

Hodge-de Rham Theory of K-Forms on Carpet Type Fractals

Jason Bello*, Yiran Li, and Robert S. Strichartz†

Abstract We outline a Hodge-de Rham theory of k -forms (for $k=0,1,2$) on two fractals: the Sierpinski Carpet (SC) and a new fractal that we call the Magic Carpet (MC), obtained by a construction similar to that of SC modified by sewing up the edges whenever a square is removed. Our method is to approximate the fractals by a sequence of graphs, use a standard Hodge-de Rham theory on each graph, and then pass to the limit. While we are not able to prove the existence of the limits, we give overwhelming experimental evidence of their existence, and we compute approximations to basic objects of the theory, such as eigenvalues and eigenforms of the Laplacian in each dimension, and harmonic 1-forms dual to generators of 1-dimensional homology cycles. On MC we observe a Poincare type duality between the Laplacian on 0-forms and 2-forms. On the other hand, on SC the Laplacian on 2-forms appears to be an operator with continuous (as opposed to discrete) spectrum. 2010 *Mathematics Subject Classification*. Primary: 28A80

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1 Introduction

There have been several approaches to developing an analogue of the Hodge-de Rham theory of k -forms on the Sierpinski gasket (SG) and other post-critically finite (pcf) fractals ([1, 8–14, 16, 17]). In this chapter we extend the approach in [1] to the Sierpinski carpet (SC) and a related fractal that we call the *magic carpet* (MC). These fractals are not finitely ramified, and this creates technical difficulties in proving that the conjectured theoretical framework is valid. On the other hand, the structure of “2-dimensional” cells intersecting along “1-dimensional” edges allows for a nontrivial theory of 2-forms. Our results are largely experimental, but they lead to a conjectured theory that is more coherent than for SG.

The approach in [1] is to approximate the fractal by graphs, define k -forms and the associated d , δ , Δ operators on them, and then pass to the limit. In the case of SC there is a natural choice of graphs. Figure 1 shows the graphs on levels 0, 1, and 2.

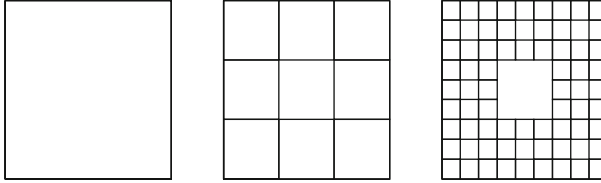


Fig. 1 The graphs approximating SC on levels 0,1, and 2

SC is defined by the self-similar identity:

$$SC = \bigcup_{j=1}^{\infty} F_j(SC)$$

where F_j is the similarity map of contraction ratio $1/3$ from the unit square to one of the eight of the nine subsquares (all except the center square) after tic-tac-toe subdivision. We define the sequence of graphs

$$\Gamma_m = \bigcup_{j=1}^{\infty} F_j(\Gamma_{m-1})$$

with the appropriate identification of vertices in $F_j(\Gamma_{m-1})$ and $F_k(\Gamma_{m-1})$. Note that a hole in SC on level m does not become visible on the graph until level $m+1$, but it will influence the definition of 2-cells. We denote by $E_0^{(m)}$ the vertices of Γ_m . A 0-form on level m is just a real-valued function $f_0^{(m)}(e_0^{(m)})$ defined on $e_0^{(m)} \in E_0^{(m)}$. We denote the vector space of 0-forms by $\Lambda_0^{(m)}$. The edges $E_1^{(m)}$ of Γ_m exist in opposite orientations $e_1^{(m)}$ and $-e_1^{(m)}$, and a 1-form (element of $\Lambda_1^{(m)}$) is a function on $E_1^{(m)}$.

satisfying

$$f_1^{(m)}(-e_1^{(m)}) = -f_1^{(m)}(e_1^{(m)}).$$

By convention we take vertical edges oriented upward and horizontal edges oriented to the right. We denote by E_2^m the squares in Γ_m that bound a cell $F_\omega(SC)$, where $\omega = (\omega_1, \dots, \omega_m)$ is a word of length m , $\omega_j = 1, 2, \dots, 8$ and $F_\omega = F_{\omega_1} \circ F_{\omega_2} \circ \dots \circ F_{\omega_m}$. Thus an element e_2^m of E_2^m consists of the subgraph of Γ_m consisting of the four vertices $\{F_\omega(e_0^0) : e_0^0 \in E_0^{(0)}\}$. In particular, there are eight elements of $E_2^{(1)}$, even though the central square is a subgraph of the same type. In general $\#E_2^{(m)} = 8^m$, and we will denote squares by the word ω that generates them. A 2-form is defined to be a function $f_2^{(m)}(\omega)$ on $E_2^{(m)}$.

The boundary of a square consists of the four edges in counterclockwise orientation. With our orientation convention the bottom and right edges will have a plus sign and the top and left edges will have minus sign. We build a signum function to do the bookkeeping: if $e_1^{(m)} \subseteq e_2^{(m)}$ then

$$\text{sgn}(e_1^{(m)}, e_2^{(m)}) = \begin{cases} +1 & \text{top and right} \\ -1 & \text{bottom and left} \\ 0 & e_1^{(m)} \text{ is not a boundary edge of } e_2^{(m)}. \end{cases}$$

It is convenient to define $\text{sgn}(e_1^{(m)}, e_2^{(m)}) = 0$ if $e_1^{(m)}$ is not a boundary edge of $e_2^{(m)}$. Similarly, if $e_1^{(m)}$ is an edge containing the vertex $e_0^{(m)}$, define

$$\text{sgn}(e_0^{(m)}, e_1^{(m)}) = \begin{cases} +1 & e_0^{(m)} \text{ is top and right} \\ -1 & e_0^{(m)} \text{ is bottom and left} \\ 0 & e_0^{(m)} \text{ is not an endpoint of } e_1^{(m)}. \end{cases}$$

It is easy to check the consistency condition

$$\sum_{e_1^{(m)} \in E_1^{(m)}} \text{sgn}(e_0^{(m)}, e_1^{(m)}) \text{sgn}(e_1^{(m)}, e_2^{(m)}) = 0 \quad (1)$$

for any fixed $e_0^{(m)}$ and $e_2^{(m)}$, since for $e_0^{(m)} \in e_2^{(m)}$ there are only two nonzero summands, one +1 and the other -1.

We may define the de Rham complex

$$0 \rightarrow \Lambda_0^{(m)} \xrightarrow{d_0^{(m)}} \Lambda_1^{(m)} \xrightarrow{d_1^{(m)}} \Lambda_2^{(m)} \rightarrow 0$$

with the operators

$$d_0^{(m)} f_0^{(m)}(e_1^{(m)}) = \sum_{e_0^{(m)} \in E_0^{(m)}} \text{sgn}(e_0^{(m)}, e_1^{(m)}) f_0^{(m)}(e_0^{(m)}) \quad (2)$$

(only two nonzero terms) and

$$d_1^{(m)} f_1^{(m)}(e_2^{(m)}) = \sum_{e_1^{(m)} \in E_1^{(m)}} \text{sgn}(e_1^{(m)}, e_2^{(m)}) f_1^{(m)}(e_1^{(m)}) \quad (3)$$

(only four nonzero terms). The relation

$$d_1^{(m)} \circ d_0^{(m)} \equiv 0 \quad (4)$$

is an immediate consequence of (1).

To describe the δ operators and the dual de Rham complex we need to choose inner products on the spaces $\Lambda_0^{(m)}, \Lambda_1^{(m)}, \Lambda_2^{(m)}$, or what is the same thing, to choose weights on $E_0^{(m)}, E_1^{(m)}, E_2^{(m)}$. The most direct choice is to weight each square $e_2^{(m)}$ equally, say

$$\mu_2(e_2^{(m)}) = \frac{1}{8^m},$$

making μ_2 a probability measure on $E_2^{(m)}$. Note that we might decide to renormalize by multiplying by a constant, depending on m , when we examine the question of the limiting behavior as $m \rightarrow \infty$. For the weighting on edges we may imagine that each square passes on a quarter of its weight to each boundary edge. Some edges bound one square and some bound two squares, so we choose

$$\mu_1(e_1^{(m)}) = \begin{cases} \frac{1}{4 \cdot 8^m} & \text{if } e_1^{(m)} \text{ bounds one square} \\ \frac{1}{2 \cdot 8^m} & \text{if } e_1^{(m)} \text{ bounds two squares.} \end{cases} \quad (5)$$

For vertices we may again imagine the weight of each square being split evenly among its vertices. A vertex may belong to 1, 2, 3, or 4 squares, so

$$\mu_0(e_0^{(m)}) = \frac{k}{4 \cdot 8^m} \quad (6)$$

if $e_0^{(m)}$ lies in k squares.

The dual de Rham complex

$$0 \leftarrow \Lambda_0^{(m)} \xleftarrow{\delta_1^{(m)}} \Lambda_1^{(m)} \xleftarrow{\delta_2^{(m)}} \Lambda_2^{(m)} \leftarrow 0$$

is defined abstractly by $\delta_1^{(m)} = d_0^{(m)*}$ and $\delta_2^{(m)} = d_1^{(m)*}$ where the adjoints are defined in terms of the inner products induced by the weights, or concretely as

$$\delta_1^{(m)} f_1^{(m)}(e_0^{(m)}) = \sum_{e_1^{(m)} \in E_1^{(m)}} \frac{\mu_1(e_1)}{\mu_0(e_0)} \text{sgn}(e_0^{(m)}, e_1^{(m)}) f_1^{(m)}(e_1^{(m)}), \quad (7)$$

$$\delta_2^{(m)} f_2^{(m)}(e_1^{(m)}) = \sum_{e_2^{(m)} \in E_2^{(m)}} \frac{\mu_2(e_2)}{\mu_1(e_1)} \text{sgn}(e_1^{(m)}, e_2^{(m)}) f_2^{(m)}(e_2^{(m)}). \quad (8)$$

There are one or two nonzero terms in (8) depending on whether $e_1^{(m)}$ bounds one or two squares. In (7) there may be 2, 3, or 4 nonzero terms, depending on the number of edges that meet at the vertex $e_0^{(m)}$. The condition

$$\delta_1^{(m)} \circ \delta_2^{(m)} = 0 \quad (9)$$

is the dual of (4).

We may then define the Laplacian

$$\begin{aligned} -\Delta_0^{(m)} &= \delta_1^{(m)} d_0^{(m)} \\ -\Delta_1^{(m)} &= \delta_2^{(m)} d_1^{(m)} + d_0^{(m)} \delta_1^{(m)} \\ -\Delta_2^{(m)} &= d_1^{(m)} \delta_2^{(m)} \end{aligned} \quad (10)$$

as usual. These are nonnegative self-adjoint operators on the associated L^2 spaces, and so have a discrete nonnegative spectrum. We will be examining the spectrum (and associated eigenfunctions) carefully to try to understand what could be said in the limit as $m \rightarrow \infty$. Also of particular interest are the harmonic 1-forms $\mathcal{H}_1^{(m)}$, solutions of $-\Delta_1^{(m)} h_1^{(m)} = 0$. As usual these can be characterized by the two equations

$$d_1^{(m)} h_1^{(m)} = 0 \quad \delta_1^{(m)} h_1^{(m)} = 0,$$

and can be put into cohomology/homology duality with the homology generating cycles in Γ_m . The Hodge decomposition

$$\Lambda_1^{(m)} = d_0^{(m)} \Lambda_0^{(m)} \oplus \delta_2^{(m)} \Lambda_2^{(m)} \oplus \mathcal{H}_1^{(m)} \quad (11)$$

shows that the eigenfunctions of $-\Delta_1^{(m)}$ with $\lambda \neq 0$ are either $d_0^{(m)} f_0^{(m)}$ for $f_0^{(m)}$ an eigenfunction of $-\Delta_0^{(m)}$, or $\delta_2^{(m)} f_2^{(m)}$ for $f_2^{(m)}$ an eigenfunction of $-\Delta_2^{(m)}$ with the same eigenvalue. Thus the nonzero spectrum of $-\Delta_1^{(m)}$ is just the union of the $-\Delta_0^{(m)}$ and $-\Delta_2^{(m)}$ spectrum.

The irregular nature of the adjacency of squares in Γ_m , with the associated variability of the weights in (6) and (5), leads to a number of complications in the behavior of $-\Delta_2^{(m)}$. To overcome these complications we have invented the fractal MC, that is obtained from SC by making identifications to eliminate boundaries. On the outer boundary of SC we identify the opposite pairs of edges with the same orientation, turning the full square containing SC into a torus. Each time we delete a small square in the construction of SC we identify the opposite edges of the deleted square with the same orientation. We may think of MC as a limit of closed surfaces

of genus $g = 1 + (1 + 8 + \dots + 8^{m-1})$, because each time we delete and “sew up” we add on a handle to the torus. This surface carries a flat metric with singularities at the corners of each deleted square (all four corners are identified). It is not difficult to show that the limit exists as a topological space, because the identifications made at level m are all points that are very close to each other, distance 3^{-m} in the Euclidean metric. The metric structure of MC is more complicated. If we consider the geodesic metric, the approximation on level m has diameter $(2/3)^m$, because we can use diagonal zig-zag lines to take advantage of the identifications that lead to the singular points to connect all vertices with a sequence of 2^m edges of size 3^{-m} . Thus it would appear that we should renormalize the metric at level m by multiplying by $(3/2)^m$ in order to obtain a metric structure in the limit. The exact nature of this metric structure remains to be investigated. Whether or not the analytic structures (energy, Laplacian, Brownian motion) on SC can be transferred to MC remains to be investigated. Our results give overwhelming evidence that this is the case.

We pass from the graphs Γ_m approximating SC to graphs $\tilde{\Gamma}_m$ approximating MC by making the same identifications of vertices and edges. The graph $\tilde{\Gamma}_1$ is shown in Fig. 2.

