circulation along the circle enclosing the vortex does not contradict the condition of irrotationality, Eq. 1.21. The vortex is the singularity at which a circulation of γ is introduced. If a closed curve does not enclose the vortex, the circulation becomes zero.

The complex velocity is given by

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = \frac{\gamma}{2\pi\mathrm{i}}\frac{1}{z - z_\mathrm{v}} \tag{2.172}$$

and by using the exponential form, W becomes

$$W = \frac{\gamma}{2\pi\mathrm{i}}\frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta} = -\mathrm{i}\frac{\gamma}{2\pi}\frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta} = (V_r - \mathrm{i}V_\theta)\mathrm{e}^{-\mathrm{i}\theta} \tag{2.173}$$

Thus, $V_r = 0$ and $V_\theta = (\gamma/2\pi)/r$. The direction of the velocity is tangential to the concentric circles and the magnitude is inversely proportional to the distance from the vortex. For positive values of γ, flow becomes counterclockwise, while for negative values of γ, flow is clockwise.

Example 2.41 In the solution to Task 3-1, let us consider uniform flow over a circular object with circulation. By adding a vortex at the origin to Eq. 2.154, it follows that

$$\Omega = -z - \frac{1}{z} - \frac{\gamma}{2\pi\mathrm{i}}\ln z$$

Since the vortex has concentric circular streamlines, the addition of the vortex to flow over a circular object does not deform the object. For $\gamma = -2\pi$, the corresponding equipotential lines and streamlines are obtained, as shown in Fig. 2.26.

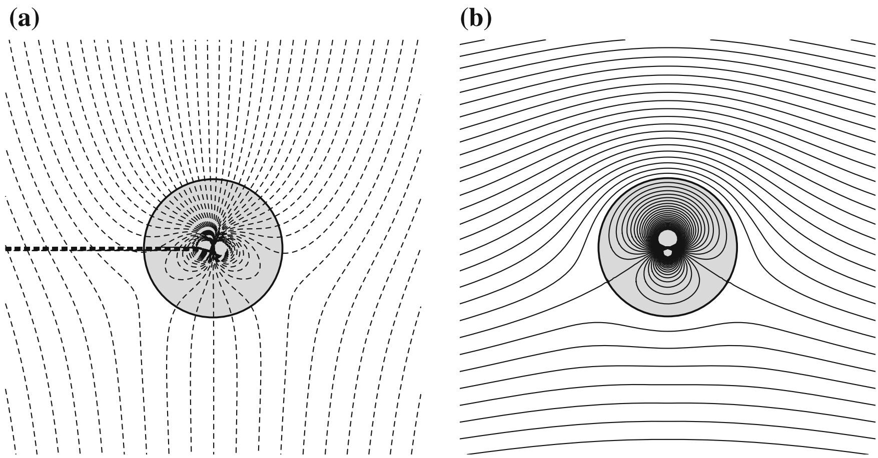


Fig. 2.26 Uniform flow over an impermeable circular object of radius $R = 1$ with circulation $\gamma = -2\pi$. **a** Equipotential lines. **b** Streamlines

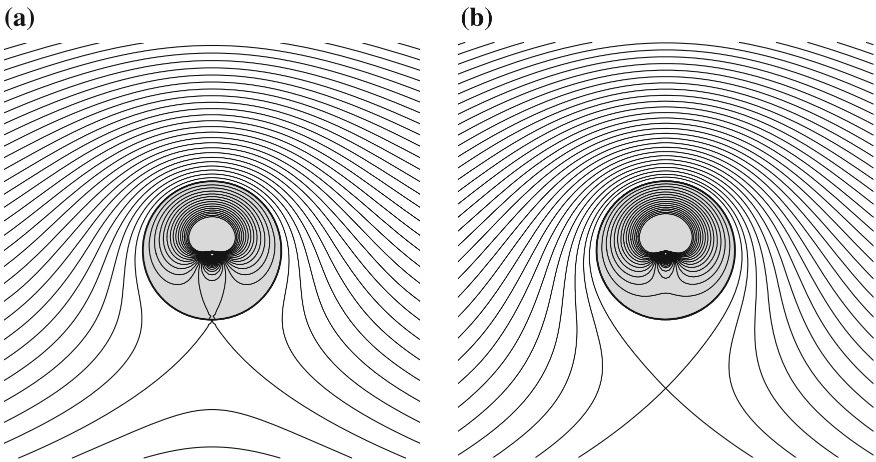


Fig. 2.27 Uniform flow over an impermeable circular object of radius $R = 1$ with circulation. **a** Circulation of $\gamma = -4\pi$. **b** Circulation of $\gamma = -5\pi$

Flow caused by the vortex is in the clockwise direction, and thus, flow above the circle is accelerated, while flow below the circle is decelerated, as can be confirmed from the dense streamlines above and the sparse streamlines below the circle.

The complex velocity is given by

$$W = -\frac{\mathrm{d}\Omega}{\mathrm{d}z} = 1 - \frac{1}{z^2} + \frac{\gamma}{2\pi \mathrm{i}}\frac{1}{z}$$

and from the solution of $W = 0$, stagnation points are found at

$$z = \frac{(\gamma/2\pi)\mathrm{i} \pm \sqrt{4 - (\gamma/2\pi)^2}}{2}$$

If $0 \leq |\gamma/2\pi| < 2$, there exist two stagnation points on the unit circle $|z| = 1$. For $\gamma = -2\pi$, for instance, the stagnation points are found at $z = \pm\sqrt{3}/2 - \mathrm{i}/2$ on the circle, as seen in Fig. 2.26. If $|\gamma/2\pi| = 2$, there is one stagnation point either at $z = +\mathrm{i}$ or $z = -\mathrm{i}$ on the unit circle. If $|\gamma/2\pi| > 2$, stagnation points lie on the imaginary axis, one outside the unit circle and the other inside the circle, which has no physical meaning. Figure 2.27 shows such flow profiles for $\gamma = -4\pi$ and $\gamma = -5\pi$.

The profiles depend on the strength of the vortex γ, and the stagnation points are found in accordance with the discussion above. For $\gamma = -4\pi$, the stagnation point is at $z = -\mathrm{i}$ on the object and for $\gamma = -5\pi$, the stagnation points are at $z = -2\mathrm{i}$ in the flow domain and at $z = -0.5\mathrm{i}$ inside the object.

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