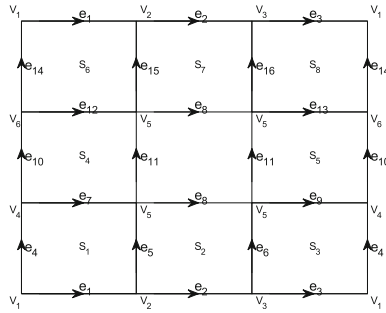


Fig. 2 $\tilde{\Gamma}_1$ with 6 vertices labeled v_j , 16 edges labeled e_j , and 8 squares labeled s_j

Each square has exactly four neighbors (not necessarily distinct) with each edge separating two squares. For example, in Fig. 2, we see that s_2 has neighbors s_1 , s_3 , and s_7 twice, as e_2 and e_8 both separate s_2 and s_7 . There are two types of vertices, that we call *nonsingular* and *singular*. The nonsingular vertices (all except v_5 in Fig. 2) belong to exactly four distinct squares and four distinct edges (two incoming and two outgoing according to our orientation choice). For example, v_1 belongs to squares s_1 , s_3 , s_6 , and s_8 and has incoming edges e_3 and e_{14} and outgoing edges e_1 and e_4 . Singular vertices belong to 12 squares (with double counting) and 12 edges (some of which may be loops). In Fig. 2 there is only one singular vertex, v_5 . It belongs to squares s_1 , s_3 , s_6 , and s_8 counted once and s_2 , s_4 , s_5 , and s_7 counted twice. In Fig. 3 we show a neighborhood of this vertex in $\tilde{\Gamma}_2$, with the incident squares and edges shown.

The definition of the de Rham complex for the graphs $\tilde{\Gamma}_m$ approximating MC is exactly the same as for Γ_m approximating SC. The difference is in the dual de Rham

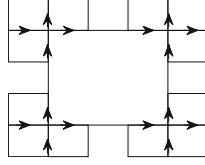


Fig. 3 A neighborhood in $\tilde{I}_2$ of vertex v_5 from Fig. 2

complex, because the weights are different. We take $\tilde{\mu}_2(e_2^{(m)}) = 1/8^m$ as before, but now $\tilde{\mu}_1(e_1^{(m)}) = \frac{1}{2 \cdot 8^m}$ because every edge bounds two squares. Finally

$$\tilde{\mu}_0(e_0^{(m)}) = \begin{cases} \frac{1}{8^m} & \text{if } e_0^{(m)} \text{ is nonsingular} \\ \frac{3}{8^m} & \text{if } e_0^{(m)} \text{ is singular} \end{cases}$$

because $e_0^{(m)}$ belongs to four squares in the first case and 12 squares in the second case. After that the definitions are the same using the new weights.

Explicitly, we have

$$-\tilde{\Delta}_2^{(m)} f_2^{(m)}(e_2^{(m)}) = \left(2 \left(\sum_{e_2^{(m)'} \sim e_2^{(m)}} f_2^{(m)}(e_2^{(m)}) - f_2^{(m)}(e_2^{(m)'}) \right) \right) \quad (12)$$

(exactly four terms in the sum), the factor 2 coming from $\tilde{\mu}_2(e_2^{(m)})/\tilde{\mu}_1(e_1^{(m)})$. Except for the factor 2 this is exactly the graph Laplacian on the 4-regular graph whose vertices are the squares in $\tilde{E}_2^{(m)}$ and whose edge relation is $e_2^{(m)'} \sim e_2^{(m)}$ if they have an edge in common (double count if there are two edges in common).

The explicit expression for $-\Delta_0^{(m)}$ is almost as simple:

$$-\tilde{\Delta}_0^{(m)} f_0^{(m)}(e_0^{(m)}) = \begin{cases} \frac{1}{2} \sum_{e_0^{(m)'} \sim e_0^{(m)}} (f_0^{(m)}(e_0^{(m)}) - f_0^{(m)}(e_0^{(m)'})) & \text{if } e_0^{(m)} \text{ is nonsingular} \\ \frac{1}{6} \sum_{e_0^{(m)'} \sim e_0^{(m)}} (f_0^{(m)}(e_0^{(m)}) - f_0^{(m)}(e_0^{(m)'})) & \text{if } e_0^{(m)} \text{ is singular.} \end{cases}$$

Note that there are four summands in the first case and 12 summands in the second case (some may be zero if there is a loop connecting $e_0^{(m)}$ to itself in the singular case). We expect that the spectra of these two Laplacians will be closely related, aside from the multiplicative factor of 4. We may define Hodge star operators from

$\Lambda_0^{(m)}$ to $\Lambda_2^{(m)}$ and from $\Lambda_2^{(m)}$ to $\Lambda_0^{(m)}$ by

$$*f_0^{(m)}(e_2^{(m)}) = \frac{1}{4} \sum_{e_0^{(m)} \subseteq e_2^{(m)}} f_0^{(m)}(e_0^{(m)}) \quad (13)$$

and

$$*f_2^{(m)}(e_0^{(m)}) = \begin{cases} \frac{1}{4} \sum_{e_2^{(m)} \supseteq e_0^{(m)}} f_2^{(m)}(e_2^{(m)}) & \text{if } e_0^{(m)} \text{ is nonsingular} \\ \frac{1}{12} \sum_{e_2^{(m)} \supseteq e_0^{(m)}} f_2^{(m)}(e_2^{(m)}) & \text{if } e_0^{(m)} \text{ is singular.} \end{cases} \quad (14)$$

Note that we do not have the inverse relation that $**$ is equal to the identity in either order. Nor is it true that the star operators conjugate the two Laplacians. However, they are approximately valid, so we can hope that in the appropriate limit there will be a complete duality between 0-forms and 2-forms with identical Laplacians. Nothing remotely like this valid for SC.

It is also easy to describe explicitly the equations for harmonic 1-forms. The condition $d_1^{(m)} h_1^{(m)}(e_2^{(m)}) = 0$ is simply the condition that the sum of the values $h_1^{(m)}(e_1^{(m)})$ over the four edges of the square is zero (with appropriate signs). Similarly the condition $\delta_1^{(m)} h_1^{(m)}(e_0^{(m)}) = 0$ means the sum over the incoming edges equals the sum over the outgoing edges at $e_0^{(m)}$. Those equations have two redundancies, since the sums $\sum_{e_2^{(m)} \in E_2^{(m)}} d_1^{(m)} f_1^{(m)}(e_2^{(m)})$ and $\sum_{e_0^{(m)} \in E_0^{(m)}} \delta_1^{(m)} f_1^{(m)}(e_0^{(m)})$ are auto-

matically zero for any 1-form $f_1^{(m)}$. Thus in $\tilde{\Lambda}_1^{(1)}$ there is a 4-dimensional space of harmonic 1-forms, and in general the dimension is $2g$, which is exactly the rank of the homology group for a surface of genus g . It is easy to identify the homology generating cycles as the edges that are identified.

The remainder of this chapter is organized as follows: In Sects. 2, 3, and 4 we give the results of our computations on SC for 0-forms, 1-forms, and 2-forms. In Sect. 5 we give the results for 0-forms and 2-forms on MC. In Sect. 6 we give the results for 1-forms on MC. We conclude with a discussion in Sect. 7 of all the results and their implications. The website [25] gives much more data than we have been able to include in this chapter, and also contains all the programs used to generate the data.

2 0-Forms on the Sierpinski Carpet

The 0-forms on SC will simply be continuous functions on SC, and we can restrict them to the vertices of Γ_m to obtain 0-forms on Γ_m . The Laplacian $-\Delta_0^{(m)}$ is exactly the graph Laplacian of Γ_m with weights on vertices and edges given by (6) and (5).

Thus

$$-\Delta_0^{(m)} f_0^{(m)}(x) = \sum_{y \sim x} c(x, y) (f(x) - f(y)) \quad (15)$$

with coefficients show in Fig. 4

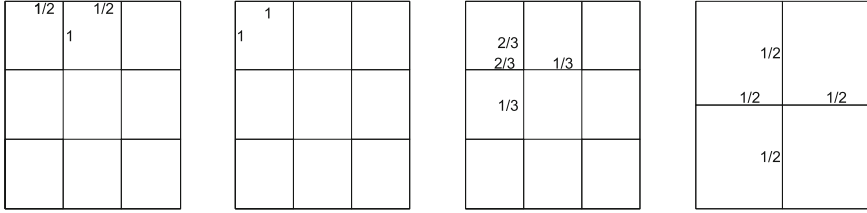


Fig. 4 Coefficients in (15)

The sequence of renormalized Laplacians

$$\{-r^m \Delta_0^{(m)}\} \quad (16)$$

for $r \approx 10.01$ converges to the Laplacian on functions [3, 5–7, 19].

In Table 1 we give the beginning of the spectrum $\{\lambda_j^{(m)}\}$ for $m = 2, 3, 4$ and the ratios $\lambda_j^{(3)}/\lambda_j^{(2)}$ and $\lambda_j^{(4)}/\lambda_j^{(3)}$. The results are in close agreement with the computations in [7] and [6], and suggest the convergence of (16).

Table 1 Eigenvalues of $-\Delta_0^{(m)}$ for $m = 2, 3, 4$ and ratios

$m = 2$	Multiplicity	$m = 3$	Multiplicity	$m = 4$	Multiplicity	$\lambda_j^{(3)}/\lambda_j^{(2)}$	$\lambda_j^{(4)}/\lambda_j^{(3)}$
0.0000	1	0.0000	1	0.0000	1		
0.0414	2	0.0041	2	0.0004	2	0.1001	0.0999
0.1069	1	0.0109	1	0.0011	1	0.1024	0.1002
0.2006	1	0.0204	1	0.0020	1	0.1017	0.0999
0.2635	2	0.0272	2	0.0027	2	0.1031	0.1002
0.2720	1	0.0284	1	0.0028	1	0.1044	0.1003
0.3927	2	0.0414	2	0.0041	2	0.1053	0.1001
0.4260	1	0.0449	1	0.0045	1	0.1055	0.1002
0.4490	1	0.0472	1	0.0047	1	0.1051	0.1001
0.5276	1	0.0560	1	0.0056	1	0.1062	0.1004
0.6375	2	0.0673	2	0.0067	2	0.1055	0.1002
0.6700	1	0.0696	1	0.0069	1	0.1038	0.0997
0.8405	2	0.0976	2	0.0099	2	0.1161	0.1014
0.8713	1	0.1009	1	0.0102	1	0.1158	0.1008
0.9102	1	0.1069	1	0.0109	1	0.1175	0.1024
0.9336	1	0.1103	1	0.0112	1	0.1181	0.1019

The convergence of (16) would imply that $\lim_{m \rightarrow \infty} r^m \lambda_j^{(m)} = \lambda_j$ gives the spectrum of the limit Laplacian $-\Delta_0$ on 0-forms on SC. In particular the values $\{r^m \lambda_j^{(m)}\}$ for small values of j (depending on m) would give a reasonable approximation of some lower portion of the spectrum of $-\Delta_0$. To visualize this portion of the spectrum we compute the eigenvalue counting function $N(t) = \#\{\lambda_j \leq t\} \approx \#\{r^m \lambda_j^{(m)} \leq t\}$ and the Weyl ratio $W(t) = \frac{N(t)}{t^\alpha}$. In Figure we display the graphs of the Weyl ratio using the $m = 1, 2, 3, 4$ approximations, with value α determined from the data to get a function that is approximately constant. As explained in [6], we expect $\alpha = \frac{\log 8}{\log r} \approx 0.9026$, which is close to the experimentally determined values. This is explained by the phenomenon called *miniaturization* as described in [7]. Every eigenfunction $u_j^{(m)}$ of $\Delta_0^{(m)}$ reappears in miniaturized form $u_k^{(m+1)}$ of $\Delta_0^{(m+1)}$ with the same eigenvalue $\lambda_k^{(m+1)} = \lambda_j^{(m)}$, so in terms of $\Delta_0^{(m)}$ we have $r^{m+1} \lambda_k^{(m+1)} = r(r^m \lambda_j^{(m)})$. We can in fact see this in Table 1. In passing from level m to level $m+1$, the number of eigenvalues is multiplied by 8, so we expect to have $N(rt) \approx 8N(t)$, and this explains why $\alpha = \frac{\log 8}{\log r}$ is the predicted power growth factor of $N(t)$. We also expect to see an approximate multiplicative periodicity in $W(t)$, namely $W(rt) \approx W(t)$. It is difficult to observe this in our data, however. We also mention that miniaturization is valid for all k -forms ($k=0, 1, 2$) on SC and MC. This is most interesting for 0-forms and 2-forms on MC as discussed in Sect. 5.

In Fig. 5, 6, and 7 we show graphs of selected eigenfunctions on levels 2, 3, 4. We only display those whose eigenspaces have multiplicity one. The convergence is visually evident. To quantify the rate of convergence we give the values of $\|f_j^{(m)}|_{E_0^{(m-1)}} - f_j^{(m-1)}\|_2^2$ in Table 2. Here the L^2 norm on $E_0^{(m-1)}$ is defined by

$$\|f\|_2^2 = \sum_{e_0^{(m-1)} \in E_0^{(m-1)}} \mu_0(e_0^{(m-1)}) |f(e_0^{(m-1)})|^2$$

and we normalize the eigenfunctions so that $\|f_j^{(m-1)}\|_2^2$ and $\|f_j^{(m)}|_{E_0^{(m-1)}}\|_2 = 1$. In

Fig. 8 we show the graph of the weyl ratio of eigenvalues of the 0 forms on different levels.

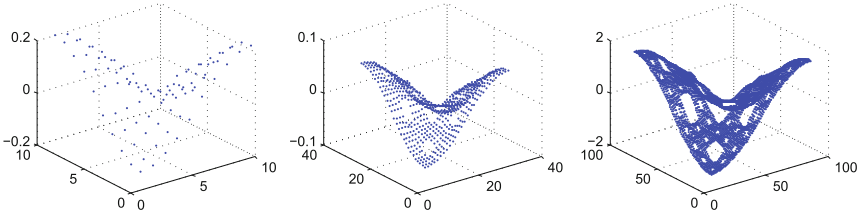


Fig. 5 Graph of eigenfunction of 4th eigenvalue on 0 forms of level 2, 3, and 4

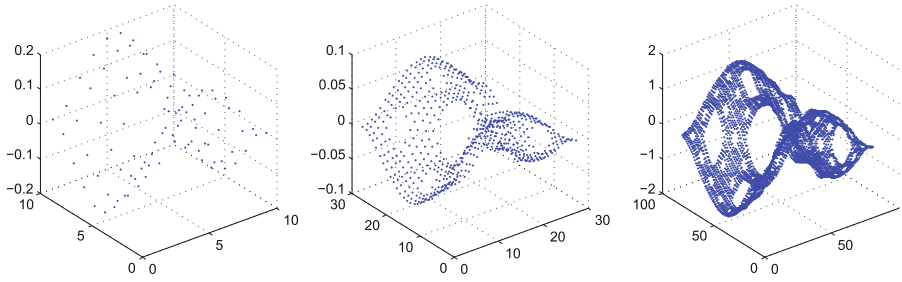


Fig. 6 Graph of eigenfunction of 5th eigenvalue on 0 forms of level 2, 3, and 4

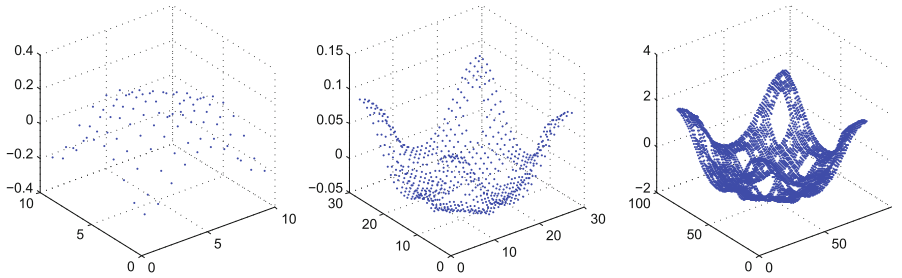


Fig. 7 Graph of eigenfunction of 8th eigenvalue on 0 forms of level 2, 3, and 4

Table 2 Values of $\|f_j^{(m)}|_{E_0^{(m-1)}} - f_j^{(m-1)}\|_2^2$, from level 3 to level 2 and level 4 to level 3 for eigenspaces of multiplicity one

Number of Eigenvalue	$m = 2$	$m = 3$	$m = 4$	Level 3 to level 2	Level 4 to level 3
1	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.1069	0.0109	0.0011	0.0001	0.0000
5	0.2006	0.0204	0.0020	0.0006	0.0001
8	0.2720	0.0284	0.0028	0.0003	0.0001
11	0.4260	0.0449	0.0045	0.0009	0.0001
12	0.4490	0.0472	0.0047	0.0013	0.0001
13	0.5276	0.0560	0.0056	0.00	0.0001
16	0.6700	0.0696	0.0069	0.0161	0.0004
19	0.8713	0.1009	0.0102	0.8818	0.0004
20	0.9102	0.1069	0.0109	0.8755	0.0001
21	0.9336	0.1103	0.0112	0.9226	0.0003

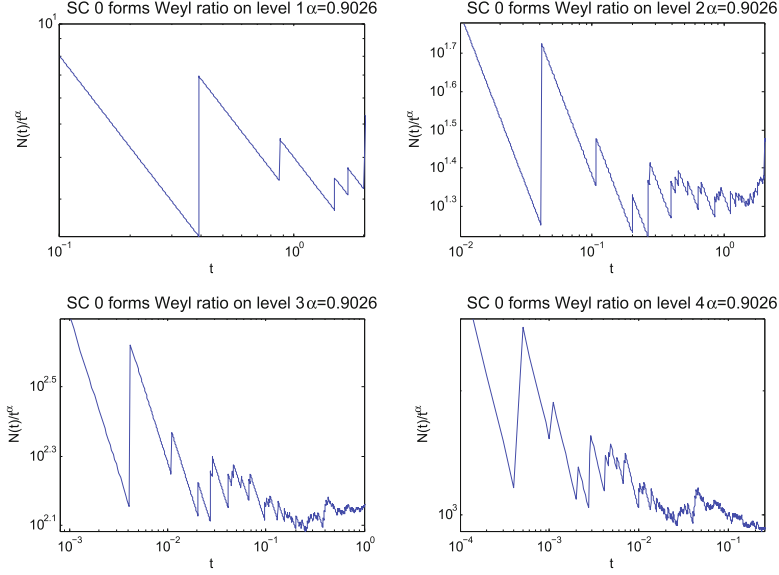


Fig. 8 Weyl ratio of SC 0-forms on level 1, 2, 3, 4 with $\alpha = 0.9026$

3 1-Forms on the Sierpinski Carpet

The 1-forms on Γ_m are functions on the edges of Γ_m , with $f_1^{(m)}(-e_1^{(m)}) = -f_1^{(m)}(e_1^{(m)})$ if $-e_1^{(m)}$ denotes the edge $e_1^{(m)}$ with opposite orientation. If L denotes any oriented path made up of edges, we may integrate $f_1^{(m)}$ over L by summing:

$$\int_L df_1^{(m)} = \sum_{e_1^{(m)} \subseteq L} f_1^{(m)}(e_1^{(m)}). \quad (17)$$

In particular, if $f_1^{(m)} = df_0^{(m)}$ then

$$\int_L df_0^{(m)} = f_0^{(m)}(b) - f_0^{(m)}(a) \quad (18)$$

where b and a denote the endpoints of L .

The Hodge decomposition splits this space $\Lambda_1^{(m)}$ of 1-forms into three orthogonal pieces.

$$\Lambda_1^{(m)} = d_0^{(m)} \Lambda_0^{(m)} \oplus \delta_2^{(m)} \Lambda_2^{(m)} \oplus \mathcal{H}_1^{(m)} \quad (19)$$

The map $d_0^{(m)} : \Lambda_0^{(m)} \rightarrow \Lambda_1^{(m)}$ has the 1-dimensional kernel consisting of constants, and the map $\delta_2^{(m)} : \Lambda_2^{(m)} \rightarrow \Lambda_1^{(m)}$ has zero kernel. A dimension count shows

$$\dim \mathcal{H}_1^{(m)} = 1 + 8 + \dots + 8^{m-1} = \frac{8^m - 1}{7}. \quad (20)$$

The Laplacian $-\Delta_1^{(m)}$ respects the decomposition, with

$$\begin{aligned} -\Delta_1^{(m)}(d_0^{(m)} f_0^{(m)}) &= d_0^{(m)}(-\Delta_0^{(m)} f_0^{(m)}) = d_0^{(m)} \delta_1^{(m)}(d_0^{(m)} f_0^{(m)}) \\ -\Delta_1^{(m)}(\delta_2^{(m)} f_2^{(m)}) &= \delta_2^{(m)}(-\Delta_2^{(m)} f_1^{(m)}) = \delta_2^{(m)} d_1^{(m)}(\delta_2^{(m)} f_2^{(m)}) \\ -\Delta_1^{(m)}|_{\mathcal{H}_1^{(m)}} &= 0. \end{aligned} \quad (21)$$

Although we may write $-\Delta_1^{(m)} = d_0^{(m)} \delta_1^{(m)} + \delta_2^{(m)} d_1^{(m)}$, in fact (21) is more informative. In particular, it shows that the spectrum of $-\Delta_1^{(m)}$ is just a union of the nonzero eigenvalues of $-\Delta_0^{(m)}$ and the eigenvalues of $-\Delta_2^{(m)}$, together with 0 with multiplicity given by (20). As such, it has no independent interest, and we will not present a table of its values. Also, when we discuss later the question of renormalizing in order to pass to the limit as $m \rightarrow \infty$, we will want to use a different factor for the two terms.

The main object of interest is the space $\mathcal{H}_1^{(m)}$ of harmonic 1-forms. We note that the dimension given by (20) is exactly equal to the number of homology generating cycles, one for each square deleted in the construction of SC up to level m . Thus the integrals

$$\int_{\gamma_j} h_1^{(m)} \quad (22)$$

as γ_j varies over the cycles and $h_1^{(m)}$ varies over a basis of $\mathcal{H}_1^{(m)}$ give a cohomology/homology pairing.

Conjecture 3.1 1 *For each m , the matrix (22) is invertible.*

If this conjecture is valid (we have verified it for $m \leq 4$) then we may define a canonical basis $h_k^{(m)}$ of $\mathcal{H}_1^{(m)}$ by the conditions

$$\int_{\gamma_j} h_k^{(m)} = \delta_{jk}. \quad (23)$$

We are particularly interested in the consistency among these harmonic 1-forms as m varies. The cycles at level m contain all the cycles from previous levels and, in addition, the $8^{(m-1)}$ cycles around the level m deleted squares. Thus we can order the cycles consistently from level to level. Also, a 1-form $f_1^{(m)}$ in $\Lambda^{(m)}$ can be restricted to a 1-form in $\Lambda^{(m-1)}$ by defining

$$Rf_1^{(m)}(e_1^{(m-1)}) = \int_{e_1^{(m-1)}} f_1^{(m)},$$

in other words summing the values of $f_1^{(m)}$ on the three level m edges that make up $e_1^{(m-1)}$. Thus we may compare the $\Lambda_1^{(m-1)}$ 1-forms $h_k^{(m-1)}$ and $Rh_k^{(m)}$ for values of k where γ_k is a level $m-1$ cycle. If these are close, we may hope to define a harmonic 1-form on SC by $\lim_{m \rightarrow \infty} h_k^{(m)}$. Note that $Rh_k^{(m)}$ will not be a harmonic 1-form in $\mathcal{H}_1^{(m-1)}$. It is easy to see that the equation $d_1^{(m)} h_k^{(m)} = 0$ and the fact that (22) is zero for all the level m cycles implies (by addition) $d_1^{(m-1)} Rh_k^{(m)} = 0$. However, there is no reason to believe that $\delta_1^{(m-1)} Rh_k^{(m)}$ should be zero.

For a graphical display of the numerical data we computed for some of the functions $h_k^{(m)}$ with $k = 1$ and their restrictions see website [25]. (We rounded the decimal expansions and multiplied by 10^4 so that all values are integers.) The condition $d_1 h_k^{(m)}(e_2^{(m)}) = 0$ says that the sum of $h_k^{(m)}$ on the four edges of the square (with appropriate $\pm$ signs) vanishes, or equivalently, the sum on the bottom and right edge equals the sum on top and left edge. A similar condition gives (23). The condition $\delta_1 h_k^{(m)}(e_0^{(m)}) = 0$ says that the weighted sum of $h_k^{(m)}$ on the incoming edges at the vertex $e_0^{(m)}$ equals the weighted sum on the outgoing edges, with weights given in Fig. 4.

To quantify the rate of convergence, we give in Table 3 the values of $\|h_k^{(m-1)} - Rh_k^{(m)}\|_2$, where we use an L^2 norm on $E_1^{(m-1)}$.

Table 3 Values of $\|h_k^{(m-1)} - Rh_k^{(m)}\|_2$

k	1	2	3	4	5	6	7	8	9
$\ h_k^{(2)} - Rh_k^{(1)}\ $	0.0303								
$\ h_k^{(3)} - Rh_k^{(2)}\ $	0.0197	0.0292	0.0383	0.0292	0.0383	0.0383	0.0292	0.0383	0.0197
$\ h_k^{(4)} - Rh_k^{(3)}\ $	0.0140	0.0206	0.0224	0.0206	0.0224	0.0224	0.0206	0.0224	0.0206

Another observation is that the size of $h_k^{(m)}$ tends to fall off as the edge moves away from the cycle γ_k . This is not a very rapid decay, however.

Since 1-forms are functions on edges, it appears difficult to display the data graphically. However, there is another point of view, of independent interest, that would enable us to “see” harmonic 1-forms graphically. Note that if $f_0^{(m)}$ is a harmonic function, then $d_0 f_0^{(m)}$ is a harmonic 1-form. This is not interesting globally, since the only harmonic functions are constant. But it is interesting locally, and we can obtain harmonic 1-forms by gluing together 1-forms $d_0 f_0^{(m)}$ for different functions $f_0^{(m)}$ that are locally harmonic. We consider *harmonic mappings* taking values in the circle $\mathbb{R}/\mathbb{Z}$. Such a mapping is represented locally by a harmonic function $f_0^{(m)}$, but when we piece the local representations globally the values may change by an additive integer constant. The additive constant will not change $d_0 f_0^{(m)}$, so this will be a global harmonic 1-form. Again, adding a global constant to $f_0^{(m)}$ will have no effect on $d_0 f_0^{(m)}$. Thus, for each basis element $h_k^{(m)}$ we can construct a harmonic

mapping $f_k^{(m)}$ by setting it equal to 0 at the lower left corner of SG, and then integrating using (18) to successively extend its values to vertices at the end of an edge where the value at the other endpoint has been determined. For each of the cycles γ_k we can find “cut line” so that the function $f_k^{(m)}$ is single valued away from the cut line, and only satisfies the equation $\delta_1^{(m)} d_0^{(m)} f_k^{(m)}(e_0^{(m)}) = 0$ at the vertices $e_0^{(m)}$ along the cut line if we use different values across the cut line.

Another fundamental question is the behavior of the restriction of a harmonic 1-form to a line segment. Suppose that L is a horizontal or vertical line segment of length one in SC (of course shorter line segments are also of interest, but the answers are expected to be the same). We regard $h_k^{(m)}$ on L as a signed measure on L via (17). Do we obtain a measure in the limit as $m \rightarrow \infty$? If so, is it absolutely continuous with respect to Lebesgue measure on L ? In terms of the restriction of the harmonic mapping $f_k^{(m)}$ to L (we choose L to avoid the cut line), these questions become: is it of bounded variation, and if so is it an absolutely continuous function? Of course we can’t answer these questions about the limit, but we can get a good sense by observing the approximations. In Fig. 9 we graph the restrictions of $h_k^{(m)}$ to L in some cases as m varies. For a quantitative approach to the first question we compute the total variation of the approximations. The results are show in Table 4. Note that in this figure and all subsequent graphs of restrictions of 1-forms to lines, we are displaying the graphs of running totals starting at the left end of the interval. Thus in the limit we would hope to get a function of bounded variation (or perhaps even an absolutely continuous function) whose derivative is a measure on the line.

Of course the same questions are of interest for 1-forms that are eigenfunctions of the Laplacian. In Fig. 10, 11 and Table 5 we give the analogous results for $d_0^{(m)} f_0^{(m)}$ and $\delta_2^{(m)} f_2^{(m)}$ where $f_0^{(m)}$ and $f_2^{(m)}$ are eigenforms for the Laplacians $-\Delta_0^{(m)}$ and $-\Delta_2^{(m)}$.

4 2-Forms on the Sierpinski Carpet

In principle, we should think of 2-forms on SC simply as measures. In the Γ_m approximation we have 8^m squares in $E_2^{(m)}$, and our 2-forms $f_2^{(m)}$ assign values to these squares, which may be identified with the m -cells in SC that lie in these squares. It is not clear a priori what class of measures we should consider; the simplest choice is the set of measures absolutely continuous with respect to the standard self-similar measure μ .

The Laplacian $-\Delta_2^{(m)}$ on $\Lambda_2^{(m)}$ is quite different from other Laplacians considered here or elsewhere. In fact, it is the sum of a difference operator and diagonal operator. For a fixed square $e_2^{(m)} \in E_2^{(m)}$ we have

$$d_1^{(m)} \delta_2^{(m)} f_2^{(m)}(e_2^{(m)}) = \sum_{e_1^{(m)} \subseteq e_2^{(m)}} \text{sgn}(e_1^{(m)}, e_2^{(m)}) \delta_2^{(m)} f_2^{(m)}(e_1^{(m)})$$

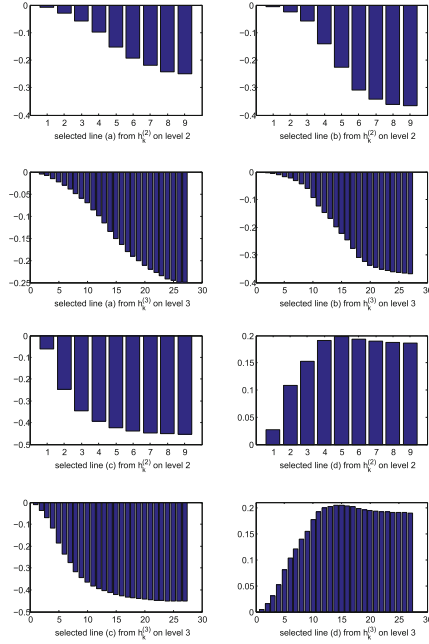


Fig. 9 Restrictions of $h_k^{(m)}$ to L with $m = 2, 3$

Table 4 Total variation of SC harmonic one forms along horizontal lines at level 1, level 2 and level 3

		Level 1	Level 2	Level 3
Line 1	Form 1	0.00	0.00	0.00
	Form 2	0.4522	0.4522	0.4522
	Form 3	0.1442	0.1442	0.1442
	Form 4	0.0478	0.0478	0.0478
Line 2	Form 1	0.3678	0.3678	0.3678
	Form 2	0.2057	0.2188	0.2188
	Form 3	0.82	0.82	0.82
	Form 4	0.0741	0.0741	0.0741
Line 3	Form 1	0.3678	0.3678	0.3678
	Form 2	0.0741	0.0741	0.0741
	Form 3	0.82	0.82	0.82
	Form 4	0.2057	0.2188	0.2188
Line 4	Form 1	0.00	0.00	0.00
	Form 2	0.0478	0.0478	0.0478
	Form 3	0.1442	0.1442	0.1442
	Form 4	0.4522	0.4522	0.4522

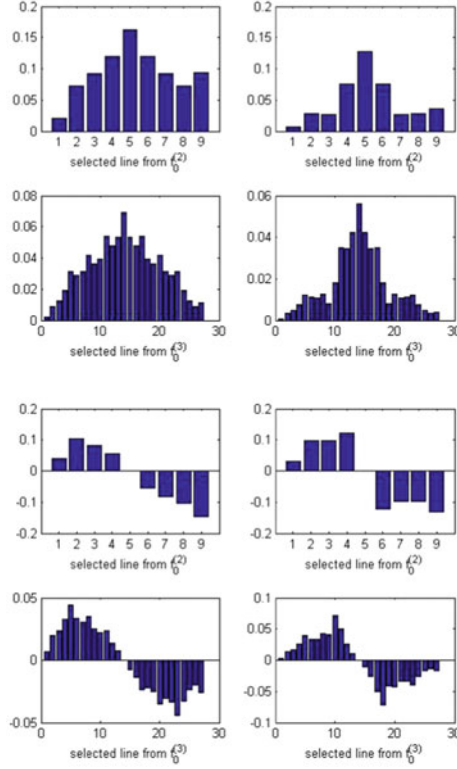


Fig. 10 Restrictions of $d_0^{(m)} f_0^{(m)}$ to L

where the sum is over the four edges of the square. However, $\delta_2^{(m)} f_2^{(m)}(e_2^{(m)})$ depends on the nature of the edge $e_1^{(m)}$, which may bound one or two squares:

$$\delta_2^{(m)} f_2^{(m)}(e_1^{(m)}) = \begin{cases} 4 \operatorname{sgn}(e_1^{(m)}, \tilde{e}_2^{(m)}) f_2^{(m)}(\tilde{e}_2^{(m)}) & \text{if } e_1^{(m)} \text{ bounds only } \tilde{e}_2^{(m)} \\ 2 \left(\operatorname{sgn}(e_1^{(m)}, \tilde{e}_2^{(m)}) f_2^{(m)}(\tilde{e}_2^{(m)}) + \operatorname{sgn}(e_1^{(m)}, \tilde{\tilde{e}}_2^{(m)}) f_2^{(m)}(\tilde{\tilde{e}}_2^{(m)}) \right) & \text{if } e_1^{(m)} \text{ bounds } \tilde{e}_2^{(m)} \text{ and } \tilde{\tilde{e}}_2^{(m)}. \end{cases} \quad (24)$$

Let $N(e_2^{(m)})$ denote the number of squares adjacent to $e_2^{(m)}$ in Γ_m (2, 3, or 4). Then

$$\begin{aligned} -\Delta_2^{(m)} f_2^{(m)}(e_2^{(m)}) &= d_1^{(m)} \delta_2^{(m)} f_2^{(m)}(e_2^{(m)}) \\ &= 4 \left(4 - N(e_2^{(m)}) \right) f_2^{(m)}(e_2^{(m)}) + 2 \sum_{\tilde{e}_2^{(m)} \sim e_2^{(m)}} \left(f_2^{(m)}(e_2^{(m)}) - f_2^{(m)}(\tilde{e}_2^{(m)}) \right) \end{aligned} \quad (25)$$

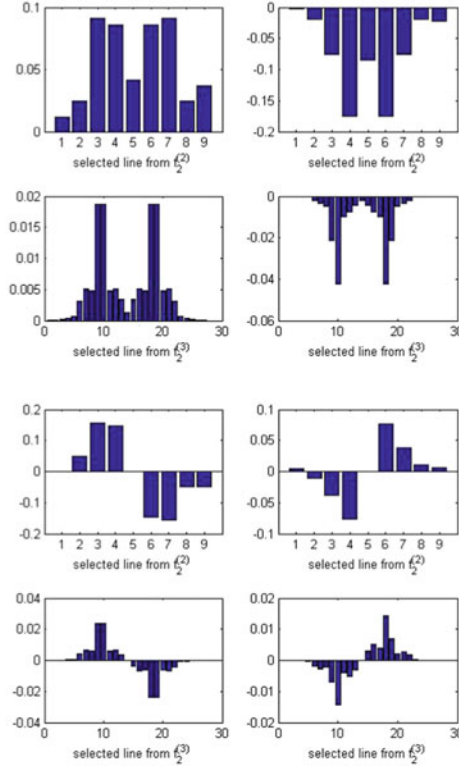


Fig. 11 Restrictions of $\delta_2^{(m)} f_2^{(m)}$ to L

Table 5 Total variation of $d_0^{(m)} f_0^{(m)}$ and $\delta_2^{(m)} f_2^{(m)}$ along horizontal lines at level 2 and level 3

D		Level 1	Level 2	Level 3	DEL		Level 1	Level 2	Level 3
Line 1	Form 1	0.8845	0.8845	0.8845	Line 1	Form 1	0.1319	0.1319	0.1319
	Form 2	0.5010	0.6201	0.6348		Form 2	0.0663	0.1232	0.1289
	Form 3	0.6758	0.8093	0.8203		Form 3	0.1701	0.1701	0.1701
	Form 4	0.7538	0.7538	0.7826		Form 4	0.0849	0.1584	0.1660
Line 2	Form 1	0.4588	0.4588	0.4588	Line 2	Form 1	0.1960	0.1960	0.1960
	Form 2	0.5010	0.7970	0.8178		Form 2	0.0658	0.1849	0.19
	Form 3	0.3688	0.4376	0.4484		Form 3	0.0861	0.0861	0.0861
	Form 4	0.5387	0.5387	0.5454		Form 4	0.0284	0.0754	0.0819
Line 3	Form 1	0.4588	0.4588	0.4588	Line 3	Form 1	0.1960	0.1960	0.1960
	Form 2	0.5010	0.7970	0.8178		Form 2	0.0658	0.1849	0.19
	Form 3	0.3688	0.4376	0.4484		Form 3	0.0861	0.0861	0.0861
	Form 4	0.5387	0.5387	0.5454		Form 4	0.0284	0.0754	0.0819
Line 4	Form 1	0.8845	0.8845	0.8845	Line 4	Form 1	0.1319	0.1319	0.1319
	Form 2	0.5010	0.6201	0.6348		Form 2	0.0663	0.1232	0.1289
	Form 3	0.6758	0.8093	0.8203		Form 3	0.1701	0.1701	0.1701
	Form 4	0.7538	0.7538	0.7826		Form 4	0.0849	0.1584	0.1660

Note that

$$-D_2^{(m)} f_2^{(m)}(e_2^{(m)}) = \sum_{\tilde{e}_2^{(m)} \sim e_2^{(m)}} \left(f_2^{(m)}(e_2^{(m)}) - f_2^{(m)}(\tilde{e}_2^{(m)}) \right)$$

is exactly the graph Laplacian for the cell graph on level m of SC. This Laplacian was first studied in [19], and it was proved in [5] that when appropriately normalized it converges to the essentially unique self-similar Laplacian on SC. This was used as the basis for extensive numerical investigations in [6]. In other words, if $f_2^{(m)}(e_2^{(m)}) = \int_{e_2^{(m)}} f_0 d\mu$ for some function f_0 in the domain of the Laplacian on SC, then $\lim_{m \rightarrow \infty} r^m D_2^{(m)} f_2^{(m)} = c(\Delta f_0) d\mu$ for $r \approx 10.01$.

However, according to (25) we have

$$-\Delta_2^{(m)} = M^{(m)} - D_2^{(m)}, \quad (26)$$

where $M^{(m)}$ is the operator of multiplication by the function ϕ_m given by

$$\phi_m = 4 \left(4 - N(e_2^{(m)}) \right).$$

Note that ϕ_m takes on values 0, 4, 8. If we were to renormalize $M^{(m)}$ by multiplying by r^m the result would surely diverge. In other words, if there is any hope of obtaining a limit for a class of measures, it would have to be $\lim_{m \rightarrow \infty} (-\Delta_2^{(m)})$ without renormalization. Of course the sequence of functions $\{\phi_m\}$ does not converge, so it seems unlikely that we could make sense of $\lim_{m \rightarrow \infty} (M^{(m)})$. Thus, although $M^{(m)}$ is clearly the major contributor to the sum (26), both operators must play a role if the limit is to exist.

We compute the distribution of the three values of $N(e_2^{(m)})$ as $e_2^{(m)}$ varies over the 8^m squares of level m . Let $n_3^{(m)}$ and $n_4^{(m)}$ denote the number of squares with N value 3 and 4, and $n_{2a}^{(m)}$ and $n_{2b}^{(m)}$ denote the number with $N = 2$, with neighbors on opposite sides ($n_{2a}^{(m)}$) or adjacent sides ($n_{2b}^{(m)}$). In fact, $n_{2b}^{(m)} = 4$ since this case only occurs at the four corners of SC. If $e_2^{(m-1)}$ is in one of those cases we may compute the N values on the eight subsquares as shown in Fig. 12.

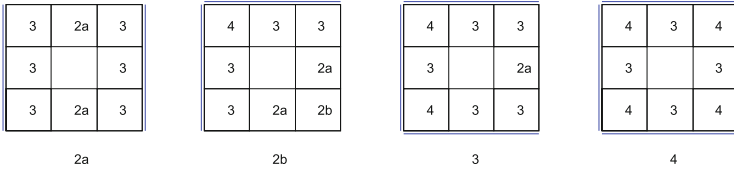


Fig. 12 Values of N on subsquares. The neighboring edges are marked by a *double line*

This gives us the recursion relation

$$\begin{pmatrix} n_{2a}^{(m)} \\ n_{2b}^{(m)} \\ n_3^{(m)} \\ n_4^{(m)} \end{pmatrix} = A \begin{pmatrix} n_{2a}^{(m-1)} \\ n_{2b}^{(m-1)} \\ n_3^{(m-1)} \\ n_4^{(m-1)} \end{pmatrix} \text{ for } A = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 4 & 5 & 4 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$

and so

$$\begin{pmatrix} n_{2a}^{(m)} \\ n_{2b}^{(m)} \\ n_3^{(m)} \\ n_4^{(m)} \end{pmatrix} = A^{m-1} \begin{pmatrix} 4 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

We also note that (1111) is a left eigenvector of A with eigenvalue 8 (in other words column sums are 8), and this implies

$$n_{2a}^{(m)} + n_{2b}^{(m)} + n_3^{(m)} + n_4^{(m)} = 8^m$$

as required. We also observe that $(1063)^T$ is the right eigenvector with eigenvalue 8, so asymptotically

$$\begin{pmatrix} n_{2a}^{(m)} \\ n_{2b}^{(m)} \\ n_3^{(m)} \\ n_4^{(m)} \end{pmatrix} \sim 8^m \begin{pmatrix} .1 \\ 0 \\ .6 \\ .3 \end{pmatrix} \text{ as } m \rightarrow \infty.$$

Thus the operator $M^{(m)}$ has eigenvalues 0, 4, 8 with multiplicities approximately $\frac{3}{10}8^m$, $\frac{6}{10}8^m$, $\frac{1}{10}8^m$.

The spectrum of the operator $-\Delta_2^{(m)}$ is quite different. Table 6 shows the entire spectrum for $m = 1, 2$ and the beginning of the spectrum for $m = 3, 4$.

In Fig. 13 we give a graphical display of these spectra. In Fig. 14 we show the graphs of some of the early eigenfunction on levels 2, 3, 4.

The data suggests that there may be a limit

$$-\Delta_2 = \lim_{m \rightarrow \infty} (-\Delta_2^{(m)}) \quad (27)$$

for a class of measures, perhaps $L^2(d\mu)$. The limit operator would be a bounded, self-adjoint operator that is bounded away from zero, hence invertible. The spectrum would be continuous (or a mix of discrete and continuous) with support on a Cantor set.

We can give some explanations as to why we might expect the following types of limits:

$$\delta_2 = \lim_{m \rightarrow \infty} \left(\frac{8}{3}\right)^m \delta_2^{(m)} \quad (28)$$

Table 6 Eigenvalues of $-\Delta_2^{(m)}$ for $m=1, 2, 3, 4$

Number of eigenvalues	$m = 1$	$m = 2$	$m = 3$	$m = 4$
1	2.0000	0.9248	0.5101	0.4505
2	2.2929	0.9497	0.5102	0.4505
3	2.2929	0.9497	0.5102	0.4505
4	3.0000	0.9789	0.5104	0.4505
5	3.0000	1.3068	0.5290	0.4511
6	3.7071	1.3506	0.5291	0.4511
7	3.7071	1.3506	0.5291	0.4511
8	4.0000	1.3989	0.5292	0.4511
9		1.5764	0.61	0.4604
10		1.6355	0.64	0.4604
11		1.6355	0.64	0.4604
12		1.7134	0.67	0.4604
13		1.8573	0.6554	0.4614
14		1.9559	0.6557	0.4614
15		1.9559	0.6557	0.4614
16		2.0000	0.6561	0.4614
17		2.0593	0.9248	0.5101
18		2.0593	0.9280	0.5102
19		2.0786	0.9280	0.5102
20		2.1187	0.9332	0.5102
21		2.1910	0.9395	0.5102
22		2.2301	0.9443	0.5102
23		2.2929	0.9450	0.5102
24		2.2929	0.9450	0.5102
		2.3272	0.9497	0.5102
26		2.3272	0.9497	0.5102
27		2.3990	0.9527	0.5103
28		2.4422	0.9544	0.5103
29		2.5000	0.9623	0.5103
30		2.5000	0.9690	0.5103
31		2.5391	0.9690	0.5103
32		2.5391	0.9755	0.5104

Table 7 Values of $\|f_j^{(m)} - f_j^{(m-1)}\|_{E_2^{(m-1)}}^2$, from level 3 to level 2 and level 4 to level 3 for eigenspaces of multiplicity one

number	$m = 2$	$m = 3$	$m = 4$	3to2	4to3
1	0.9248	0.5101	0.4505	0.0726	0.0196
4	0.9789	0.5104	0.4505	0.0532	0.0190
5	1.3068	0.5290	0.4511	0.11	0.0114
8	1.3989	0.5292	0.4511	0.0670	0.0106
9	1.5764	0.61	0.4604	0.1709	0.0334
12	1.7134	0.67	0.4604	0.1591	0.0317
13	1.8573	0.6554	0.4614	0.2796	0.0243

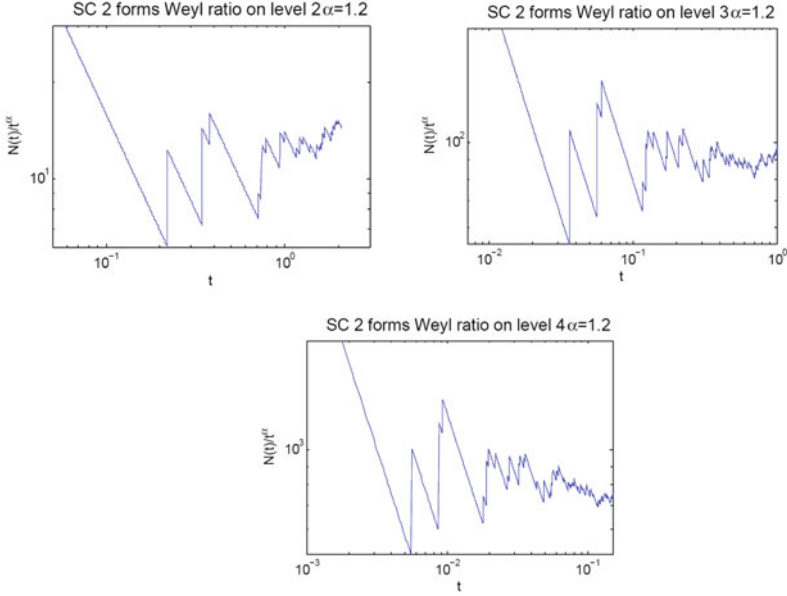


Fig. 13 Weyl ratio for spectra of SC 2 forms with $\alpha=1.2$

$$d_1 = \lim_{m \rightarrow \infty} \left(\frac{3}{8} \right)^m d_1^{(m)}$$

which are consistent with (27). Suppose $f_2 = f d\mu$ for a reasonable function f , and define $f_2^{(m)}(e_2^{(m)}) = \int_{e_2^{(m)}} f d\mu$. Then $f_2^{(m)}(e_2^{(m)})$ is on the order of 8^{-m} . In the definition of $\delta_2^{(m)} f_2^{(m)}$ in (24) we note that when $e_1^{(m)}$ bounds only one square, $\delta_2^{(m)} f_2^{(m)}(e_1^{(m)})$ is also on the order of 8^{-m} , so multiplying by $(8/3)^m$ gives a value on the order of 3^{-m} , which is reasonable for a measure on a line segment L containing $e_1^{(m)}$. On the other hand, if $e_1^{(m)}$ bounds two squares, then the sgn function has opposite signs so $(8/3)^m \delta_2^{(m)} f_2^{(m)}(e_1^{(m)})$ is close to zero. If we assume the function f is continuous then the limit in (28) will give a measure on L equal to $4f|_L dt$ on the portion of L with squares on only one side, and zero on the portion of L with squares on both sides.

Next suppose the f_1 is a measure on each line L in SC that has

$$|f_1(e_1^{(m)})| \leq c3^{-m}. \quad (29)$$

Fix a square $e_2^{(m)}$ on level n , and write it as a union of 8^{m-n} squares on level m . We want to define

$$d_1 f_1(e_2^{(n)}) = \lim_{m \rightarrow \infty} \left(\frac{3}{8} \right)^m \sum_{e_2^{(m)} \subseteq e_2^{(n)}} d_1^{(m)} f_1^{(m)}(e_2^{(m)}). \quad (30)$$

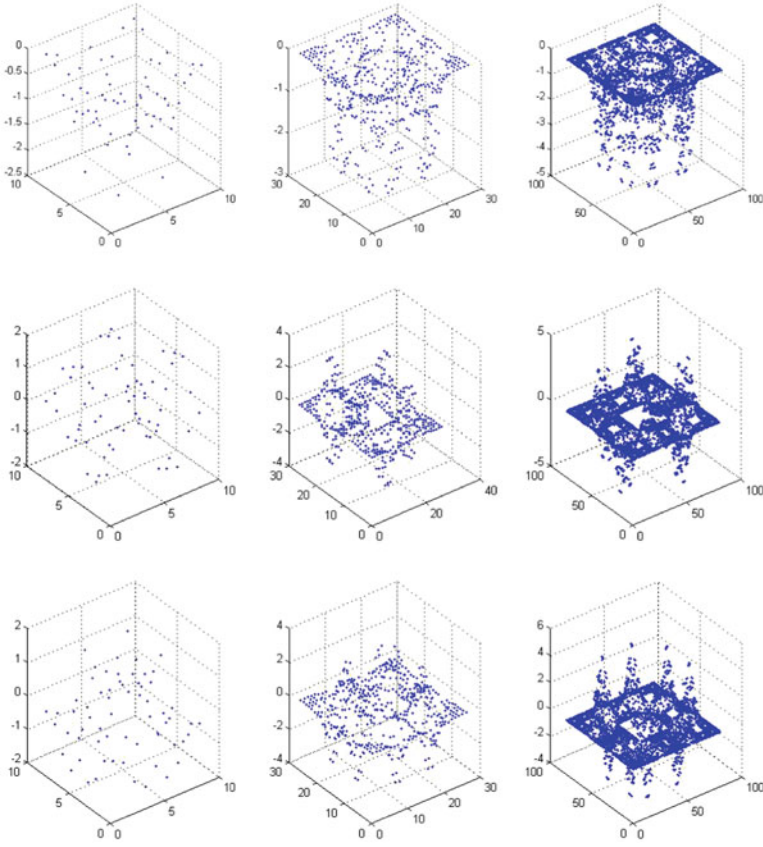


Fig. 14 Graphs of early eigenfunctions of SC 2 forms on level 2, 3, 4

Does this make sense? Because of the cancellation from the *sgn* function on opposite sides of an edge, $\sum_{e_2^{(m)} \subseteq e_2^{(n)}} d_1^{(m)} f_1^{(m)}(e_2^{(m)})$ is just the measure of the boundary of $e_2^{(n)}$ decomposed to level m . This boundary is the union of the four edges $e_1^{(n)}$ on the outside of the square, the four edges around the inner deleted square on level $n+1$, and in general the $4 \cdot 8^{k-1}$ edges around the 8^k deleted squares on level $n+k$, for $k \leq m-n$. (see Fig. 15 for $m = n+2$) Using the estimate (29) we have

$$\left| \left(\frac{3}{8} \right)^m \sum_{e_2^{(m)} \subseteq e_2^{(n)}} d_1^{(m)} f_1^{(m)}(e_2^{(m)}) \right| \leq 4c \left(\frac{3}{8} \right)^m \left(\frac{1}{3^n} + \frac{8}{3^{n+1}} + \frac{8^2}{3^{n+2}} + \dots + \frac{8^m}{3^m} \right) \leq c.$$

Thus the terms on the right side of (30) are uniformly bounded, so it is not unreasonable to hope that the limit exists.

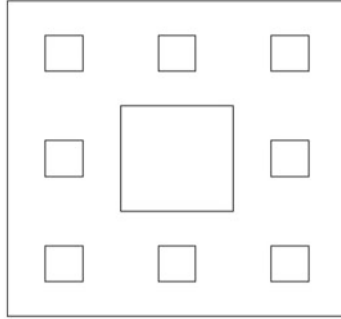


Fig. 15 Edges in sum for $m = n + 2$

5 0-Forms and 2-Forms on the Magic Carpet

The vertices $\tilde{E}_0^{(m)}$ of $\tilde{\Gamma}_m$ split into singular $\tilde{E}_{0s}^{(m)}$ and nonsingular vertices $\tilde{E}_{0n}^{(m)}$. Let $V_m = \#\tilde{E}_0^{(m)}$, $S_m = \#\tilde{E}_{0s}^{(m)}$, and $N_m = \#\tilde{E}_{0n}^{(m)}$. Then $S_m = 1 + 8 + 8^2 + \dots + 8^{m-1} = (8^m - 1)/7$, since each time we remove a square and identify its boundaries we create a single singular vertex. To compute the other two counts, we note that each of the 8^m squares has 4 vertices, and singular vertices arise in 12 different ways, while nonsingular vertices arise in 4 different ways. Thus

$$12S_m + 4N_m = 8^m$$

Solving for N_m we obtain

$$N_m = \frac{4 \cdot 8^m + 3}{7}, V_m = \frac{5 \cdot 8^m + 2}{7}.$$

Asymptotically, one-fifth of all vertices are singular.

The Laplacian $-\tilde{\Delta}_0^{(m)}$ given by (1) has V_m eigenvalues, starting at $\lambda_0 = 0$ corresponding to the constants. In Table 8 we give the beginning of the spectrum for $m = 1, 2, 3, 4$ along with the ratios from levels 3 to 2 and 4 to 3.

We note that the ratios are around $r = 6. \dots$, so we expect

$$-\tilde{\Delta}_0 = \lim_{m \rightarrow \infty} r^m \left(-\tilde{\Delta}_0^{(m)} \right)$$

to define a Laplacian on MC. At present there is no proof that MC has a Laplacian, so our data is strong experimental evidence that $-\tilde{\Delta}_0$ exists. What is striking is that

Table 8 Beginning of the spectrum of $-\tilde{\Delta}_0^{(m)}$ and ratios

λ_1	λ_2	λ_3	λ_4	λ_2/λ_3	λ_3/λ_4
0.0000	0.0000	0.0000	0.0000		
1.5000	0.2877	0.0458	0.0071	6.2809	6.4099
1.7047	0.4458	0.0734	0.0114	6.0699	6.4402
2.5000	0.4458	0.0734	0.0114	6.0699	6.4402
2.5000	0.4496	0.0764	0.0120	5.8860	6.3642
3.1287	0.8333	0.1449	0.0230	5.7521	6.2957
	0.8595	0.1538	0.0244	5.5870	6.2971
	0.8595	0.1538	0.0244	5.5870	6.2971
	0.9862	0.1645	0.07	5.9938	6.4006
	1.0967	0.1854	0.0289	5.9150	6.4218
	1.1838	0.2175	0.0346	5.4441	6.2913
	1.3014	0.2386	0.0372	5.4555	6.4138
	1.3014	0.2386	0.0372	5.4555	6.4138
	1.5000	0.2684	0.0420	5.5881	6.3862
	1.5000	0.2684	0.0420	5.5881	6.3862
	1.5000	0.2701	0.0428	5.5541	6.3056
	1.5793	0.2849	0.0454	5.5438	6.2771
	1.5793	0.2877	0.0458	5.4898	6.2809
	1.6912	0.3407	0.0552	4.9636	6.1695
	1.7047	0.3869	0.0626	4.4058	6.1772
	1.8770	0.3922	0.0633	4.7861	6.1940
	1.8913	0.3922	0.0633	4.8228	6.1940
	1.9661	0.4169	0.0685	4.7164	6.0849
	1.9661	0.43	0.0705	4.5462	6.1319

$r < 8$, in contrast to the factor of ≈ 10.01 for SC. Since the measure renormalization factor is 8^m for both carpets, we conclude that the energy renormalization factor for MC would have to be less than one. In Euclidean spaces or manifolds, this happens in dimensions greater than two. Note that the unrenormalized energy on level m would be

$$\tilde{E}^{(m)}(f_0) = \frac{1}{2 * 8^m} \sum_{e_1^{(m)} \in \tilde{E}_1^{(m)}} |d_0 f_0(e_1^{(m)})|^2,$$

so $\tilde{\mathcal{E}}^{(m)} = r^m \tilde{E}^{(m)}$ means that the graph energy $\sum_{e_1^{(m)} \in \tilde{E}_1^{(m)}} |d_0 f_0(e_1^{(m)})|^2$ is multiplied

by $(r/8)^m$ before taking the limit.

In Fig. 16 we show the graphs of selected eigenfunctions on levels 2, 3, 4. Again we quantify the rate of convergence as in the case of SC by giving in Table 9 the values of $\|f_j^{(m)}|_{E_0^{(m-1)}} - f_j^{(m-1)}\|_2^2$.

We give similar data for the spectrum of $-\tilde{\Delta}_2^{(m)}$ defined by (12). In order to make the comparison with Table 8 clear, we give Table 10 the eigenvalues of $-\tilde{\Delta}_2^{(m)}$ multiplied by 0.3221 and ratios. In Table 11 we show quantitative rates of convergence.

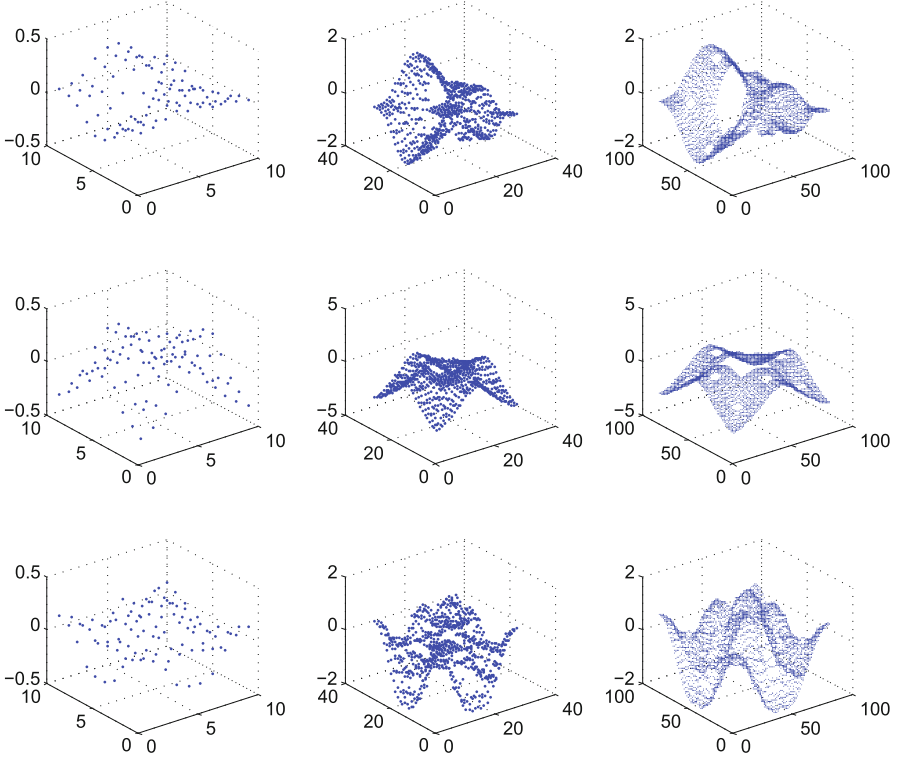


Fig. 16 Graphs of selected eigenfunctions of MC 0 forms on levels 2, 3, 4

Table 9 Values of $\|f_j^{(m)}|_{E_0^{(m-1)}} - f_j^{(m-1)}\|_2^2$

number of eigenvalues	$m = 2$	$m = 3$	$m = 4$	3 to 2	4 to 3
1	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.2877	0.0458	0.0071	0.0000	0.0000
5	0.4496	0.0764	0.0120	0.0001	0.0000
6	0.8333	0.1449	0.0230	0.0002	0.0000
9	0.9862	0.1645	0.07	0.0004	0.0000
10	1.0967	0.1854	0.0289	0.0008	0.0000

In Fig. 17 we show graphs of eigenfunctions. In Fig. 18 we show graphs of $*f_0^{(m)}$ when $f_0^{(m)}$ is an eigenfunction in Fig. 16, and in Fig. 19 we show graphs of $*f_2^{(m)}$ when $f_2^{(m)}$ is an eigenfunction in Fig. 17. In Fig. 20 and 21, we give the Weyl ratios of the eigenvalues of the 0 forms and 2 forms.

Because miniaturization holds we expect a value $\alpha = \frac{\log 8}{\log r} \approx \frac{\log 8}{\log 6}$. We have renormalized the eigenvalues as $\{r^m \lambda_j^{(m)}\}$, with the expectation that in the limit as $m \rightarrow \infty$ we obtain eigenvalues of $-\tilde{\Delta}_0$ and $-\tilde{\Delta}_2$ on MC.

Table 10 Eigenvalues of $-\tilde{\Delta}_2^{(m)}$ multiplied by 0.3221 and ratios

$m = 1$	$m = 2$	$m = 3$	$m = 4$	λ_2/λ_3	λ_3/λ_4
0.0000	0.0000	0.0000	0.0000		
1.2885	0.2841	0.0463	0.0071	6.1409	6.4749
2.1054	0.4449	0.0721	0.0112	6.1710	6.4514
2.1054	0.4449	0.0721	0.0112	6.1710	6.4514
2.5770	0.4889	0.0776	0.0119	6.2970	6.5040
3.8655	0.9193	0.1504	0.0232	6.1139	6.4904
4.3371	0.9553	0.1580	0.0244	6.0455	6.4640
4.3371	0.9553	0.1580	0.0244	6.0455	6.4640
	0.9653	0.1623	0.03	5.9494	6.4158
	1.0110	0.1792	0.0283	5.6430	6.3393
	1.2077	0.2192	0.0345	5.5097	6.3554
	1.2135	0.2226	0.0356	5.4505	6.32
	1.2135	0.2226	0.0356	5.4505	6.32
	1.2885	0.2679	0.0418	4.8091	6.4058
	1.5007	0.2679	0.0418	5.6010	6.4058
	1.5562	0.2699	0.04	5.7662	6.3573
	1.5562	0.2841	0.0452	5.4771	6.2923
	1.6334	0.2878	0.0463	5.6757	6.2202
	1.7959	0.3541	0.0560	5.0715	6.3241
	1.9528	0.3946	0.0628	4.9493	6.2829
	1.9528	0.3969	0.0630	4.9204	6.3040
	2.0288	0.3969	0.0630	5.1118	6.3040
	2.0840	0.4358	0.0693	4.78	6.2886
	2.1054	0.4410	0.0712	4.7737	6.1952
	2.1054	0.4410	0.0712	4.7737	6.1952
	2.1700	0.4449	0.0721	4.8774	6.1710
	2.1700	0.4449	0.0721	4.8774	6.1710
	2.3521	0.4529	0.0728	5.1929	6.2232
	2.3579	0.4847	0.0770	4.8648	6.2915
	2.4038	0.4889	0.0776	4.9172	6.2970
	2.4038	0.4971	0.0796	4.8356	6.2415
	2.4760	0.4971	0.0796	4.9807	6.2415
	2.4760	0.5364	0.0870	4.6156	6.1657

Table 11 Values of $\|f_j^{(m)}|_{E_2^{(m-1)}} - f_j^{(m-1)}\|_2^2$

Number of eigenvalues	$m = 2$	$m = 3$	$m = 4$	3 to 2	4 to 3
1	0	0	0	0	0
2	0.2205	0.0359	0.0055	0.0006	0.0001
5	0.3794	0.0602	0.0093	0.0012	0.0002
6	0.7135	0.1167	0.0180	0.0051	0.0003
9	0.7492	0.19	0.0196	0.0056	0.0004
10	0.7846	0.1391	0.0219	0.0018	0.0004
11	0.9373	0.1701	0.0268	0.0350	0.0011

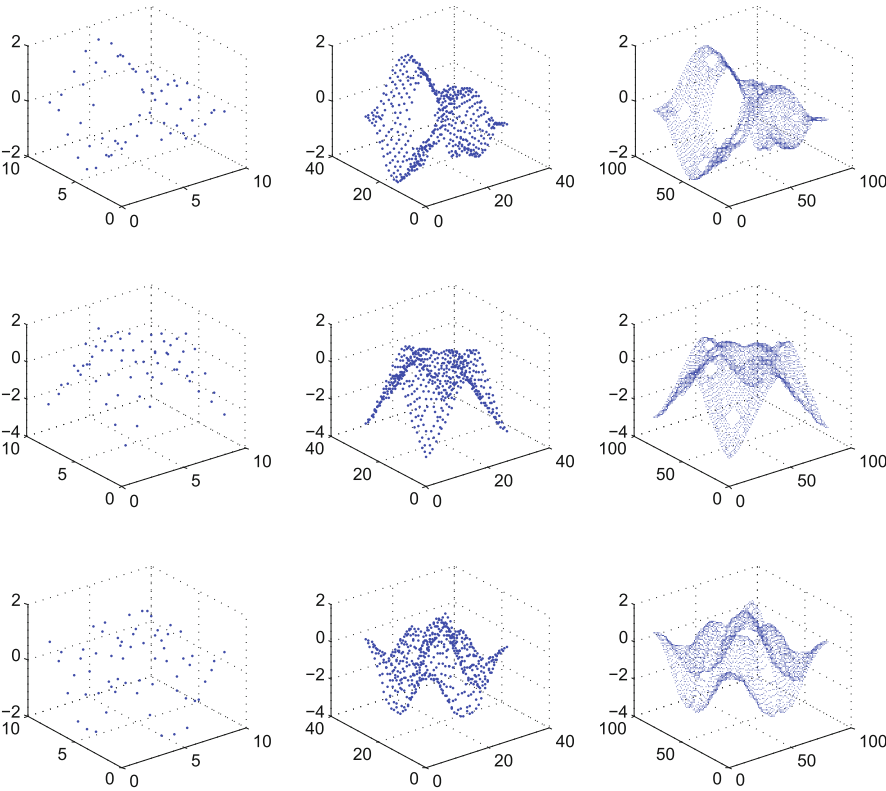


Fig. 17 Graphs of selected eigenfunctions of MC 2 forms on levels 2, 3, 4

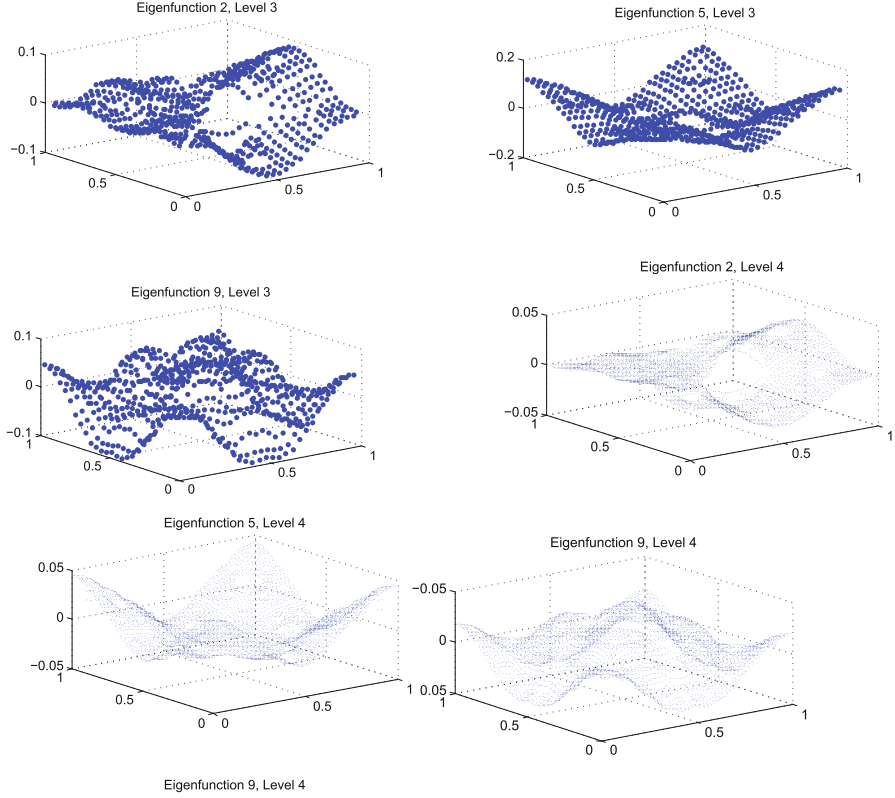


Fig. 18 Graphs of $*f_0^{(m)}$ when $f_0^{(m)}$ is an eigenfunction in Fig. 16

6 1-Forms on the Magic Carpet

We indicate briefly how the theory differs from SC. We have the analogs of (17), (18) and (19), but the dimension count is different because $\tilde{\delta}_2^{(m)}$ also has a 1-dimensional kernel, namely the constants. There are exactly $2 \cdot 8^m$ edges, since each edge is the boundary of exactly two squares, so

$$\dim \mathcal{H}_1^{(m)} = 2 \cdot 8^m - 8^m - \frac{5 \cdot 8^m + 1}{7} + 2 = \frac{2 \cdot 8^m + 12}{7}.$$

On the other hand, the surface approximating MC on level m has genus $g = (8^m + 6)/7$, and the cycles generating the homology are in one-to-one correspondence with the horizontal and vertical identified edges. The analog of Conjecture 3.1 is true, in fact it is a well-known result in topology.

In Fig. 22, 23, and 24 we show the values of $h_1^{(2)}$, $h_1^{(3)}$ and the restriction of $h_1^{(3)}$ to $\tilde{E}_1^{(2)}$ where γ_1 is the top and bottom horizontal line. There are some surprising

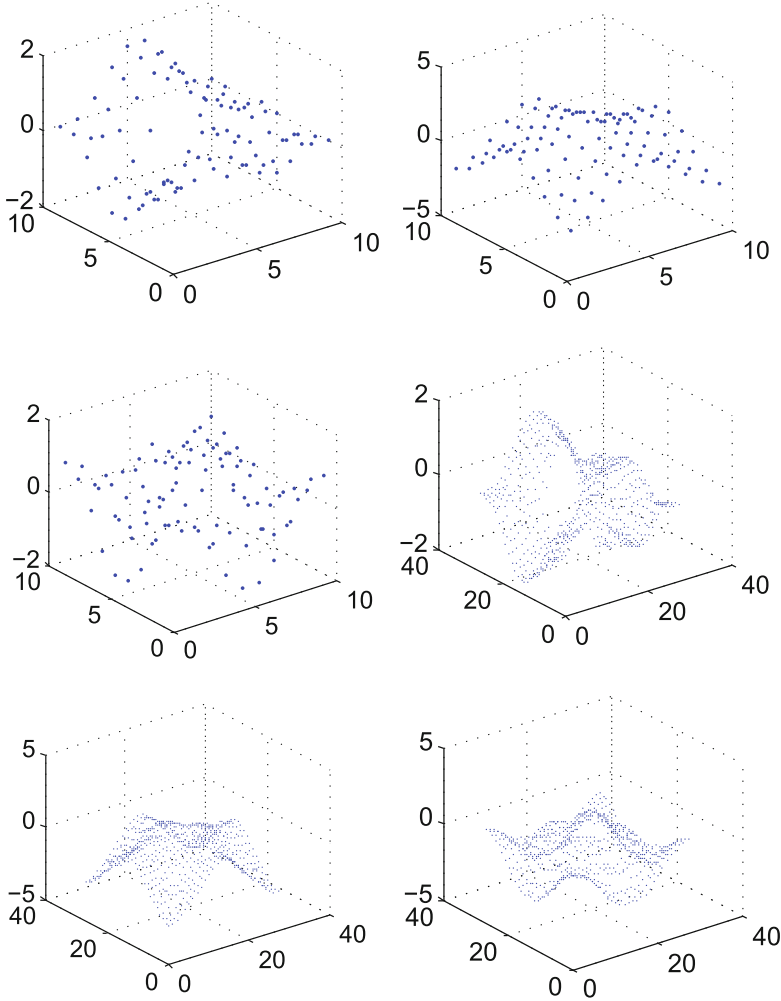
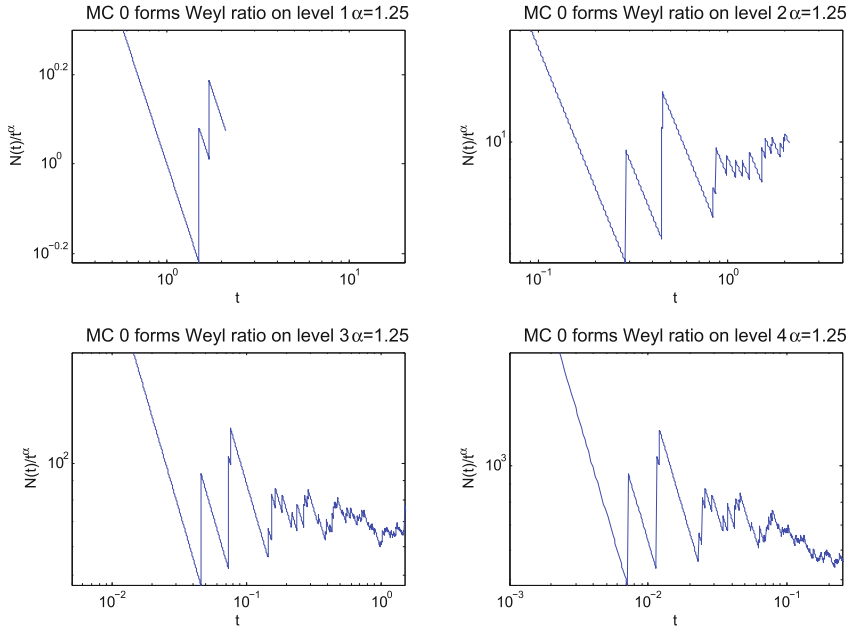
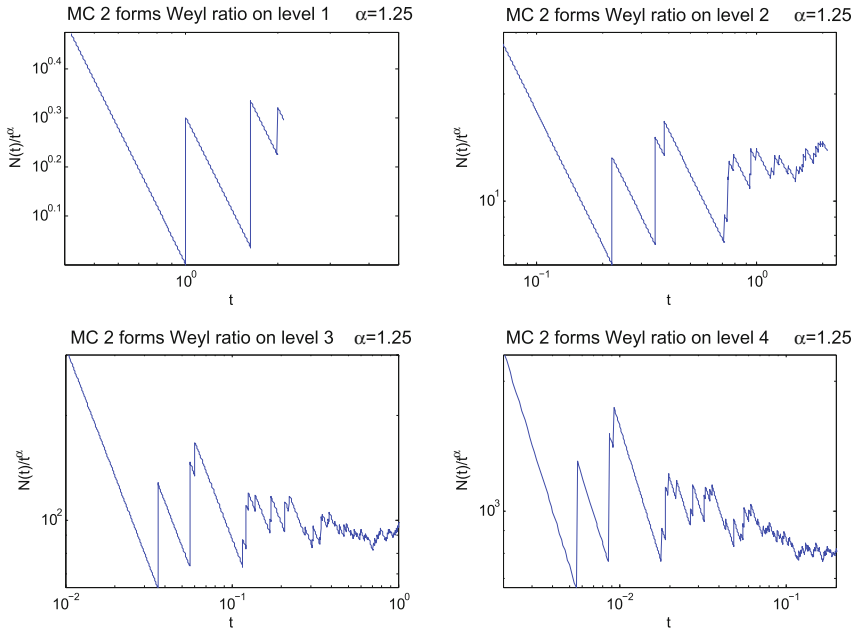


Fig. 19 Graphs of $*f_2^{(m)}$ when $f_2^{(m)}$ is an eigenfunction in Fig. 17

features of these 1-forms that may be explained by symmetry. Let R_H denote the horizontal reflection and R_V the vertical reflection about the center. Note that $R_H\gamma_1 = -\gamma_1$ because the orientation is reversed, while $R_V\gamma_1 = \gamma_1$. Since $-h_1^{(m)}(R_Hx)$ and $h_1^{(m)}(R_Vx)$ are harmonic 1-forms with the same integrals around cycles as $h_1^{(m)}$, it follows by uniqueness that $-h_1^{(m)}(R_Hx) = h_1^{(m)}(x)$ and $h_1^{(m)}(R_Vx) = h_1^{(m)}(x)$. This implies that $h_1^{(m)}$ vanishes identically along the cycle consisting of the vertical edges of the large square, and indeed any square that is symmetric with respect to R_H . Certain other vanishings of $h_1^{(m)}$ are accidental to the level m , and do not persist

**Fig. 20** Weyl ratio of MC 0 forms eigenvalues**Fig. 21** Weyl ratio of MC 2 forms eigenvalues

when m increases. For example, every cycle of level m contains just a single edge, so vanishing of the integral forces vanishing on the edge. Sometimes this has a ripple effect. For example, consider the region in the top center of the level 2 graph shown in Fig. 25, where the horizontal symmetry has been used in labeling edges. We obtain the equation $2a + b = 0$ from the equation $\tilde{d}_1^{(2)} \tilde{h}_1^{(2)} = 0$ on the small square, and the equation $2c + b = 0$ from $\int_\gamma \tilde{h}_1^{(2)} = 0$ on the cycle along the top of the big square. Finally, the equation $a + c = 0$ comes from $\tilde{\delta}_1^{(2)} \tilde{h}_1^{(2)} = 0$ at the indicated point. These yield $a = b = c = 0$ that we see in Fig. 22, 23 and 24.

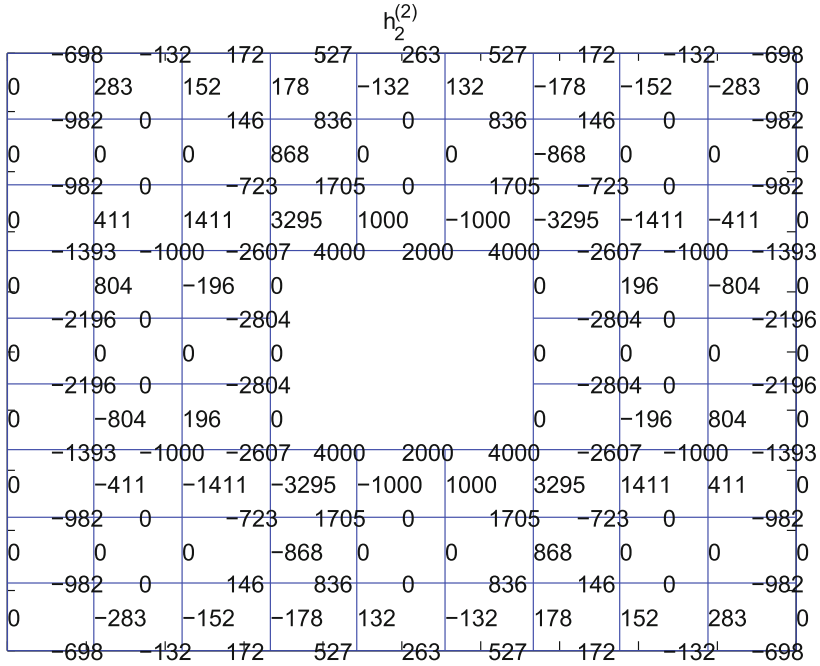


Fig. 22 Values of $h_2^{(2)}$

To quantify the rate of convergence we give in Table 12 the values of $\|h_k^{(m)} - Rh_k^{(m)}\|_2$ analogous to Table 3.

7 Discussion

We consider first SC. In Sect. 1 we defined the operators $d_0^{(m)}, d_1^{(m)}, \delta_1^{(m)}, \delta_2^{(m)}, \Delta_0^{(m)}, \Delta_1^{(m)}, \Delta_2^{(m)}$ on the graphs Γ_m approximating SC. In light of the computational results

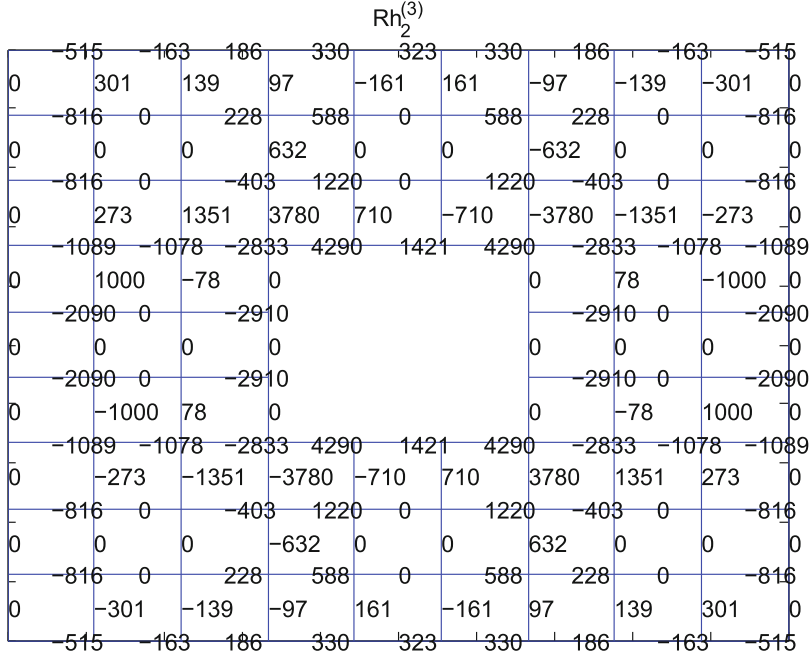


Fig. 23 The restriction of $Rh_2^{(3)}$ to $\tilde{E}_1^{(1)}$ where γ_1 is the *top* and *bottom* horizontal line

reported in Sects. 2, 3, and 4 we may speculate on how it may be possible to pass to the limit as $m \rightarrow \infty$ to obtain corresponding operators on SC.

The simplest case to consider is d_0 . We note that (2) is consistent from level to level without renormalization. In other words, if f_0 is any continuous function on SC whose restriction to any line segment in SC is of bounded variation, then

$$d_0 f_0(L_{ab}) = f_0(b) - f_0(a), \quad (31)$$

where L_{ab} is a horizontal or vertical line segment joining a to b , defines a measure on all such line segments with

$$d_0 f_0(e_1^{(m)}) = d_0^{(m)} f_0^{(m)}(e_1^{(m)}) \quad (32)$$

where $f_0^{(m)} = f_0$ restricted to $E_0^{(m)}$. This does not settle the question of what is the most suitable domain for the space of 0-forms f_0 , and then what the corresponding space of 1-forms will be the range under d_0 .

The renormalization of $\Delta_0^{(m)}$ suggested in (16) requires that we renormalize $\delta_1^{(m)}$ also by the factor of r^m , and so we would like to define

$$\delta_1 f_1(x) = \lim_{m \rightarrow \infty} r^m \delta_1^{(m)} f_1^{(m)}(x) \quad (33)$$

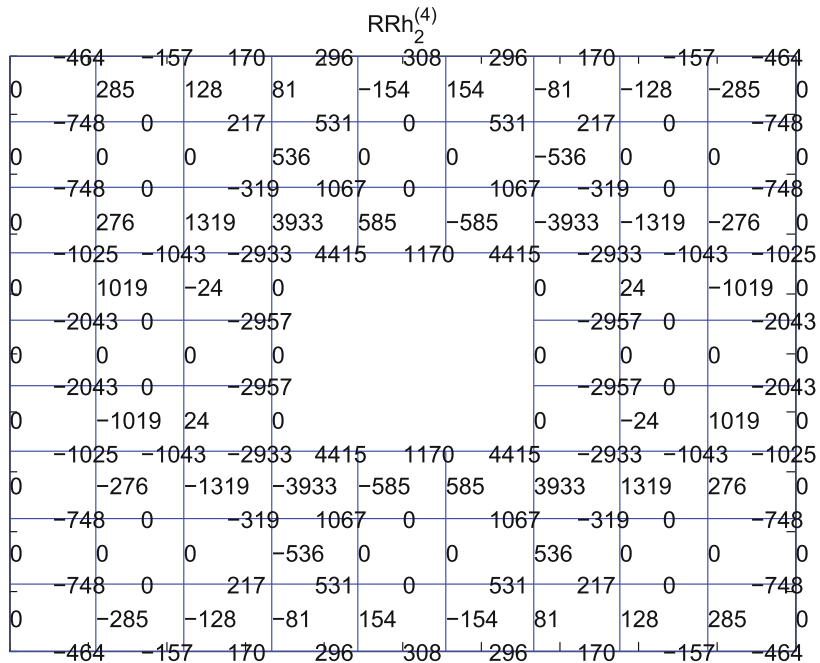


Fig. 24 The restriction of $RRh_2^{(4)}$ to $\tilde{E}_1^{(1)}$ where γ_1 is the *top* and *bottom* horizontal line

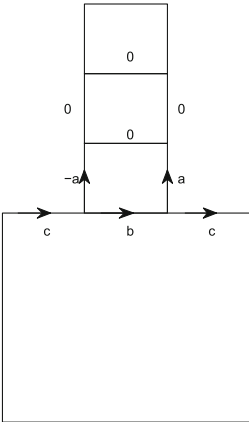


Fig. 25 A figure of the region in the top center of the level 2 graph where the horizontal symmetry has been used in labeling edges

Table 12 Values of $\|h_k^{(m)} - Rh_k^{(m)}\|_2$

Number	1	2	3	4	5	6	7	8	9	10
$\ h_k^{(2)} - Rh_k^{(1)}\ $	0.1549	0.4647								
$\ h_k^{(3)} - Rh_k^{(2)}\ $	0.1476	0.1581	0.2419	0.2039	0.2419	0.2669	0.2669	0.2419	0.2039	0.2419
$\ h_k^{(4)} - Rh_k^{(3)}\ $	0.1485	0.1349	0.1452	0.1468	0.1452	0.1458	0.1458	0.1452	0.1468	0.1452

for $x \in \bigcup_m E_0^{(m)}$, with $f_1^{(m)} = f_1$ restricted to $E_1^{(m)}$ and $\delta_1^{(m)}$ given by To make sense of this we need to find a precise domain for the 1-forms f_1 so that the limit exists and is a continuous function on the dense set $\bigcup_m E_0^{(m)}$, so that $\delta_1 f_1$ extends to a function on SC. Then we would want to characterize the range of δ_1 . We would specifically like to have the domain of δ_1 contain the range of d_0 , so that

$$\Delta_0 = \delta_1 d_0$$

might be defined directly without limits. The results of [5] would then imply that this Laplacian must agree, up to a constant multiple, with any other symmetric Laplacian defined on functions on SC.

The numerical evidence reported in Sect. 2 gives very strong support for the existence of $-\Delta_0$ as the limit in (16). The spectrum behaves well, and Table 2 indicates convergence of the eigenfunctions (note that we only computed changes in eigenfunctions coming from eigenvalues of multiplicity one because it is easier to make sure we are comparing the same eigenfunctions; once we normalize the eigenfunction to have L^2 norm equal to one, it is uniquely determined up to a $\pm$ sign). So we have strong evidence that the limit in (33) exists when $f_1 = d_1 f_0$ for f_0 an eigenfunction, and hence any linear combination of eigenfunctions. This does not, of course, give any “generic” space of 1-forms for which $\delta_1 f_1$ exists. Because the factors of 8^m in the definitions 6 and 5 of the weights in (7), only the sizes of $f_1^{(m)}(e_1^{(m)})$ can create decay, and under the assumption that the measures comprising f_1 are absolutely continuous, this would only create a decay rate of 3^{-m} , not nearly enough to compensate for r^m in (33). In other words, it is only cancellation of positive and negative terms that can lead to the existence of the limit in (33).

The situation for 2-forms is discussed in Sect. 4 The evidence is not as clearcut, but the suggestion is that $-\Delta_2$ may exist as a nonrenormalized limit, while δ_2 and d_1 require renormalization factors $8/3$ and $3/8$. We would then still have $d_1 d_0 = 0$ and $\delta_1 \delta_0 = 0$ because (4) and (9) hold on each level, not because of limits of nonzero terms tending to zero. If this is correct, the spectrum of $-\Delta_2$ would not be discrete, and most likely would be entirely continuous.

For 1-forms, we would expect the Hodge decomposition (11) on the approximate levels to converge to a Hodge decomposition

$$\Lambda_1 = d_0 \Lambda_0 \oplus \delta_2 \Lambda_2 \oplus \mathcal{H}_1 \quad (34)$$

for the appropriate spaces $\Lambda_0, \Lambda_1, \Lambda_2$ of forms on SC, where $\mathcal{H}_1$ denotes the harmonic 1-forms, with

$$-\Delta_1 = \begin{cases} d_0 \delta_1 & \text{on } d_0 \Lambda_0 \text{ or an appropriate subspace} \\ \delta_2 d_1 & \text{on } \delta_2 \Lambda_2 \text{ or an appropriate subspace} \\ 0 & \text{on } \mathcal{H}_1. \end{cases}$$

For example, any L^2 0-form may be written

$$f_0 = \sum_k c_k (f_k)_0$$

where $\{(f_k)_0\}$ is an orthonormal basis of 0-eigenforms with

$$-\Delta_0 (f_k)_0 = \lambda_k (f_k)_0,$$

where

$$\sum_k |c_k|^2 = \|f_0\|_2^2 < \infty. \quad (35)$$

However, to define $-\Delta_0 f_0$ in L^2 via

$$-\Delta_0 f_0 = \sum_k \lambda_k c_k (f_k)_0$$

we need the condition

$$\sum_k \lambda_k^2 |c_k|^2 < \infty, \quad (36)$$

which is stronger than (35).

On the other hand, $\{\lambda_k^{-\frac{1}{2}} d_0 (f_k)_0\}$ is an orthonormal set of 1-forms, since

$$\langle d_0 (f_k)_0, d_0 (f_k)_0 \rangle_1 = \langle (f_k)_0, \delta_1 d_0 (f_k)_0 \rangle_0 = \lambda_k,$$

and this might be considered an orthonormal basis for the $d_0 \Lambda_0$ part of the Hodge decomposition. But if

$$f_1 = \sum_k c_k \lambda_k^{-\frac{1}{2}} d_0 (f_k)_0,$$

then to define $\delta_1 f_1$ via

$$\delta_1 f_1 = \sum_k c_k \lambda_k^{-\frac{1}{2}} \delta_1 d_0 (f_k)_0 = \sum_k c_k \lambda_k^{\frac{1}{2}} (f_k)_0$$

we would require

$$\sum_k |c_k|^2 \lambda_k < \infty,$$

a condition intermediate between (35) and (36). This would also allow us to define

$$-\Delta_1 f_1 = d_0 \delta_1 f_1 = \sum_k c_k \lambda_k^{\frac{1}{2}} d_0(f_k)_0$$

as an L^2 1-form.

The situation for the $\delta_2 \Lambda_2$ part of the Hodge decomposition would be similar, except that we would have a spectral resolution rather than an infinite series of eigenfunctions, and because the operator $-\Delta_2$ is presumably bounded, we would not encounter different conditions to define $\delta_2 f_2$ and $-\Delta_2 f_2$.

What can be said about $\mathcal{H}_1$? The evidence presented in Sect. 3, in particular Table 3, suggests that limits of sequences of 1-forms in $\mathcal{H}_1^{(m)}$ will exist and give us 1-forms in $\mathcal{H}_1$, namely

$$h_k = \lim_{m \rightarrow \infty} h_k^{(m)}$$

where $h_k^{(m)}$ is defined by the cohomology/homology duality (23), keeping the cycle γ_k the same at all large enough levels m . In particular we would have

$$\int_{\gamma_j} h_k = \delta_{jk}$$

as γ_j ranges over all homology generating cycles at all levels. By taking finite linear combinations we would get 1-forms with prescribed integrals over cycles, provided all but a finite number of integrals are zero. A more challenging question is whether there is a reasonable class of harmonic 1-forms expressible as infinite linear combinations of $\{h_k\}$.

The evidence presented in Figs. 9, 10 and 11 and Tables 4 and 5 supports the conjecture that all our 1-forms restrict to measures on line segments. For harmonic 1-forms it seems almost certain that the measures are absolutely continuous. This appears to hold in all cases.

Next we consider MC. The results reported in Sect. 5 strongly support the conjecture that Laplacians $-\tilde{\Delta}_0$ and $-\tilde{\Delta}_2$ exist in the limit, and have the identical spectrum (up to a constant). Unfortunately, we are not able to get a good estimate of the Laplacian renormalization factor r , other than to say it is approximately equal to 6. The definition (31) and (32) for d_0 is the same as for SC, and similarly (33) would presumably define δ_1 with the MC value of r . In this case we would also define d_1 without renormalization,

$$d_1 f_1(e_2^{(k)}) = \lim_{m \rightarrow \infty} d_1^{(m)} f_1^{(m)}(e_2^{(k)}),$$

where $d_1^{(m)}$ is defined by (3), and $f_1^{(m)}$ is the restriction to $E_1^{(m)}$ of f_1 . Indeed the limit in (33) exists trivially because $d_1^{(m)} f_1^{(m)}(e_2^{(k)})$ is constant for $m \geq k$. This follows by

induction from the identity

$$d_1^{(k+1)} f_1^{(k+1)}(e_2^{(k)}) = d_1^{(k)} f_1^{(k)}(e_2^{(k)}). \quad (37)$$

To see why (37) holds we note that $e_2^{(k)}$ splits into 8 $(k+1)$ -cells. Now the edges of these smaller cells come in two varieties, those that partition the edges bounding $e_2^{(k)}$ and pairs of interior edges. But each pair includes opposite orientations, so the sum of $f_1^{(k+1)}$ over the pair is zero. This of course uses the fact that the central edges in MC are identified in pairs, so nothing like this is true for SC.

The renormalization factor for $-\tilde{\Delta}_2$ would then have to be the renormalization factor defining δ_2 ,

$$\delta_2 f_2(e_1^{(k)}) = \lim_{m \rightarrow \infty} r^m \delta_2^{(m)} f_2^{(m)}(e_1^{(k)}). \quad (38)$$

Since $\delta_2^{(m)} f_2^{(m)}(e_1^{(k)})$ involves the difference between the f_2 measure of “thickenings” of the edge on either side, it seems difficult to explain why the limit (38) should exist, or for what class of 2-forms.

It is rather simple to understand how to obtain the Hodge $*$ -operators on 0-forms and 2-forms on MC as renormalized limits of 13 and 14. Indeed we should define

$$*f_0 = \lim_{m \rightarrow \infty} *8^m f_0^{(m)}$$

and

$$*f_2 = \lim_{m \rightarrow \infty} *8^m f_2^{(m)}$$

If f_0 is continuous on MC then $*f_0 = f_0 d\mu$, where μ is the standard measure on MC with $\mu(e_2^{(m)}) = 8^{-m}$ for every m -cell $e_2^{(m)}$, since $*f_2^{(m)}(e_2^{(m)})$ simply averages f_0 over the four vertices of $e_2^{(m)}$. Similarly, if $f_2 = g d\mu$ for some continuous function g , then $*f_2 = g$ because $*f_2^{(m)}(e_0^{(m)})$ averages g over a small neighborhood of $e_0^{(m)}$. In particular we have $**$ equal to the identity in both directions.

It seems plausible that the $*$ -operations should conjugate the two Laplacians, at least up to a constant multiple. Our data strongly supports this conjecture. The eigenvalues match well when multiplied by an experimentally determined constant. We don’t have any intuitive explanation for this particular constant, however. The graphs of the eigenfunctions seem qualitatively similar, and so do the graphs of the $*$ -operators at level m applied to eigenfunctions. Thus we have evidence for Poincaré duality on MC. Since MC is a limit of surfaces without boundary (but with singular points for the geometry) this is perhaps not surprising.

The apparent fact that the Laplacian renormalization factor r is small than 8 (the measure renormalization factor) has important implications. In Euclidean space this property only appears in dimensions higher than two. In particular, if it were possible to define an energy ε on MC so that $-\tilde{\Delta}_0$ is derived from ε and μ by the weak formulation

$$\int (-\tilde{\Delta}_0 f) g d\mu = \mathcal{E}(f, g),$$

then functions of finite energy would not necessarily be continuous. Also the value of α in the Weyl ratio will be greater than one.

For 1-forms on MC we also expect a Hodge decomposition (34). The operator d_0 intertwines $-\tilde{\Delta}_0$ on Λ_0 with $-\tilde{\Delta}_1$ on the $d_0\Lambda_0$ portion of Λ_1 , and similarly δ_2 intertwines $-\tilde{\Delta}_2$ on Λ_2 with $-\tilde{\Delta}_1$ on the $\delta_2\Lambda_2$ portion of Λ_1 . Note that d_0 annihilates constants and δ_2 annihilates μ , so the spectrum of $-\tilde{\Delta}_1$ consists exactly of the nonzero portions of the spectra of $-\tilde{\Delta}_0$ and $-\tilde{\Delta}_2$ together with the 0-eigenspace $\mathcal{H}_1$. Because the eigenvalues of $-\tilde{\Delta}_2$ and $-\tilde{\Delta}_1$ are proportional by a nontrivial factor, we do not see the eigenspaces doubling in multiplicity.

The story for the space of harmonic 1-forms $\mathcal{H}_1$ on MC is almost identical to SC, except the homology is different so the total number arising at each level is different.

A different approach to constructing a Laplacian on functions on MC by using a Peano curve is given in [20].

It should be possible to define a Hodge *-operator from 1-forms to 1-forms as a limit of such operators on the graph approximations. We have not been successful in carrying this through.

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